UNIVERSITAT DE BARCELONA

Master Thesis

Master Advanced Mathematics

Interpolation Theory and Applications to the Boundedness of Operators in Analysis

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Barcelona, June 21, 2016

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Acknowledgments

To begin I want to thank to my supervisor F. Javier Soria for his support and all the help brought during this course and during the production of this final project. Without his help this work could not emerge. Also, for all the hours that he spent on revising the different versions of this work, helping in my English, LaTeX, and mathematical problems.

Finally, I want to thank to my master's colleagues and friends for their unconditional support during all this year, for the greatest moments that we live together, for still here in the bad moments, for made this course to be more tolerable and for made me a better person because without all these moments the life could have been harder.

Introduction

The objective of this work is to introduce some results and applications of Interpolation Theory (as a reference we use the books [4] and [3]) .

The interpolation theory was aimed in the two classical theorems: The Riesz-Thorin Interpolation Theorem that motivates the complex interpolation and was proved by Riesz in 1927 but only for the lower-triangle case, and the general case by Thorin in 1938; and the Marcinkiewicz Interpolation Theorem that motivates the real interpolation and was proved by Marcinkiewicz in 1939.

In the first chapter, we will introduce some tools in complex, functional and harmonic analysis that will be useful in the following chapters to state and prove theorems.

In the second chapter, we will give the statements and proofs of the two classical theorems and see several applications of the Marcinkiewicz theorem for the Fourier Transform for L^p spaces and $L^p(\omega)$ with $p \in [1, 2]$ and $\omega(\theta) = |\theta|^{-n(2-p)}$ being a weight on $\mathbb{R}^n$.

In the third chapter, we will introduce the real interpolation methods, in particular, we will study the $K-$ and $J-$ functionals and the interpolation spaces generated by this functionals, giving the definitions and some properties of those methods and those spaces. Also we will see that the spaces generated by the $K-$functionals are the same than the spaces generated by the $J-$functional. Finally, we will see the Reiteration Theorem which tells us that interpolate two interpolation spaces is the same that interpolate the original spaces.

In the fourth chapter, we will introduce the complex interpolation methods, in particular, we will study the $C^\theta-$ and $C_\theta-$ functionals and the interpolation spaces generated by these functionals, giving the definitions and some properties of those methods and spaces. Also we will see that in this case, the spaces generated by the $C^\theta-$functionals are not the same than the spaces generated by the $C_\theta-$functional, but there are some inclusions between them. Finally, we will see the Reiteration Theorem which tells us that interpolate two interpolation spaces is the same that interpolate the original spaces.

In the fifth chapter, we will see some applications of those methods in some functional spaces. For example, we will interpolate L^p spaces and see that we obtain the Lorentz spaces, also we will interpolate the Hardy spaces.

In the last chapter, we will apply those methods to the boundedness of operators between some Banach spaces. For example, we will use them in the case of the Fourier Multipliers and the Hilbert Transform.

Chapter 1

Basic Notions and Preliminary Results

In this chapter we will introduce a few results of Complex Analysis, Functional Analysis and Harmonic Analysis which will be useful in the next chapters.

1.1 Complex Analysis

In this section we introduce some tools in order to prove the Riesz-Thorin Interpolation Theorem 2.1.1, in particular, we will need the Hadamard Three Line Theorem and the Phragmén-Lindelöf Principle. Also, we will define what is a conformal mapping, the Poisson kernel and give an expression for the Poisson kernel in the strip $\{z \in \mathbb{C} : 0 \leqslant \Re z \leqslant 1\}$.

The aim of the Phragmén-Lindelöf Principle is to generalize on the horizontal strip of the complex plane, the maximum modulus principle, which does not apply to unbounded regions.

Theorem 1.1.1 (Phragmén-Lindelöf Principle)**.** *Let f be a holomorphic function on the horizontal strip*

$$\left\{z : -\frac{\pi}{2} \leqslant \Im(z) \leqslant \frac{\pi}{2}\right\}.$$

If

$$|f(z)| \lesssim e^{\cosh C\Re(z)}$$

for some constant $0 \leqslant C < 1$ and $|f(z)| \leqslant 1$ on the edges of the strip. Then, $|f(z)| \leqslant 1$ in the interior of the strip.

Proof. This proof reduces to the maximum modulus principle. Fix D such that $C < D < 1$ and fix $\varepsilon > 0$. The function

$$F_\varepsilon(z) = f(z)/e^{\varepsilon \cosh Dz}$$

is bounded by 1 on the edges of the strip, and in the interior goes to 0 uniformly in y as $x \to \pm\infty$. Then, on a rectangle

$$R_{T_\varepsilon} = \left\{z : -\frac{\pi}{2} \leqslant \Im(z) \leqslant \frac{\pi}{2},\ -T_\varepsilon \leqslant x \leqslant T_\varepsilon\right\}$$

the function F_ε is bounded by 1 on the edges.

Then, the maximum modulus principle implies that F_ε is bounded by 1 in the whole rectangle. That is, for each z_0 fixed in the strip,

$$|f(z_0)| \leqslant \exp(\varepsilon \cosh D\Re(z_0)).$$

We can let $\varepsilon \to 0^+$, giving $|f(z_0)| \leqslant 1$. ■

Now, we can prove the Hadamard Three Line Theorem which says that if we have an holomorphic function inside a strip of the form $\{z_1 + iz_2 : a \leqslant z_1 \leqslant b\}$ in the complex plane, and this function is continuous on the whole strip then the logarithm of $M(z_1) = \sup_{z_2} |f(z_1 + iz_2)|$ is a convex function in the interval $[a, b]$.

Theorem 1.1.2 (Hadamard Three Line Theorem)**.** *Let $f(z)$ be a bounded function of $z = z_1 + iz_2$ defined on the strip $\{z_1 + iz_2 : a \leqslant z_1 \leqslant b\}$ holomorphic in the interior and continuous on the whole strip. If we define*

$$M(z_1) = \sup_y |f(z_1 + iz_2)|$$

then $\log(M(z_1))$ is a convex function in $[a, b]$. That is, if $z_1 = ta + (1-t)b$ with $t \in [0, 1]$ then

$$M(z_1) \leqslant M(a)^t M(b)^{1-t}.$$

Proof. We can assume that the interval $[a, b]$ is $[0, 1]$, this assumption only change some constants in the proof and reduces the notation. Then, by hypothesis we have that $|f(yi)| \leqslant M(0)$ and $|f(1 + yi)| \leqslant M(1)$.

Let ε be positive and λ be a real number. Define

$$F_\varepsilon(z) := \exp(\varepsilon z^2 + \lambda z)f(z).$$

Where $z = z_1 + z_2 i \in \mathbb{C}$ with $z_1 \in [0, 1]$. Notice that we have

$$F_\varepsilon(z) := \exp(\varepsilon(z_1^2 - z_2^2) + \lambda z_1)\exp(i(\varepsilon(2z_1z_2) + \lambda z_2))f(z).$$

Since z_1, $f(z)$ and $\exp(i(\varepsilon(2z_1z_2) + \lambda z_2))$ are bounded we have that

$$F_\varepsilon(z) \to 0 \quad \text{as} \quad z_2 \to \pm\infty.$$

We also have that

$$|F_\varepsilon(iz_2)| = |e^{-z_2^2\varepsilon}||e^{i\lambda z_2}||f(iz_2)| \leqslant |f(z_2 i)| \leqslant M(0),$$

and that

$$\begin{aligned}|F_\varepsilon(1 + iz_2)| &= |\exp(\varepsilon(1 - z_2^2) + \lambda)||\exp(i(\varepsilon 2z_2 + \lambda y))||f(z_2 i)| \\ &\leqslant |\exp(\varepsilon(1 - z_2^2) + \lambda)||f(1 + z_2 i)| \\ &\leqslant e^{\varepsilon+\lambda}|f(1 + z_2 i)| \leqslant e^{\varepsilon+\lambda}M(1).\end{aligned}$$

Now using Theorem 1.1.1 with $\Re(z) = \varepsilon(z_1^2 - z_2^2) + \lambda z_1$, we obtain

$$|F_\varepsilon(z)| \leqslant \max(M(0), e^{\varepsilon+\lambda}M(1)).$$

Hence,

$$|F_\varepsilon(z_1 + z_2 i)| \leqslant \exp(-\varepsilon(z_1^2 - z_2^2)) \max(M(0)e^{-z_1\lambda}, e^{\varepsilon+\lambda(1-z_1)}M(1)).$$

This holds for any z. Now, taking $\varepsilon \to 0$ we obtain that

$$|F(z)| \leqslant \max(M(0)e^{-z_1\lambda}, e^{\lambda(1-z_1)}M(1)).$$

The right hand side is as small as possible when $M(0)e^{-z_1\lambda} = e^{\lambda(1-z_1)}M(1)$. So, we get

$$|F(z_1 + z_2 i)| \leqslant M(0)^{1-z_1}M(1)^{z_1}. \tag{1.1}$$

Taking $z_1 = ta + (1-t)b = 1-t$ with $t \in [0,1]$ we can write (1.1) as

$$|F(z_1 + z_2 i)| \leqslant M(0)^t M(1)^{1-t}$$

as we want. ∎

1.1.1 Poisson Kernel in the Strip

In this section we will give some notions that will be useful in the Section 4.3 and in the Section 5.2. We will begin by defining conformal maps.

Definition 1.1.3. Let Ω be a domain in $\mathbb{C}$ and $f \in \mathrm{Hol}(\mathbb{C})$. We say that f is a conformal mapping if $f'(z) \neq 0$ for all $z \in \Omega$.

Recall that if $z = x + iy$ then

$$f'(z) = \frac{\partial f}{\partial z}(z) = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)(z).$$

Now we are going to give some expressions of the Poisson kernel in the unit disk and in the strip S. First we will see an expression for the unit disk, this will be used in the Section 5.2.

Definition 1.1.4. Let $\mathbb{D}$ be the unit disk in the complex plane, and let $0 < r \leqslant 1$. We define a Poisson kernel in $\mathbb{D}$ as

$$P_r(z, e^{it}) = \frac{r^2 - |z|^2}{|re^{it} - z|^2}.$$

Then, we define the Poisson integral of a function f in the unit disk as

$$\frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})P_r(z, e^{it})d\theta.$$

Now, we are going to see two expressions of the Poisson kernel in the strip S.

Definition 1.1.5. Let $s + it \in S$. We define a Poisson kernel in S as

$$P_0(s + it, \tau) = \frac{e^{-\pi(\tau - t)}\sin \pi s}{\sin^2 \pi s + (\cos \pi s - e^{-\pi(\tau-t)})^2}.$$

Definition 1.1.6. Let $s + it \in S$. We define a Poisson kernel in S as

$$P_1(s + it, \tau) = \frac{e^{-\pi(\tau - t)} \sin \pi s}{\sin^2 \pi s + (\cos \pi s + e^{-\pi(\tau - t)})^2}.$$

Those expressions will we used in the Section 4.3, and some properties of those expressions will be seen in the mentioned section. Also, it can be proved that the Poisson kernel in the unit disk and those expressions are equivalent under some conformal mapping from $\mathbb{D}$ to S (see [10, Chapter 14, Theorem 14.8]).

1.2 Functional Analysis

In this section we will introduce the most useful topics in functional analysis, in particular, we will see how integrate functions with values in Banach spaces. Also, we will see the Completeness Theorem in Banach spaces.

1.2.1 Completeness

When we study if a normed space is complete it will be useful to work with series instead of sequences. The following theorem tells us that in normed spaces the convergence of Cauchy sequences is equivalent to proving that all absolutely convergent series are convergent.

Theorem 1.2.1 (Completeness)**.** *Let E be a normed space. Then the following are equivalent:*

1. *E is a Banach space*
2. *all absolutely convergent series are convergent.*

Proof. $\boxed{(2) \Rightarrow (1)}$ Let $(x_n)_n \subset E$ be a Cauchy sequence such that

$$\text{If } \varepsilon = 1 \Rightarrow \exists n_0 : \|x_m - x_n\| < 1 \text{ if } m, n \geqslant n_0,$$
$$\text{If } \varepsilon = 2^{-k} \Rightarrow \exists n_0 : \|x_m - x_n\| < 2^{-k} \text{ if } m, n \geqslant n_k.$$

Take

$$\left\{ \begin{aligned} y_0 &= x_{n_0} \\ y_k &= x_{n_k} - x_{n_{k-1}} \end{aligned} \right\} \Rightarrow \|y_k\| \leqslant 2^{-k}.$$

Then, $\sum_k y_k$ is absolutely convergent, so $\sum_k y_k$ is convergent. By definition it is equivalent to say that

$$\left(\sum_{k=1}^{N} y_k \right)_N$$

is convergent in E. Therefore, $(x_n)_n$ has a partial which is convergent. But, as we can do it for all the partials of $(x_n)_n$ we have that $(x_n)_n$ is convergent.

$\boxed{(1) \Rightarrow (2)}$ Let $\sum_{j=1}^{\infty} x_j$ be an absolutely convergent series, and assume that $m > n$. Define

$$S_n = \sum_{j=1}^{n} x_j.$$

Then,

$$\|S_m - S_n\| = \left\| \sum_{j=n}^{m} x_j \right\| \leqslant \sum_{j=n}^{m} \|x_j\| \to 0, \quad \text{as } m, n \uparrow \infty.$$

So, $(S_n)_n$ is a Cauchy sequence which implies that $(S_n)_n$ is convergent to $S = \sum_{j=1}^{\infty} x_j$. Consequently, $\sum_{j=1}^{\infty} x_j$ is a convergent series. ■

1.2.2 Weak Topologies

In this section we will introduce what the weak topologies for a normed space X and its dual X'. Also, we will see the Banach-Alaoglu Theorem which deals with weak compactness of the unit ball in X'.

We will begin by defining the dual space of a normed space and a seminorm, since the weak topology is defined in terms of the dual space and seminorms.

Definition 1.2.2. Let X be a normed space in the field $\mathbb{F}$, we define its dual space, X', as the set of ω satisfying that

$$\omega : X \to \mathbb{F}$$

such that ω is linear and continuous. We define the norm in X' as

$$\|\omega\|_{X'} = \sup_{\|x\|_X = 1} |\omega(x)|.$$

Usually we will denote $\omega(x)$ as $\langle \omega, x \rangle$ which means the action of ω over x.

Definition 1.2.3. Let V be a vector space in the field $\mathbb{F}$, then a function $\sigma : V \to \mathbb{F}$ is called a seminorm if satisfies

1. $\sigma(x) \geqslant 0$ for all $x \in V$,
2. $\sigma(x + y) \leqslant \sigma(x) + \sigma(y)$ for all $x, y \in V$,
3. $\sigma(\lambda x) = |\lambda| \sigma(x)$ for all $x \in V$ and for all $\lambda \in \mathbb{F}$.

Notice that can happens that $\sigma(x) = 0$ and $x \neq 0$. Now we are going to define the weak and the weak* topologies.

Definition 1.2.4. Let X be a normed space and X' its dual, we call the weak topology in X to the topology induced by the family of seminorms of the form

$$\sigma_\omega(x) = |\langle \omega, x \rangle|$$

where $x \in X$ and $\omega \in X'$.

Definition 1.2.5. Let X be a normed space and X' its dual, let $(x_n)_n \subset X$ and $x \in X$. We say that x_n converges in the weak topology (or converges weakly) if

$$\omega(x_n) \to \omega(x) \quad \forall \omega \in X'.$$

We denote this convergence as $x_n \xrightarrow{w} x$.

Remark 1.2.6.

(i) In the literature the weak convergence can be found as $x_n \rightharpoonup x$.

(ii) A subset B of X is called weakly closed if B is closed with the weak topology. The same happens with the notions of weakly compact, weakly open and weak closure.

Definition 1.2.7. Let X be a normed space, $\varphi \in X'$ and let $(\varphi_n)_n \subset X'$ be a sequence in X'. Then, we say that φ_n converges to φ in the weak* topology if

$$\lim_{n \uparrow \infty} \varphi_n(x) = \varphi(x) \quad \forall x \in X.$$

We will write this as

$$\varphi_n \xrightarrow{w^*} \varphi.$$

Also we need to define what is a reflexive Banach space.

Definition 1.2.8. Let X be a Banach space and X' its dual. We say that X is reflexive if X is isomorphic and isometric to the dual of X', this is the bidual of X. This means that there exists an isomorphism φ from X to $(X')' = X''$ such that for all $x \in X$ we have that

$$\|x\|_X = \|\varphi(x)\|_{X''}.$$

Finally, we are going to prove the Banach-Alaoglu Theorem, but in order to do this we need the Tychonov Theorem.

Theorem 1.2.9 (Tychonov's Theorem)**.** *Let $\{X_\alpha : U_\alpha\}$ be a family of compact spaces. Then $\prod_\alpha X_\alpha$ endowed with the product topology is compact.*

The proof can be found in [5, Chapter 1, section 8].

Theorem 1.2.10 (Banach-Alaoglu Theorem)**.** *Let X be a normed space and X' its dual. Then, the unit ball of X' is weak* compact.*

Proof. Let X be a normed space and X' its dual and let $B^* = \{T \in X' : \|T\| \leqslant 1\}$. If $T \in B^*$, then $T(x) \in [-\|x\|, \|x\|]$ for all $x \in X$. Consider the cartesian product

$$P = \prod_{x \in X} [-\|x\|, \|x\|].$$

A point in P is a function $f : X \to \mathbb{R}$ such that $f(x) \in [-\|x\|, \|x\|]$, and P is the collection of all such functions. The set B^* is a subset of P and inherits the product topology of P. On the other hand, since $B^* \subset X'$ we have that B^* also inherits the weak* topology of X'. Then we have to prove the following things:

(i) These two topologies coincide on B^*.

(ii) B^* is closed in its relative product topology.

Let us prove (i), every weak* open neighborhood of a point $T_0 \in X'$ contains an open set of the form

$$\mathcal{O} = \{T \in X' : |T(x_j) - T_0(x_j)| < \delta \text{ for some } \delta > 0, \text{and for finite } x_j, j = 1, \cdots, n\}.$$

Likewise, every neighborhood of $T_0 \in P$ open in the product topology of P contains an open set of the form

$$\mathcal{V} = \{f \in P : |f(x_j) - T_0(x_j)| < \delta \text{ for some } \delta > 0, \text{and for finite } x_j, j = 1, \cdots, n\}.$$

These open sets form a base for the corresponding topologies. Since $B^* = P \cap X'$, we have that

$$\mathcal{O} \cap B^* = \mathcal{V} \cap B^*.$$

These intersections form a base for the corresponding relative topologies inherited by B^*. Therefore, the weak* topology and the product topology coincide in B^*.

In order to prove (ii), let f_0 be in the closure of B^* in the relative product topology. Fix $x, y \in X$ and $\alpha, \beta \in \mathbb{R}$ and consider the three points

$$a_1 = x, \quad x_2 = y, \quad x_3 = \alpha x + \beta y.$$

For $\varepsilon > 0$, the sets

$$\mathcal{V}_\varepsilon = \{f \in P : |f(x_j) - f_0(x_j)| < \varepsilon \text{ for } j = 1, 2, 3\}$$

are open neighborhoods of f_0. Since they intersect B^*, there exists $T \in B^*$ such that

$$|f_0(x) - T(x)| < \varepsilon, \quad |f_0(y) - T(y)| < \varepsilon$$

and since T is lineal,

$$|f_0(\alpha x + \beta y) - \alpha T(x) - \beta T(y)| < \varepsilon.$$

Using this three inequalities we have that

$$|f_0(\alpha x + \beta y) - \alpha f_0(x) - \beta f_0(y)| < (1 + |\alpha| + |\beta|)\varepsilon.$$

So, we have that f_0 is linear and as it holds for any $\varepsilon > 0$, we have that $f_0 \in B^*$. Since the intervals $[-\|x\|, \|x\|]$ are compact in the euclidean topology, by the Tychonov's Theorem 1.2.9 we have that P is compact in the product topology. But since by (ii) B^* is closed in P with this topology, we have that B^* is compact with the product topology. Now by (i) the product topology and the weak* topology coincide in B^*, therefore B^* is compact with the weak* topology. ■

Corollary 1.2.11. *If X is a reflexive Banach space, then if we apply the Banach-Alaoglu Theorem 1.2.10 we will have that the unit ball of X is weak compact.*

1.2.3 Bochner Integral

In this section we give a vision on vector calculus. In fact, we will define the Bochner integral and give a generalization of the Riemann-Stieltjes integral for functions with values in Banach spaces.

