# A DYNAMICAL EQUIVALENT TO THE EQUILATERAL LIBRATION POINTS OF THE EARTH-MOON SYSTEM

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(Received: 11 April, 1990; accepted: 7 December, 1990)

**Abstract.** Consider the Earth-Moon-particle system as a Restricted Three Body Problem. There are two equilateral libration points. In the actual world system, those points are no longer relative equilibrium points mainly due to the effect of the Sun and to the noncircular motion of the Moon around the Earth. In this paper we present the problem as a perturbation of the RTBP and we look for the dynamical equivalent of $L_{4,5}$. It turns out to be a quasiperiodic orbit. It is obtained for a simplified model but the procedure to obtain it is general and can be carried out with an additional computational effort.

**Key words:** Quasiperiodic perturbations – quasiperiodic solutions – libration points – algebraic manipulators

## 1. Introduction

Relative equilibrium solutions or libration points are well known in Celestial Mechanics. They or nearby orbits can be useful for space missions. However it turns out that the actual solar system is more complex. We can define geometrical libration points. For instance, for the Earth-Moon system we can define points $L_{4,5}$ which belong to the instantaneous plane of motion of the Moon around the Earth and such that the distances to Earth and to Moon are equal to the actual distance from the Moon to the Earth. The effects of the remaining bodies, specially the Sun, and the noncircular (even non elliptical!) motion of the Moon around the Earth, prevent this point to be a relative equilibrium one. Here we write the full problem as a perturbation of the RTBP. We look for a dynamical equivalent of the libration points, that is, for a solution of the equations of motion which has, as basic frequencies, the ones of the perturbing bodies. This can be done in several ways. The more general one consists in taking the Hamiltonian with quasiperiodic time-dependent coefficients, then performing canonical time-dependent transformations ignoring (to some order) the temporal dependence and looking for the equilibrium point of the transformed (autonomous up to some order) Hamiltonian system. Another way consists of looking directly for the solutions as a quasiperiodic function of time with basic frequencies the ones of the coefficients of the equations. The first method is more general allowing to compute, not only the equivalent of the libration points but also the equivalent of the periodic orbits and the tori which are found around $L_{4,5}$ in the spatial RTBP. However we have chosen the second approach which is more direct. In the subsequent sections we present the equations of motion, the simplified model used in this paper, the related expansions, the method used to obtain an approximate quasiperiodic solution and the corresponding manipulator. We end with the results obtained and a comparison against direct numerical integration of the model starting at the same point that the semianalytical solution. The local behaviour around that solution is also discussed showing that it appears to be mildly unstable.

Previous studies concerning this topic can be found in (Tapley, Schultz, 1970) where only numerical simulations are presented. An analytical approach can be found in (Kamel, Breakwell, 1970). Partial results can also be found in (Gómez, Llibre, Martínez, Simó, 1987), work done under ESOC contract 6139/84/D/JS(SC). If we consider only the Elliptic Restricted Three Body Problem for the Earth-Moon system skipping the influence of the Sun there are some results. The earliest ones can be found in (Szebehely, 1967), p. 599.

## 2. Equations of motion

It is known that, taking a system of reference with the origin at the center of masses of the solar system and axes parallel to the ecliptic ones, the equation of the motion are:

$$\ddot{\mathbf{R}} = \sum_{A\in\{S,E,M,P_1,\dots,P_k\}} \frac{GM_A(\mathbf{R}_A - \mathbf{R})}{|\mathbf{R}_A - \mathbf{R}|^3},$$

where $G$ is the gravitational constant, $\mathbf{R}$ the position vector of the particle, $\mathbf{R}_A$ the position vector of the body of mass $M_A$, and $A$ ranges over the Sun, Earth, Moon and the planets. We ignore all the non Newtonian forces and those not coming from the solar system.

However, this system of reference is not the best for this problem. Let us consider instead a system of reference with the origin in one of the instantaneous equilateral points and the axes defined by the unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ given as follows:

$$\mathbf{e}_1 = \frac{\mathbf{r}_{EM}}{|\mathbf{r}_{EM}|}, \quad \mathbf{e}_3 = \frac{\mathbf{r}_{EM} \wedge \dot{\mathbf{r}}_{EM}}{|\mathbf{r}_{EM} \wedge \dot{\mathbf{r}}_{EM}|}, \quad \mathbf{e}_2 = \mathbf{e}_3 \wedge \mathbf{e}_1,$$

where $\mathbf{r}_{EM}(t)$ is the position vector of the Moon with respect to the Earth.

In order to satisfy Kepler's third law we modify the mass of the Earth, considering the remaining mass as a perturbation.

The units of length, time and mass are normalized so that the angular velocity of rotation, the sum of the masses of the Earth and the Moon and $G$ are all equal to one. With this, the distance between the Earth and the Moon is also equal to one. (For a detailed description of the system see (Gómez, Llibre, Martínez, Simó, 1987)).

Now, taking into account the complete solar system, the Lagrangian of the problem and the equations of the motion can be written.

Then, the terms which contain Legendre polynomials (except these coming from the RTBP) are expanded as power series in $x$, $y$, $z$. Its coefficients are known functions of the positions of the bodies of the solar system. For a short time interval it can be assumed that those positions, and therefore the coefficients, are quasiperiodic functions of time.

A computation of these coefficients using a Fourier analysis shows that the relevant frequencies are the ones of the following four angles:

1. The mean longitude of the Moon (equal to 1, because of the choice of the units).
2. The mean longitude of the lunar perigee.
3. The mean longitude of the ascending node of the Moon.
4. The mean elongation of the Sun.

All the contributions with amplitude less than $5 \cdot 10^{-4}$ are dropped in order to keep a manageable number of terms. This leads to the fact that the perturbations coming from the planets, the radiation pressure and the aspherical terms coming from the Earth and Moon can be neglected (for more details about all the process of computation see (Gómez, Llibre, Martínez, Simó, 1987)).

