STABILITY AND ASYMPTOTIC BEHAVIOUR OF THE VERTICAL FAMILY OF PERIODIC ORBITS AROUND $\mathbf{L_5}$ OF THE RESTRICTED THREE BODY PROBLEM

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Abstract

In this note we present some numerical results about a family of periodic orbits of the Restricted Three Body Problem (RTBP). The family considered is one of the Lyapunov families related to the equlibrium point L_5. More concretely, we deal with the family related to the vertical oscillations around this point.

Here we present a study of the normal behaviour of this family for several values of the mass parameter μ. We focus on the case in which μ tends to zero (note that $\mu = 0$ is a degenerate case), and we identify the orbits for $\mu = 0$ (they are Keplerian orbits around the primary) that give rise to the vertical family when $\mu \neq 0$.

The vertical family

Let us consider an infinitesimal particle moving under the gravitational attraction of two bodies (called primaries), that are revolving in circular orbits around their centre of masses. Let us take a rotating system of reference such that the origin is at the centre of masses. The coordinates (x, y) are taken into the plane of motion of the primaries, with the x axis defined by the line containing the primaries, and oriented from the smaller primary to the biggest one, and the orthogonal y axis is oriented by the sense of rotation of the primaries. The z axis is taken as the orthogonal direction to this plane, oriented by the angular momentum of the rotating bodies. This (non-inertial) rotating frame is called the synodic system of reference. In order to simplify the equations of motion of the infinitesimal particle, it is usual to take normalized units of time, mass and distance, such that the sum of the masses of the primaries, the distance between them, and the gravitational constant are all equal to one. With these units, the rotating period of the primaries is 2π. Let us call μ to the mass of the smaller primary (the mass of the bigger primary is then $1-\mu$), $0 < \mu \leq 1/2$. The value μ is called the mass parameter of the system. Hence, the primaries are located at the points $(\mu - 1, 0, 0)$ and $(\mu, 0, 0)$, and the motion of the third particle in the gravitational vector field generated by the primaries is described by the Hamiltonian (see [6] for more details):

$$H(x, y, z, p_x, p_y, p_z) = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) + y p_x - x p_y - \frac{1-\mu}{r_1} - \frac{\mu}{r_2},$$

where $p_x = \dot{x} - y$, $p_y = \dot{y} + x$, and $p_z = \dot{z}$, are the conjugate momenta of x, y and z, and where $r_1^2 = (x-\mu)^2 + y^2 + z^2$ and $r_2^2 = (x - \mu + 1)^2 + y^2 + z^2$. The system of differential equations associated to this Hamiltonian is called the spatial Restricted Three Body Problem (RTBP from now on). The RTBP has five equilibrium points. Three of

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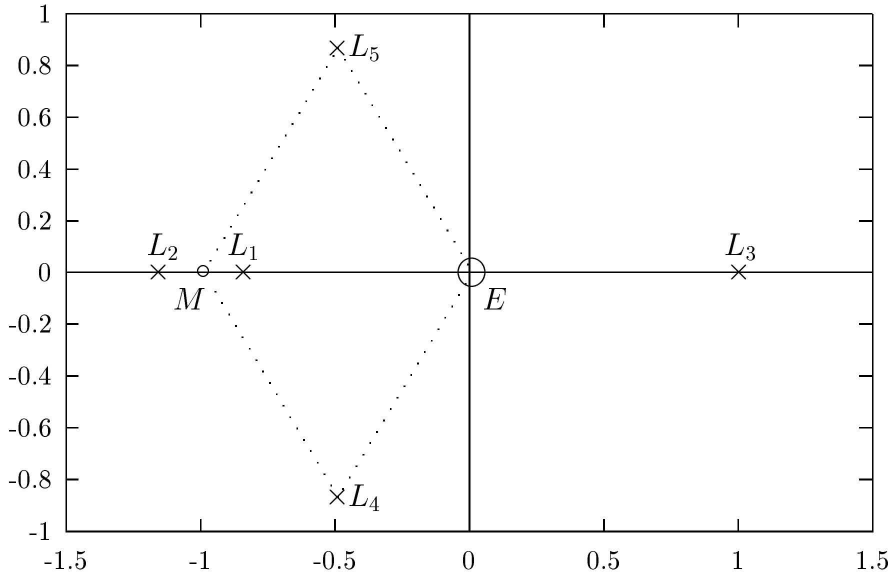