Since the J-method 3.1.2 is defined in terms of the Bochner integral, we need to see how to integrate a function that takes values in a Banach space. We are going to follow the book [12, Chapter V, Section 5].

The aim of the Bochner integral is to extend the Lebesgue integral to functions that take values in a Banach space. The way to define this integral is the usual, we start integrating a simple function, and later we take the limit of integrals of simple functions.

So, let us start defining the integral for simple functions.

Definition 1.2.12. Let $x(s)$ be a simple function defined on a measure space $(S, \mathcal{F}, \mu)$ with values in a Banach space X. That is,

$$x(s) = \begin{cases} x_i \neq 0, & s \in B_i \in \mathcal{F} \\ 0, & s \in S \setminus \cup_i B_i \end{cases}$$

where $B_i \cap B_j = \varnothing$ for all $i \neq j \in \{1, \cdots, n\}$ and $\mu(B_i) < \infty$ for all $i \in \{1, \cdots, n\}$. Then, we define the integral as

$$\int_S x(s)\mu(ds) := \sum_{i=1}^{n} x_i \mu(B_i).$$

The following definition will be useful to prove that this integral is well defined.

Definition 1.2.13. Let $x(s)$ be a function defined on a measure space $(S, \mathcal{F}, \mu)$ with values in a Banach space X. $x(s)$ is said to be a strongly $\mathcal{F}$ -measurable if there exists a sequence of simple functions convergent to $x(s)$ μ-a.e. (i.e. except in sets of measure 0) on S.

Definition 1.2.14. A function $x(s)$ defined on a measure space $(S, \mathcal{F}, \mu)$ with values in a Banach space X is said to be Bochner integrable, if there exists a sequence of simple functions $\{x_n(s)\}$ which is s-convergent (convergent in S) to $x(s)$ $\mu-$a.e. in such a way that

$$\lim_{n \uparrow \infty} \int_S \|x(s) - x_n(s)\| \mu(ds) = 0. \tag{1.2}$$

For any set $B \in \mathcal{F}$, the Bochner integral of $x(s)$ over B is defined as

$$\int_B x(s)\mu(ds) = S - \lim_{n \uparrow \infty} \int_S \chi_B(s) x_n(s) \mu(ds) \tag{1.3}$$

where $\chi_B(s)$ is the characteristic function of the set B. i.e.

1. $\chi_B(s) \equiv 1$ if $s \in B$;
2. $\chi_B(s) \equiv 0$ if $s \in S \setminus B$.

Lemma 1.2.15. *The Bochner integral is well-defined.*

Proof. We have to see that (1.3) exists and that this value does not depend on $\{x_n(s)\}$.

First note that (1.2) makes sense because $x(s)$ is strongly $\mathcal{F}$-measurable. From the inequality

$$\begin{aligned}\left\|\int_B x_n(s)\mu(ds) - \int_B x_k(s)\mu(ds)\right\|_X &= \left\|\int_B x_n(s) - x_k(s)\mu(ds)\right\|_X \\ &\leqslant \int_B \|x_n(s) - x_k(s)\|_X \mu(ds) \\ &\leqslant \int_S \|x_n(s) - x_k(s)\|_X \mu(ds) \\ &\leqslant \int_S \|x_n(s) - x(s)\|_X \mu(ds) \\ &\quad + \int_S \|x(s) - x_k(s)\|_X \mu(ds),\end{aligned}$$

and this tends to 0. Since X is a Banach space we have that

$$S - \lim_{n\uparrow\infty} \int_S C_B(s)x_n(s)\mu(ds)$$

exists. Now, we will see the independence of $\{x_n(s)\}$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences such that

$$\begin{aligned}x_n(s) &\to x(s)\\ y_n(s) &\to x(s)\end{aligned}$$

but satisfying that

$$\begin{aligned}S - \lim_{n\uparrow\infty} \int_S C_B(s)x_n(s)\mu(ds) &= a\\ S - \lim_{n\uparrow\infty} \int_S C_B(s)y_n(s)\mu(ds) &= b.\end{aligned}$$

Then, taking (z_n) such that

$$z_n(s) = \begin{cases} x_n(s), & \text{if } n \text{ is odd}\\ y_n(s), & \text{if } n \text{ is even,}\end{cases}$$

we have that (z_n) converges to $x(s)$, and

$$S - \lim_{n\uparrow\infty} \int_S C_B(s)z_n(s)\mu(ds)$$

has to be convergent. Therefore $a = b$ and this implies that this integral does not depend on the sequence. ∎

The following proposition will be useful in the proof of Proposition 3.1.20.

Proposition 1.2.16. *Let T be a bounded linear operator on a Banach space X into a Banach space Y. If $x(s)$ takes values in X and is a Bochner integrable function, then $Tx(s)$ takes values in Y and, also, is Bochner integrable. Moreover,*

$$\int_B Tx(s)\mu(ds) = T\left(\int_B x(s)\mu(ds)\right).$$

Proof. Let a sequence of simple functions $\{y_n(s)\}$ satisfying

$$\|y_n(s)\|_X \leqslant \|x(s)\|_X(1+n^{-1})$$

and

$$S-\lim_{n\uparrow\infty} y_n(s) = x(s) \quad \mu-\text{a.e.}$$

Then, by linearity and continuity of T, we have that

$$\int_B Ty_n(s)\mu(ds) = T\left(\int_B y_n(s)\mu(ds)\right).$$

Also, by the continuity of T,

$$\|Ty_n(s)\|_X \leqslant \|T\|_{X\to Y}\|y_n(s)\|_X \leqslant \|T\|_{X\to Y}\|x(s)\|_X(1+n^{-1}).$$

And

$$S-\lim_{n\uparrow\infty} Ty_n(s) = Tx(s) \quad \mu-\text{a.e.}$$

Hence $Tx(s)$ is Bochner integrable and

$$\begin{aligned}\int_B Tx(s)\mu(ds) &= S-\lim_{n\uparrow\infty}\int_B Ty_n(s)\mu(ds) = S-\lim_{n\uparrow\infty} T\left(\int_B y_n(s)\mu(ds)\right)\\ &= T\left(\int_B x(s)\mu(ds)\right).\end{aligned}$$

■

As happens when we integrate functions with values in $\mathbb{R}$ or in $\mathbb{C}$, we can extend the Bochner integral in the way that instead of integrate with respect to $\mu(ds)$ we integrate with respect to other function which takes values in a Banach space, defining the vector-valued Stieltjes Integral. This generalization will be useful when we prove the Theorem 4.1.10.

1.2.4 Fréchet Spaces and the Big Theorems

In this section we will define the Fréchet Spaces and we will see some of the most important theorems in the functional analysis.

In the following lemma we will define what is a Fréchet norm.

Lemma 1.2.17. *Given an increasing sequence of seminorms as in Definition 1.2.3,*

$$p_1(x) \leqslant p_2(x) \leqslant \cdots \leqslant p_n(x) \leqslant p_{n+1}(x),$$

such that $p_k(x) = 0$ for all k implies $x = 0$, then

$$\|x\| = \sum_{k=1}^{n} = \frac{p_k(x)}{2^k(1+p_k(x))}$$

is a Fréchet norm, that is

(a) $\|x+y\| \leqslant \|x\| + \|y\|$,

(b) $\|-x\| = \|x\|$,

(c) $\|x\| = 0 \Rightarrow x = 0$.

Proof. Since p_k are seminorms we have that $p_k(-x) = p_k(x)$ for all k, then (b) is satisfied. Also, we have that $p_k(x) \geqslant 0$, then the only way that $\|x\| = 0$ is that $p_k(x) = 0$ for all k but, by hypothesis, this implies that $x = 0$. Therefore, (c) holds. In order to prove (a) we will use that

$$\varphi(t) = \frac{t}{t+1}$$

is an increasing function with respect to t, and that p_k are seminorms, then we have that

$$\frac{p_k(x+y)}{1+p_k(x+y)} \leqslant \frac{p_k(x)+p_k(y)}{1+p_k(x)+p_k(y)} \leqslant \frac{p_k(x)}{1+p_k(x)} + \frac{p_k(y)}{1+p_k(y)}.$$

This implies (a). ∎

Now we are going to define the Fréchet spaces.

Definition 1.2.18. A Fréchet space is a topological vector space endowed with a Fréchet norm so that it is complete.

The first theorem that we will see is the Baire's Theorem, this theorem deals with the union of open and dense sets, and will be useful for the proof of the Open Mapping Theorem.

Theorem 1.2.19 (Baire's Theorem)**.** *If $(G_n)_n$ is a sequence of open and dense sets in a complete metric space, E, then $\bigcup_n G_n$ is also dense.*

Proof. Let G be an open set, we want to prove that $G \cap (\cap_n G_n) \neq \varnothing$. Since G_1 is dense we have that there exists a ball $B(x_1, r_1)$ with $r_1 < 1$ and $x_1 \in G \cap G_1$ such that $Cl(B(x_1, r_1)) \subset G \cap G_1$. Then, since G_2 is dense we have that there exists a ball $B(X_2, r_2)$ with $r_2 < 1/2$ and $x_2 \in B(x_1, r_1) \cap G_2$ such that $Cl(B(x_2, r_2)) \subset B(x_1, r_1) \cap G_2$. Now, iterating this we obtain that for all n

$$Cl(B(x_n, r_n)) \subset B(x_{n-1}, r_{n-1}) \cap G_n, \quad \text{with } r_n < 1/n.$$

Now, if $p, q \geqslant n$ then the distance between x_p and x_q is less than or equal to $2/n$. Therefore the sequence $(x_p)_p \subset E$ is a Cauchy sequence, so there exists $x = \lim_p x_p$.

On the other hand, by construction we have that $x_k \in Cl(B(x_n, r_n))$ for all $k \geqslant n$ this implies that $x \in Cl(B(x_n, r_n)) \subset G_n$ for all n. Hence, $x \in \cap_n G_n$. But, in particular, $x \in Cl(B(x_n, r_n))$ so $x \in G \cap (\cap_n G_n)$. ∎

Corollary 1.2.20. *If $(F_n)_n$ is a countable collection of closed sets in a complete metric space such that the interior of F_n is the empty set, $\mathring{F}_n = \varnothing$ for all n, then $\bigcup_n \mathring{F}_n = \varnothing$.*

The next theorem is the Open Mapping Theorem and is, maybe, one of the most useful theorems in functional analysis.

Theorem 1.2.21 (Open Mapping Theorem)**.** *Let E and F be two Fréchet spaces. If $T : E \to F$ is a linear continuous operator so that $T(E) = F$ then T is open, i.e. for all G open set in E $T(G)$ is open in F.*

If T is also injective, then T^{-1} is also continuous.

The proof of this theorem can be found in [9, Theorem 2.11 and Corollary 2.12].

Theorem 1.2.22 (Closed Graph Theorem)**.** *Let E and F be two Fréchet spaces. Then a linear map $T : E \to F$ is continuous if and only if* $\mathrm{Graph}(T) = \{(x, Tx) : x \in E\}$ *is closed in $E \times F$.*

Proof. Assume that T is continuous, and take a sequence (x_n, Tx_n) convergent to (x, y) we want to see that $y = Tx$, but since T is continuous we have that if $x_n \to x$ then $Tx_n \to Tx$. Therefore $y = Tx$ because $(x_n, Tx_n) \to (x, y)$ implies that $x_n \to x$ and that $Tx_n \to y$.

In order to see the other implication, since $\mathrm{Graph}(T)$ is closed then it is a Fréchet space. Let us consider the mappings

$$\pi_1 : \mathrm{Graph}(T) \to E,$$
$$\pi_2 : \mathrm{Graph}(T) \to F.$$

Since π_1 is linear, continuous and exhaustive we have that by the Open Mapping Theorem 1.2.21 π_1^{-1} is continuous. So

$$Tx = (\pi_2 \circ \pi_1^{-1})(x)$$

is continuous. ∎

1.3 Harmonic Analysis

In this section we will study the results in harmonic analysis that will be of interest for our work.

1.3.1 The Weak-L^p Spaces

In this section we will define the weak-L^p spaces, but in order to define these spaces first we need to introduce the distribution function and the non-increasing rearrangement of f, this topic will be important for give the statement and the proof of the Marcinkiewicz Interpolation Theorem and the next chapters.

So, let us define the distribution function of f.

Definition 1.3.1. Let (X, μ) be a measure space, and let $f \in \mathcal{M}(X)$, where $\mathcal{M}(X)$ is the space of measurable functions.

Fix $t > 0$ and consider the level set

$$f_t = \{x \in X : |f(x)| > t\}.$$

Then, we define the distribution function of f as

$$\lambda_f(t) = \mu(f_t).$$

Now, we are going to see some properties of the distribution function.

Remarks 1.3.2. (i) $\lambda_f \in [0, \infty]$.

(ii) If $s < t$ and $x \in f_t$, then $|f(x)| > t > s$, in other words, $f_t \subset f_s$. Hence, $\lambda_f(s) \geqslant \lambda_f(t)$. Therefore, λ_f is a decreasing function.

(iii) Let $\alpha \in \mathbb{C}$, $E \in \sigma(X)$, and take $f = \alpha\chi_E \in \mathcal{M}(X)$. Then, $\lambda_f(t) = \mu(E)\chi_{[0,|\alpha|]}(t)$.

(iv) Let $f_1, f_2 \in \mathcal{M}(X)$. Then, $\lambda_{f_1+f_2}(t_1 + t_2) \leqslant \lambda_{f_1}(t_1) + \lambda_{f_2}(t_2)$. In fact, this follows because

$$\{x \in X : |f_1(x) + f_2(x)| > t_1 + t_2\} \subset \{x : |f_1(x)| > t_1\} \cup \{x : |f_2(x)| > t_2\}.$$

(v) If $0 < |f_1| \leqslant |f_2|$, then $\lambda_{f_1}(t) \leqslant \lambda_{f_2}(t)$. So, the distribution function is a increasing function as a function of f.

The next remark is important in order to the proof of Proposition 1.3.7.

Remark 1.3.3. Let (X, μ) be a measure space, and let $f \in \mathcal{M}(X)$. Then, λ_f is right-continuous.

Proof. Recall

$$f_t = \{x \in X : |f(x)| > t\},$$

and fix $t_0 > 0$. The sets f_t are increasing as t decrease, and

$$f_{t_0} = \bigcup_{t>t_0} f_t = \bigcup_{n=1}^{\infty} f_{t_0+\frac{1}{n}}.$$

Hence, we can apply the Monotone Convergence Theorem, (MCT),

$$\begin{aligned}\lim_n \lambda_f\left(t_0 + \frac{1}{n}\right) &= \lim_n \mu\left(f_{t_0+\frac{1}{n}}\right) \\ &\overset{MCT}{=} \mu\left(\bigcup_{n=1}^{\infty} f_{t_0+\frac{1}{n}}\right) = \mu(f_{t_0}) = \lambda_f(t_0).\end{aligned}$$

∎

The following proposition gives us a characterization of the L^p norm of f in the space (X, μ).

Proposition 1.3.4. *For any $0 < p < \infty$, the following equality holds*

$$\int_X |f|^p d\mu = p\int_0^\infty t^{p-1}\lambda_f(t)dt.$$

Proof. Assume that μ is smooth enough in order to be able to apply Fubini.

$$\begin{aligned}\int_0^\infty t^{p-1}\lambda_f(t)dt &= \int_0^\infty t^{p-1}\int_{\{x\,:\,|f(x)|>t\}} d\mu dt \\ &\overset{\text{Fubini}}{=} \int_X \int_0^{|f(x)|} t^{p-1}dtd\mu = \int_X \frac{1}{p}t^p\Big|_0^{|f(x)|}d\mu = \frac{1}{p}\int_X |f(x)|^p d\mu.\end{aligned}$$

∎

The following definition that we need is the non-increasing rearrangement of f (or decreasing rearrangement of f).

Definition 1.3.5. We define the non-increasing rearrangement of f as

$$f^*(t) = \inf\{s > 0 : \lambda_f(s) \leqslant t\}.$$

The non-increasing rearrangement of f is also known as the right-inverse of the distribution function of f.

Now let us see some properties of $f^*(t)$.

Remarks 1.3.6. (i) If λ_f is a bijection then $f^* = \lambda_f^{-1}$.

(ii) If $t_1 > t_2$, then $f^*(t_2) \geqslant f^*(t_1)$, since $\lambda_f(s) \leqslant t_2 < t_1$, where s is the infimum of $\{s > 0 : \lambda_f(s) \leqslant t_2\}$

(iii) Let $\alpha \in \mathbb{C}$, $E \in \sigma(X)$, and take $f = \alpha\chi_E \in \mathcal{M}(X)$. Then, $f^*(t) = |\alpha|\chi_{(0,\mu(E))}(t)$.

(iv) Let $f_1, f_2 \in \mathcal{M}(X)$. Then, $(f_1 + f_2)^*(t_1 + t_2) \leqslant f_1^*(t_1) + f_2^*(t_2)$.

We can characterize the $L^p(X)$ norm of f using the non-increasing rearrangement of f.

Proposition 1.3.7. *Let $0 < p < \infty$ and $f \in \mathcal{M}(X)$. Then,*

$$\|f\|_p^p = \int_X |f|^p = \int_0^\infty (f^*(t))^p dt.$$

Before we prove this result we introduce what means that two functions are equimeasurable.

Definition 1.3.8. We say that f_1 and f_2 are equimeasurable if

$$\lambda_{f_1} \equiv \lambda_{f_2}.$$

Remark 1.3.9. By Proposition 1.3.4 that f_1 and f_2 are equimeasurable implies that

$$\|f_1\|_p = \|f_2\|_p \quad \forall 0 < p < \infty.$$

Proof of Proposition 1.3.7: We have to see that

$$\lambda_f \equiv \lambda_{f^*}.$$

Because, if this happens then by Proposition 1.3.4 we obtain

$$\int_X |f|^p = p\int_0^\infty t^{p-1}\lambda_f(t)dt = p\int_0^\infty t^{p-1}\lambda_{f^*}(t)dt = \int_0^\infty (f^*(t))^p dt.$$

So, let us see that $\lambda_f \equiv \lambda_{f^*}$.

As f is measurable we can take a sequence of a nonnegative simple functions $(f_n)_n$ such that $f_n \uparrow |f|$. Then, if we are able to prove that for each n

$$\lambda_{f_n}(t) = \lambda_{f_n^*}(t),$$

using the monotone convergence theorem we will have that

$$\lambda_f(t) = \lim_n \lambda_{f_n}(t) = \lim_n \lambda_{f_n^*}(t) = \lambda_{f^*}(t).$$

Then, we verify that $\lambda_{f_n}(t) = \lambda_{f_n^*}(t)$ for each n.

Fix n, then

$$f_n(x) = \sum_{j=1}^{r} a_j \chi_{E_j}(x).$$

Where E_j are disjoint measurable sets in X. Call

$$\begin{aligned} m_j &= \sum_{i=1}^{j} \mu(E_i), \text{ if } j \geqslant 1 \\ m_0 &= 0. \end{aligned}$$

Now, we can observe that $f^*(t) = 0$ if $t \geqslant m_r$, $f^*(t) = a_r$ if $m_r > t \geqslant m_{r-1}$, and so on.

Then,

$$f_n^*(t) = \sum_{j=1}^{r} a_j \chi_{[m_{j-1}, m_j)}(t).$$

Note that, the coefficients of f_n and f_n^* are the same a_j. So, if $|f_n(x)| < s$ implies that $f_n^*(t) < s$. And, as by definition of m_j, the measure of E_j is the measure of $m_j - m_{j-1}$. we have that $\lambda_{f_n}(t) = \lambda_{f_n^*}(t)$. ■

Definition 1.3.10. Let $T_n : B \to \mathcal{M}(X)$ be a sequence of lineal operators. We define the maximal operator

$$T^* f(x) = \sup_{n \in \mathbb{N}} |T_n f(x)|.$$

Remark 1.3.11. T^* is sublinear, this means that T^* satisfies

(i) $T^*(f + g)(x) \leqslant T^* f(x) + T^* g(x)$ (sub-additive),

(ii) $T^*(\alpha f)(x) = |\alpha| T^* f(x)$ (homogeneous).

The following definition is the main estimates in the Marcinkiewicz Interpolation Theorem.

Definition 1.3.12. Given $1 \leqslant p \leqslant \infty$, we define the weak-type (p,∞) space as follows

$$L^{p,\infty} = \{f \in \mathcal{M}(\mathbb{R}^n) : \|f\|_{p,\infty} = \sup_{t>0} t \lambda_f^{1/p}(t) < \infty\}.$$

$L^{p,\infty}$ is also called weak-L^p space.