The equations of the motion are

$$\begin{aligned}
\ddot{x} &= P(7)\left[-\frac{x-x_E}{r_{PE}^3}(1-\mu_M) - \frac{x+x_E}{r_{PM}^3}\mu_M - x_E(1-2\mu_M)\right] + P(1) + \\
&\quad + P(2)x + P(3)y + P(4)z + P(5)\dot{x} + P(6)\dot{y}, \\
\ddot{y} &= P(7)\left[-\frac{y-y_E}{r_{PE}^3}(1-\mu_M) - \frac{y-y_E}{r_{PM}^3}\mu_M - y_E\right] + P(8) + P(9)x + \\
&\quad + P(10)y + P(11)z + P(12)\dot{x} + P(13)\dot{y} + P(14)\dot{z}, \\
\ddot{z} &= P(7)\left[-\frac{z}{r_{PE}^3}(1-\mu_M) - \frac{z}{r_{PM}^3}\mu_M\right] + P(15) + P(16)x + P(17)y + \\
&\quad + P(18)z + P(19)\dot{y} + P(20)\dot{z},
\end{aligned}$$

where $r_{PE}$, $r_{PM}$ denote the distances from the particle to the Earth and Moon, respectively, given by $r_{PE}^2 = (x - x_E)^2 + (y - y_E)^2 + z^2$, $r_{PM}^2 =$

$(x + x_E)^2 + (y - y_E)^2 + z^2$. We recall $x_E = -1/2$, $y_E = -\sqrt{3}/2$ for $L_4$ and $x_E = -1/2$, $y_E = \sqrt{3}/2$ for $L_5$. The functions $P(i)$ are defined as

$$P(i) = A_{i,0} + \sum_{j=1}^{m} A_{i,j} \cos\theta_j + \sum_{j=1}^{m} B_{i,j} \sin\theta_j,$$

with $\theta_j = \nu_j t_n + \varphi_j$ and the value $t_n$ denotes the normalized time.

In order to find a proper method to deal with this problem the following simplifications are introduced:

1. The problem is considered planar.
2. Only the two more relevant frequencies are retained, namely
   a) Frequency of the mean elongation of the Sun ($\psi$).
   b) Difference between the frequencies of the mean longitude of the Moon and the mean longitude of the lunar perigee ($M$).

After these simplifications we obtain a system that contains the same basic difficulties as the original one, but it is easier to compute.

The study of the complete system will be done in a further work.

Finally, introducing a new notation, the equations that we have studied are:

$$\begin{aligned} \ddot{x} &= q_0 \left[ -\frac{x - x_E}{r_{PE}^3}(1-\mu) - \frac{x + x_E}{r_{PM}^3}\mu - x_E(1-2\mu) \right] + q_1 + q_2 x + q_3 y + \\ &\quad + q_4 \dot{x} + q_5 \dot{y}, \\ \ddot{y} &= q_0 \left[ -\frac{y - y_E}{r_{PE}^3}(1-\mu) - \frac{y - y_E}{r_{PM}^3}\mu - y_E \right] + q_6 + q_7 y + q_8 \dot{x} + q_9 \dot{y}, \end{aligned} \tag{2.1}$$

where $x_E$, $y_E$, $r_{PE}$ and $r_{PM}$ are defined as above, $q_0, \ldots, q_9$ are functions of time:

$$q_i = \sum_{j=0}^{6} A_{ij} \cos\theta_j + B_{ij} \sin\theta_j,$$

with $\theta_0 = 0$, $\theta_1 = 2\psi - M$, $\theta_2 = \psi$, $\theta_3 = M$, $\theta_4 = 2\psi$, $\theta_5 = 2M$ and $\theta_6 = 2\psi + M$. The values of $\psi$ and $M$ are given by $\psi = 0.9251959855t + 5.0920835091$ and $M = 0.9915452215t + 2.2415337977$. We recall that the origin of time is in the year 2000.0, and that $2\pi$ units of it are equivalent to a sidereal periode of the Moon. The coefficients $A_{ij}$, $B_{ij}$ are given in Table I.

TABLE I
Fourier coefficients of the perturbations.

| $i$ | $j$ | $A_{ij}$ | $B_{ij}$ | $i$ | $j$ | $A_{ij}$ | $B_{ij}$ | $i$ | $j$ | $A_{ij}$ | $B_{ij}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 1.0047 | 0.0000 | 3 | 4 | 0.0000 | -0.0165 | 7 | 1 | 0.0335 | 0.0000 |
| 0 | 1 | 0.0315 | 0.0000 | 3 | 6 | 0.0000 | -0.0018 | 7 | 2 | -0.0009 | 0.0000 |
| 0 | 2 | -0.0008 | 0.0000 | 4 | 1 | 0.0000 | -0.0169 | 7 | 3 | 0.0165 | 0.0000 |
| 0 | 3 | 0.1644 | 0.0000 | 4 | 3 | 0.0000 | -0.1079 | 7 | 4 | 0.0102 | 0.0000 |
| 0 | 4 | 0.0266 | 0.0000 | 4 | 4 | 0.0000 | -0.0295 | 7 | 5 | 0.0135 | 0.0000 |
| 0 | 5 | 0.0134 | 0.0000 | 4 | 5 | 0.0000 | -0.0088 | 7 | 6 | 0.0023 | 0.0000 |
| 0 | 6 | 0.0042 | 0.0000 | 4 | 6 | 0.0000 | -0.0038 | 8 | 0 | -2.0000 | 0.0000 |
| 1 | 0 | 0.0014 | 0.0000 | 5 | 0 | 2.0000 | 0.0000 | 8 | 1 | -0.0382 | 0.0000 |
| 1 | 1 | 0.0000 | 0.0016 | 5 | 1 | 0.0382 | 0.0000 | 8 | 2 | 0.0011 | 0.0000 |
| 1 | 4 | 0.0000 | -0.0142 | 5 | 2 | -0.0011 | 0.0000 | 8 | 3 | -0.2176 | 0.0000 |
| 1 | 6 | 0.0000 | -0.0016 | 5 | 3 | 0.2176 | 0.0000 | 8 | 4 | -0.0429 | 0.0000 |
| 2 | 0 | 1.0076 | 0.0000 | 5 | 4 | 0.0429 | 0.0000 | 8 | 5 | -0.0148 | 0.0000 |
| 2 | 1 | 0.0315 | 0.0000 | 5 | 5 | 0.0148 | 0.0000 | 8 | 6 | -0.0053 | 0.0000 |
| 2 | 2 | -0.0008 | 0.0000 | 5 | 6 | 0.0053 | 0.0000 | 9 | 1 | 0.0000 | -0.0164 |
| 2 | 3 | 0.1650 | 0.0000 | 6 | 0 | 0.0025 | 0.0000 | 9 | 2 | 0.0000 | 0.0005 |
| 2 | 4 | 0.0266 | 0.0000 | 6 | 1 | 0.0017 | 0.0000 | 9 | 3 | 0.0000 | -0.1079 |
| 2 | 5 | 0.0135 | 0.0000 | 6 | 4 | -0.0143 | 0.0000 | 9 | 4 | 0.0000 | -0.0295 |
| 2 | 6 | 0.0043 | 0.0000 | 6 | 6 | -0.0016 | 0.0000 | 9 | 5 | 0.0000 | -0.0088 |
| 3 | 1 | 0.0000 | 0.0018 | 7 | 0 | 1.0076 | 0.0000 | 9 | 6 | 0.0000 | -0.0038 |

