ON THE TRIANGULAR POINTS OF THE SUN-JUPITER SYSTEM

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Abstract. We focus on the dynamics of a small particle near the Lagrangian points of the Sun-Jupiter system. To try to account for the effect of Saturn, we develop a specific model (a restricted four body problem) based on the computation of a true solution of the planar three-body problem for Sun, Jupiter and Saturn. Then, we study the dynamics of this model near the triangular points. The tools are based on computing, up to high order, suitable normal forms and first integrals.

1. Introduction

It is a known fact that Trojan asteroids move in a neighbourhood of the triangular points of the Sun-Jupiter system. Our final goal is to study the dynamical properties of their orbits. In order to achieve this, we are going to construct a new model, more sophisticated than the Sun-Jupiter RTBP, that tries to be closer to the real system.

Before discussing the model, let us explain an easy numerical experiment. We integrate the orbit of 588-Achilles using the JPL Ephemeris in two cases: In the first one, we assume that the gravitational forces coming from all the planets are acting on the asteroid, and in the second one, we consider only the actions of Sun and Jupiter on the asteroid, while these two (main) bodies are moving according to JPL Ephemeris model. The result of the integration in a short time interval is shown in Figure 1. From this plot, we can see that what really matters in order to study the dynamics of the Trojan asteroids is the Sun-Jupiter relative motion. Thus, if we want to build models to study the Trojans, we have to try to simulate in a more realistic way this relative motion.

Then, a natural improvement to the Sun-Jupiter RTBP is to include the effect of Saturn on the motion of Sun and Jupiter. We develop a model where the Sun, Jupiter and Saturn move in a periodic solution of the (full) planar three body problem, with a (relative) period close to the real one. Then, it is possible to write the equations of motion of a fourth massless particle that moves under the attraction of those three. This is a restricted four body problem and we have call it Bicircular Coherent Problem (BCCP, for short). This talk is devoted to study the triangular points of the RTBP using this model. The details can be found in (Gabern and Jorba, 2001).

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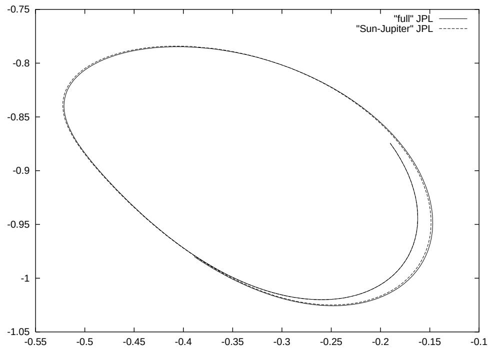

Figure 1. Projection in the (x,y)-plane of the 588-Achilles orbit in the JPL Ephemeris model, when all the forces act on the asteroid (continous line) and when only the direct actions of the Sun and Jupiter on the asteroid are considered (dashed line).

2. The BCCP model

It is possible to find, in a rotating reference frame, periodic solutions of the planar three body Sun-Jupiter-Saturn problem by means of a continuation method using the masses of the planets as parameters (Gabern and Jorba, 2001). The relative Jupiter-Saturn period can be chosen as the actual one, and its related frequency is $\omega_{sat} = 0.597039074021947$.

Assuming that these three main bodies move in this periodic orbit, it is possible to write the Hamiltonian for the motion of a fourth massless particle as:

$$\begin{aligned} H &= \frac{1}{2}\alpha_1(\theta)(p_x^2+p_y^2+p_z^2)+\alpha_2(\theta)(xp_x+yp_y+zp_z)+\alpha_3(\theta)(yp_x-xp_y) \\ &\quad +\alpha_4(\theta)x+\alpha_5(\theta)y-\alpha_6(\theta)\left[\frac{1-\mu}{q_S}+\frac{\mu}{q_J}+\frac{m_{sat}}{q_{sat}}\right], \end{aligned} \tag{1}$$

where $q_S^2=(x-\mu)^2+y^2+z^2$, $q_J^2=(x-\mu+1)^2+y^2+z^2$ and $q_{sat}^2=(x-\alpha_7(\theta))^2+(y-\alpha_8(\theta))^2+z^2$. The functions $\alpha_i(\theta)$ are periodic functions in $\theta=\omega_{sat}t$ and can be computed with a Fourier analysis of the periodic solution of the three body problem.

At that point, we want to mention that a Bicircular Coherent problem was already developed by Andreu (1998) for the Earth-Moon-Sun case to study the dynamics near the Eulerian points.

