Equidistribution and β-ensembles

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J. Marzo (Universitat de Barcelona)

T. Carroll, College Cork; **X. Massaneda** UB; **J. Ortega-Cerdà** UB

“Well distributed” points

Take N_k points in $\mathbb{S}^2$

$$\mathcal{Z}_k = \{z_j^{(k)} \in \mathbb{S}^2 : 1 \leq j \leq N_k\}, \quad k \geq 1.$$

Define a family of points $\mathcal{Z} = \{\mathcal{Z}_k\}_{k \geq 1}$

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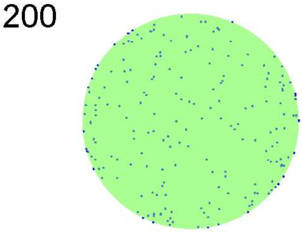

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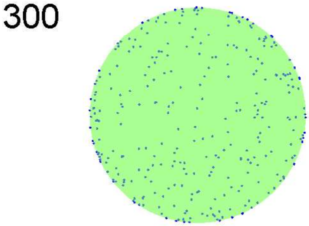

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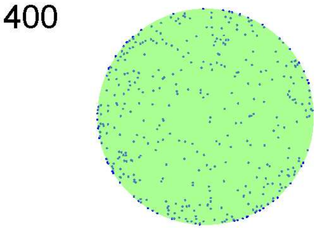

Let μ the normalized surface measure in $\mathbb{S}^2$. The triagular family $\mathcal{Z}$ is asymptotically equidistributed when

$$\mu^{(k)} = \frac{1}{N_k} \sum_{j=1}^{N_k} \delta_{z_j^{(k)}} \xrightarrow{w} \mu, \quad k \to \infty.$$

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Recall that for measures $\{\nu^{(k)}\}_k, \nu$ we say that $\nu^{(k)} \xrightarrow{w} \nu$ iff

$$\int_{\mathbb{S}^2} f(z) d\nu^{(k)}(z) \to \int_{\mathbb{S}^2} f(z) d\nu(z), \quad k \to \infty,$$

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In our case, $\mathcal{Z}$ is asymptotically equidistributed iff

$$\frac{1}{N_k} \sum_{j=1}^{N_k} f(z_j^{(k)}) \to \int_{\mathbb{S}^2} f(z) d\mu(z), \quad k \to \infty,$$

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The space $\mathcal{P}(\mathbb{S}^2)$ of probability measures with the weak convergence can be metrizised by the Kantorovich-Wasserstein distance:

$$\nu^{(k)} \xrightarrow{w} \nu,\ k \to \infty, \ \text{ iff }\ W_1(\nu^{(k)}, \nu) \to 0,\ k \to \infty.$$

Monte Carlo Integration

Take $\mathcal{Z}_k$ independent uniformly distributed points in $\mathbb{S}^2$ and define the empirical measure

$$\mu^{(k)} = \frac{1}{\#\mathcal{Z}_k} \sum_{z \in \mathcal{Z}_k} \delta_z.$$

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Given any $f \in \mathcal{C}(\mathbb{S}^2)$ we have that, by the strong law of large numbers, almost surely

$$\int_{\mathbb{S}^2} f(z) d\mu^{(k)}(z) = \frac{1}{\#\mathcal{Z}_k} \sum_{z \in \mathcal{Z}_k} f(z) \to \int_{\mathbb{S}^2} f(z) d\mu(z), \quad k \to \infty.$$

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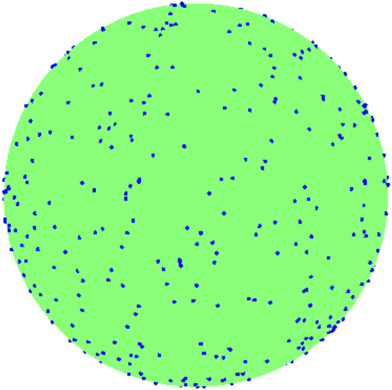
In fact, almost surely, $W_1(\mu^{(k)}, \mu) \to 0, \ k \to \infty.$