## 3. Expansion of the equations

We consider the potential $V = \frac{1-\mu}{r_{PE}} + \frac{\mu}{r_{PM}}$ and let $\varphi$ be the angle between the vectors $(x_E, y_E)$ and $(x, y)$. We denote by $\rho^2$ the term $x^2 + y^2$. Then

$$\frac{1}{r_{PE}} = \frac{1}{\sqrt{1 - 2\rho\cos\varphi + \rho^2}} = \sum_{n=0}^{\infty} \rho^n P_n(\cos\varphi) \tag{3.1}$$

where $P_n$ is the Legendre polinomial of degree $n$:

$$P_n(\omega) = \frac{1}{2^n n!}\frac{d^n}{d\omega^n}(\omega^2 - 1)^n = \sum_{k=0}^{[\frac{n}{2}]} \alpha_{n,k}\omega^{n-2k}.$$

Here $[\frac{n}{2}]$ denotes the integer part of $\frac{n}{2}$ and

$$\alpha_{n,k} = \frac{(-1)^k}{2^n n!}\binom{n}{k}\frac{(2n-2k)!}{(n-2k)!}.$$

Substituting these expressions in (3.1) one obtains

$$\begin{aligned} \frac{1}{r_{PE}} &= \sum_{n=0}^{\infty} \rho^n \sum_{n=0}^{[\frac{n}{2}]} \alpha_{n,k} \cos^{n-2k} \varphi = \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{[\frac{n}{2}]} \alpha_{n,k} (x x_E + y y_E)^{n-2k} (x^2 + y^2)^k. \end{aligned} \tag{3.2}$$

Finally, in order to obtain the expansion for $L_4$ (the $L_5$ case can be done with an identical process) we put $x_E = -\frac{1}{2}$, $y_E = -\frac{\sqrt{3}}{2}$ and (3.2) becomes

$$\frac{1}{r_{PE}} = \sum_{n=0}^{\infty} \sum_{k=0}^{[\frac{n}{2}]} \beta_{n,k} (x + \sqrt{3} y)^{n-2k} (x^2 + y^2)^k, \tag{3.3}$$

where

$$\beta_{n,k} = \frac{(-1)^{n-k} (2n - 2k)!}{4^{n-k} (n-k)! k! (n-2k)!}.$$

To perform the expansion (3.3) through the construction of an algebraic manipulator it is better to change the order of the sums:

$$\frac{1}{r_{PE}} = \sum_{k=0}^{[\frac{N}{2}]} (x^2 + y^2)^k \sum_{n=2k}^{N} \beta_{n,k} (x + \sqrt{3} y)^{n-2k},$$

where $N$ denotes the expansion order wanted. With a similar computation it is obtained that

$$\frac{1}{r_{PM}} = \sum_{k=0}^{[\frac{N}{2}]} (x^2 + y^2)^k \sum_{n=2k}^{N} \beta_{n,k} (-x + \sqrt{3} y)^{n-2k}$$

and, finally,

$$\begin{aligned} V &= \frac{1-\mu}{r_{PE}} + \frac{\mu}{r_{PM}} = \\ &= \sum_{k=0}^{[\frac{N}{2}]} (x^2 + y^2)^k \sum_{n=2k}^{N} \beta_{n,k} \left[ (1-\mu)(x + \sqrt{3} y)^{n-2k} + \mu(-x + \sqrt{3} y)^{n-2k} \right]. \end{aligned}$$

Now, using a program which takes advantage of the particularities of the latter expression, the expansions of $\frac{\partial V}{\partial x}$ and $\frac{\partial V}{\partial y}$ are obtained. Thereafter, the

TABLE II
Coefficients of the expansions in (3.4)

| $i$ | $j$ | $\bar{\alpha}_{ij}$ | $\bar{\beta}_{ij}$ |
|---|---|---|---|
| 0 | 0 | 0.00000000D+00 | 0.00000000D+00 |
| 1 | 0 | −0.25000000D+00 | 0.12674707D+01 |
| 0 | 1 | 0.12674707D+01 | 0.12500000D+01 |
| 2 | 0 | 0.12806055D+01 | −0.32475953D+00 |
| 1 | 1 | −0.64951905D+00 | −0.40247600D+01 |
| 0 | 2 | −0.20123800D+01 | −0.97427858D+00 |
| 3 | 0 | −0.11562500D+01 | −0.13202820D+01 |
| 2 | 1 | −0.39608460D+01 | 0.38437500D+01 |
| 1 | 2 | 0.38437500D+01 | 0.71295227D+01 |
| 0 | 3 | 0.23765076D+01 | 0.93750000D−01 |
| 4 | 0 | −0.43830247D+00 | 0.19282597D+01 |
| 3 | 1 | 0.77130388D+01 | 0.32777402D+01 |
| 2 | 2 | 0.49166103D+01 | −0.14005255D+02 |
| 1 | 3 | −0.93368364D+01 | −0.84611433D+01 |
| 0 | 4 | −0.21152858D+01 | 0.11163609D+01 |