Figure 1: The collinear and triangular points on the RTBP Earth-Moon: $\mu = \mu_{EM} \approx 0.012150582$.

them, L_1, L_2 and L_3 (collinear or Euler points) are on the x axis, and the other two, L_4 and L_5 (triangular or Lagrangian points) are on the (x, y) plane, forming an equilateral triangle with the primaries (see Figure 1). They are given, in the phase space, by

$$L_4 = (-\frac{1}{2} + \mu, -\frac{\sqrt{3}}{2}, 0, \frac{\sqrt{3}}{2}, -\frac{1}{2} + \mu, 0), \; L_5 = (-\frac{1}{2} + \mu, \frac{\sqrt{3}}{2}, 0, -\frac{\sqrt{3}}{2}, -\frac{1}{2} + \mu, 0).$$

In this contribution we are going to focus on the point L_5 (the results also hold for L_4 due to the symmetries of the RTBP). First, let us consider the linearized system around the point L_5. The corresponding linear matrix has the following eigenvalues: $\pm i$, where $i = \sqrt{-1}$ is the imaginary unity, and

$$\pm\lambda^{\pm} = \pm\sqrt{-\frac{1}{2} \pm \frac{1}{2}\sqrt{1 - 27\mu(1-\mu)}}.$$

The real projection of the vectorial subspace spanned by the eigenvectors of eigenvalue i or $-i$ is the "vertical" direction (z, p_z). It means that this linear system has vertical oscillations with normal frequency 1. The well-known Lyapunov centre theorem (see [4]) alows to prove that for any value of μ, $0 < \mu \leq 1/2$, there exists a "vertical" (Lyapunov) family of periodic orbits emanating from L_5. This is usually called the vertical family of periodic orbits of L_5. From the expression of $\pm\lambda^{\pm}$, the linear character of the point L_5 can be easily discussed. For small values of μ these eigenvalues are purely imaginary and different, but if we increase μ, λ^+ and λ^- collide when we reach $\mu = \mu_R$, where $\mu_R \equiv \frac{1}{2}(1 - \sqrt{23/27}) \approx 0.03852$ is called the Routh mass parameter. This collision produces a bifurcation on the linear stability of L_5, and the linearized system becomes unstable for $\mu_R \leq \mu \leq 1/2$.

Of course, if $\mu \neq \mu_R$, the linear stability of the vertical family is "the same" as L_5, at least for moderate amplitudes. What happens if we continue increasing this amplitude