3. Diffusion in a neighbourhood of the triangular points

It is very interesting to determine zones of effective stability (that is, stability for very long time spans) around the triangular points. We will use the BCCP model and different techniques to estimate these zones. The tools used are direct numerical simulation, normal forms and first integrals. It is enough to focus on the study of a single Lagrangian point, L_5 for instance, because of a particular symmetry of the Hamiltonian (1).

In Figure 2 (top left), we can see the (x,y) projection of a region around L_5, computed by a simple numerical integration of the BCCP. The time to escape from this region is larger than 1 Myr.

3.1. Semi-analytical local study around L_5

In the BCCP system, the RTBP L_5 point is replaced by a periodic orbit. The linear dynamics of this orbit is totally elliptic.

In order to make a local study around the periodic orbit that replaces L_5, we write the Hamiltonian (1) in a more convenient way by means of a composition of three linear changes of variables: a periodic translation (to see the periodic orbit as a fixed point), a symplectic Floquet transformation (to remove the linear time-dependence) and a complexification (to diagonalize the second degree of the Hamiltonian); and expanding it in Fourier-Taylor series. The real linear behaviour is given by $H_2 = \frac{1}{2}\omega_1(x_1^2+y_1^2)+\frac{1}{2}\omega_2(x_2^2+y_2^2)+\frac{1}{2}\omega_3(x_3^2+y_3^2)$ where the frequencies are $\omega_1 = -0.08047340341466$, $\omega_2 = 0.99668687782956$ and $\omega_3 = 1.00006744139040$. Thus, it is possible to write the expanded Hamiltonian (in complex variables) as:

$$H(q,p,\theta,p_\theta) = \omega_{sat}\, p_\theta + H_2(q,p) + \sum_{n\geq 3} H_n(q,p,\theta) \tag{2}$$

3.1.1. *Truncated normal form*

Using the Lie series method (the computer implementation has been done in C++, using the methods explained by Jorba (1999)), we transform the expanded Hamiltonian to a truncated normal form up to degree 16 in the (q,p) variables:

$$H = \omega_{sat}\, p_\theta + \sum_{n=2}^{16} H_n(qp) + \sum_{n\geq 17} H_n(q,p,\theta)$$

Bounding the norm of the remainder, it is possible to determine a zone of effective stability (for a time span of $\sim$ 5000 Myrs.) around the periodic orbit. The x-y projection of a slice for $t = 0$ of this zone is plotted in Figure 2 (top right).

3.1.2. *Approximate first integrals*

Given the expanded Hamiltonian (2) we look for a function

$$F(q,p,\theta) = \sum_{n\geq 2} F_n(q,p,\theta)$$

such that $\{H,F\} = 0$. This equation gives a recursive way of computing F. It is solved up to order 16 and F_2 is chosen to be $F_2 = iq_1p_1 + iq_2p_2 + iq_3p_3$.

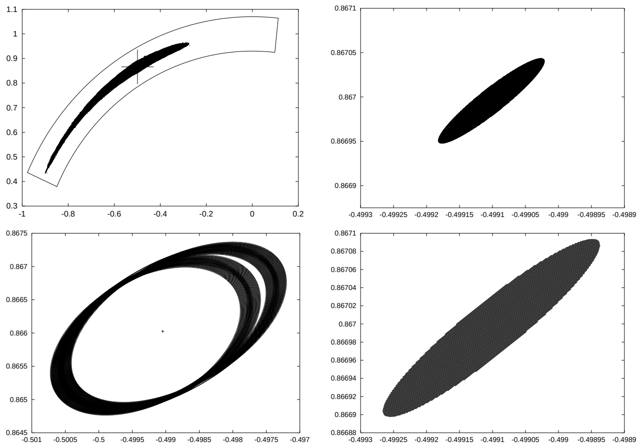

Figure 2. Top left: Numerical estimation of the stability region for the BCCP, for $t = 0$. Top right: Region of effective stability for $t = 0$ from the normal form computation. Bottom left: Zone of effective stability around the periodic orbit computed with the first integral. Bottom right: A slice for $t = 0$ of the previous plot. The axis are the x and y synodical coordinates of the RTBP for all the plots.

Estimating the norm of $\dot{F}$ (that is, the part of $\{H,F\}$ that is not exactly zero), we can also found (see Celletti and Giorgilli, 1991) a region of effective stability around the periodic orbit. The x-y projection of this zone and a slice of it for $t = 0$ are plotted in Figure 2 (bottom right and left, respectively).

4. Conclusions

We have constructed a model that tries to account the effect of Saturn on the Sun-Jupiter system and using semi-analytical tools we have studied the effective stability around the triangular points. The study of the normal form of the Hamiltonian gives us a smaller zone of stability than the first integral, but it can provide much more information of the dynamics around those points.

References

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