differential equations have been expanded, so that (2.1) becomes

$$\begin{aligned} \ddot{x} &= q_0 \left[ \sum_{i+j\le L} \bar{\alpha}_{ij} x^i y^j \right] + q_1 + q_2 x + q_3 y + q_4 \dot{x} + q_5 \dot{y} \\ \ddot{y} &= q_0 \left[ \sum_{i+j\le L} \bar{\beta}_{ij} x^i y^j \right] + q_6 + q_7 x + q_8 \dot{x} + q_9 \dot{y} \end{aligned} \tag{3.4}$$

where the coefficients $\bar{\alpha}_{ij}$ and $\bar{\beta}_{ij}$ are given in Table II. From now on we shall take $L = 4$ and, as we shall see later, this will be enough to have an accuracy similar to the accuracy of the perturbations.

## 4. The method

We want to find a quasiperiodic solution of the preceeding system, i.e., a solution expressed in terms of cosine and sine of angles depending on $\psi$ and $M$ in the following form:

$$\begin{aligned} x &= \sum_{j,k=-\infty}^{\infty} x_1(j,k)\cos(j\psi + kM) + x_2(j,k)\sin(j\psi + kM), \\ y &= \sum_{j,k=-\infty}^{\infty} y_1(j,k)\cos(j\psi + kM) + y_2(j,k)\sin(j\psi + kM). \end{aligned} \tag{4.1}$$

The problem is now to find the coefficients $x_1$, $x_2$, $y_1$, $y_2$ of these series that satisfy the equation of the motion.

Our approach to the problem is semianalytical, this is, we shall not find exactly these coefficients but some numerical aproximations of them.

The method that we have used is essentially to substitute expresions (4.1) of $x$ and $y$ in the equations (3.4), make the computation of the operations analytically (by means of an algebraic manipulator) and then solve the resultant system numerically.

The first problem of this method is to find a good expression of the series $x$, $y$ that allows us to reduce the number of coefficients to a manageable one without loosing significant information.

The biggests coefficients of these series are always located near the diagonal $(n\psi, -nM)$ and that suggested us to use the following expressions for $x$ and $y$:

$$\begin{aligned}
x &= x(1) + \sum_{i=1}^{N_M}(x(2i)\cos(iM) + x(2i+1)\sin(iM)) + \\
&\quad + \sum_{j=1}^{N_\psi}\Big( \sum_{i=-N_M-j}^{N_M-j} (x(4j(N_M+1)+2i)\cos(j\psi+iM) + \\
&\quad + x(4j(N_M+1)+2i+1)\sin(j\psi+iM))\Big), \\
y &= y(1) + \sum_{i=1}^{N_M}(y(2i)\cos(iM) + y(2i+1)\sin(iM)) + \\
&\quad + \sum_{j=1}^{N_\psi}\Big( \sum_{i=-N_M-j}^{N_M-j} (y(4j(N_M+1)+2i)\cos(j\psi+iM) + \\
&\quad + y(4j(N_M+1)+2i+1)\sin(j\psi+iM))\Big),
\end{aligned} \tag{4.2}$$

where $N_M$ and $N_\psi$ are some values that we can fix arbitrarily, and they represent, respectively the "dispersion" allowed for the coefficients from the diagonal and the longitude of it.

Now, the problem can be reduced to the search of the solution of the following system

$$\left.\begin{array}{l} G_1(x,y) = f_1(x,y,\dot{x},\dot{y}) - \ddot{x} = 0 \\ G_2(x,y) = f_2(x,y,\dot{x},\dot{y}) - \ddot{y} = 0 \end{array}\right\}, \tag{4.3}$$

where

$$f_1(x,y,\dot{x},\dot{y}) = q_0\left[\sum_{i+j\le L} \bar{\alpha}_{ij}x^iy^j\right] + q_1 + q_2x + q_3y + q_4\dot{x} + q_5\dot{y},$$

$$f_2(x,y,\dot{x},\dot{y}) = q_0 \left[ \sum_{i+j\leq L} \bar{\beta}_{ij} x^i y^j \right] + q_6 + q_7 x + q_8 \dot{x} + q_9 \dot{y}.$$

Because $x$ and $y$ are considered as a function of its coefficients, we can see our problem as the search of the zero of a function:

$$\begin{aligned} F: \mathbf{R}^{2p} &\longrightarrow \mathbf{R}^{2p} \\ (x(1),\ldots,x(p),y(1),\ldots,y(p)) &\longmapsto (f_1 - \ddot{x}, f_2 - \ddot{y}) \end{aligned} \tag{4.4}$$

where $p = (2N_\psi + 1)(2N_M + 1)$ and in the computation of $f_1$ and $f_2$ we keep the same kind of terms used for $x$, $y$.

The numerical method that we have used to solve this equation is a Newton continuation method.

Thus, the general scheme of the method is the following:

We consider a $m$-th approximation of the system (4.4), $F^{(m)}$. Then, we compute the Jacobian of the system as usual, using the following expressions

$$\begin{aligned} \frac{\partial F_k^{(m)}}{\partial x(j)} &= \frac{\partial F_k^{(m)}}{\partial x} \cdot \frac{\partial x}{\partial x(j)} + \frac{\partial F_k^{(m)}}{\partial \dot{x}} \cdot \frac{\partial \dot{x}}{\partial x(j)} + \frac{\partial F_k^{(m)}}{\partial \ddot{x}} \cdot \frac{\partial \ddot{x}}{\partial x(j)}, \\ \frac{\partial F_k^{(m)}}{\partial y(j)} &= \frac{\partial F_k^{(m)}}{\partial y} \cdot \frac{\partial y}{\partial y(j)} + \frac{\partial F_k^{(m)}}{\partial \dot{y}} \cdot \frac{\partial \dot{y}}{\partial y(j)} + \frac{\partial F_k^{(m)}}{\partial \ddot{y}} \cdot \frac{\partial \ddot{y}}{\partial y(j)}. \end{aligned}$$

The relationship between $\psi$, $M$ and $t$ and (4.2) allows finally to write the corresponding expressions for $\frac{\partial \dot{x}}{\partial x(j)}, \frac{\partial \ddot{x}}{\partial x(j)}, \frac{\partial \dot{y}}{\partial y(j)}, \frac{\partial \ddot{y}}{\partial y(j)}$, in each case.