Remarks 1.3.13. (i) $L^{p,\infty}$ is a linear space and $\|\cdot\|_{p,\infty}$ is a quasi-norm, i.e.

(a) $\|f\|_{p,\infty} \geqslant 0$ and $\|f\|_{p,\infty} = 0 \Leftrightarrow f \equiv 0$,

(b) $\|\alpha f\|_{p,\infty} = |\alpha| \|f\|_{p,\infty}$,

(c) $\|f + g\|_{p,\infty} \leqslant C_p (\|f\|_{p,\infty} + \|g\|_{p,\infty})$ with $C_p > 1$.

Moreover, $(L^{p,\infty}, \|\cdot\|_{p,\infty})$ is a quasi-Banach space.

(ii) If $p > 1$, then $L^{p,\infty}$ is a Banach space.

(iii) $L^p \subsetneq L^{p,\infty}$, this inclusion is the Chebyshev's inequality.

Let us see an example of function that belongs in $L^{p,\infty}$ but not in L^p.

Example 1.3.14. Let $f_\alpha(x) = |x|^{1-\alpha}$ with $\alpha > 0$. Then, it is clear that $f_\alpha \notin L^p$. Now, compute the distribution function of f_α.

$$\lambda_{f_\alpha}(t) = \left|\{x \in \mathbb{R}^n : |x|^{-\alpha} > t\}\right| = \left|\{x \in \mathbb{R}^n : |x| > t^{-\frac{1}{\alpha}}\}\right| = C_n t^{-\frac{n}{\alpha}}.$$

Then, putting this in $\|f_\alpha\|_{p,\infty}$.

$$\|f_\alpha\|_{p,\infty} = \sup_{t>0} t \cdot t^{-\frac{n}{\alpha p}} < \infty \Leftrightarrow \frac{n}{\alpha p} = 1 \Leftrightarrow \alpha = \frac{n}{p}.$$

Therefore, $|x|^{-n/p}$ belongs in $L^{p,\infty}$ but not in L^p.

Now we will define what is a weak type (p,p) operator and see a simpler example of a weak type $(1,1)$ operator.

Definition 1.3.15. Let $1 \leqslant p < \infty$. We say that T is a weak type (p,p) operator, if

$$T : L^p \to L^{p,\infty}.$$

If

$$T : L^p \to L^p.$$

We say that T is a strong type (p,p) operator.

Now let us give a simpler example of a weak type $(1,1)$ operator.

Definition 1.3.16. Let $f \in L^1_{\text{loc}}(\mathbb{R}^+)$. We define the Hardy operator as

$$Sf(t) = \frac{1}{t}\int_0^t f(s)ds.$$

Before proving that is a weak-type $(1,1)$ we need the Minkowski's integral inequalities.

Theorem 1.3.17 (Minkowski's integral inequalities)**.** *Let $F : X \times Y \to \mathbb{R}^+$, $1 \leqslant p \leqslant \infty$. Then,*

$$\left(\int_Y \left(\int_X F(x,y)dx\right)^p dy\right)^{\frac{1}{p}} \leqslant \int_X \left(\int_Y F(x,y)^p dy\right)^{\frac{1}{p}} dx.$$

Proof. If $p = 1$ the theorem holds by Fubini.

If $p = \infty$, in this case we have to change the integrals by the essential supremum and since

$$\sup_j \sup_i \alpha_{i,j} = \sup_i \sup_j \alpha_{i,j},$$

if we can take $\alpha_{i,j} = F(x,y)$, $\sup_i = \sup_{x\in X}$ and $\sup_j = \sup_{y\in Y}$, then we have the equality.

If $1 < p < \infty$, then via Hölder's inequality we have that

$$\|f\|_p = \sup_{g \in L^{p'}} \frac{|\int fg|}{\|g\|_{p'}}.$$

Now, if we define

$$f(y) := \int_X F(x,y)dx$$

then

$$\|f\|_p = \sup_{\|g\|_{p'} \leqslant 1} \left| \int_Y f(y)g(y)dy \right| = \sup_{\|g\|_{p'} \leqslant 1} \left| \int_Y \int_X F(x,y)dxg(y)dy \right|.$$

Applying Fubini we have

$$\|f\|_p \leqslant \sup_{\|g\|_{p'} \leqslant 1} \int_X \int_Y F(x,y)|g(y)|dydx \overset{\text{Hölder}}{\leqslant} \sup_{\|g\|_{p'} \leqslant 1} \int_X \|g\|_{p'} \left(\int_Y F(x,y)^p dy \right) dx.$$

And now, since we are taking $\|g\|_{p'} \leqslant 1$ we can take out the supremum and we are done.

$$\|f\|_p \leqslant \sup_{\|g\|_{p'} \leqslant 1} \int_X \|g\|_{p'} \left(\int_Y F(x,y)^p dy \right) dx = \int_X \left(\int_Y F(x,y)^p dy \right) dx.$$

■

Once we have proved the Minkowski's integral inequality, we are able to prove the Hardy's inequality that says that the Hardy operator is a strong type (p,p) operator if $p > 1$ and a weak type operator if $p = 1$.

Theorem 1.3.18 (Hardy's inequalities)**.** *If* $1 < p \leqslant \infty$ *then*

$$S : L^p \to L^p.$$

Moreover,

$$S : L^1 \to L^{1,\infty}.$$

Proof. If $p = \infty$, then

$$|Sf(t)| \leqslant \frac{1}{t} \int_0^t |f(s)|ds \leqslant \frac{1}{t} \int_0^t \|f\|_\infty ds = \|f\|_\infty.$$

If $1 < p < \infty$, then

$$Sf(t) = \frac{1}{t} \int_0^t f(s)ds = \frac{1}{t} \int_0^1 tf(tr)dr = \int_0^1 f(tr)dr.$$

Thus,

$$\|Sf(t)\|_p = \left(\int_0^\infty \left| \int_0^1 f(rt)dr \right|^p dt \right)^{\frac{1}{p}}.$$

Applying the Minkowski's integral inequality, we get

$$\begin{aligned}\|Sf(t)\|_p &\leqslant \int_0^1 \left(\int_0^\infty f^p(rt)dt\right)^{\frac{1}{p}} dr = \int_0^1 \left(\int_0^\infty f^p(s)\frac{ds}{r}\right)^{\frac{1}{p}} dr \\ &= \|f\|_p \int_0^1 \frac{dr}{r^{\frac{1}{p}}} = \frac{r^{\frac{1}{p'}}|_0^1}{\frac{1}{p'}}\|f\|_p = p'\|f\|_p.\end{aligned}$$

So, we have proved that S is a strong type (p,p) operator if $1 < p \leqslant \infty$.

Now, let us see it is a weak type $(1,1)$ operator. For this purpose we first check that exists $f \in L^1$ such that $Sf \notin L^1$.

Let $f = \chi_{(0,1)}$, then

$$Sf(t) = \frac{1}{t}\int_0^t \chi_{(0,1)}(s)ds = \begin{cases} 1, & 0 < t < 1 \\ \frac{1}{t}, & t \geqslant 1 \end{cases}$$

So, $Sf(t) \notin L^1$.

Now take $f \in L^1$ and we want to see that Sf belongs in $L^{1,\infty}$. So, we have to compute the λ_{Sf}.

$$\lambda_{Sf}(t) = \left|\left\{s > 0 : \frac{1}{s}\left|\int_0^s f(r)dr\right| > t\right\}\right| \leqslant \frac{\|f\|_1}{t} \Rightarrow \sup t\frac{\|f\|_1}{t} = \|f\|_1.$$

Therefore, $\|Sf\|_{1,\infty} \leqslant \|f\|_1$. Then S is a weak type $(1,1)$ operator. ■

1.3.2 Lebesgue Differentiation Theorem

In this section we will prove the Lebesgue differentiation theorem that is an analogous version of the Fundamental Calculus Theorem, and it will be useful when we want to prove that the Fourier multipliers, $\mathcal{M}_2$, are the functions of $L^\infty(\mathbb{R}^n)$.

In order to prove the Lebesgue differentiation theorem we need to define the Hardy-Littlewood maximal function.

Definition 1.3.19. Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, we define the Hardy-Littlewood maximal function of f as

$$Mf(x) := \sup_{r>0} \frac{1}{|B(x,r)|}\int_{B(x,r)} |f(y)|dy$$

where $B(x,r)$ is the ball of radius r and center x.

Remark 1.3.20. The Hardy-Littlewood maximal function of f satisfies the following properties:

(a) Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, then $Mf(x) \geqslant 0$ and there exists $x \in \mathbb{R}^n$ such that $Mf(x) = 0$ if and only if $f \equiv 0$ a.e. x.

(b) $f, g \in L^1_{\text{loc}}(\mathbb{R}^n)$, then $M(f+g)(x) \leqslant Mf(x) + Mg(x)$ and $M(\alpha g)(x) = |\alpha|Mg(x)$ for all $\alpha \in \mathbb{R}^n$.

(c) If $f \in L^\infty(\mathbb{R}^n)$ then for all $x \in \mathbb{R}^n$ we have that $Mf(x) \leqslant \|f\|_\infty$. So, we have that $\|Mf\|_\infty \leqslant \|f\|_\infty$ and that

$$\|M\| = \sup_{f \in L^\infty(\mathbb{R}^n)} \frac{\|Mf\|_\infty}{\|f\|_\infty} = 1.$$

(d) If $f \in L^1(\mathbb{R}^n) \backslash \{0\}$ then $Mf \notin L^1(\mathbb{R}^n)$.

The following theorem shows that the Hardy-Littlewood maximal function, is a continuous operator from $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$, the proof of this theorem can be found in [3, Chapter 3, pg 119].

Theorem 1.3.21 (Hardy-Littlewood Theorem)**.** *Let M be the Hardy-Littlewood maximal function, then*

$$M : L^1(\mathbb{R}^n) \to L^{1,\infty}(\mathbb{R}^n).$$

Theorem 1.3.22 (Lebesgue Differentiation Theorem)**.** *If $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ then*

$$\lim_{r \downarrow 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)dy = f(x) \quad a.e. \ x \in \mathbb{R}^n$$

where $B(x,r)$ is the ball of radius r and center x.

Proof. Let Q_j be cubes in $\mathbb{R}^n$ such that Q_j are disjoint and $\mathbb{R}^n = \bigcup_j Q_j$. Then, it suffices to prove the result for $f\chi_{Q_j} \in L^1(\mathbb{R}^n)$. Observe that if $g \in C(\mathbb{R}^1) \cap L^1(\mathbb{R}^n)$ then, by the Fundamental Calculus Theorem, we have that

$$\lim_{r \downarrow 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} g(y)dy = g(x) \quad \forall x \in \mathbb{R}^n. \tag{1.4}$$

Assume that $f \in L^1(\mathbb{R}^n)$, we only need to prove that for a given $j \in \mathbb{N}$ the set

$$A_j := \left\{ x \in \mathbb{R}^n : \limsup_{r \to 0} \left| \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)dy - f(x) \right| > \frac{1}{j} \right\}$$

has measure 0. Take $\varepsilon > 0$ and $g \in C(\mathbb{R}^1) \cap L^1(\mathbb{R}^n)$ such that $\|g - f\|_1 < \varepsilon$. Define $h = f - g$ and put $f = h + g$, by (1.4) we can rewrite A_j as

$$A_j := \left\{ x \in \mathbb{R}^n : \limsup_{r \to 0} \left| \frac{1}{|B(x,r)|} \int_{B(x,r)} h(y)dy - h(x) \right| > \frac{1}{j} \right\}.$$

But, by the inclusion $\{a + b > t\} \subset \{a > t/2\} \cup \{b > t/2\}$ we have that

$$|A_j| \leqslant \left| \left\{ x \in \mathbb{R}^n : Mh(x) > \frac{1}{2j} \right\} \right| + \left| \left\{ x \in \mathbb{R}^n : |h(x)| > \frac{1}{2j} \right\} \right|.$$

Now, using the Hardy-Littlewood Theorem 1.3.21 and the Chebyshev theorem we have that

$$|A_j| \leqslant \left| \left\{ x \in \mathbb{R}^n : Mh(x) > \frac{1}{2j} \right\} \right| + \left| \left\{ x \in \mathbb{R}^n : |h(x)| > \frac{1}{2j} \right\} \right| \leqslant K2j\varepsilon + 2j\varepsilon = 2j\varepsilon(K+1),$$

where K is the constant such that $\|Mh\|_{1,\infty} \leqslant K\|h\|_1$. Moreover, since $2j(K+1)$ is fixed and independent of ε we have that

$$|A_j| \leqslant 2j\varepsilon(K+1) \to 0, \quad \text{as } \varepsilon \to 0.$$

∎

1.3.3 Fourier Transform

In this section we will introduce the Fourier Transform and some of its properties for functions of $L^1(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n)$.

Let us begin by defining the Fourier Transform for functions in $L^1(\mathbb{R}^n)$.

Definition 1.3.23. For all $f \in L^1(\mathbb{R}^n)$ we define its Fourier Transform as

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}dx$$

with $\xi \in \mathbb{R}^n$.

Proposition 1.3.24. *The Fourier transform is a continuous map from $L^1(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$.*

Proof. Let $f \in L^1(\mathbb{R}^n)$, then we have that

$$\sup_{\xi\in\mathbb{R}^n} |\hat{f}(\xi)| \leqslant \sup_{\xi\in\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x)||e^{-ix\cdot\xi}|dx = \sup_{\xi\in\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x)|dx = \sup_{\xi\in\mathbb{R}^n} \|f\|_1 = \|f\|_1.$$

■

Theorem 1.3.25 (Hat Theorem). *If $f, g \in L^1(\mathbb{R}^n)$, then*

$$\int_{\mathbb{R}^n} \hat{f}(\xi)g(\xi)d\xi = \int_{\mathbb{R}^n} f(x)\hat{g}(x)dx.$$

Proof. The proof of this theorem follows by applying Fubini's Theorem to the definition of $\mathscr{F}$. In fact, by definition of $\hat{f}$ we have that

$$\int_{\mathbb{R}^n} \hat{f}(\xi)g(\xi)d\xi = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}dx\right) g(\xi)d\xi.$$

Now, since $f, g \in L^1$ we can apply Fubini and we obtain that

$$\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(x)e^{-ix\cdot\xi}dx\right) g(\xi)d\xi = \int_{\mathbb{R}^n} f(x) \left(\int_{\mathbb{R}^n} g(\xi)e^{-ix\cdot\xi}d\xi\right) dx = \int_{\mathbb{R}^n} f(x)\hat{g}(x)dx.$$

■

Now, we will define the convolution of two functions.

Definition 1.3.26. Let $f, g \in L^1$ and $\Omega = \mathbb{R}^n$, then we define the convolution of f and g as

$$(f * g)(x) = \int_{\mathbb{R}^n} g(x-y)f(y)dy = \Lambda_f(\tau_x\tilde{g}),$$

where $\tilde{g}(z) = g(-z)$ and $\tau_x g(y) = g(y-x)$.

Remark 1.3.27. If $f, g \in L^1(\mathbb{R}^n)$, then

(a) $(\widehat{\tau_x f})(\xi) = \hat{f}(\xi)e^{-ix\cdot\xi}$, where $\tau_x f(y) = f(y-x)$.

(b) $(e^{ix\cdot y}f(y))\hat{}(\xi) = \tau_x\hat{f}(\xi)$.

(c) $(\widehat{f * g})(\xi) = \hat{f}(\xi)\hat{g}(\xi)$.

(d) If $\lambda > 0$ and $h(x) = f(x/\lambda)$ then $\hat{h}(\xi) = \lambda^n\hat{f}(\lambda\xi)$.

1.3.4 Schwarz Class

In this section we will introduce the class of Schwarz functions and we will see some properties of this class, in particular, we will apply the Fourier Transform to this class and use some of these results to prove some properties of $\mathscr{F}$ in L^1.

Let P be a polynomial of n variables of the form

$$P(\xi) = \sum_{\alpha} C_\alpha \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}.$$

Let $D_\alpha = i^{-|\alpha|} D^\alpha$, i.e.

$$D_\alpha = \prod_{j=1}^{n} \left(\frac{\partial}{i \partial x_j} \right)^{\alpha_j}.$$

Then, $P(D) = \sum_\alpha C_\alpha D_\alpha$ and $P(-D) = \sum_\alpha C_\alpha (-1)^{|\alpha|} D_\alpha$. This definition of $P(D)$ will be the main definition in the chapter of PDE's. Now we are going to define the space S.

Definition 1.3.28. A function $f \in C^\infty(\mathbb{R}^n)$ belongs in the space $S_n (= S)$ if for all $N \in \mathbb{N}$ we have that

$$P_N(f) = \sup_{\substack{x \in \mathbb{R}^n \\ |\alpha| \leqslant N}} (1 + |x|^2)^N |D_\alpha f(x)| < \infty.$$

Remark 1.3.29.

(a) If $f \in S$ and Q is any polynomial then

$$|D_\alpha f(x)| \lesssim |Q(x)|^{-1} \quad \forall x \in \mathbb{R}^n.$$

(b) $(S, \{P_n\}_n)$ is a Fréchet space.

Theorem 1.3.30.

(a) If P is a polynomial and $g \in S$ then the following mappings are linear and continuous

$$\begin{aligned} S &\to S \\ f &\mapsto Pf \\ f &\mapsto gf \\ f &\mapsto D_\alpha f. \end{aligned}$$

(b) If $f \in S$ then $(P(D)f)\hat{} = P\hat{f}$ and $(Pf)\hat{} = P(-D)\hat{f}$.

(c) The mapping $\mathscr{F} : S \to S$ is linear and continuous.

Proof. We will begin by prove (a). First $Pf \in C^\infty$, and for all $N \in \mathbb{N}$ by Leibniz Formula we have that

$$\sup_{\substack{x \in \mathbb{R}^n \\ |\alpha| \leqslant N}} (1 + |x|^2)^N |D_\alpha (Pf)(x)| \leqslant C P_{N+M}(f) \leqslant \infty$$

where $M = deg(P)$. To see that $gf \in S$ we have to use the same argument. Now since S is a Fréchet space the continuity follows from the Closed Graph Theorem 1.2.22. In fact

$$f_n \to f \text{ in } S \Rightarrow f_n(x) \to f(x) \forall x,$$

and

$$gf_n \to h \text{ in } S \Rightarrow g(x)f_n(x) \to h(x) \forall x.$$

That $f \mapsto D_\alpha f$ is continuous follows since S is Fréchet and $f \in C^\infty$.