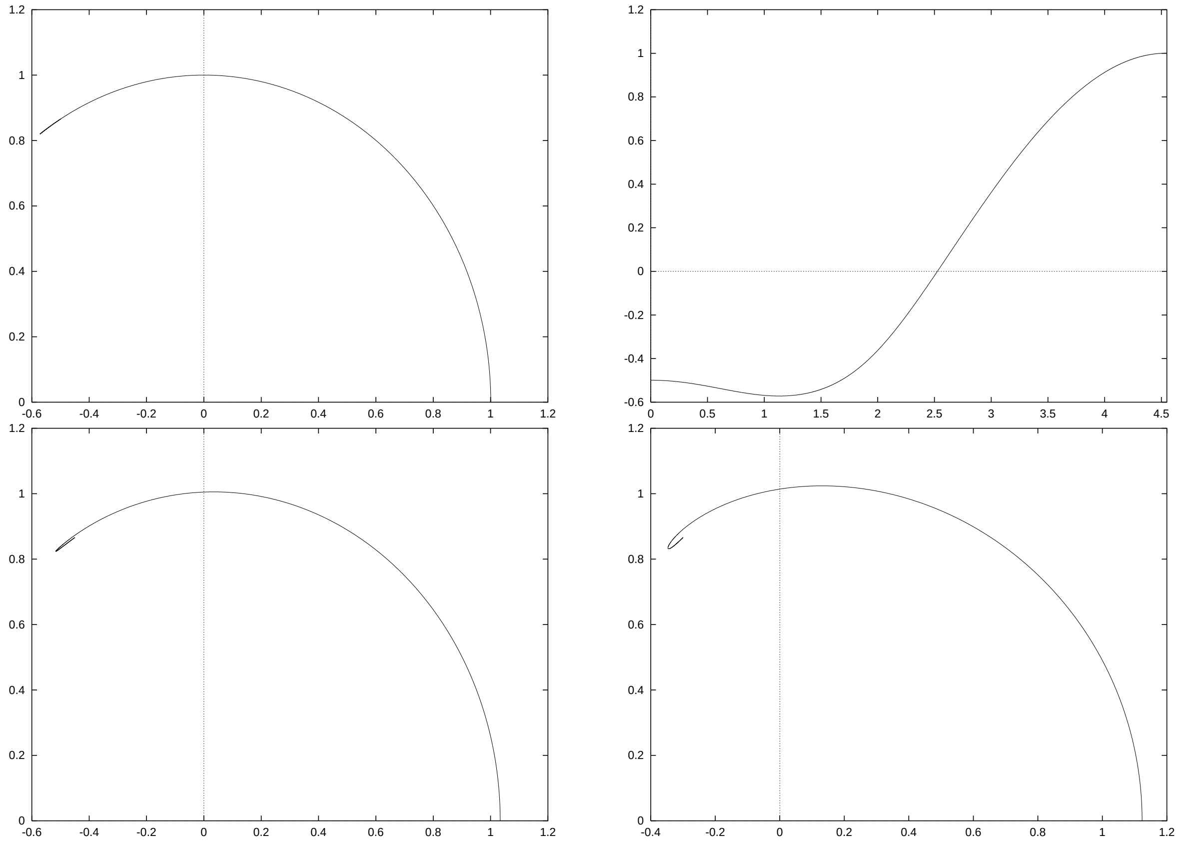

Figure 2: Results obtained by numerical continuation of the vertical family for some values of μ. Up: Left, the projection (x, y) of the curve determined by the Poincaré section of the orbits of the family by Π_0. Right, the graphic of the variable x in this section as function of the arc parameter on the section curve. Note that in this second figure we see a turning point for the x variable that it is not visible in the first one with the actual scale (it is necessary to magnify the plot). Down: the same projection (x, y) as the first figure, but now for $\mu = 0.2$ and $\mu = 0.05$. For these values of μ, we see the turning point near L_5.

is one of the subjects we are going to discuss. Note that the linear character of these periodic orbits can be determined from the monodromy matrix of the orbit (that is, the matrix solution of the variational equations of the orbit, with initial condition the identity matrix, and integrated up to time one period). This matrix always has (at least) a couple of eigenvalues 1, one coming from the periodic character of the orbit, and the other one due to the Hamiltonian character of the system (note that the monodromy matrix is a symplectic matrix). As a periodic orbit with zero amplitude we take the point L_5. Hence, as the vertical oscillations of the linearized system at this point have limit frequency 1, we take as period for this limit orbit 2π. Then, the monodromy matrix of L_5 is given by the exponential matrix of 2π times the differential matrix of the RTBP at L_5, and the non-trivial eigenvalues of this matrix are $\exp(\pm 2\pi\lambda^{\pm})$. As the linear stability character for such periodic orbits is fulfilled when all these non-trivial eigenvalues are different and with modulus one, it implies linear stability for the orbits of small amplitude when $0 < \mu < \mu_R$.