Actually, we took a first degree aproximation to the system, and the solution of this system is found in one iteration. The following step consisted in adding the next order terms. By means of a continuation method, using as the initial condition of every iteration the solution obtained in the preceeding one it has been possible to get the solution when quadratic terms were included in the equation. The third degree terms and the quartic ones could be added without needing of continuation, and finally we have obtained a solution of the fourth degree aproximation to the system that is also a good aproximation to the solution of the system. We shall see this further on.

Before finishing this point we must make an important remark. Using the method explained below, the matrix of the system we must solve in every step was of $2p \times 2p$. Because the final values for $N_\psi$ and $N_M$ are, respectively, 70 and 30 and not all the coefficients are significant, we need to solve a smaller system containing only the most significant coefficients.

The central point of this is how to choose these coefficients. After each iteration we have two series, $\tilde{x}$, $\tilde{y}$, which are approximations to the solutions of the system. If we substitute now these series $\tilde{x}$, $\tilde{y}$ in $F^{(m)}$ we obtain a series $F^{(m)}(\tilde{x},\tilde{y})$ from which we can choose its biggests coefficients and add them to the ones that take part into the Newton method. In this way the list of elements to keep for $x$ and $y$ is modified on line as required.

It is important to note that all the operations involved in the computation of the function have been made with all the coefficients, and it is only in the resolution of the linear system where we use the selected ones. This is because we can not choose, a priori, which ones are the most significant and, on the other hand, the operations involved in the function (such as product of series) produced many new components. If we had worked only with a small number of coefficients in all the operations we would have lost much information. With this method all the operations are done to lose as little information as possible. Using the results above, we corrected the coefficients dynamically and accurately.

## 5. The manipulator

In order to do all the computations in an easy way we have implemented an algebraic manipulator that can deal with Fourier series of the type explained below. This manipulator has to do four main operations: product and derivatives (of first and second orders) of the series and conversions between series with selected coefficients and series with all the coefficients, in both ways.

These four operations have been increased with two more: the square of a series and the product of a series by an unitary series (only one coefficient different from 0). These last operations were not necessary, but they must be done very often and, in order to optimize the computations, we built them.

The nucleus of the manipulator consists in two routines that allow us to separate the operations explained before from the actual disposition of the terms inside the vectors that we used to store them. These routines are, in fact, the only ones that know this disposition. One of these routines returns, given an index of the vector, the coefficient of $\psi$ and $M$ and if the component is either a sine or a cosine. The other routine returns the index of the vector of a $\sin(j, \psi + kM)$ or $\cos(j\psi + kM)$.

The other routines of the manipulator simply apply the formulae for each operation and they call these routines to know the information about the computation of the vectors they are working with.

There is another routine of the manipulator which compares all the components of the vector with a threshold value, dropping all of them that are less than this tolerance. This routine is called after each operation of the manipulator and it allows us to pay attention only to the meaningful components, preventing the program from an uncontrolled growth of the vectors.

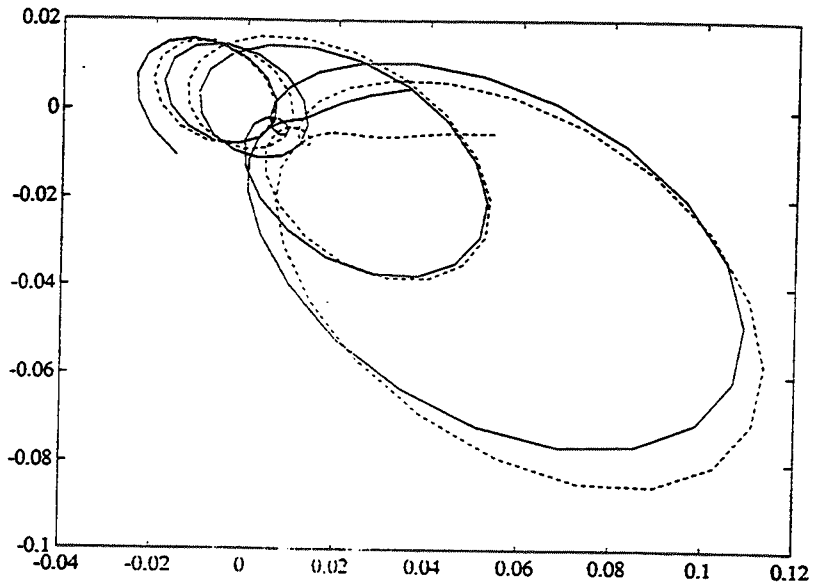


Fig. 1. Continuous line: semianalytical solution of the linearized system. Dotted line: numerical solution of (2.1) with the same initial conditions as the semianalytical one. Initial epoch: year 2000.0. Time interval: 90 days. Projections on the $(x, y)$ plane with normalized units (1 unit = distance Earth-Moon at the epoch). A polygonal is plotted with a time interval of 1 day between consecutive vertices.