Independent uniform points have clumping



529 random uniform points

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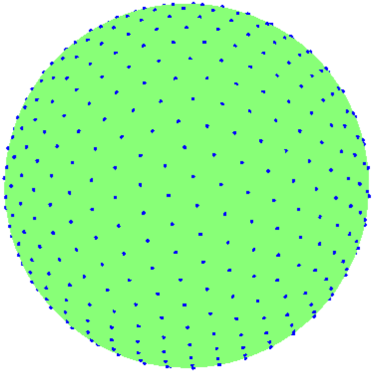

Rob Womersley web http://web.maths.unsw.edu.au/
529 minimal Coulomb energy points

Objective

Quantify the weak convergence (and therefore the "regularity" of $\mathcal{Z}$)

$$\mu^{(k)} = \frac{1}{N_k} \sum_{j=1}^{N_k} \delta_{z_j^{(k)}} \xrightarrow{w} \mu, \quad k \to \infty,$$

in terms of the decay of Kantorovich-Wasserstein distance

$$W_1(\mu^{(k)}, \mu) \to 0, \ k \to \infty,$$

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Previous work in RMT

S. Dallaporta, E. S. Meckes, M. W. Meckes, Z. D. Bai, T. Tao and V. Vu, E. Sandier and S. Serfaty, N. Rougerie and S. Serfaty...

Determinantal point process (Macchi 70's)

Let μ be a normalized measure in X.

Given a function (kernel) $K : X \times X \longrightarrow \mathbb{C}$ such that:

- $K(x,y) = \overline{K(y,x)}$
- Reproducing property

$$\int_X K(x,y)K(y,z)d\mu(y) = K(x,z)$$

- Trace

$$\int_X K(x,x)d\mu(x) = n$$

Then

$$f(x_1,\ldots,x_n) = \frac{1}{n!}\det(K(x_i,x_j))_{1\leq i,j\leq n}$$

is a density function (w.r.t. μ) in X.

Determinantal point process

Take $\phi_1, \ldots, \phi_n$ ON system in $L^2(X)$ then

$$K(x, y) = \sum_{i=1}^{n} \phi_i(x)\overline{\phi_i(y)},$$

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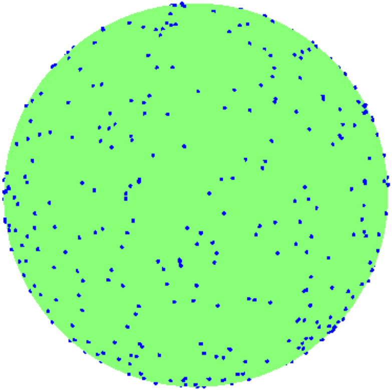
Random matrix theory, Mathematical Physics, Machine learning, Numerical integration...

By the HKPV (Ben Hough-Krishnapur-Peres-Virág) algorithm these processes are “easy” to sample.



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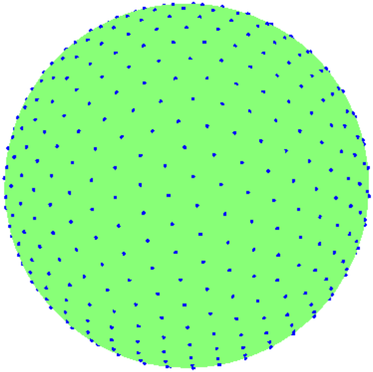

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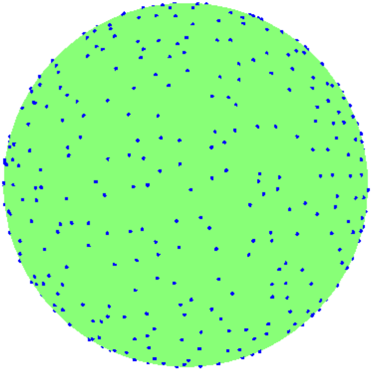