Now we are going to prove (b), but we can reduce to prove for $P(x) = x_1$ by symmetry and iteration. First we will see that $S \subset L^1$. Let $g \in S$ then

$$\int_{\mathbb{R}^n} |g(x)|dx = \int_{\mathbb{R}^n} (1+|x|^2)^N |g(x)| \frac{1}{(1+|x|^2)^N} dx \leqslant P_N(g) \int_{\mathbb{R}^n} \frac{1}{(1+|x|^2)^N} dx.$$

But this type of integrals are finite if and only if $2N > n$. So taking N big enough we have that

$$\int_{\mathbb{R}^n} |g(x)|dx < \infty.$$

So, since $S \subset L^1$ we have that

$$(P(D)f)\widehat{\ }(\xi) = \int_{\mathbb{R}^n} P(D)f(x)e^{-ix\cdot\xi}dx = \frac{1}{i}\int_{\mathbb{R}^n} \frac{df}{dx_1}(x)e^{-ix\cdot\xi}dx.$$

Now using Fubini and taking $\bar{x} = (x_2, \cdots, x_n)$, we obtain that

$$\frac{1}{i}\int_{\mathbb{R}^n} \frac{df}{dx_1}(x)e^{-ix\cdot\xi}dx = \frac{1}{i}\int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}} \frac{df}{dx_1}(x)e^{-ix_1\xi_1}dx_1\right) e^{-i\bar{x}\cdot\bar{\xi}}d\bar{x}.$$

Now integrating by parts

$$\begin{aligned}\frac{1}{i}\int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}} \frac{df}{dx_1}(x)e^{-ix_1\xi_1}dx_1\right) e^{-i\bar{x}\cdot\bar{\xi}}d\bar{x} &= \int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}} \xi_1 \frac{df}{dx_1}(x)e^{-ix_1\xi_1}dx_1\right) e^{-i\bar{x}\cdot\bar{\xi}}d\bar{x} \\ &= \xi_1 \frac{1}{i}\int_{\mathbb{R}^n} \frac{df}{dx_1}(x)e^{-ix\cdot\xi}dx = \xi_1\hat{f}(\xi) \\ &= P(\xi)\hat{f}(\xi).\end{aligned}$$

Therefore, we have that $(P(D)f)\widehat{\ } = P\hat{f}$. Now we will see that $(Pf)\widehat{\ } = P(-D)\hat{f}$. Let $t = (t_1, \cdots, t_n)$ and $t' = (t_1 + \varepsilon, \cdots, t_n)$. Then

$$\frac{\hat{f}(t') - \hat{f}(t)}{i\varepsilon} = \int_{\mathbb{R}^n} x_1 f(x) \frac{e^{-ix_1\varepsilon} - 1}{i\varepsilon x_1} e^{-ix\cdot t}dx.$$

But,

$$\lim_{\varepsilon\to 0} \frac{\hat{f}(t') - \hat{f}(t)}{i\varepsilon} = \frac{1}{i}\frac{d}{dx_1}\hat{f}(t).$$

In the other hand, since $x_1 f(x) \in L^1(\mathbb{R}^n)$ and

$$\frac{e^{-ix_1\varepsilon} - 1}{i\varepsilon x_1}$$

is bounded, we can apply the Dominated Convergence Theorem, and we arrive at the fact that there exists

$$\frac{d\hat{f}}{dx_1}$$

and

$$\frac{1}{i}\frac{d}{dx_1}\hat{f}(t) = -(Pf)\hat{}.$$

Then

$$\widehat{Pf} = -\frac{1}{i}\frac{d}{dx_1}\hat{f}(t) = P(-D)f.$$

It remains to see (c), let $f \in S$ and define

$$g(x) = (-1)^{|\alpha|}x^\alpha f(x) \in S$$

then $\hat{g}(\xi) = D^\alpha \hat{f}(\xi)$. Therefore, $P(\xi)\hat{g} = (P(D)g)\hat{}(\xi) \in L^\infty$, then $\hat{f} \in S$. The linearity of $\mathscr{F}$ follows from definition and the continuity follows from the Closed Graph Theorem 1.2.22 using that if $f_i \to f$ in S then $f_i \to f$ in L^1 and therefore $\hat{f}_i(\xi) \to f(\xi)$ for all ξ. ∎

Remark 1.3.31. The function $\phi(x) = \exp(-|x|^2/2) \in S$ and $\phi = \hat{\phi}$.

Now, we are going to see two Inversion Theorems, the first version deals with functions in S and the second deals with functions in L^1.

Theorem 1.3.32 (Inversion Theorem)**.**

(a) If $g \in S$ then

$$g(x) = \int_{\mathbb{R}^n} \hat{g}(\xi)e^{ix\cdot\xi}d\xi.$$

(b) The Fourier Transform $\mathscr{F}$ is an injective and continuous mapping from S to S, it has period 4 and its inverse

$$\mathscr{F}^{-1} : S \to S$$

is continuous.

Proof. We will begin by proving (a), let $\phi(x) = \exp(-|x|^2/2)$ and $g \in S$, then by the Hat Theorem 1.3.25

$$\int_{\mathbb{R}^n} g\left(\frac{t}{\lambda}\right)\hat{\phi}(t)dt = \int_{\mathbb{R}^n} \lambda^{-n}\hat{g}(\lambda t)\phi(t)dt = \int_{\mathbb{R}^n} \hat{g}(t)\phi\left(\frac{t}{\lambda}\right)dt.$$

Using again the Dominated Convergence Theorem, the Remark 1.3.31 and letting $\lambda \to \infty$ we obtain that

$$g(0) = g(0)\int_{\mathbb{R}^n}\hat{\phi} = \phi(0)\int_{\mathbb{R}^n}\hat{g}(x)dx = \int \hat{g}(x)dx.$$

So, we have (a) proved for $x = 0$. Now consider $x \neq 0$, then

$$g(x) = (\tau_{-x}g)(0) = \int_{\mathbb{R}^n}(\tau_{-x}g)\hat{}(y)dy = \int_{\mathbb{R}^n}\hat{g}(y)e^{ix\cdot y}dy.$$

Now we are going to see (b), but by (a) by know that $\mathscr{F}$ is one-to-one, also we proved the continuity in last Theorem, so if we see that $\mathscr{F}^2 g = \tilde{g}(= g(-x))$ then we will have that $\mathscr{F}^4 g = g$ and that $\mathscr{F}^{-1} = \mathscr{F}^3$ which is continuous. But we have that

$$g(-x) = (\tau_x g)(0) = \int_{\mathbb{R}^n} (\tau_x g)\hat{}(y)dy = \int_{\mathbb{R}^n} \hat{g}(y)e^{-ix\cdot y}dy = \mathscr{F}^2 g(x).$$

■

Corollary 1.3.33. *If $f, g \in S$ then we have*

(a) $f * g \in S$

(b) $(\widehat{f * g})(\xi) = \hat{f}(\xi)\hat{g}(\xi)$.

Proof. We will begin by proving (a), since $f, g \in S$ we have that $\hat{f}, \hat{g} \in S$, this implies that $\hat{f}\hat{g} = (f * g)\hat{} \in S$. Therefore, $f * g \in S$. The proof of (b) is the same that for $L^1(\mathbb{R}^n)$. ■

The following theorem deals with functions in $L^1(\mathbb{R}^n)$.

Theorem 1.3.34 (Inversion Theorem). *If f and $\hat{f}$ are in $L^1(\mathbb{R}^n)$, then*

$$f(x) = \int_{\mathbb{R}^n} \hat{f}(\xi)e^{i\xi x}d\xi \quad a.e.\ x.$$

Proof. Let

$$f_0(x) = \int_{\mathbb{R}^n} \hat{f}(t)e^{ixt}dt$$

and $g \in S$. Then

$$\int_{\mathbb{R}^n} f_0(x)\hat{g}(x)dx = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \hat{f}(t)e^{ixt}dt\right)\hat{g}(x)dx.$$

Using Fubini and the Hat Theorem 1.3.25 we arrive at

$$\begin{aligned}\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \hat{f}(t)e^{ixt}dt\right)\hat{g}(x)dx &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \hat{g}(x)dxe^{ixt}\right)\hat{f}(t)dt \\ &= \int_{\mathbb{R}^n} g(t)\hat{f}(t)dt = \int_{\mathbb{R}^n} f(t)\hat{g}(t)dt.\end{aligned}$$

Since $\mathscr{F} : S \to S$ we can write $\hat{g}$ as g. Then

$$\int_{\mathbb{R}^n} (f_0 - f)(t)g(t)dt = 0 \quad \forall g \in S.$$

Then, by density $f_0(t) = f(t)$ a.e.t. ■

Theorem 1.3.35 (Plancherel's Theorem). *If $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, then $\|f\|_2 = \|\hat{f}\|_2$.*

Proof. Let $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, then we have that

$$\|f\|_2^2 = \int_{\mathbb{R}^n} |f(x)|^2 dx = (\tilde{f} * \bar{f})(0),$$

where $\tilde{f}(x) = f(-x)$ and $\bar{f}$ is the conjugate of f. Since $f, \tilde{f} \in L^1(\mathbb{R}^n)$, by the Remark 1.3.27, we have that $g = \tilde{f} * \bar{f}$ is continuous and is in $L^1(\mathbb{R}^n)$. Moreover

$$\hat{g}(\xi) = \hat{\tilde{f}}(\xi)\hat{\bar{f}}(\xi) = \tilde{\hat{f}}(\xi)\bar{\tilde{\hat{f}}}(\xi) = |\tilde{\hat{f}}(\xi)|^2.$$

Now, we can apply the inversion Theorem 1.3.34 and obtain that

$$g(x) = \int_{\mathbb{R}^n} |\tilde{\hat{f}}(\xi)|^2 e^{ix\cdot\xi} d\xi \quad \text{a.e. } x.$$

In particular, for $x = 0$, we have that

$$\|f\|_2^2 = g(0) = \int_{\mathbb{R}^n} |\tilde{\hat{f}}(\xi)|^2 d\xi = \|\hat{f}\|_2^2.$$

Then, we have that $\|f\|_2 = \|\hat{f}\|_2$. ∎

Proposition 1.3.36. *The space $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$.*

Proof. Let $f \in L^2(\mathbb{R}^n)$ and let $f_T(x) = f(x)\chi_{B(0,T)}(x)$, we are going to see that $f_T \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Since $\|f_T\|_2 \leqslant \|f\|_2$ we have that $f_T \in L^2(\mathbb{R}^n)$, also since $f \in L^2(\mathbb{R}^n)$ we have that $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ so $f_T \in L^1(\mathbb{R}^n)$. Therefore $f_T \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Notice that

$$\lim_{T\uparrow\infty} |f_T(x) - f(x)| = 0 \quad \text{a.e.} x \in \mathbb{R}^n.$$

Even more, since $|f_T(x) - f(x)| \leqslant |f(x)|$ we can dominate $|f_T(x) - f(x)|^2$ by $|f(x)|^2$. Then we can apply the Dominated Convergence Theorem to

$$\lim_{T\uparrow\infty} \int_{\mathbb{R}^n} |f_T(x) - f(x)|^2 dx = \int_{\mathbb{R}^n} \lim_{T\uparrow\infty} |f_T(x) - f(x)|^2 dx = 0.$$

With this we can conclude that $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$. ∎

Remark 1.3.37. Let $f \in L^2(\mathbb{R}^n)$ since, by Proposition 1.3.36 $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, we have that if $(f_j)_j \subset L^1 \cap L^2$ so that $f = L^2 - \lim f_j$, then by Plancherel's Theorem 1.3.35, we have that

$$\|f_j - f_i\|_2 = \|\hat{f}_j - \hat{f}_i\|_2 \Rightarrow \exists L^2 - \lim \hat{f}_j.$$

Hence, we can define the Fourier Transform in $L^2(\mathbb{R}^n)$ as

$$\mathscr{F}(f)(\xi) = L^2 - \lim \hat{f}_j,$$

such that $f_j \to f$ in $L^2(\mathbb{R}^n)$.

Remark 1.3.38. (a) The Fourier Transform in $L^2(\mathbb{R}^n)$ is well defined.

(b) If $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, then $\mathscr{F}(f) = \hat{f}$.

(c) If $f \in L^2(\mathbb{R}^n)$ and $f_r = f\chi_{B_r(0)} \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, then

$$f_r \to f \quad \text{in } L^2 \Rightarrow \mathscr{F} f = L^2 - \lim_{r\uparrow\infty} \hat{f}_r = L^2 - \lim_{r\uparrow\infty} \int_{B_r(0)} f(x)e^{-ix\xi}dx.$$

For $n > 1$ it still unknown if

$$\mathscr{F} f(\xi) = \lim_{R\uparrow\infty} \int_{B_R(0)} f(x)e^{-ix\cdot\xi}dx \quad \text{a.e. } x.$$

As a consequence of Plancherel's Theorem 1.3.35 and Remark 1.3.37 we have the following theorem.

Theorem 1.3.39 (Parseval's Theorem)**.** *If $f \in L^2(\mathbb{R}^n)$ then $\|\mathcal{F}(f)\|_2 = \|f\|_2$.*

Proof. By Remark 1.3.37 the Fourier transform extends in a uniquely way in $L^2(\mathbb{R}^n)$. Moreover we have that, if $g_i \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ satisfying that $L^2 - \lim_{i\uparrow\infty} g_i(x) = f(x)$, then, by Plancherel's Theorem 1.3.35, we have that

$$\|\mathcal{F}(f)\|_2 = \|\lim_{i\uparrow\infty} \hat{g}_i\|_2 = \lim_{i\uparrow\infty} \|g_i\|_2 = \|f\|_2.$$

∎

1.4 Distribution Theory

In this section we are going to define what is a distribution and a tempered distribution, also we will see that if $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ then we can construct a distribution that acts by integration.

Let Ω be an open set in $\mathbb{R}^n$ and for each compact set $K \subset \Omega$ ($K \in \mathcal{K}(\Omega)$), consider the subspace

$$D_K = \{f \in C^\infty(\Omega) : \text{supp}(f) \subset K\}$$

with the topology induced by $C^\infty(\Omega)$. Recall that

$$(C^\infty(\Omega), \{P_{K_j,j}\}_j)$$

is a Fréchet space (see Section 1.2.4), where $K_j \subset \overset{\circ}{K}_{j+1} \subset \cdots \subset \Omega$, $\bigcup K_j = \Omega$ and

$$P_{K_j,j}(f) = \sup_{\substack{|\alpha|\leqslant j \\ x\in K_j}} |D^\alpha f(x)|.$$

Since D_K is closed in $C^\infty(\Omega)$, we have that D_K is also a Fréchet space.

Observe that in this subspace the topology is also given by the following family of seminorms:

$$\|f\|_N := \sup_{\substack{|\alpha|\leqslant N \\ x\in\Omega}} |D^\alpha f(x)|.$$

Now we are going to define the test functions space.

Definition 1.4.1. The test space is $D(\Omega) = \bigcup_{K \in \mathcal{K}(\Omega)} D_K$, that is the set of all $C^\infty(\Omega)$ functions with compact support in Ω.

In general, $D(\Omega)$ is not a Fréchet space.

Definition 1.4.2. Let $\varphi, (\varphi_j)_j \subset D(\Omega)$ we say that $\varphi_j \to \varphi$ if and only if there exist $K \in \mathcal{K}(\Omega)$ and $j_0 \in \mathbb{N}$ such that $\forall j \geqslant j_0$ we have that $\varphi_j \in D_K$ and $\varphi_j \to \varphi$ in D_K.

Now we can define what is a distribution.

Definition 1.4.3. A distribution, Λ, is a linear and continuous functional over $D(\Omega)$ such that $\Lambda \in D'(\Omega)$ and

$$\Lambda : D(\Omega) \to \mathbb{K}$$

in the following sense:

$\forall K \in \mathcal{K}(\Omega)$ there exist $N \in \mathbb{N}$ and $C > 0$ such that

$$|\Lambda(\varphi)| \leqslant C\|\varphi\|_N, \quad \forall \varphi \in D_K.$$

And we will say that $(\Lambda_j)_j \subset D'(\Omega)$ converges to $\Lambda \in D'(\Omega)$ if and only if

$$\Lambda_j(\varphi) \to \Lambda(\varphi), \quad \forall \varphi \in D_K.$$

Remark 1.4.4. The map

$$\begin{aligned} L^1_{\text{loc}}(\Omega) &\to D'(\Omega) \\ f &\to \Lambda_f \end{aligned}$$

where

$$\Lambda_f(\varphi) = \int_{\mathbb{R}^n} f(x)\varphi(x)dx$$

is injective but not exhaustive.

1.4.1 Tempered Distributions

The aim of the tempered distributions is that if $\varphi \in D$ and $\hat{\varphi}$ is its Fourier Transform then we never have that $\hat{\varphi} \in D$. So, in general, we cannot apply the Fourier Transform to distributions.

Definition 1.4.5. A tempered distribution u is an element of the dual space S'.

Remark 1.4.6. Let $u \in S'$ as $D \hookrightarrow S$ we have that $u|_D$ is a distribution.

Theorem 1.4.7. *If P is a polynomial, $g \in S$ and $u \in S'$ then $D^\alpha u$, Pu and gu are also tempered distributions.*

The proof of this theorem can be found in [9, Theorem 7.13].

Definition 1.4.8. If $u \in S'$ we define $\hat{u}(\varphi) := u(\hat{\varphi})$ where $\varphi \in S$.

Now we are going to see that this definition is consistent when $f \in L^1(\mathbb{R}^n)$ and $u \in S'$ is of the form

$$u_f(\phi) = \int_{\mathbb{R}^n} f(x)\phi(x)dx.$$

Let $\varphi \in S$, then by definition we have that

$$\hat{u}_f(\varphi) = u_f(\hat{\varphi}) = \int_{\mathbb{R}^n} f(x)\hat{\varphi}(x)dx.$$

Using the Hat Theorem 1.3.25 we arrive at

$$\hat{u}_f(\varphi) = u_f(\hat{\varphi}) = \int_{\mathbb{R}^n} f(x)\hat{\varphi}(x)dx = \int_{\mathbb{R}^n} \varphi(x)\hat{f}(x)dx = u_{\hat{f}}(\varphi).$$

Remark 1.4.9. The map $\mathscr{F} : S' \to S'$ is continuous, bijective, has period 4 and its inverse is also continuous.

The proof of this result can be found in [9, Theorem 7.15].

Example 1.4.10. $\hat{1} = \delta_0$ and $\hat{\delta_0} = 1$.

Let $\varphi \in S(\mathbb{R})$, by the Definition 1.4.8 we have that

$$\hat{1}(\varphi) = 1(\hat{\varphi}) = \int_{\mathbb{R}} \hat{\varphi}(x)dx = \int_{\mathbb{R}} \hat{\varphi}(x)e^{ix\cdot 0}dx.$$

And, by the Inversion Theorem 1.3.32 we obtain that

$$\hat{1}(\varphi) = \int_{\mathbb{R}} \hat{\varphi}(x)e^{ix\cdot 0}dx = \varphi(0) = \delta_0(\varphi).$$

Therefore, $\hat{1} = \delta_0$ in the sense of distributions. Now, again by the Definition 1.4.8 we have that

$$\hat{\delta_0}(\varphi) = \delta_0(\hat{\varphi}) = \hat{\varphi}(0) = \int_{\mathbb{R}^n} \varphi(x)dx = 1(\varphi).$$

Hence, $\hat{\delta_0} = 1$ in the sense of distributions.

Chapter 2

Classical Methods in the Interpolation Theory

In this chapter we study the classical methods in the interpolation theory, these are the Riesz-Thorin Theorem and the Marcinkiewicz Theorem. These results provided the impetus for the study of the interpolation theory, the proof of the first theorem gives the idea behind the complex interpolation method, meanwhile the proof of the second theorem provides the construction of the real interpolation method.

2.1 Riesz-Thorin Theorem

The first theorem that we will prove is the Riesz-Thorin Theorem. For this theorem we assume that the scalars are complex numbers.

Theorem 2.1.1 (Riesz-Thorin interpolation Theorem)**.** *Let (U, μ) and (V, ν) be two measurable spaces. Assume that $p_0 \neq p_1$, $q_0 \neq q_1$, and that*

$$T : L^{p_0}(U) \to L^{q_0}(V)$$

is bounded with norm M_0, and that

$$T : L^{p_1}(U) \to L^{q_1}(V)$$

is also bounded with norm M_1. Then

$$T : L^{p}(U) \to L^{q}(V)$$

is bounded with norm

$$M \leqslant M_0^{1-\theta} M_1^{\theta}$$

provided that $0 < \theta < 1$ and

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}. \tag{2.1}$$

Notice that the points $(1/p, 1/q)$ described in (2.1) can be geometrically interpreted as the points in the line with end points $(\frac{1}{p_0}, \frac{1}{q_0})$ and $(\frac{1}{p_1}, \frac{1}{q_1})$.

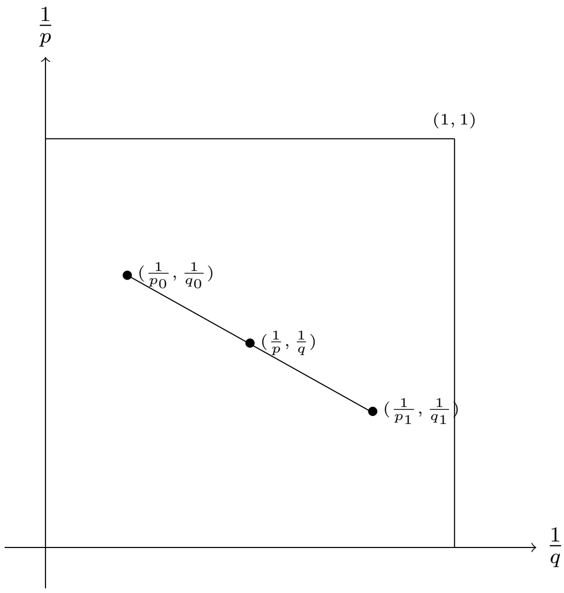

Figure 2.1: Geometric interpretation of (2.1)

Proof. Let

$$\frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1},\quad \frac{1}{q}=\frac{1-\theta}{q_0}+\frac{\theta}{q_1}.$$

And let $\frac{1}{q'}=1-\frac{1}{q}$. Then, by Hölder's inequality,

$$M=\sup\left\{\left|\int_V Tf(y)g(y)d\nu\right| : \|f\|_p=\|g\|_{q'}=1\right\}.$$

Since $p<\infty$, $q'<\infty$ we can assume that $f\in L^p$ and $g\in L^{q'}$ are bounded with compact supports.