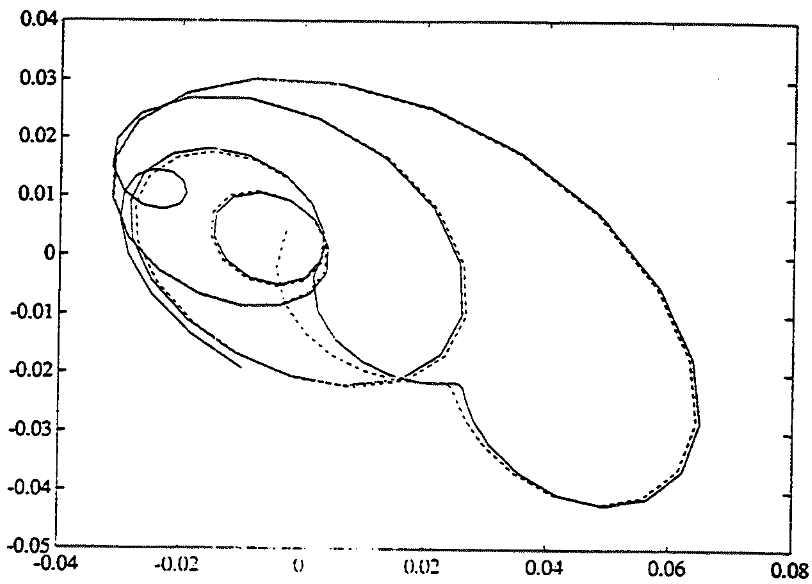


Fig. 2. Same as Figure 1 but the semianalytical solution corresponds to the system up to second order.

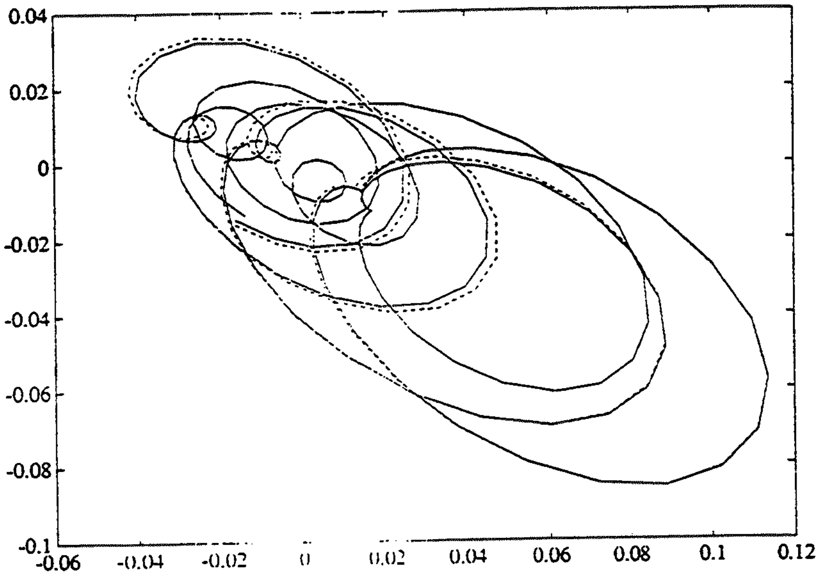


Fig. 3. Same as Fig. 1 but the semianalytical solution corresponds to the system up to third order. Time interval: 192 days.

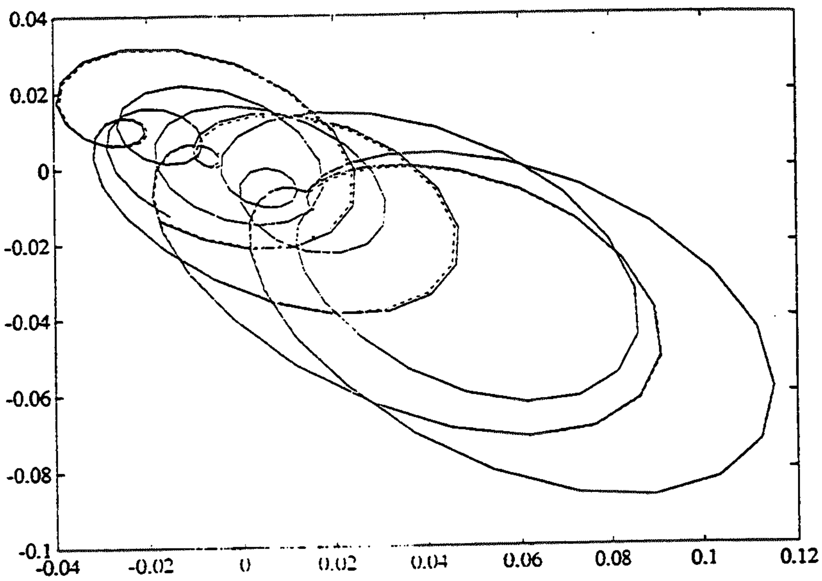


Fig. 4. Same as Fig. 1 but the semianalytical solution corresponds to the system up to fourth order. Time interval: 192 days.

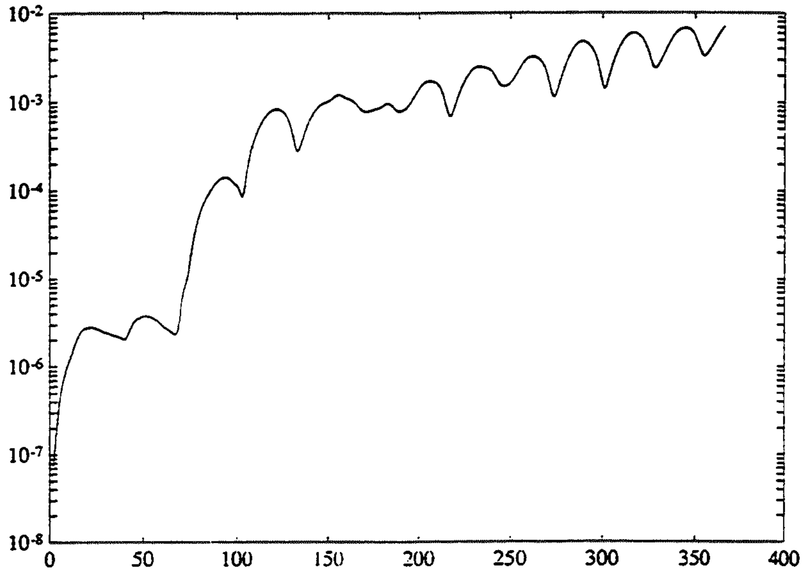


Fig. 5. Euclidean norm of the error between the solutions plotted in Fig. 4. in the phase space versus time in days.