529 points spherical ensemble

Spherical ensemble

Let A, B be $n \times n$ random matrices with i.i.d. complex Gaussian entries. The generalized eigenvalues associated to (A, B), i.e. the eigenvalues of $A^{-1}B$, form a determinantal point process in $\mathbb{C}$ with kernel

$$K_n(z, w) = \frac{n(1 + z\overline{w})^{n-1}}{(1 + |z|^2)^{(n-1)/2}(1 + |w|^2)^{(n-1)/2}},$$

with respect to the measure $\frac{dz}{\pi(1+|z|^2)^2}$. The eigenvalues have joint probability density

$$f(z_1, \ldots, z_n) = \frac{1}{n!} \det(K_n(z_i, z_j)) \prod_{l=1}^{n} \frac{1}{\pi(1 + |z_l|^2)^2}$$

$$= \frac{1}{Z_n} \prod_{l=1}^{n} \frac{1}{(1 + |z_l|^2)^{n+1}} \prod_{i<j} |z_i - z_j|^2,$$

with respect to the Lebesgue measure in $\mathbb{C}^n$.

Spherical ensemble

Using the stereographic projection, the joint density (with respect to the product area measure in the product of spheres) is

$$f(x_1, \ldots, x_n) = C_n \prod_{i<j} \|x_i - x_j\|^2, \quad x_1, \ldots, x_n \in \mathbb{S}^2.$$

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Observe that the mode of this point process is a set of Fekete points $(x_1^*, \ldots, x_n^*)$ on the sphere

$$\sup_{x_1, \ldots, x_n} \prod_{i<j} \|x_i - x_j\|^2 = \prod_{i<j} \|x_i^* - x_j^*\|^2,$$

iff

$$\sum_{i<j} \log \frac{1}{\|x_i^* - x_j^*\|} = \inf_{x_1, \ldots, x_n} \sum_{i<j} \log \frac{1}{\|x_i - x_j\|}.$$

Spherical ensemble

This point process was considered (without a random matrix model) by Caillol (81), and Forrester, Jancovici and Madore (92) as the model of one-component plasma on a sphere.

Bordenave (11) proved the universality of the spectral distribution of the $n \times n$ matrix $A^{-1}B$ with respect to other i.i.d. random distribution of entries. As an outcome, he proved that μ is the weak limit of the spectral measures (so $\lim_n W_1(\mu_n, \mu) = 0$).

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We want to estimate the speed of the convergence of the empirical measure $\mu_n = \frac{1}{n}\sum_{j=1}^{n} \delta_{x_j}$ towards the surface mesure μ.

The Kantorovich-Wasserstein distance is defined on the probability measures over a compact metric space (X, d) and metrizes the weak convergence of measures:

$$W_1(\mu, \sigma) = \inf_{\rho} \iint_{X \times X} d(x, y) d\rho(x, y),$$

where the infimum is taken over all admissible transport plans ρ, i.e., probability measures $\rho \in \mathcal{P}(X \times X)$ with marginal measures μ and σ respectively.

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Dual formulation

$$W_1(\mu, \sigma) = \sup_{f} \left| \int_X f(x) d(\mu - \sigma)(x) \right|,$$

where f belongs to $\mathrm{Lip}_{1,1}(X)$ i.e. $|f(x) - f(y)| \leq d(x, y)$ for $x, y \in X$.

Lower bound

Given a set of n distinct points $Z = \{x_1, \ldots, x_n\} \subset \mathbb{S}^2$. Let $\mu_n = \frac{1}{n}\sum_{j=1}^{n} \delta_{x_j}$. Then

$$W_1(\mu_n, \mu) \geq \frac{c}{\sqrt{n}}$$

for some constant $c > 0$.

Take $2n$ disjoint spherical caps in $\mathbb{S}^2$ of radius $cn^{-1/2}$ for some $c > 0$. At least n of these caps (say) $B_1, \ldots, B_n$ do not have points from Z. Let $\frac{1}{2}B_j$ be the spherical cap with the same center than B_j and half the radius. The function $f(x) = d(x, Z)$ belongs to $\mathrm{Lip}_{1,1}(\mathbb{S}^2)$

$$W_1(\mu_n, \mu) \geq \int_{\mathbb{S}^2} f(x) d\mu(x) \geq \int_{\cup_{j=1}^{n} \frac{1}{2}B_j} f(x) d\mu(x)$$