For $0\leqslant \Re(z)\leqslant 1$, we put

$$\frac{1}{p(z)}=\frac{1-z}{p_0}+\frac{z}{p_1},\quad \frac{1}{q'(z)}=\frac{1-z}{q'_0}+\frac{z}{q'_1},$$

and

$$\varphi(z)=\varphi(x,z)=|f(x)|^{\frac{p}{p(z)}}\frac{f(x)}{|f(x)|},\quad x\in U,$$
$$\psi(z)=\psi(y,z)=|g(y)|^{\frac{q'}{q'(z)}}\frac{g(y)}{|g(y)|},\quad y\in V.$$

Now, we will see that $\varphi(z)\in L^{p_j}(U)$.

$$\begin{aligned}\|\varphi\|_p^p=\int_U|\varphi(x,z)|^{p_j}d\mu&=\int_U|f(x)|^{p_jp/p(z)}\frac{|f(x)|^{p_j}}{|f(x)|^{p_j}}d\mu\\&=\int_U|f(x)|^{p_jp/p(z)}d\mu\leqslant\int_U|f(x)|^{\Re(p_jp/p(z))}d\mu.\end{aligned}$$

Since, $0 \leqslant \Re(z) \leqslant 1$, we have that $p_j/\Re(p(z)) \leqslant 1$. Then, we obtain

$$\begin{aligned}\int_U |\varphi(x,z)|^{p_j} d\mu &\leqslant \int_U |f(x)|^{\Re(p_j p/p(z))} d\mu \\ &\leqslant \int_U |f(x)|^p d\mu \leqslant \|f\|_p^p.\end{aligned}$$

And the same argument can be used to see that $\psi(z) \in L^{q_j'}(V)$.

Since $\varphi(z) \in L^{p_j}(U)$, then $T\varphi \in L^{q_j}(V)$ with $j = 0, 1$. Also, we can check that $\varphi'(z) \in L^{p_j}(U)$, $\psi'(z) \in L^{q_j'}(V)$ and thus also that $(T\varphi)'(z) \in L_{q_j}(V)$ if $(0 < \Re(z) < 1)$. This implies the existence of

$$F(z) = \int_V T\varphi(y)\psi(y) d\nu, \quad 0 \leqslant \Re(z) \leqslant 1.$$

Even more, we have that F is an analytic function on the open strip $0 < \Re(z) < 1$, and bounded and continuous on the closed strip $0 \leqslant \Re(z) \leqslant 1$.

Also, by definition of φ and ψ we have that

$$\begin{aligned}\|\varphi(it)\|_{p_0} &= \||f|^{p/p_0}\|_{p_0} &&= \|f\|_p^{p/p_0} = 1, \\ \|\varphi(1+it)\|_{p_1} &= \||f|^{p/p_1}\|_{p_1} &&= \|f\|_p^{p/p_1} = 1,\end{aligned}$$

and the same for ψ

$$\|\psi(it)\|_{q_0'} = \|\psi(1+it)\|_{q_1'} = 1.$$

Therefore, we obtain

$$\begin{aligned}|F(it)| &\overset{\text{Hölder}}{\leqslant} \|T\varphi(it)\|_{p_0} \|\psi(it)\|_{q_0'} \leqslant M_0, \\ |F(1+it)| &\overset{\text{Hölder}}{\leqslant} \|T\varphi(1+it)\|_{p_1} \|\psi(1+it)\|_{q_1'} \leqslant M_1.\end{aligned}$$

Moreover, since $p(\theta) = p$ and $q'(\theta) = q'$, we have

$$\varphi(\theta) = f, \quad \psi(\theta) = g,$$

and so,

$$F(\theta) = \int_V Tf(y)g(y) d\nu.$$

Using now Theorem 1.1.2 we obtain

$$\left| \int_V Tf(y)g(y) d\nu \right| \leqslant M_0^{1-\theta} M_1^{\theta},$$

or what is the same (taking supremum in the both sides and using that $M_0^{1-\theta} M_1^{\theta}$ is constant)

$$M \leqslant M_0^{1-\theta} M_1^{\theta}.$$

■

2.2 The Marcinkiewicz Theorem

In this section we give the statement and the proof of the Marcinkiewicz Interpolation Theorem. As we said before this theorem contains the main ideas used in the real interpolation method. We are going to use some results seen in Section 1.3.

Also, it is important to note that from now the functions f can take values in $\mathbb{R}$ and in $\mathbb{C}$ as a difference with the Riesz-Thorin Theorem 2.1.1 where the values had to be complex.

Another important difference between these two theorems is that now, in the hypothesis, we replace the strong spaces (L^p) for the weak spaces who are largest spaces. Therefore, this theorem can be used where Theorem 2.1.1 fails.

So, let us give the statement of the Marcinkiewicz Interpolation Theorem.

Theorem 2.2.1 (The Marcinkiewicz Interpolation Theorem)**.** *Let (U, μ) and (V, ν) be two measurable spaces. Assume that $p_0 \neq p_1$, $q_0 \neq q_1$, and that*

$$T : L^{p_0}(U) \to L^{q_0,\infty}(V)$$

is bounded with norm M_0^*, and that

$$T : L^{p_1}(U) \to L^{q_1,\infty}(V)$$

is also bounded with norm M_1^*.
Let

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1},$$

and assume that

$$p \leqslant q. \tag{2.2}$$

Then,

$$T : L^p(U) \to L^q(V)$$

with norm M satisfying

$$M \leqslant C_\theta (M_0^*)^{1-\theta}(M_1^*)^\theta.$$

Before we start with the proof we pay attention with the statement. Notice that we have one hypothesis more than in the Theorem 2.1.1 that is the restriction (2.2). Moreover, notice that in this theorem M satisfies

$$M \leqslant C_\theta (M_0^*)^{1-\theta}(M_1^*)^\theta$$

while

$$M \leqslant (M_0^*)^{1-\theta}(M_1^*)^\theta$$

this is because if the scalars are real then we can only prove the convexity inequality

$$M \leqslant C_\theta (M_0^*)^{1-\theta}(M_1^*)^\theta.$$

Now, let us prove the theorem but only for $q_0 = p_0$ and $p_1 = q_1$. The general case can be found in [13, Theorem 4.6, p.112].

Proof. Let $p_0 = q_0$ and that $p_1 = q_1$ then we define p as

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$$

and we have that $p = q$.

Let $f \in L^p$, we want to see that Tf belongs in L^p, that is, there exists $M < \infty$ such that

$$\|Tf\|_p \leqslant M\|f\|_p.$$

Define

$$f_0(x) = f_0(t,x) = \begin{cases} f(x), & \text{if } x \in E \\ 0, & \text{otherwise} \end{cases}$$

where $E \subset \{x \in U : |f(x)| \geqslant f^*(t)\}$.

Define also $f_1(x) = f(x) - f_0(x)$. Then, $f_0 \in L^{p_0}(U)$ and $f_1 \in L^{p_1}(U)$.

Recall that we want to see that $\|Tf\|_p \leqslant M\|f\|_p$, and that by can write $\|Tf\|_p$ as follows

$$\|Tf\|_p^p = \int_0^\infty ((Tf)^*(t))^p dt.$$

Also, we have that

$$(Tf)^*(t) = (Tf_0 + Tf_1)^*(t) \leqslant (Tf_0)^*(t/2) + (Tf_1)^*(t/2).$$

Therefore,

$$\|Tf\|_p^p = \int_0^\infty ((Tf)^*(t))^p dt \leqslant \int_0^\infty ((Tf_0)^*(t/2))^p dt + \int_0^\infty ((Tf_1)^*(t/2))^p dt.$$

And, by the hypothesis of T, we have that

$$(Tf_j)^*(t/2) \leqslant M_j^* t^{-1/p_j} \|f_j\|_{p_j} \quad \text{with } j = 0, 1.$$

Thus,

$$\|Tf\|_p^p \leqslant \int_0^\infty (M_0^* t^{-1/p_0} \|f_0\|_{p_0})^p dt + \int_0^\infty (M_1^* t^{-1/p_1} \|f_1\|_{p_1})^p dt.$$

So,

$$\begin{aligned} \|Tf\|_p^p &\leqslant (M_0^*)^p \int_0^\infty (t^{-1/p_0} \|f_0\|_{p_0})^p dt + (M_1^*)^p \int_0^\infty (t^{-1/p_1} \|f_1\|_{p_1})^p dt \\ &= (M_0^*)^p I_0 + (M_1^*)^p I_1. \end{aligned}$$

Now, we will study I_0 and I_1 separately. We start with I_0.

$$I_0 = \int_0^\infty (t^{-1/p_0} \|f_0\|_{p_0})^p dt = \int_0^\infty \left(t^{-p_0/p_0} \|f_0\|_{p_0}^{p_0} \right)^{p/p_0} dt.$$

Applying the definition of f_0 and $\|\cdot\|_{p_0}$ we obtain

$$I_0 = \int_0^\infty \left(t^{-p_0/p_0} \int_0^t (f^*(s))^{p_0} ds \right)^{p/p_0} dt.$$

Then, we have that

$$I_0 = \int_0^\infty \left(\int_0^t (f^*(s))^{p_0} \frac{ds}{t} \right)^{p/p_0} dt.$$

And taking $\sigma = s/t$, we obtain

$$I_0 = \int_0^\infty \left(\int_0^1 (f^*(t\sigma))^{p_0} d\sigma \right)^{p/p_0} dt.$$

Now, in order to be able to apply Minkowski's integral inequalities we take $(I_0)^{1/p}$, notice that $\|Tf\|_p \leqslant (I_0 + I_1)^{1/p} \leqslant I_0^{1/p} + I_1^{1/p}$.

$$I_0^{1/p} = \left(\int_0^\infty \left(\int_0^1 (f^*(t\sigma))^{p_0} d\sigma \right)^{p/p_0} dt \right)^{1/p} \leqslant \left(\int_0^1 \left(\int_0^\infty (f^*(t\sigma))^{p} dt \right)^{p_0/p} d\sigma \right)^{1/p_0}.$$

Doing the change of variables $t = \frac{s}{\sigma}$.

$$I_0^{1/p} \leqslant \left(\int_0^1 \left(\int_0^\infty (f^*(s))^{p} \frac{ds}{\sigma} \right)^{p_0/p} d\sigma \right)^{1/p_0}.$$

Using Theorem 1.3.7, we obtain that

$$\begin{aligned} \left(\int_0^1 \left(\int_0^\infty (f^*(s))^{p} \frac{ds}{\sigma} \right)^{p_0/p} d\sigma \right)^{1/p_0} &= \left(\int_0^1 \|f\|_p^{p_0} \frac{d\sigma}{\sigma^{p_0/p}} \right)^{1/p_0} \\ &= \|f\|_p \left(\int_0^1 \frac{d\sigma}{\sigma^{p_0/p}} \right)^{1/p_0} \end{aligned}$$

Since $p_0 \leqslant p$, we have that

$$\left(\int_0^1 \frac{d\sigma}{\sigma^{p_0/p}} \right)^{1/p_0} = \left(\frac{\sigma^{1-p_0/p}}{1-p_0/p} \right)^{p_0} \Bigg|_0^1 = \left(\frac{1}{1-p_0/p} \right)^{p_0} = \left(\frac{p}{p-p_0} \right)^{p_0} = C_p.$$

Therefore, we obtain that

$$I_0^{1/p} \leqslant C_p \|f\|_p.$$

As we have the bound for I_0 let us study the I_1 integral.

$$I_1 = \int_0^\infty (t^{-1/p_1} \|f_1\|_{p_1})^p dt$$

Let $\varphi = |f|^{p_1}$ and $\eta = \frac{p}{p_1} < 1$, then $\varphi^* = (f^*)^{p_1}$ and I_1 becomes

$$I_1 = \int_0^\infty \left(t^{-1} \int_t^\infty \varphi^*(s) ds \right)^{\eta} dt.$$

Since $t^{-1}\int_t^\infty \varphi^*(s)ds$ and $\varphi^*(t)$ are positive and decreasing functions of t we can apply the integral test for convergence, using the dyadic partition in the two intervals, we have

$$\int_0^\infty \left(t^{-1}\int_t^\infty \varphi^*(s)ds\right)^\eta dt \leqslant C \sum_{v=-\infty}^{\infty}\left(2^v \sum_{m\geqslant v}\varphi^*(2^m)2^m\right)^\eta 2^v.$$

Since $(x+y)^\eta \leqslant x^\eta + y^\eta$, we can estimate the right hand side by a constant multiplied by

$$\sum_v \sum_{m\geqslant v} 2^{(1-\eta)v}(\varphi^*(2^m))^\eta 2^{m\eta} = \sum_m \left(2^{m\eta}(\varphi^*(2^m))^\eta \sum_{v\geqslant m} 2^{v(1-\eta)}\right).$$

But, note that

$$\sum_m \left(2^{m\eta}(\varphi^*(2^m))^\eta \sum_{v\geqslant m} 2^{v(1-\eta)}\right) \leqslant \sum_m 2^{m\eta}(\varphi^*(2^m))^\eta \leqslant \sum_m 2^m(\varphi^*(2^m))^\eta.$$

Therefore, we obtain

$$\int_0^\infty \left(t^{-1}\int_t^\infty \varphi^*(s)ds\right)^\eta dt \leqslant \sum_m 2^m(\varphi^*(2^m))^\eta.$$

And applying again the integral test for convergence we deduce

$$\int_0^\infty \left(t^{-1}\int_t^\infty \varphi^*(s)ds\right)^\eta dt \leqslant \sum_m 2^m(\varphi^*(2^m))^\eta \leqslant C\int_0^\infty (\varphi^*(s))^\eta ds.$$

Now, applying the definition of η and φ^*, we obtain

$$I_1 \leqslant C\int_0^\infty (\varphi^*(s))^\eta ds = C\int_0^\infty (f^*(s))^{\frac{p_1 p}{p_1}} ds = C\int_0^\infty (f^*(s))^p ds.$$

Therefore, we have that

$$I_1 \leqslant C\int_0^\infty (f^*(s))^p ds = C\|f\|_p^p.$$

Hence,

$$I_1^{1/p} \leqslant C^{1/p}\|f\|_p.$$

Then, putting the two bounds in $\|Tf\|_p$ we obtain

$$\|Tf\|_p \leqslant (M_0^* C_p + M_1^* C^{1/p})\|f\|_p = M\|f\|_p.$$

Now, we go to see that $M \leqslant C_\theta (M_0^*)^{1-\theta}(M_1^*)^\theta$.

First, we express $M_0^* C_p + M_1^* C^{1/p}$ in terms of $(M_0^*)^{1-\theta}$ and $(M_1^*)^\theta$.

$$\begin{aligned} M_0^* C_p + M_1^* C^{1/p} \leqslant\; & C_p \frac{(M_0^*)^{1-\theta}(M_0^*)^\theta (M_1^*)^\theta}{(M_1^*)^\theta} \\ & + C^{1/p}\frac{(M_1^*)^{1-\theta}(M_0^*)^{1-\theta}(M_1^*)^\theta}{(M_0^*)^{1-\theta}}. \end{aligned}$$

As p depends of θ we can take $C'_\theta = \max(C_p, C^{1/p})$, then

$$\begin{aligned} M_0^* C_p + M_1^* C^{1/p} &\leqslant C'_\theta \left(\frac{(M_0^*)^{1-\theta}(M_0^*)^{\theta}(M_1^*)^{\theta}}{(M_1^*)^{\theta}} + \frac{(M_1^*)^{1-\theta}(M_0^*)^{1-\theta}(M_1^*)^{\theta}}{(M_0^*)^{1-\theta}} \right) \\ &= C'_\theta (A+B). \end{aligned}$$

So, we want to bound $A+B$ by $K_\theta (M_0^*)^{1-\theta}(M_1^*)^{\theta}$, and considering $C'_\theta \cdot K_\theta = C_\theta$, we will obtain that $M \leqslant C_\theta (M_0^*)^{1-\theta}(M_1^*)^{\theta}$.

But,

$$A + B \leqslant (M_0^*)^{1-\theta}(M_1^*)^{\theta} + \max(M_0^*, M_1^*),$$

and as M_0^* and M_1^* are finite and greater than 0 there exist $K_\theta > 0$ such that

$$\max(M_0^*, M_1^*) \leqslant K_\theta (M_0^*)^{1-\theta}(M_1^*)^{\theta}.$$

So,

$$A + B \leqslant (M_0^*)^{1-\theta}(M_1^*)^{\theta} + K_\theta (M_0^*)^{1-\theta}(M_1^*)^{\theta} \leqslant (K_\theta + 1)(M_0^*)^{1-\theta}(M_1^*)^{\theta}.$$

Therefore, taking $C_\theta = C'_\theta \cdot (1 + K_\theta)$, we have that $M \leqslant C_\theta (M_0^*)^{1-\theta}(M_1^*)^{\theta}$, as we want. ■

Before finish this chapter we will see an application of Marcinkiewicz interpolation theorem, this will be the interpolation of the Fourier Transform operator.

2.3 An Application of Marcinkiewicz Theorem

As we mentioned before in this section we will see an application of Marcinkiewicz interpolation theorem for the Fourier Transform operator.

Assume that we are in $(\mathbb{R}^n, dx)$, where dx denotes the Lebesgue measure in $\mathbb{R}^n$. Denote by L^p the L^p-space of $(\mathbb{R}^n, dx)$, and let ω be a weight function on $\mathbb{R}^n$, that is, a positive, measurable function on $\mathbb{R}^n$ and such that $\omega \in L^1_{loc}(\mathbb{R}^n)$. Then, we denote by $L^p(\omega)$ the L^p-space with respect to ωdx.

In fact, we will see two things:

1. The Fourier Transform goes from L^p to $L^{p'}$ if $1 \leqslant p \leqslant 2$.
2. The Fourier Transform goes from L^p to $L^p(\omega)$ if $1 \leqslant p \leqslant 2$, where, in this case, $\omega(\xi) = |\xi|^{-n(2-p)}$.

First we see that the Fourier transform goes from L^p to $L^{p'}$ if $1 \leqslant p \leqslant 2$.

To see the proof of this we use that as we saw in Section 1.3.3, the Fourier transform ($\mathscr{F}$) goes from L^1 to L^∞, and also is an isometry from L^2 to L^2. Then, by Marcinkiewicz interpolation Theorem 2.2.1 we have that $\mathscr{F}$ goes from L^p to L^q with p and q satisfying

$$\begin{aligned} \frac{1}{p} &= \frac{1-\theta}{p_0} + \frac{\theta}{p_1} = 1 - \theta + \frac{\theta}{2} = \frac{2-\theta}{2}, \\ \frac{1}{q} &= \frac{1-\theta}{q_0} + \frac{\theta}{q_1} = \frac{1-\theta}{\infty} + \frac{\theta}{2} = \frac{\theta}{2}, \end{aligned}$$

with $0 < \theta < 1$.

But notice, that

$$\frac{1}{p} + \frac{1}{q} = \frac{2 - \theta + \theta}{2} = \frac{2}{2} = 1.$$

Therefore, $q = p'$.

Also, since $M_1 = 1$ because the Fourier Transform is an isometry from L^2 to L^2, we obtain that

$$M \leqslant C_\theta M_0^{1-\theta} M_1^\theta = C_\theta M_0^{1-\theta}.$$

And since M_0 is also 1, we have that

$$M \leqslant C_\theta M_0^{1-\theta} = C_\theta.$$

Notice that we can apply the Theorem 2.2.1 because the Fourier transform is a strong type $(1, \infty)$ and is also a strong type $(2, 2)$. Then, in particular, it is a weak type $(1, \infty)$ and is also a weak type $(2, 2)$. We have that $p \leqslant q$. So, we are in the hypothesis of the Marcinkiewicz interpolation theorem.

Now, let us see that if ω is a weight in $\mathbb{R}^n$, then the Fourier Transform goes from L^p to $L^p(\omega)$ if $1 \leqslant p \leqslant 2$.

Theorem 2.3.1. *Assume that* $1 \leqslant p \leqslant 2$. *Then*

$$\|\mathscr{F} f\|_{L^p(|\xi|^{-n(2-p)})} \leqslant \|f\|_p.$$

Here, $\|\cdot\|_{L^p(|\xi|^{-n(2-p)})}$ denotes the norm in $L^p(|\xi|^{-n(2-p)})$.