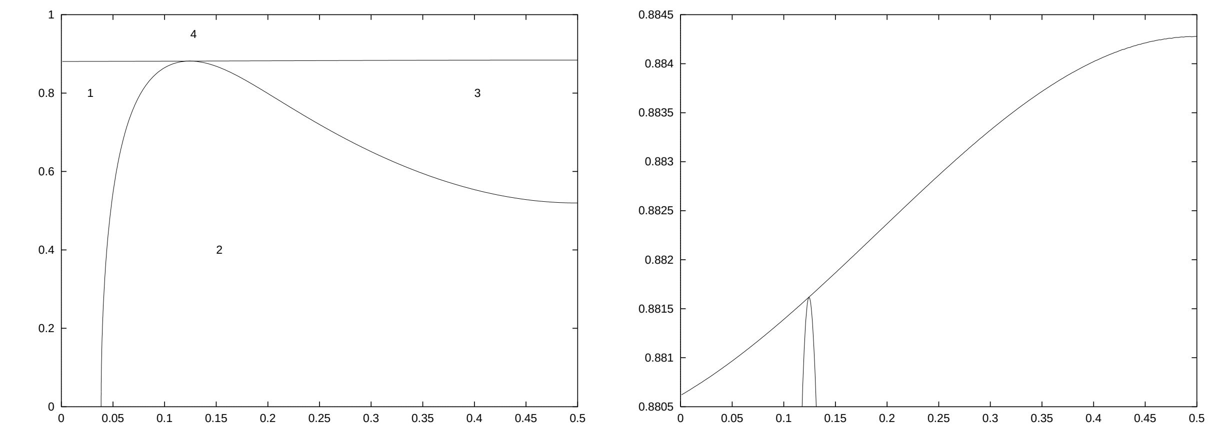

Figure 3: Left, the curves corresponding to the changes on the linear character of the orbits of the vertical family described in the text. The variables in the plots are μ (horizontal axis) and the value of $\dot{z}$ (vertical axis) on the Poincaré section by Π_0. Excluding the bifurcation curves, the non-trivial eigenvalues of the monodromy matrix of the corresponding periodic orbit are all different, and can be described, according to the region, as follows: 1. two couples of conjugate eigenvalues of modulus 1, 2. two conjugate eigenvalues outside $\mathbb{S}^1$ and the corresponding inverse ones, 3. two couples of positive eigenvalues γ, $1/\gamma$, 4. two conjugate eigenvalues of modulus 1 and a couple of positive eigenvalues γ, $1/\gamma$. Right, the upper curve with a more suitable scale for the $\dot{z}$ variable.

Numerical Continuation of the vertical family

In order to study the linear stability of the orbits of the vertical family when we increase the amplitude, as a first step we have done a numerical continuation of such orbits for several values of μ. To illustrate these results, in Figure 2 we present results obtained working with $\mu = \mu_{SJ}$, $\mu = 0.05$ and $\mu = 0.2$. Here, $\mu_{SJ} = 9.538736 \times 10^{-4}$ is the mass parameter corresponding to the Sun-Jupiter system. It has been chosen as an example of mass parameter with $\mu < \mu_R$. The other parameters are bigger than μ_R, but as it is displayed in Figure 3, for such values of μ the linear character of the orbits of the vertical family has different behaviour, even when we continue the family for moderate amplitudes.

The method used to compute such orbits is a standard continuation method on the fixed points of the Poincaré section of the flow of the RTBP (for a fixed value of μ) by the hyperplane $\Pi_0 \equiv \{z = 0\}$. Then, we look for the intersections of the orbits of the vertical family with Π_0, when $p_z > 0$. Note that, in principle, we only can ask this Poincaré section to be well defined for the orbits of the vertical family with moderate (but non-vanishing) vertical amplitude, but during the numerical computations we have checked that it works as "global" Poincaré section for the whole family. For more details about these computations, we refer to [5].

Numerical continuation of the stability bifurcations

Simultaneously with the numerical continuation of the vertical family for the chosen values of μ, we can also compute the non-trivial eigenvalues of the monodromy matrix along the family. From the Hamiltonian character of the system, these four eigenvalues can be written as $\exp(\pm i\omega_1)$ and $\exp(\pm i\omega_2)$, where ω_1 and ω_2 can be complex numbers. We call such non-trivial eigenvalues the normal eigenvalues of the orbit, and the real parts of ω_1

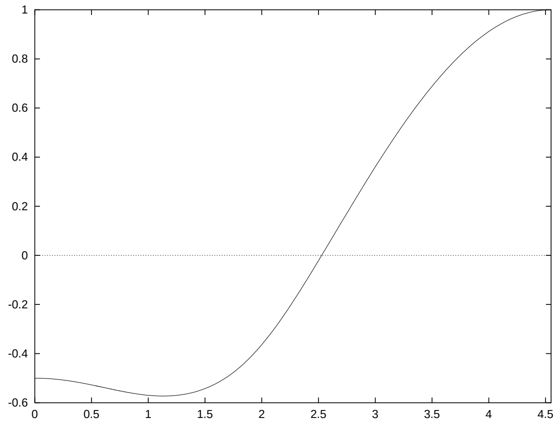

Figure 4: The "vertical family" at the limit $\mu = 0$. Here we plot the graphic of the variable x on the section by Π_0 as function of the arc parameter on this section curve.

and ω_2 are usually called normal frequencies of the orbit.