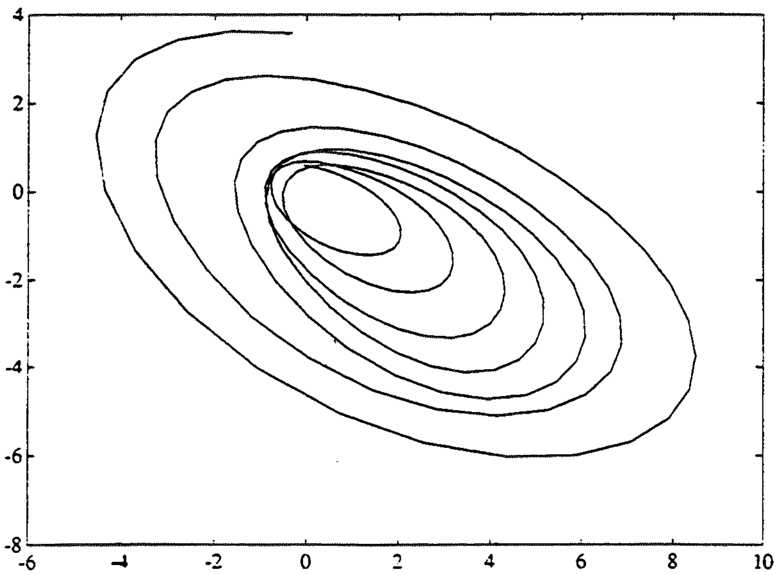


Fig. 6. Projection on the $(x, y)$ plane of the unstable eigenvector of the pseudo-monodromy matrix for the time span of Fig. 4. The variational equations associated to (2.1) and the initial conditions of Fig. 4 have been used. The initial vector is unitary (in the phase space).

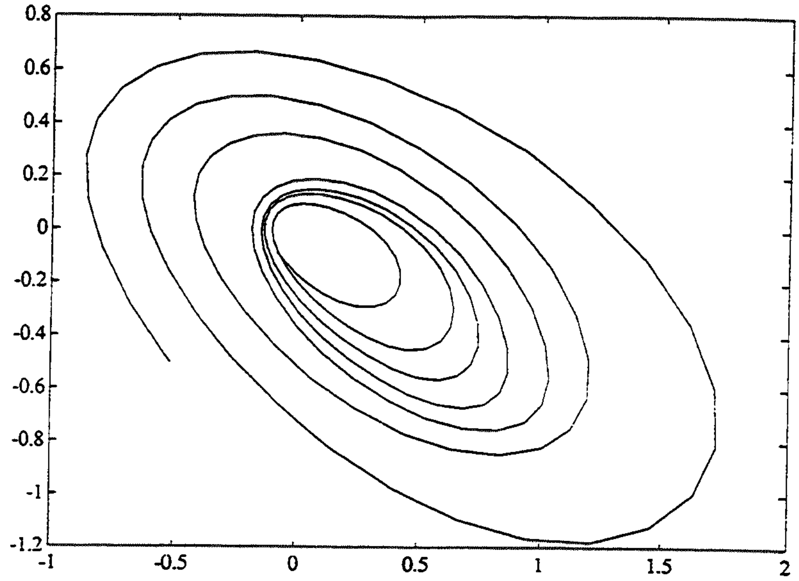


Fig. 7. Same as Fig. 6 but for the stable eigenvector.

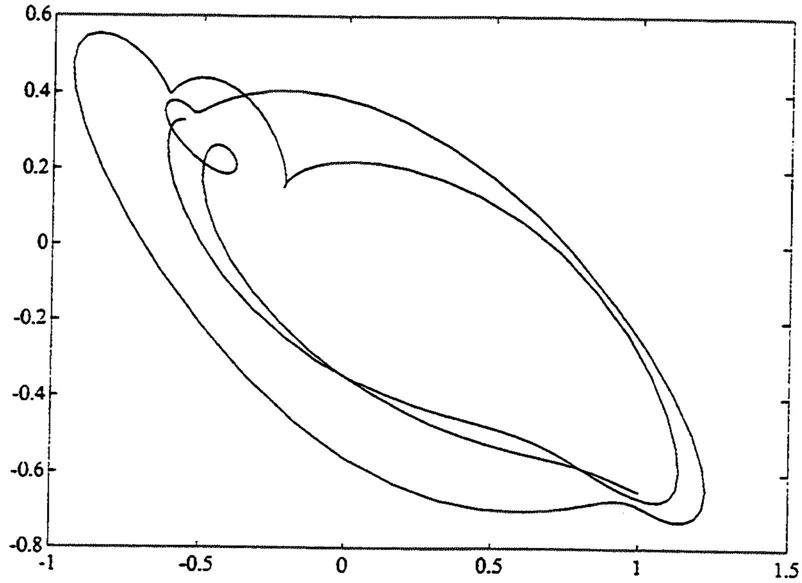


Fig. 8. Same as Fig. 6 but for the real part of the eigenvector with complex eigenvalue (of modulus close to 1).