Proof. Consider the map

$$(TF)(\xi) = |\xi|^n \hat{f}(\xi).$$

Then, using Parseval's Identity, for all $f \in L^2$ we have that

$$\begin{aligned}\|Tf\|_{L^2(|\xi|^{-n})}^2 &= \int_{\mathbb{R}^n} |Tf(\xi)|^2 |\xi|^{-2n} d\xi \\ &= \int_{\mathbb{R}^n} |\xi|^{2n} |\hat{f}(\xi)|^2 |\xi|^{-2n} d\xi \\ &= \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 d\xi = \|\hat{f}\|_2^2 = \|f\|_2^2.\end{aligned}$$

Therefore, T is a strong type $(2, 2)$ operator, and so T is a weak type $(2, 2)$ operator. Then, in order to apply the Theorem 2.2.1 we have to see that T is a weak type $(1, 1)$ operator because if we see it, then $p = q$ and T is a strong type (p, p) operator.

Let us see that T is a weak type $(1, 1)$ operator. If $f \in L^1$, then

$$\|Tf\|_{L^{1,\infty}(|\xi|^{-2n})} = \sup_{t>0} t\lambda_{Tf}(t).$$

Now, using that the measure of a set E with respect to ωdx is $\int_E \omega dx$, we have that

$$\begin{aligned}\lambda_{Tf}(t) &= \mu\{\xi \in \mathbb{R}^n : \ |\xi|^n |\hat{f}(\xi)| > t\} \\ &= \int_{\{\xi \in \mathbb{R}^n : \ |\xi|^n |\hat{f}(\xi)| > t\}} |\xi|^{-2n} d\xi.\end{aligned}$$

Assume that $\|f\|_1 = 1$, then since $\|\hat{f}\|_\infty \leqslant \|f\|_1 = 1$ we have that

$$\sup_{\xi} |\hat{f}(\xi)| \leqslant 1.$$

Therefore, $|\hat{f}(\xi)| \leqslant 1$. Putting this in λ_{Tf} we obtain

$$\begin{aligned}
\lambda_{Tf}(t) &= \int_{\{\xi\in\mathbb{R}^n :\, |\xi|^n|\hat{f}(\xi)|>t\}} |\xi|^{-2n} d\xi \\
&\leqslant \int_{\{\xi\in\mathbb{R}^n :\, |\xi|^n>t\}} |\xi|^{-2n} d\xi \\
&= \int_{|\xi|^n>t} |\xi|^{-2n} d\xi \leqslant Ct^{-1}.
\end{aligned}$$

So, we obtain that

$$\|Tf\|_{L^{1,\infty}(|\xi|^{-2n})} = \sup_{t>0} t\lambda_{Tf}(t) \leqslant \sup_{t>0} tCt^{-1} = C.$$

And since $\|f\|_1 = 1$, we have that $\|Tf\|_{1,\infty} \leqslant C\|f\|_1$.

Therefore, we are in the hypothesis of Marcinkiewicz interpolation theorem and then

$$T : L^p \to L^p(|\xi^{-2n}|) \quad \text{with } 1 \leqslant p \leqslant 2.$$

Notice that when we take $f \in L^p$ and compute $\|Tf\|^p_L(|\xi|^{-2n})$ what we have is

$$\begin{aligned}
\|Tf\|_{L^p(|\xi|^{-2n})} &= \int_{\mathbb{R}^n} |\xi|^{np}|\hat{f}(\xi)||\xi|^{-2n} d\xi \\
&= \int_{\mathbb{R}^n} |\xi|^{-n(2-p)}|\hat{f}(\xi)| d\xi \\
&= \|\hat{f}\|_{L^p(|\xi|^{-n(2-p)})}.
\end{aligned}$$

Therefore, for $1 \leqslant p \leqslant 2$ we have that

$$\|\mathscr{F}f\|_{L^p(|\xi|^{-n(2-p)})} \leqslant \|f\|_p.$$

∎

Chapter 3

Real Interpolation

In this chapter we will study the real interpolation methods and also some properties of the real interpolation spaces. As we said in the previous chapter, those methods are inspired in the proof of the Marcinkiewicz Theorem 2.2.1.

3.1 Real Interpolation Methods

In this section we will study the main subject of this chapter, that is the Real Interpolation Methods. In particular, we will study the J-method and the K-method.

But, before to see the results we have to introduce the compatible couple of quasi-Banach spaces.

Definition 3.1.1 (Compatible couple of quasi-Banach spaces)**.** Let A_0 and A_1 be two quasi-Banach spaces. We say that $\bar{A} = (A_0, A_1)$ is a compatible couple if there exists a Banach space $\mathcal{A}$ such that A_0, $A_1 \hookrightarrow \mathcal{A}$.

With this definition we can define the different methods that will describe the diverse interpolation spaces.

3.1.1 K-method

In this section we will study the K-method, but first we define when a belongs in $A_0 + A_1$, where (A_0, A_1) is a compatible couple of quasi-Banach spaces.

Definition 3.1.2. If (A_0, A_1) is a compatible couple of quasi-Banach spaces, we say that $a \in A_0 + A_1$ if there exist $a_0 \in A_0$ and $a_1 \in A_1$ such that $a = a_0 + a_1$.

The K-method is based in the Peetre's K-functional which is defined as follows.

Definition 3.1.3 (Peetre's K-functional)**.** Let $\bar{A} = (A_0, A_1)$ be a compatible couple of quasi-Banach spaces, and let $a \in A_0 + A_1$. We define the Peetre's K-functional as follows

$$K(t, a; A_0, A_1) = \inf_{a=a_0+a_1} \left\{ \|a_0\|_{A_0} + t\|a_1\|_{A_1} \right\}, \text{ with } t > 0.$$

Let us see one property of the Peetre's K-functional.

Remark 3.1.4. Fix $a \in A_0 + A_1$, then $K(t, a; A_0, A_1)$ is an increasing function with respect to t.

Proof. Let $t < s$ and $a = a_0 + a_1$ be an arbitrary decomposition of a. Then,

$$K(t, a; A_0 + A_1) \leqslant \|a_0\|_{A_0} + t\|a_1\|_{A_1} \leqslant \|a_0\|_{A_0} + s\|a_1\|_{A_1}.$$

And, as it holds for any decomposition of a, in particular it holds for the infimum. So,

$$K(t, a; A_0 + A_1) \leqslant \|a_0\|_{A_0} + t\|a_1\|_{A_1} \leqslant K(s, a; A_0 + A_1).$$

■

Now, we are going to see that, in fact, $K(t, \cdot; A_0, A_1)$ is a norm in the space $A_0 + tA_1$.

Proposition 3.1.5. *$K(t, \cdot; A_0, A_1)$ is a norm in $A_0 + tA_1$.*

Proof. We have to see that

1. $K(t, a; A_0, A_1) \geqslant 0$ for all $a \in A_0 + A_1$ and is 0 if and only if $a = 0$,
2. $K(t, a + b; A_0, A_1) \leqslant K(t, a; A_0, A_1) + K(t, b; A_0, A_1)$ for all $a, b \in A_0 + A_1$,
3. $K(t, \lambda a; A_0, A_1) = |\lambda| K(t, a; A_0, A_1)$ for all $a \in A_0 + A_1$ and for all $\lambda \in \mathbb{R}$.

By definition,

$$K(t, a; A_0, A_1) = \inf_{a = a_0 + a_1} \left\{ \|a_0\|_{A_0} + t\|a_1\|_{A_1} \right\},\ t > 0.$$

Then, $K(t, a; A_0, A_1) \geqslant 0$, and if $a = 0$, then $K(t, a; A_0, A_1) = 0$. So, let us see the converse implication.

Let $a \in A_0 + A_1$ such that $K(t, a; A_0, A_1) = 0$. As $K(t, a; A_0, A_1)$ is an infimum, we have that for all $k \in \mathbb{N}$, there exist elements $a_0^k \in A_0$ and $a_1^k \in A_1$, so that $a = a_0^k + a_1^k$ and

$$\|a_0^k\|_{A_0} + t\|a_1^k\|_{A_1} \leqslant K(t, a; A_0, A_1) + \frac{1}{k} = \frac{1}{k}.$$

Letting k tends to infinity, as $t > 0$, we have that

$$\|a_0^k\|_{A_0} \to 0$$
$$\|a_1^k\|_{A_1} \to 0.$$

And as A_0 and A_1 are quasi-Banach spaces this implies that

$$a_0^k \to 0$$
$$a_1^k \to 0.$$

Then, the sequence

$$\left(a_0^k + a_1^k\right)_k.$$

is convergent to 0 in $A_0 + tA_1$. Hence, $a = 0$.

Now, let us see that $K(t, a+b; A_0 + A_1) \leqslant K(t, a; A_0 + A_1) + K(t, b; A_0 + A_1)$ for all $a, b \in A_0 + A_1$.

Let $a, b \in A_0 + tA_1$, let $\varepsilon > 0$ and let $a = a_0 + a_1$ and $b = b_0 + b_1$ be a decomposition of a and b satisfying that

$$\begin{aligned} \|a_0\|_{A_0} + t\|a_1\|_{A_1} &\leqslant (1+\varepsilon)K(t, a; A_0 + A_1), \\ \|b_0\|_{A_0} + t\|b_1\|_{A_1} &\leqslant (1+\varepsilon)K(t, b; A_0 + A_1). \end{aligned}$$

Then, we have that

$$\begin{aligned} K(t, a+b; A_0 + A_1) &\leqslant \|a_0 + b_0\|_{A_0} + t\|a_1 + b_1\|_{A_1} \\ &\leqslant \|a_0\|_{A_0} + t\|a_1\|_{A_1} + \|b_0\|_{A_0} + t\|b_1\|_{A_1} \\ &\leqslant (1+\varepsilon)K(t, a; \bar{A}) + (1+\varepsilon)K(t, b; \bar{A}) \\ &= (1+\varepsilon)\left(K(t, a; \bar{A}) + K(t, b; \bar{A})\right). \end{aligned}$$

And, as it holds for any $\varepsilon > 0$, we can make $\varepsilon \downarrow 0$ and obtain that

$$K(t, a+b; A_0 + A_1) \leqslant K(t, a; \bar{A}) + K(t, b; A_0 + A_1).$$

Finally, we will see $K(t, \lambda a; A_0, A_1) = |\lambda| K(t, a; A_0, A_1)$ for all $a \in A_0 + A_1$ and for all $\lambda \in \mathbb{R}$.

Let $a \in A_0 + A_1$ and $\lambda \in \mathbb{R}$. Then, $\lambda a = \lambda a_0 + \lambda a_1$.

So, we have that

$$\begin{aligned} K(t, \lambda a; A_0, A_1) &= \inf\left\{\|\lambda a_0\|_{A_0} + t\|\lambda a_1\|_{A_1} : \lambda a = \lambda(a_0 + a_1)\right\} \\ &= \inf\left\{|\lambda|(\|a_0\|_{A_0} + t\|a_1\|_{A_1}) : \lambda a = \lambda(a_0 + a_1)\right\} \\ &= |\lambda| \inf_{a = a_0 + a_1} \left\{\|a_0\|_{A_0} + t\|a_1\|_{A_1}\right\} = |\lambda| K(t, a; A_0, A_1). \end{aligned}$$

Therefore, the Peetre's K-functional is a norm in $A_0 + tA_1$. ■

For simplicity, from now we call $K(t, a; A_0, A_1)$ as $K(t, a; \bar{A})$.

Remark 3.1.6. Fix $a \in A_0 + A_1$, then $K(t, a; \bar{A})$ is a concave function with respect to t.

Proof. Let $x, y > 0$ and $0 \leqslant t \leqslant 1$, let $z = tx + (1-t)y$. Take $a = a_0 + a_1$ be an arbitrary decomposition of a. Then,

$$\begin{aligned} tK(x, a; \bar{A}) + (1-t)K(y, a; \bar{A}) &\leqslant t\left(\|a_0\|_{A_0} + x\|a_1\|_{A_1}\right) \\ &\quad + (1-t)\left(\|a_0\|_{A_0} + y\|a_1\|_{A_1}\right) \\ &= \|a_0\|_{A_0} + z\|a_1\|_{A_1}. \end{aligned}$$

As it holds for any decomposition of a, in particular holds for the infimum, that is $K(z, a; \bar{A})$. Then,

$$tK(x, a; \bar{A}) + (1-t)K(y, a; \bar{A}) \leqslant K(z, a; \bar{A}).$$

■

With this norm we have the following proposition that gives us a bound for operators.

Proposition 3.1.7. *Let $\bar{A}$ and $\bar{B}$ be two compatible couples of quasi-Banach spaces. If $T : A_j \to B_j$ is bounded with norm M_j for $j = 0, 1$, then*

$$K(t, Ta, \bar{B}) \leqslant M_0 K\left(\frac{M_1}{M_0} t, a; \bar{A}\right).$$

Proof. Let $a \in A_0 + A_1$, $\varepsilon > 0$ and $t > 0$. Then, there exist $a_0 \in A_0$ and $a_1 \in A_1$ such that

$$\|a_0\|_{A_0} + t\frac{M_1}{M_0}\|a_1\|_{A_1} \leqslant (1+\varepsilon) K\left(t\frac{M_1}{M_0}, a; \bar{A}\right). \tag{3.1}$$

Thus, $Ta = Ta_0 + Ta_1 \in B_0 + B_1$. So,

$$\begin{aligned} K(t, Ta, \bar{B}) \leqslant \|Ta_0\|_{B_0} + t\|Ta_1\|_{B_1} &\leqslant M_0\|a_0\|_{A_0} + M_1 t\|a_1\|_{A_1} \\ &= M_0\left[\|a_0\|_{A_0} + \frac{M_1}{M_0} t\|a_1\|_{A_1}\right]. \end{aligned}$$

And applying (3.1) we obtain

$$K(t, Ta, \bar{B}) \leqslant M_0\left[\|a_0\|_{A_0} + \frac{M_1}{M_0} t\|a_1\|_{A_1}\right] \leqslant M_0(1+\varepsilon) K\left(t\frac{M_1}{M_0}, a; \bar{A}\right).$$

And since $K(t, Ta, \bar{B})$ does not depend on ε if we make ε tend to 0, we have

$$K(t, Ta, \bar{B}) \leqslant M_0 K\left(t\frac{M_1}{M_0}, a; \bar{A}\right).$$

■

With all those things we can define the real interpolation spaces obtained with the K-method.

Definition 3.1.8. Let $\bar{A} = (A_0, A_1)$ be a compatible couple of quasi-Banach spaces. We define the real interpolation space $(A_0, A_1)_{\theta,p}$ as follows

$$(A_0, A_1)_{\theta,p} = \left\{ a \in A_0 + A_1 : \ \|a\|_{(A_0,A_1)_{\theta,p}} = \left(\int_0^\infty t^{-p\theta} K(t, a; \bar{A})^p \frac{dt}{t}\right)^{1/p} < \infty \right\}.$$

Here we consider the cases $0 < \theta < 1$, $1 \leqslant p \leqslant \infty$ and $0 \leqslant \theta \leqslant 1$, $p = \infty$.

We have two important properties of the K-functional and other relation of this functional with this norm.

Properties 3.1.9.

1. For any $a \in A_0 + A_1$ we have that

$$K(t, a; \bar{A}) \leqslant \max(1, t/s) K(s, a; \bar{A}).$$

2. $K(s, a; \bar{A}) \leqslant \gamma_{\theta,p} s^\theta \|a\|_{(A_0,A_1)_{\theta,p}}$.

Proof. Start with the first property.

If $t/s < 1$, then $s > t$ and as $K(t, a; \bar{A})$ is increasing with respect to t we have that $K(t, a; \bar{A}) \leqslant K(s, a; \bar{A})$.

If $t/s > 1$, take $a = a_0 + a_1$ be any decomposition of a, then

$$\|a_0\|_{A_0} + t\|a_1\|_{A_1} = \|a_0\|_{A_0} + st/s\|a_1\|_{A_1} \leqslant \frac{t}{s}\left(\|a_0\|_{A_0} + s\|a_1\|_{A_1}\right).$$

Taking infimums in both sides we obtain that

$$K(t, a; \bar{A}) \leqslant \max(1, t/s) K(s, a; \bar{A}).$$

Now, let us see the second property. To prove it we will use the previous property but write in the form

$$\min(1, t/s) K(s, a; \bar{A}) \leqslant K(t, a; \bar{A}).$$

Using this we arrive at

$$\begin{aligned}
\left(\int_0^\infty t^{-\theta p} \min(1, t/s)^p K(s, a; \bar{A})^p dt/t\right)^{1/p} &= K(s, a; \bar{A}) \left(\int_0^\infty t^{-\theta p} \min(1, t/s)^p dt/t\right)^{1/p} \\
&\leqslant \left(\int_0^\infty t^{-\theta p} K(t, a; \bar{A})^p dt/t\right)^{1/p} \\
&= \|a\|_{(A_0, A_1)_{\theta,p}}
\end{aligned}$$

So, we want to bound

$$\left(\int_0^\infty t^{-\theta p} \min(1, t/s)^p dt/t\right)^{1/p}.$$

Let $r = t/s$, then

$$\left(\int_0^\infty t^{-\theta p} \min(1, t/s)^p dt/t\right)^{1/p} = s^{-\theta} \left(\int_0^\infty r^{-\theta p} \min(1, r)^p dr/r\right)^{1/p}.$$

But,

$$\left(\int_0^\infty r^{-\theta p} \min(1, r)^p dr/r\right)^{1/p} = \left(\frac{1}{p\theta(1-\theta)}\right)^{1/p} = \frac{1}{\gamma_{\theta,p}}.$$

Therefore, we have that

$$\frac{K(s, a; \bar{A})}{s^\theta \gamma_{\theta,p}} \leqslant \|a\|_{(A_0, A_1)_{\theta,p}}.$$

■

Proposition 3.1.10. *If A_0 and A_1 are quasi-Banach spaces, then $(A_0, A_1)_{\theta,p}$ is a Banach space with norm $\|\cdot\|_{(A_0,A_1)_{\theta,p}}$.*

Proof. Note, that in the earlier prove we do not use that $\|\cdot\|_{(A_0,A_1)_{\theta,p}}$ is a norm. So, in order to see that $(A_0, A_1)_{\theta,p}$ is a Banach space with the norm $\|\cdot\|_{(A_0,A_1)_{\theta,p}}$ we have to prove the following things.

1. $\|\cdot\|_{(A_0,A_1)_{\theta,p}}$ is a norm,
2. $\left((A_0,A_1)_{\theta,p}, \|\cdot\|_{(A_0,A_1)_{\theta,p}}\right)$ is complete.

Note that

$$\|\cdot\|_{(A_0,A_1)_{\theta,p}} = \|t^{-\theta}K(t,\cdot;\bar{A})\|_{L^p(dt/t)},$$

where $L^p(dt/t)$ is the L^p space with respect to the Lebesgue measure with weight t^{-1}. But, in Proposition 3.1.5 we proved that $K(t,\cdot;\bar{A})$ is a norm in A_0+tA_1. So,

$$\|\cdot\|_{(A_0,A_1)_{\theta,p}} = \|t^{-\theta}(\|\cdot\|_{A_0+tA_1})\|_{L^p(dt/t)}.$$

Then, that for all $a \in (A_0,A_1)_{\theta,p}$ we have that $\|a\|_{(A_0,A_1)_{\theta,p}}$ is positive and that is 0 if and only if $a=0$ is clear.

Let us see the triangular inequality. Take $a,b \in (A_0,A_1)_{\theta,p}$.

$$\begin{aligned}
\|a+b\|_{(A_0,A_1)_{\theta,p}} &= \|t^{-\theta}(\|a+b\|_{A_0+A_1})\|_{L^p(dt/t)}\\
&\leqslant \|t^{-\theta}(\|a\|_{A_0+A_1} + \|b\|_{A_0+A_1})\|_{L^p(dt/t)}\\
&\leqslant \|t^{-\theta}(\|a\|_{A_0+A_1})\|_{L^p(dt/t)} + \|t^{-\theta}(\|b\|_{A_0+A_1})\|_{L^p(dt/t)}\\
&= \|a\|_{(A_0,A_1)_{\theta,p}} + \|b\|_{(A_0,A_1)_{\theta,p}}.
\end{aligned}$$

Then, $\|a+b\|_{(A_0,A_1)_{\theta,p}} \leqslant \|a\|_{(A_0,A_1)_{\theta,p}} + \|b\|_{(A_0,A_1)_{\theta,p}}$.