$$\geq \frac{c}{2\sqrt{n}} n\mu(\frac{1}{2}B_1) \sim \frac{1}{\sqrt{n}}.$$

Upper bound

Theorem

If μ_n is the empirical measure corresponding to the spherical ensemble, then

$$\mathbb{E}W_1(\mu_n, \mu) = O(1/\sqrt{n}).$$

Moreover, almost surely:

$$W_1(\mu_n, \mu) = O(\sqrt{\log n}/\sqrt{n}).$$

Idea of the proof I. Transport plan

We build the transport plan explicitly (Lev and Ortega-Cerdà (10))

$$\rho(z,w) = \frac{1}{n}\sum_{j=1}^{n} \delta_{z_j}(w) K_n(z,z_j)\ell_j(z) d\mu(z),$$

and use the fast decay of the kernel for $z, w \in \mathbb{C}$

$$|K_n(z,w)|^2 = n^2 \left(1 - \frac{|z-w|^2}{(1+|z|^2)(1+|w|^2)}\right)^{n-1}$$

$$\lesssim n^2 \exp\left(-Cn\frac{|z-w|^2}{(1+|z|^2)(1+|w|^2)}\right) = C_1 n^2 \exp\left(-C_2 n\, d(z,w)^2\right),$$

to estimate

$$\mathbb{E} W_1(\mu,\sigma) \le \mathbb{E} \iint_{X\times X} d(x,y) d|\rho|(x,y) \lesssim \frac{1}{\sqrt{n}}.$$

Idea of the proof II. Concentration of measure

Theorem (Pemantle-Peres)

Let Z be a determinantal point process of n points. Let f be a Lipschitz-1 functional defined in the set of finite counting measures (with respect to the total variation distance). Then

$$\mathbb{P}(f - \mathbb{E}f \geq a) \leq 3\exp\left(-\frac{a^2}{16(a+2n)}\right)$$

$$f(\sigma) = nW_1(\frac{1}{n}\sigma, \mu)$$

is Lipschitz-1

$$|f(\sigma) - f(\sigma')| \leq nW_1(\frac{1}{n}\sigma, \frac{1}{n}\sigma') \leq \|\sigma - \sigma'\|_{TV}$$

Idea of the proof III. Borel-Cantelli

Then

$$\mathbb{P}\Big(W_1(\mu_n, \mu) > \frac{11\sqrt{\log n}}{\sqrt{n}}\Big) \lesssim \frac{1}{n^2}.$$

By Borel-Cantelli we get a.s.

$$W_1(\mu_n, \mu) \leq \frac{10\sqrt{\log n}}{\sqrt{n}},$$

for n big enough.

Compact complex manifolds

(X,ω) be a n-dimensional compact complex manifold endowed with a smooth Hermitian metric ω and let (L,ϕ) be a holomorphic line bundle with a positive Hermitian metric ϕ.
Denote by $H^0(X,L)$ the global holomorphic sections. If $s \in H^0(X,L)$ then $|s(x)|_\phi$ is the pointwise norm on the fiber induced by ϕ.
Given a basis $s_1,\ldots,s_N$ of $H^0(X,L)$ we define $\det(s_i(x_j))$ as a section of $L^{\boxtimes N}$ over X^N by the identities

$$\det(s_i(x_j)) = \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \bigotimes_{i=1}^{N} s_i(x_{\sigma_i}).$$

Definition

Given $\beta > 0$. A β-ensemble is an N point random process on X which has joint distribution given by

$$\frac{1}{Z_N} | \det(s_i(x_j))|_\phi^\beta \, d\mu(x_1) \otimes \cdots \otimes d\mu(x_N).$$

Theorem

Consider the empirical measure μ_N associated to the β-ensemble and let $\nu = \frac{(i\partial\bar{\partial}\phi)^n}{\int_X (i\partial\bar{\partial}\phi)^n}$ be the equilibrium measure. Then

$$\mathbb{E} W_1(\mu_N, \nu) \leq C/\sqrt{N}.$$

Results in terms of large deviations Robert Berman (14), Tien-Cuong Dinh and Viet-Anh Nguyen (16).