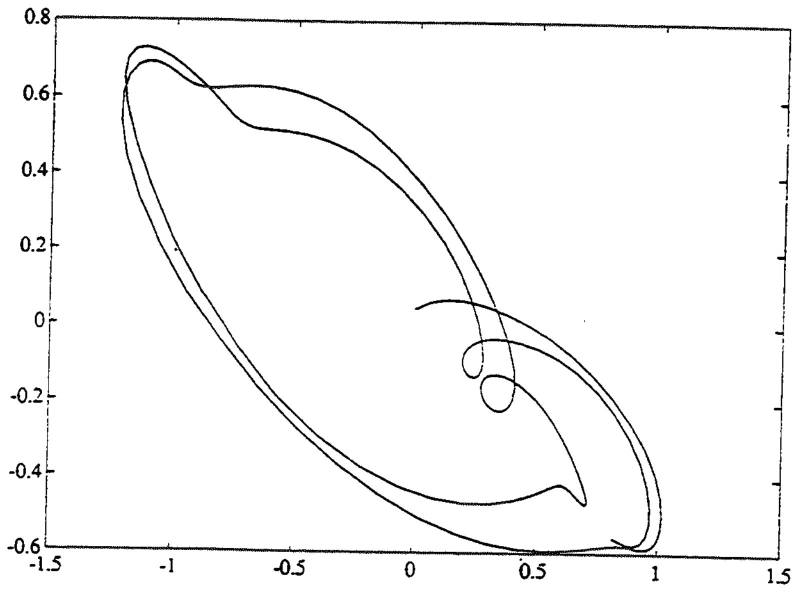


Fig. 9. Same as Fig. 8 but for the imaginary part.

## 6. Results

In this part the results obtained with these computations are presented. The dominant coefficients of the fourth order solution are given in Table III.

First we check the quality of the linear approximation. We consider the semianalytical solution of the linearized system and we take the value of this solution at $t = 0$ as an initial condition for numerical integration of the model equations (2.1). This is shown on Figure 1. The same process is made for the semianalytical solutions of the equations of 2nd, 3th and 4th order and the results are shown in Figures 2, 3 and 4. These plots show that the solution of the linear and quadratic systems are not good approximations to the numerical solution of the model. For this reason we consider the euclidean norm of the difference between numerical and semianalytical solutions. This allows to see that, in the quartic case, we have a degree of accuracy similar to the accuracy of the perturbations. Plots of the error in this case can be seen in Figure 5. Note that the error shows an exponential behaviour due to the existence of an unstable direction.

To compute the local behaviour associated to this orbit we observe that, after 192 days, the solution passes near (in the four-dimensional phase space) the initial conditions and it suggests to compute a pseudo-monodromy matrix, obtain its eigenvalues (shown in Table IV) and use the corresponding eigenvectors to get a linear approximation to the invariant manifolds. Figures 6 to 9 represent, in linear approximation, the behaviour inside each one of these manifolds. These results seem to show the existence of three invari-

TABLE III

Fourier coefficients of the quasiperiodic final solution which satisfy that $\sqrt{x(i,j,k)^2 + y(i,j,k)^2} \geq 5 \cdot 10^{-4}$. The values of $i$ and $j$ are the coefficients of $\psi$ and $M$ (see section 2), and $k$ indicates if it refers to a cosine or a sine (0 = cosine, 1 = sine).

| $i$ | $j$ | $k$ | $x(i,j,k)$ | $y(i,j,k)$ |
|---|---|---|---|---|
| 0 | 0 | 0 | 0.115257D−01 | −0.877293D−02 |
| 0 | 1 | 0 | 0.226069D−01 | −0.132180D−01 |
| 0 | 1 | 1 | −0.529571D−02 | −0.117557D−01 |
| 0 | 2 | 0 | 0.165964D−02 | −0.121897D−02 |
| 0 | 2 | 1 | −0.322429D−03 | −0.114187D−02 |
| 2 | −4 | 0 | −0.571189D−03 | 0.595989D−03 |
| 2 | −4 | 1 | −0.335783D−03 | −0.421593D−03 |
| 2 | −3 | 0 | −0.795810D−02 | 0.600651D−02 |
| 2 | −3 | 1 | −0.362805D−02 | −0.425411D−02 |
| 2 | −2 | 0 | −0.294875D−01 | 0.162597D−01 |
| 2 | −2 | 1 | 0.989340D−02 | −0.984482D−02 |
| 2 | −1 | 0 | −0.285269D−02 | 0.133790D−01 |
| 2 | −1 | 1 | 0.200583D−01 | −0.760429D−02 |
| 2 | 0 | 0 | −0.191681D−02 | 0.148359D−01 |
| 2 | 0 | 1 | 0.174897D−01 | −0.314588D−02 |
| 2 | 1 | 0 | 0.137323D−04 | 0.135286D−02 |
| 2 | 1 | 1 | 0.141434D−02 | −0.435312D−03 |
| 4 | −5 | 0 | 0.526449D−03 | −0.239412D−04 |
| 4 | −5 | 1 | −0.424884D−03 | 0.689134D−03 |
| 4 | −4 | 0 | 0.132698D−02 | −0.226398D−03 |
| 4 | −4 | 1 | −0.359551D−02 | 0.273513D−02 |
| 4 | −3 | 0 | 0.728474D−03 | −0.335751D−03 |
| 4 | −2 | 1 | 0.563396D−03 | −0.136330D−03 |

ant manifolds associated with this orbit. One of them is slightly unstable and gives this nature to the orbit, another one is stable and the last one looks like a central manifold.

## 7. Conclusion

It has been shown that the equilateral libration points of the Earth-Moon system using the RTBP have a dynamical equivalent in the real problem close to a quasiperiodic solution which reaches a distance from the instantaneous geometrical libration point of 0.134 nondimensional units ($\approx 5 \cdot 10^4$ Km.). This is the maximum distance using the coefficients of the fourth order solution and a time span of 100 years (from 1950.0 to 2050.0). The orbit is mildly unstable, the errors increasing, in a exponential way, by less than 1% per day. To keep a spacecraft close to this orbit station keeping would

TABLE IV
Eigenvalues of the pseudo-monodromy matrix

| | | |
|---|---|---|
| −0.562891D+00 | + | 0.821943D+00$\sqrt{-1}$ |
| −0.562891D+00 | − | 0.821943D+00$\sqrt{-1}$ |
| 0.607630D+01 | | |
| 0.170180D+00 | | |

be necessary but only small manoeuvres, once every year would be required.

## Acknowledgements

The work of the second author has been partially supported by a GPC fund from MEC (Spain) and the work of the third author has been partially supported by a CICYT Grant PB86-0527 (Spain).

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