Now, let us see that for all $a \in (A_0,A_1)_{\theta,p}$ and for all $\lambda \in \mathbb{R}$ we have that

$$\|a\lambda\|_{(A_0,A_1)_{\theta,p}} = |\lambda|\|a\|_{(A_0,A_1)_{\theta,p}}.$$

Let $a \in (A_0,A_1)_{\theta,p}$ and $\lambda \in \mathbb{R}$, then

$$\begin{aligned}
\|a\lambda\|_{(A_0,A_1)_{\theta,p}} &= \|t^{-\theta}(\|a\lambda\|_{A_0+A_1})\|_{L^p(dt/t)}\\
&= \|t^{-\theta}(\|a\|_{A_0+A_1}|\lambda|)\|_{L^p(dt/t)}\\
&= |\lambda|\|t^{-\theta}(\|a\|_{A_0+A_1})\|_{L^p(dt/t)}\\
&= |\lambda|\|a\|_{(A_0,A_1)_{\theta,p}}.
\end{aligned}$$

Therefore, $\|a\lambda\|_{(A_0,A_1)_{\theta,p}} = |\lambda|\|a\|_{(A_0,A_1)_{\theta,p}}$ and, hence, $\|\cdot\|_{(A_0,A_1)_{\theta,p}}$ is a norm.

So, we have to see the completeness of $\left((A_0,A_1)_{\theta,p}, \|\cdot\|_{(A_0,A_1)_{\theta,p}}\right)$. By, the Theorem 1.2.1 it is enough to see that every absolute convergent series is a convergent series in $\left((A_0,A_1)_{\theta,p}, \|\cdot\|_{(A_0,A_1)_{\theta,p}}\right)$.

Let $\sum_{n=1}^{\infty}\|a_n\|_{(A_0,A_1)_{\theta,p}}$ be an absolute convergent series in $(A_0,A_1)_{\theta,p}$, then we want to see that $\sum_0^{\infty} a_n$ is convergent in $(A_0,A_1)_{\theta,p}$.

In order to simplify the notation we put the space in the superindex instead of in the subindex.

Let $a_n^0 + a_n^1$ be a decomposition of a_n satisfying that

$$\|a_n^0\|_{A_0} + \|a_n^1\|_{A_1} \leqslant K(1,a_n;\bar{A}) + 2^{-n}. \tag{3.2}$$

By the Properties 3.1.9 we have that $K(1, a_n; \bar{A}) \leqslant \gamma_{\theta,p}\|a_n\|_{(A_0,A_1)_{\theta,p}}$. Then, we have that

$$\sum_{n=1}^{\infty} K(1, a_n; \bar{A}) \leqslant \gamma_{\theta,p} \sum_{n=1}^{\infty} \|a_n\|_{(A_0,A_1)_{\theta,p}},$$

which is convergent in $\mathbb{R}$, so $\sum_{n=1}^{\infty} K(1, a_n; \bar{A})$ is convergent in $\mathbb{R}$. By (3.2) we have that

$$\sum_{n=1}^{\infty} \|a_n^0\|_{A_0} \leqslant \sum_{n=1}^{\infty} K(1, a_n; \bar{A}) + 1.$$

And the same for $\sum_{n=1}^{\infty} \|a_n^1\|_{A_1}$. Then, we have that $\sum_{n=1}^{\infty} \|a_n^0\|_{A_0}$ and $\sum_{n=1}^{\infty} \|a_n^1\|_{A_1}$ are convergent in $\mathbb{R}$. Since, A_0 and A_1 are quasi-Banach we have that $\sum_{n=1}^{\infty} a_n^0$ and $\sum_{n=1}^{\infty} a_n^1$ are convergent in A_0 and A_1 respectively. Therefore,

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_n^0 + \sum_{n=1}^{\infty} a_n^1$$

is convergent in $(A_0, A_1)_{\theta,p}$. ∎

The first theorem that we meet with these definitions is the Interpolation theorem that tells us that if an operator is continuous between two compatible couples of quasi-Banach spaces then, it is continuous between the interpolation spaces of the two couples. Moreover, its norm in the interpolation spaces is $\|T\|_{(A_0,A_1)_{\theta,p}} \leqslant M_0^{1-\theta} M_1^{\theta}$.

Theorem 3.1.11 (Interpolation theorem)**.** *Let $\bar{A} = (A_0, A_1)$ and $\bar{B} = (B_0, B_1)$ be two compatible couples of quasi-Banach spaces. Let*

$$T : A_j \to B_j, \ j = 0, 1$$

be continuous with norm M_j. Then,

$$T : (A_0, A_1)_{\theta,p} \to (B_0, B_1)_{\theta,p}$$

is continuous with norm M, for $0 < \theta < 1$ and $1 \leqslant p < \infty$.

Moreover,

$$M \leqslant M_0^{1-\theta} M_1^{\theta}.$$

Proof. Let $a \in (A_0, A_1)_{\theta,p}$. By the previous proposition we have that

$$\begin{aligned}\|Ta\|_{(B_0,B_1)_{\theta,p}} &= \left(\int_0^{\infty} t^{-p\theta} K(t, Ta, \bar{B})^p \frac{dt}{t} \right)^{1/p} \\ &\leqslant M_0 \left(\int_0^{\infty} t^{-p\theta} K\left(\frac{M_1}{M_0} t, a; \bar{A} \right)^p \frac{dt}{t} \right)^{1/p}.\end{aligned}$$

Taking $s = \frac{M_1}{M_0} t$ we have that $\frac{ds}{s} = \frac{dt}{t}$ and that $t^{-p\theta} = s^{-p\theta}(M_0/M_1)^{-p\theta}$. Putting this in the last expression we obtain

$$M_0 \left(\int_0^{\infty} t^{-p\theta} K\left(\frac{M_1}{M_0} t, a; \bar{A} \right)^p \frac{dt}{t} \right)^{1/p} = M_0 \frac{M_0^{-\theta}}{M_1^{-\theta}} \left(\int_0^{\infty} s^{-p\theta} K(s, a; \bar{A})^p \frac{ds}{s} \right)^{1/p}.$$

And by definition of $\|a\|_{(A_0,A_1)_{\theta,p}}$ we have that

$$M_0\frac{M_0^{-\theta}}{M_1^{-\theta}}\left(\int_0^\infty s^{-p\theta}K(s,a;\bar{A})^p\frac{ds}{s}\right)^{1/p}=M_0^{1-\theta}M_1^{\theta}\|a\|_{(A_0,A_1)_{\theta,p}}.$$

Therefore,

$$\|Ta\|_{(B_0,B_1)_{\theta,p}}\leqslant M_0^{1-\theta}M_1^{\theta}\|a\|_{(A_0,A_1)_{\theta,p}}$$

and $M\leqslant M_0^{1-\theta}M_1^{\theta}$. ■

Because of this bound, we say that the K-functor is an exact interpolation functor.

We can generalize this method in many ways, but the most useful is the discrete K-functional, where t is changed for a discrete variable n using the relation $t=2^n$.

Denote by $\lambda^{\theta,q}$ the space of all sequences $(\alpha_n)_{n\in\mathbb{Z}}$, such that

$$\|(\alpha_n)_n\|_{\lambda^{\theta,q}}=\left(\sum_{n\in\mathbb{Z}}(2^{-n\theta}|\alpha_n|)^q\right)^{1/q}<\infty.$$

The next lemma says us that we can characterize the elements in $(A_0,A_1)_{\theta,p}$ via the sequence $\alpha_n=K(2^n,\cdot;\bar{A})$.

Lemma 3.1.12. *Let (A_0,A_1) be compatible couple of quasi-Banach spaces.*

If $a\in A_0+A_1$ and we take $\alpha_n=K(2^n,a;\bar{A})$, then $a\in(A_0,A_1)_{\theta,p}$ if and only if $(\alpha_n)_{n\in\mathbb{Z}}\in\lambda^{\theta,p}$. Even more, we have

$$2^{-\theta}(\log(2))^{1/p}\|(\alpha_n)_n\|_{\lambda^{\theta,p}}\leqslant\|a\|_{(A_0,A_1)_{\theta,p}}\leqslant 2(\log(2))^{1/p}\|(\alpha_n)_n\|_{\lambda^{\theta,p}}.$$

Proof. First note that we can write $\|a\|_{(A_0,A_1)_{\theta,p}}$ as

$$\|a\|_{(A_0,A_1)_{\theta,p}}=\left(\sum_{n\in\mathbb{Z}}\int_{2^n}^{2^{n+1}}(t^{-\theta}K(t,a;\bar{A}))^p\frac{dt}{t}\right)^{1/p}.$$

because this is the dyadic partition of the interval $(0,\infty)$.

Now, since $K(t,a;\bar{A})$ is increasing and concave with respect to t, if we take $2^n\leqslant t\leqslant 2^{n+1}$, then

$$K(2^n,a;\bar{A})\leqslant K(t,a;\bar{A})\leqslant K(2^{n+1},a;\bar{A})\leqslant 2K(2^n,a;\bar{A}).$$

Hence, $t^{-\theta}\in[2^{-n\theta-\theta},2^{-\theta}]$ and using the concavity with respect to t we obtain that

$$K(t\cdot t^{-\theta},a;\bar{A})\leqslant t^{-\theta}K(t,a;\bar{A})\leqslant 2^{-n\theta}2K(2^n,a;\bar{A}).$$

And that,

$$t^{-\theta}K(t,a;\bar{A})\geqslant 2^{-\theta}2^{-n\theta}K(2^n,a;\bar{A}).$$

Thus,

$$2^{-\theta}2^{-n\theta}K(2^n,a;\bar{A})\leqslant t^{-\theta}K(t,a;\bar{A})\leqslant 2\cdot 2^{-n\theta}K(2^n,a;\bar{A}).$$

And by definition of α_n, it is

$$2^{-\theta}2^{-n\theta}\alpha_n\leqslant t^{-\theta}K(t,a;\bar{A})\leqslant 2\cdot 2^{-n\theta}\alpha_n.$$

Applying this to $\|a\|_{(A_0,A_1)_{\theta,p}}$ we obtain the following:

$$\begin{aligned}
\|a\|_{(A_0,A_1)_{\theta,p}} &= \left(\sum_{n\in\mathbb{Z}}\int_{2^n}^{2^{n+1}}(t^{-\theta}K(t,a;\bar{A}))^p\frac{dt}{t}\right)^{1/p} \\
&\leqslant \left(\sum_{n\in\mathbb{Z}}\int_{2^n}^{2^{n+1}}(2\cdot 2^{-n\theta}K(2^n,a;\bar{A}))^p\frac{dt}{t}\right)^{1/p} \\
&= 2\left(\sum_{n\in\mathbb{Z}}2^{-np\theta}K(2^n,a;\bar{A})^p\int_{2^n}^{2^{n+1}}\frac{dt}{t}\right)^{1/p} \\
&= 2\left(\sum_{n\in\mathbb{Z}}2^{-np\theta}K(2^n,a;\bar{A})^p\log\left(\frac{2^{n+1}}{2^n}\right)\right)^{1/p} \\
&= 2(\log(2))^{1/p}\|(\alpha_n)_n\|_{\lambda^{\theta,p}}.
\end{aligned}$$

Doing the same argument with $2^{-\theta}2^{-n\theta}\alpha_n \leqslant t^{-\theta}K(t,a;\bar{A})$, we conclude that

$$2^{-\theta}(\log(2))^{1/p}\|(\alpha_n)_n\|_{\lambda^{\theta,p}} \leqslant \|a\|_{(A_0,A_1)_{\theta,p}} \leqslant 2(\log(2))^{1/p}\|(\alpha_n)_n\|_{\lambda^{\theta,p}}.$$

Then, $\|(\alpha_n)_n\|_{\lambda^{\theta,q}}$ is finite if and only if $\|a\|_{(A_0,A_1)_{\theta,p}}$ is finite. So, we have proved that $a \in (A_0,A_1)_{\theta,p}$ if and only if $(\alpha_n)_{n\in\mathbb{Z}} \in \lambda^{\theta,p}$. ■

3.1.2 J-Method

In this section we will study the J-method. Instead of starting with the space $A_0 + A_1$ starts with the space $A_0 \cap A_1$. Therefore, we start this section defining the space $A_0 \cap A_1$.

Definition 3.1.13. Let (A_0, A_1) be a compatible couple of quasi-Banach spaces. Then, $a \in A_0 \cap A_1$ if $\|a\|_{A_0} < \infty$ and $\|a\|_{A_1} < \infty$.

And we define the J-functor as follows.

Definition 3.1.14 (J-functor)**.** Let $\bar{A} = (A_0, A_1)$ be a compatible couple of quasi-Banach spaces and let $a \in A_0 \cap A_1$. We define the J-functor as

$$J(t,a;A_0,A_1) = \max(\|a\|_{A_0}, t\|a\|_{A_1}), \quad t > 0.$$

And, by simplicity we denote $J(t,a;A_0,A_1)$ by $J(t,a;\bar{A})$.

As in the K-method, the J-functional is a norm in $A_0 \cap A_1$.

Proposition 3.1.15. *Let $\bar{A} = (A_0, A_1)$ be a compatible couple of quasi-Banach spaces. Then*

$$J(t,\cdot;\bar{A})$$

is a norm in $A_0 \cap A_1$.

Proof. Let $\bar{A} = (A_0, A_1)$ be a compatible couple of quasi-Banach spaces and let $a \in A_0 \cap A_1$. And fix $t > 0$. As

$$J(t, a; \bar{A}) = \max(\|a\|_{A_0}, t\|a\|_{A_1}).$$

Then, $J(t, a; \bar{A})$ is positive and is 0 if and only if $\|a\|_{A_0} = \|a\|_{A_1} = 0$. And, as A_0 and A_1 are quasi-Banach spaces this implies that $a = 0$, hence $J(t, a; \bar{A}) = 0$ if and only if $a = 0$.

Let $\lambda \in \mathbb{R}$, then

$$J(t, a\lambda; \bar{A}) = \max(\|\lambda a\|_{A_0}, t\|a\lambda\|_{A_1}).$$

And, as $\|\cdot\|_{A_0}$ and $\|\cdot\|_{A_1}$ are quasi-norms, then $\|\lambda a\|_{A_0} = |\lambda|\|a\|_{A_0}$ and $t\|a\lambda\|_{A_1} = |\lambda|t\|a\|_{A_1}$. So,

$$J(t, a\lambda; \bar{A}) = |\lambda| \max(\|a\|_{A_0}, t\|a\|_{A_1}) = |\lambda| J(t, a; \bar{A}).$$

Let $a, b \in A_0 \cap A_1$. Then, as $\|\cdot\|_{A_0}$ and $\|\cdot\|_{A_1}$ are quasi-norms, then $\|a + b\|_{A_0} \leqslant \|a\|_{A_0} + \|b\|_{A_0}$ and $\|a + b\|_{A_1} \leqslant \|a\|_{A_1} + \|b\|_{A_1}$. Therefore,

$$J(t, a + b; \bar{A}) \leqslant J(t, a; \bar{A}) + J(t, b; \bar{A}).$$

■

Using this proposition, the J-functor is a positive function of t. Also, it is increasing as a function of t because if $t\|a\|_{A_1} \geqslant \|a\|_{A_0}$, then $J(t, a; \bar{A}) = t\|a\|_{A_1}$. So, it is a polynomial of t of degree 1 and positive coefficients.

And, using a similar argument as in the Remark 3.1.6 we can prove that this functor is a concave function of t.

The following lemma gives us a relation between $J(t, a; \bar{A})$ and $J(s, a; \bar{A})$; and between $J(s, a; \bar{A})$ and $K(t, a; \bar{A})$.

Lemma 3.1.16. *Let $\bar{A} = (A_0, A_1)$ be a compatible couple of quasi-Banach spaces and let $a \in A_0 \cap A_1$. Then,*

$$J(t, a; \bar{A}) \leqslant \max(1, t/s) J(s, a; \bar{A}),$$
$$K(t, a; \bar{A}) \leqslant \min(1, t/s) J(s, a; \bar{A}).$$

Proof. First, let us see $J(t, a; \bar{A}) \leqslant \max(1, t/s) J(s, a; \bar{A})$.

If $t/s \leqslant 1$, then $t \leqslant s$ and $\max(1, t/s) = 1$. And, as $J(t, a; \bar{A})$ is an increasing function of t, then $J(t, a; \bar{A}) \leqslant J(s, a; \bar{A})$.

If $t/s \geqslant 1$, then $t \geqslant s$ and $\max(1, t/s) = t/s$. So, we have to see that

$$J(t, a; \bar{A}) \leqslant \frac{t}{s} J(s, a; \bar{A}).$$

$$\begin{aligned} J(t,a;\bar{A}) = J\left(t\frac{s}{s},a;\bar{A}\right) &= \max\left(\|a\|_{A_0}, s\frac{t}{s}\|a\|_{A_1}\right) \\ &= \max\left(\|a\|_{A_0}, s\|a\frac{t}{s}\|_{A_1}\right) \\ &\leqslant \max\left(\|a\frac{t}{s}\|_{A_0}, s\|a\frac{t}{s}\|_{A_1}\right) \\ &= J(s,a(t/s);\bar{A}). \end{aligned}$$

And, by Proposition 3.1.15, $J(s,a(t/s);\bar{A}) = (s/s)J(t,a;\bar{A})$. Then, we have proved that $J(t,a;\bar{A}) \leqslant \max(1,t/s)J(s,a;\bar{A})$.

Now, we are going to see that $K(t,a;\bar{A}) \leqslant \min(1,t/s)J(s,a;\bar{A})$.

First, since $a \in A_0 \cap A_1$ we have that

$$K(t,a;\bar{A}) \leqslant \min(\|a\|_{A_0}, t\|a\|_{A_1}).$$

Because, as $a \in A_0 \cap A_1$ we can consider the decompositions $a = a_0 + 0$ and $a = 0 + a_1$. So,

$$\inf_{a=a_0+a_1} \{\|a_0\|_{A_0} + t\|a_1\|_{A_1}\} \leqslant \min(\|a\|_{A_0}, t\|a\|_{A_1}).$$

Now, consider $t \geqslant s$, so $\min(1,t/s) = 1$ and we have three cases:

- If $\|a\|_{A_0} \leqslant s\|a\|_{A_1} \leqslant t\|a\|_{A_1}$, then

  $$K(t,a;\bar{A}) \leqslant \|a\|_{A_0} \leqslant J(s,a;\bar{A}).$$

- If $s\|a\|_{A_1} \leqslant \|a\|_{A_0} \leqslant t\|a\|_{A_1}$, then

  $$K(t,a;\bar{A}) \leqslant \|a\|_{A_0},$$

  and $J(s,a;\bar{A}) = \|a\|_{A_0}$. Therefore, $K(t,a;\bar{A}) \leqslant J(s,a;\bar{A})$.

- If $t\|a\|_{A_1} \leqslant \|a\|_{A_0}$, then

  $$K(t,a;\bar{A}) \leqslant t\|a\|_{A_1} \leqslant \|a\|_{A_0},$$

  and $J(s,a;\bar{A}) = \|a\|_{A_0}$. Therefore, $K(t,a;\bar{A}) \leqslant J(s,a;\bar{A})$.

And, finally take $t < s$, this implies that $\min(1,t/s) = t/s$. Again, we have three cases

- If $\|a\|_{A_0} \leqslant t\|a\|_{A_1}$, then

  $$\begin{aligned} K(t,a;\bar{A}) &\leqslant \|a\|_{A_0}, \\ J(t,a;\bar{A}) &= t\|a\|_{A_1} = \frac{ts\|a\|_{A_1}}{s} = (t/s)J(s,a;\bar{A}). \end{aligned}$$

  So, $K(t,a;\bar{A}) \leqslant J(t,a;\bar{A}) = (t/s)J(s,a;\bar{A})$.