Summarized with the three cases considered before, we can describe the three different (generic) behaviours for the linear character of the orbits of the vertical family, at least if we only worry about the bifurcations on the linear character that appear "not too far" from L_5:

(i) For $\mu = \mu_{SJ}$, the orbits are linearly stable for small amplitudes, but as we increase the vertical amplitude, one of the normal frequencies takes the value zero, and produces the collision of four eigenvalues of the monodromy matrix at 1. After this collision, the vertical family becomes unstable.

(ii) For $\mu = 0.05$ we have, for small amplitudes, four complex eigenvalues out of $\mathbb{S}^1$. In this case, the first bifurcation we found is a collision of ω_1 and ω_2 (that is, at this singular value we have a double eigenvalue of modulus one and the corresponding conjugate ones). After this collision the vertical family becomes ("unexpected") linearly stable.[1] The following bifurcation on the linear character of the family is the the same described for $\mu = \mu_{SJ}$, that turns the family into unstable.

(iii) For $\mu = 0.2$, there are no linearly stable periodic orbits as the first bifurcation described for $\mu = 0.05$ does not appear. In this case, the first change on the linear character of the orbits that we found is the collision of four eigenvalues at 1.

When continuing the family there are other bifurcations on the linear character, but not another stability changes on the orbits. Then, we have deduced the equations on the coefficients of the monodromy matrix that define these stability changes, and we have continued numerically the bifurcation curves with respect to μ. The results obtained are plotted in Figure 3.

[1]It shows that there are stable motions not too far from L_5 for some range of values of $\mu \geq \mu_R$, for which the Lagrangian point are unstable. This has been used in [3] (see also [2]) to show, using a numerical computation of a normal form, the existence of regions of effective stability around one of these periodic orbits.

Asymptotic behaviour of the vertical family when $\mu \to 0$

As it has been explained before, it is not difficult to continue numerically, for any $0 < \mu \leq 1/2$, the vertical family of L_5. Thus, a natural question is what happens to this family when $\mu \to 0^+$. The RTBP for $\mu = 0$ is a "degenerate case": it is not a 3-bodies problem, but only a 2-bodies one, where the small primary is neglected. This system is obtained by applying the rotating change of coordinates to a 2-bodies problem (it is in this case the sidereal system), where one of the masses is assumed to be zero (it is equivalent to say that this 2-bodies problem is reduced to the centre of masses). Then, the periodic orbits of this system are the periodic ones of the 2-bodies problem with period $2\pi/m$. Such condition meets circular and elliptic orbits, but note that the vertical family is obtained by continuation, when $\mu \neq 0$, of some circular 2π periodic orbits. This is a consequence of the symmetries of the orbits of the vertical family for $\mu \neq 0$. The circular orbits of the RTBP for $\mu = 0$, with period 2π, span a 2-parameter family of orbits, and we can ask which is the curve in the parameters space (or the corresponding 1-parameter family of 2π-periodic circular orbits) that give rise (asymptotically) to the orbits of the vertical family when $\mu \to 0$. The answer can be given in the following way: let us put $\dot{\zeta} = f(\zeta, \mu)$, for the system of differential equations of the RTBP, and let us take $\zeta_0(t)$ a 2π-periodic and circular solution of the system for $\mu = 0$. We denote by $M(t)$ and $m(t)$, the solutions of the variationals equations of the orbit with respect to ζ and with respect to μ, that is:

$$
\begin{aligned}
\dot{M}(t) &= Df(\zeta_0(t), 0)M(t), \ M(0) = Id_6, \\
\dot{m}(t) &= Df(\zeta_0(t), 0)m(\theta) + \frac{\partial f}{\partial \mu}(\zeta_0(t), 0), \ m(0) = 0.
\end{aligned}
$$

Then, the necessary condition (that suffices if we do not have higher order degeneracy) that allows this orbit to be continued in a regular way with respect to μ, is that

$$
-m(2\pi) + 2\pi\alpha\dot{\zeta}_0(0) \in \mathrm{range}(M(2\pi) - Id_6),
$$

for certain α. As the 2-bodies problem is an integrable Hamiltonian system, it allows to prove that $M(2\pi)$ has a sole eigenvalue 1 with multiplicity 6. In fact, it is possible to check that $M(2\pi)$ has 5 independent eigenvectors with eigenvalue 1, and only one Jordan box of size 2. As the eigenvector associated to this Jordan box can be computed explicitly, this condition for persistence of the orbit can be explicitly formulated, and continued with respect to the parameters on the family, taking as initial condition the parameters corresponding to L_5 (see Figure 4).

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