- If $t\|a\|_{A_1} \leqslant \|a\|_{A_0} \leqslant s\|a\|_{A_1}$, then

  $$\begin{aligned} K(t,a;\bar{A}) &\leqslant t\|a\|_{A_1}, \\ J(t,a;\bar{A}) &= \|a\|_{A_0} \geqslant K(t,a;\bar{A}). \end{aligned}$$

But, notice that writing t as ts/s we can get

$$J(t,a;\bar{A}) = \max\left(\|a\|_{A_0}, s\|ta/s\|_{A_1}\right) \leqslant J(s,ta/s;\bar{A}).$$

And, by Proposition 3.1.15, $J(s,a(t/s);\bar{A}) = (t/s)J(s,a;\bar{A})$. So, we have that

$$K(t,a;\bar{A}) \leqslant (t/s)J(s,a;\bar{A}).$$

- If $s\|a\|_{A_1} \leqslant \|a\|_{A_0}$, then

$$\begin{aligned} K(t,a;\bar{A}) &\leqslant t\|a\|_{A_1}, \\ J(t,a;\bar{A}) &= \|a\|_{A_0} \geqslant K(t,a;\bar{A}). \end{aligned}$$

  And, using the same argument that in the case $t\|a\|_{A_1} \leqslant \|a\|_{A_0} \leqslant s\|a\|_{A_1}$, we arrive at

$$K(t,a;\bar{A}) \leqslant (t/s)J(s,a;\bar{A}).$$

Therefore, we have that $K(t,a;\bar{A}) \leqslant \min(1,t/s)J(s,a;\bar{A})$. ■

Now, we are going to define the interpolation spaces obtained via the J-functor. But, first we introduce some notation in order to distinguish the spaces generated by the K- and the J-functors.

From now, the spaces

$$(A_0,A_1)_{\theta,p} = \left\{ a \in A_0 + A_1 : \ \|a\|_{(A_0,A_1)_{\theta,p}} = \left(\int_0^\infty t^{-p\theta} K(t,a;\bar{A})^p \frac{dt}{t}\right)^{1/p} < \infty \right\}$$

will be denoted by $(A_0,A_1)^K_{\theta,p}$ and its norm will we denote by $\|\cdot\|_{(A_0,A_1)_{\theta,p,K}}$, and the real interpolation spaces generated by the J-method will be denoted by $(A_0,A_1)^J_{\theta,p}$ and its norm will we denote by $\|\cdot\|_{(A_0,A_1)_{\theta,p,J}}$.

So, let us define the $(A_0,A_1)^J_{\theta,p}$ spaces.

Definition 3.1.17. Let $\bar{A} = (A_0,A_1)$ be a compatible couple of quasi-Banach spaces. We define the spaces $(A_0,A_1)^J_{\theta,p}$, as the elements in $A_0 + A_1$, such that:

- There exists a measurable function u with values in $A_0 \cap A_1$, satisfying that

$$a = (A_0 + A_1) - \lim_{k\uparrow\infty} \int_{\frac{1}{k}}^k \frac{u(t)}{t} dt = \int_0^\infty \frac{u(t)}{t} dt. \tag{3.3}$$

  Notice that this integral is a limit of Bochner integrals as in Definition 1.2.3.

- $$\left(\int_0^\infty t^{-p\theta} J(t,u(t);\bar{A})^p \frac{dt}{t}\right)^{1/p} < \infty. \tag{3.4}$$

Here we consider the cases $0 < \theta < 1$, $1 \leqslant p \leqslant \infty$ and $0 \leqslant \theta \leqslant 1$, $p = 1$.

We define the norm in $(A_0, A_1)^J_{\theta,p}$ as

$$\|a\|_{(A_0,A_1)_{\theta,p,J}} = \inf_u \left(\int_0^\infty t^{-p\theta} J(t, u(t); \bar{A})^p \frac{dt}{t} \right)^{1/p},$$

where the infimum is taken over all u such that (3.3) and (3.4) hold.

As in the K-method we have the following proposition that tells us that the $(A_0, A_1)^J_{\theta,p}$ spaces are Banach spaces.

Proposition 3.1.18. *If $\bar{A} = (A_0, A_1)$ is a compatible couple of quasi-Banach spaces, then $(A_0, A_1)^J_{\theta,p}$ is a Banach space with norm $\|\cdot\|_{(A_0,A_1)_{\theta,p,J}}$.*

Proof. Using a similar argument that in Lemma 3.1.10 we have that $\|\cdot\|_{(A_0,A_1)_{\theta,p,J}}$ is a norm. So, let us prove that the space $(A_0, A_1)^J_{\theta,p}$ is complete with the norm $\|\cdot\|_{(A_0,A_1)_{\theta,p,J}}$. Let $(a_n)_n$ be a Cauchy sequence in $(A_0, A_1)^J_{\theta,p}$, then

$$\|a_n - a_m\|_{(A_0,A_1)_{\theta,p,J}} \to 0$$

as $n, m \to \infty$. But,

$$\|a_n - a_m\|_{(A_0,A_1)_{\theta,p,J}} = \inf_u \left(\int_0^\infty t^{-p\theta} J(t, u_n(t) - u_m(t); \bar{A})^p \frac{dt}{t} \right)^{1/p}$$

where u_n is a sequence of measurable functions such that

$$a_n = \int_0^\infty \frac{u_n(t)dt}{t} \quad \text{(with convergence in } A_0 + A_1).$$

This means that,

$$J(t, u_n(t) - u_m(t); \bar{A}) = \max(\|u_n(t) - u_m(t)\|_{A_0}, t\|u_n(t) - u_m(t)\|_{A_1}) \to 0$$

as $n, m \to \infty$. Hence

$$\|u_n(t) - u_m(t)\|_{A_0} \to 0,$$
$$\|u_n(t) - u_m(t)\|_{A_1} \to 0.$$

And, as A_0 and A_1 are quasi-Banach spaces there exist $b \in A_0$ and $c \in A_1$ such that $\|u_n(t) - b\|_{A_0} \to 0$ and $\|u_n(t) - c\|_{A_1} \to 0$ as $n \to \infty$.

Now, if we are able to prove that $b = c$, then $u_n(t)$ has a limit in $A_0 \cap A_1$ and taking

$$a = \int_0^\infty \lim_n \frac{u_n(t)dt}{t}$$

we have that $\|a_n - a\|_{(A_0,A_1)_{\theta,p,J}} \to 0$ as $n \to \infty$.

In order to prove that $b = c$ let $u_{n_k}(t)$ be a subsequence with limit b and $u_{n_l}(t)$ be a subsequence with limit c, and assume that $b \neq c$. Then,

$$J(t, b - c; \bar{A}) > 0.$$

Thus,

$$J(t, u_{n_k}(t) - u_{n_l}(t); \bar{A}) \to J(t, b - c; \bar{A}) > 0.$$

This is a contradiction with the that $(u_n)_n$ is a Cauchy sequence, so $b = c$. ■

Proposition 3.1.19. *Let $a \in A_0 \cap A_1$, then*

$$\|a\|_{(A_0,A_1)_{\theta,p,J}} \leqslant Cs^{-\theta} J(s,a;\bar{A})$$

where C is independent of θ and p.

Proof. Let $a \in A_0 \cap A_1$, then by Proposition 3.1.16 we have that

$$\begin{aligned}
\|a\|_{(A_0,A_1)_{\theta,p,J}} &= \inf_u \left(\int_0^\infty t^{-p\theta} J(t,u(t);\bar{A})^p \frac{dt}{t} \right)^{1/p} \\
&\leqslant \inf_u \left(\int_0^\infty t^{-p\theta} \max(1,t/s)^p J(s,u(t);\bar{A})^p \frac{dt}{t} \right)^{1/p} \\
&\leqslant \left(\int_0^\infty t^{-p\theta} \max(1,t/s)^p J(s,u(t);\bar{A})^p \frac{dt}{t} \right)^{1/p}.
\end{aligned}$$

Now, as it happens for all $u(t)$ such that

$$a = \int_0^\infty u(t) \frac{dt}{t}$$

we can take

$$u(t) = (\log(2))^{-1} a \chi_{(1,2)}(t)$$

and we obtain that

$$\begin{aligned}
\|a\|_{(A_0,A_1)_{\theta,p,J}} &\leqslant \left(\int_0^\infty t^{-p\theta} \max(1,t/s)^p J(s,u(t);\bar{A})^p \frac{dt}{t} \right)^{1/p} \\
&= \left(\int_1^2 t^{-p\theta} \max(1,t/s)^p J(s,(\log(2))^{-1}a;\bar{A})^p \frac{dt}{t} \right)^{1/p}.
\end{aligned}$$

But notice that we can decompose the interval $(1,2)$ in the intervals $(1,s)$ and $(s,2)$ where the values of $\max(1,t/s)$ are 1 and t/s respectively. Thus,

$$\begin{aligned}
\|a\|_{(A_0,A_1)_{\theta,p,J}} &\leqslant (\log(2))^{-1} \left(\int_1^2 t^{-p\theta} \max(1,t/s)^p J(s,a;\bar{A})^p \frac{dt}{t} \right)^{1/p} \\
&= (\log(2))^{-1} \left(\int_1^s t^{-p\theta} J(s,a;\bar{A})^p \frac{dt}{t} + \frac{1}{s^p} \int_s^2 t^{-p\theta} t^p J(s,(a;\bar{A})^p \frac{dt}{t} \right)^{1/p}.
\end{aligned}$$

As $J(s,(a;\bar{A})^p$ does not depend on t, we can take it out and obtain:

$$\|a\|_{(A_0,A_1)_{\theta,p,J}} \leqslant (\log(2))^{-1} J(s,a;\bar{A}) \left(\int_1^s t^{-p\theta} \frac{dt}{t} + \frac{1}{s^p} \int_s^2 t^{-p\theta} t^p \frac{dt}{t} \right)^{1/p}.$$

But, notice that $t^{-p\theta}/t$ is positive, so

$$\int_1^s t^{-p\theta} \frac{dt}{t} \leqslant \int_1^2 t^{-p\theta} \frac{dt}{t} \leqslant 2^p.$$

For the other integral, as we are in $(s, 2)$ we have that $t^p \leqslant 2^p$ and $t^{-p\theta} \leqslant s^{-p\theta}$. So,

$$\frac{1}{s^p}\int_s^2 t^{-p\theta}t^p\frac{dt}{t} \leqslant \frac{2^p s^{-p\theta}}{s^p}\int_s^2 \frac{dt}{t}.$$

Again, as t is positive we have that

$$\frac{2^p s^{-p\theta}}{s^p}\int_s^2 \frac{dt}{t} \leqslant \frac{2^p s^{-p\theta}}{s^p}\int_1^2 \frac{dt}{t} = \frac{2^p}{s^{p(\theta+1)}}\log(2).$$

And, as $s \geqslant 1$ and $\log(2) < 1$, we arrive at

$$\frac{2^p}{s^{p(\theta+1)}}\log(2) \leqslant \frac{2^p}{s^{p\theta}}.$$

Therefore,

$$\begin{aligned}
\|a\|_{(A_0,A_1)_{\theta,p,J}} &\leqslant (\log(2))^{-1}J(s,a;\bar{A})\left(\int_1^s t^{-p\theta}\frac{dt}{t} + \frac{1}{s^p}\int_s^2 t^{-p\theta}t^p\frac{dt}{t}\right)^{1/p} \\
&\leqslant (\log(2))^{-1}J(s,a;\bar{A})\left(2^p + \frac{2^p}{s^{p\theta}}\right)^{1/p} \\
&\leqslant (\log(2))^{-1}J(s,a;\bar{A})2\left(1 + s^{-p\theta}\right)^{1/p}.
\end{aligned}$$

But, we have that

$$\left(1 + s^{-p\theta}\right)^{1/p} \leqslant 1 + s^{-\theta}.$$

Then,

$$\begin{aligned}
\|a\|_{(A_0,A_1)_{\theta,p,J}} &\leqslant (\log(2))^{-1}J(s,a;\bar{A})2\left(1 + s^{-p\theta}\right)^{1/p} \\
&\leqslant (\log(2))^{-1}J(s,a;\bar{A})2 + (\log(2))^{-1}J(s,a;\bar{A})2s^{-\theta}.
\end{aligned}$$

Notice that now there exists some constant k such that it is independent of θ and p; and satisfies that

$$\begin{aligned}
\|a\|_{(A_0,A_1)_{\theta,p,J}} &\leqslant (\log(2))^{-1}J(s,a;\bar{A})2 + (\log(2))^{-1}J(s,a;\bar{A})2s^{-\theta} \\
&\leqslant k(\log(2))^{-1}J(s,a;\bar{A})2s^{-\theta}.
\end{aligned}$$

Calling $C = 2(\log(2))^{-1}k$, we arrive at

$$\|a\|_{(A_0,A_1)_{\theta,p,J}} \leqslant CJ(s,a;\bar{A})s^{-\theta}.$$

■

Now we will prove that as the K-method, the J-method is an exact functor of exponent θ and that the J-method can be discretized in the same way that the K-functor.

Proposition 3.1.20. *Let $\bar{A} = (A_0, A_1)$ and $\bar{B} = (B_0, B_1)$ be two compatible couples of quasi-Banach spaces. Let*

$$T : A_j \to B_j,\ j = 0, 1$$

be continuous with norm M_j. Then,

$$T : (A_0, A_1)^J_{\theta,p} \to (B_0, B_1)^J_{\theta,p}$$

is continuous with norm M. Moreover,

$$M \leqslant M_0^{1-\theta} M_1^{\theta}.$$

Proof. Let $a \in (A_0, A_1)^J_{\theta,p}$, then

$$Ta = T\left(\int_0^\infty u(t)\frac{dt}{t}\right).$$

But, as T is bounded in B_j and u is measurable we can commute it with the integral and obtain that

$$Ta = \int_0^\infty Tu(t)\frac{dt}{t}.$$

This follows from Proposition 1.2.16. Using this, we have that

$$\begin{aligned} J(t, Tu(t); \bar{B}) &= \max(\|Tu(t)\|_{B_0}, t\|Tu(t)\|_{B_1}) \\ &\leqslant \max(M_0\|u(t)\|_{A_0}, tM_1\|u(t)\|_{A_1}) \\ &\leqslant M_0 \max(\|u(t)\|_{A_0}, tM_1/M_0\|u(t)\|_{A_1}) \\ &= M_0 J(tM_1/M_0, u(t); \bar{A}). \end{aligned}$$

So, we have that

$$\int_0^\infty t^{-\theta p} J(t, Tu(t); \bar{B})^p \frac{dt}{t} \leqslant M_0^p \int_0^\infty t^{-\theta p} J(tM_1/M_0, u(t); \bar{A})^p \frac{dt}{t}.$$

Taking $s = tM_1/M_0$ in the second integral we obtain

$$\begin{aligned} \int_0^\infty t^{-\theta p} J(t, Tu(t); \bar{B})^p \frac{dt}{t} &\leqslant M_0^p \int_0^\infty t^{-\theta p} J(tM_1/M_0, u(t); \bar{A})^p \frac{dt}{t} \\ &\leqslant M_0^p \int_0^\infty \left(\frac{M_0}{M_1}\right)^{-\theta p} s^{-\theta p} J(s, u(s); \bar{A})^p \frac{ds}{s} \\ &= (M_0^{1-\theta} M_1^{\theta})^p \int_0^\infty s^{-\theta p} J(s, u(s); \bar{A})^p \frac{ds}{s}. \end{aligned}$$

Now, taking infimums in both sides we arrive at

$$\|Ta\|^p_{(B_0,B_1)_{\theta,p,J}} \leqslant (M_0^{1-\theta} M_1^{\theta})^p \|a\|^p_{(A_0,A_1)_{\theta,p,J}}.$$

■

Proposition 3.1.21. *Let $\theta \in (0,1)$, if $p \in (1,\infty]$ and $\theta \in [0,1]$, if $p = 1$. Then $a \in (A_0, A_1)^J_{\theta,p}$ if and only if there exist $u_n \in A_0 \cap A_1$, $n \in \mathbb{Z}$, with*

$$a = \sum_{n\in\mathbb{Z}} u_n \tag{3.5}$$

with convergence in $A_0 + A_1$, and such that $J(2^n, u_n) \in \lambda^{\theta,p}$. Moreover

$$\|a\|_{(A_0,A_1)_{\theta,p,J}} \sim \inf_{u_n} \|J(2^n, u_n)\|_{\lambda^{\theta,p}},$$

where the infimum is extended over all sequences $\{u_n\}$ satisfying (3.5).

Proof. Let $a \in (A_0, A_1)^J_{\theta,p}$. Then,

$$a = \int_0^\infty u(t)\frac{dt}{t}.$$

So, we can take as $\{u_n\}$

$$u_n = \int_{2^n}^{2^{n+1}} u(t)\frac{dt}{t}.$$

Then (3.5) holds because this is the dyadic partition of $(0,\infty)$. Even more,

$$\begin{aligned}\|J(2^n, u_n; \bar{A})\|^p_{\lambda^{\theta,p}} &= \sum_{n\in\mathbb{Z}} \left(2^{-n\theta} J(2^n, u_n; \bar{A})\right)^p \\ &\leqslant C \sum_{n\in\mathbb{Z}} \int_{2^n}^{2^{n+1}} \left(t^{-\theta} J(t, u(t); \bar{A})\right)^p \frac{dt}{t}.\end{aligned}$$

Taking infimums we arrive at

$$\inf_{u_n} \|J(2^n, u_n; \bar{A})\|^p_{\lambda^{\theta,p}} \leqslant C\|a\|^p_{(A_0,A_1)_{\theta,p,J}}.$$

For the converse implication, assume that $a = \sum_n u_n$ and $J(2^n, u_n; \bar{A}) \in \lambda^{\theta,p}$. Taking

$$u(t) = \frac{u_n}{\log 2}, \quad 2^n \leqslant t \leqslant 2^{n+1},$$

we obtain that

$$a = \sum_n u_n = \sum_n \int_{2^n}^{2^{n+1}} \frac{u(t)}{\log(2)}\frac{dt}{t} = \int_0^\infty u(t)\frac{dt}{t}.$$

Also, we have that

$$\begin{aligned}\int_0^\infty \left(t^{-\theta} J(t, u(t); \bar{A})\right)^p \frac{dt}{t} &= \sum_{n\in\mathbb{Z}} \int_{2^n}^{2^{n+1}} \left(t^{-\theta} J(t, u(t); \bar{A})\right)^p \frac{dt}{t} \\ &\leqslant C_2 \sum_{n\in\mathbb{Z}} \left(2^{-n\theta} J(2^n, u_n; \bar{A})\right)^p.\end{aligned}$$

Again, taking infimums we arrive at

$$C_2\|a\|^p_{(A_0,A_1)_{\theta,p,J}} \leqslant \inf_{u_n} \|J(2^n, u_n; \bar{A})\|^p_{\lambda^{\theta,p}}.$$

Then, as we have

$$\begin{cases} \inf_{u_n} \|J(2^n, u_n; \bar{A})\|^p_{\lambda^{\theta,p}} & \leqslant C\|a\|^p_{(A_0,A_1)_{\theta,p,J}} \\ C_2\|a\|^p_{(A_0,A_1)_{\theta,p,J}} & \leqslant \inf_{u_n} \|J(2^n, u_n; \bar{A})\|^p_{\lambda^{\theta,p}}, \end{cases}$$

we conclude that

$$\|a\|_{(A_0,A_1)_{\theta,p,J}} \sim \inf_{u_n} \|J(2^n, u_n)\|_{\lambda^{\theta,p}}.$$

■

3.2 The Equivalence Theorem

In this section we will prove that the spaces generated by the $K-$ and the $J-$ methods are equivalent for the θ and p where the two methods are defined. But for this purpose we need the following lemma that gives us a bound for the $J-$functional by the $K-$functional.

Lemma 3.2.1 (The fundamental lemma of interpolation theory)**.** *Assume that*

$$\min(1, 1/t)K(t, a; \bar{A}) \to 0$$

as $t \downarrow 0$ or as $t \uparrow \infty$.

Then, for any $\varepsilon > 0$, there is a representation

$$a = \sum_n u_n, \quad \textit{with convergence in } A_0 + A_1$$

such that

$$J(2^n, u_n; \bar{A}) \leqslant (\gamma + \varepsilon)K(2^n, a; \bar{A}),$$

where γ is a universal constant less than or equal to 3.

Proof. Let $a \in A_0 + A_1$. For every integer n, there exists a decomposition $a = a_{0,n} + a_{1,n}$, such that for a given $\varepsilon > 0$

$$\|a_{0,n}\|_{A_0} + 2^n\|a_{1,n}\|_{A_1} \leqslant (1 + \varepsilon)K(2^n, a; \bar{A}). \tag{3.6}$$

Thus, since

$$\min(1, 1/t)K(t, a; \bar{A}) \to 0$$

as $t \downarrow 0$ or as $t \uparrow \infty$, we obtain that

$$\|a_{0,n}\|_{A_0} \to 0, \quad \text{as } n \to -\infty,$$
$$\|a_{1,n}\|_{A_1} \to 0, \quad \text{as } n \to \infty.$$

Take

$$u_n = a_{0,n} - a_{0,n-1} = a_{1,n-1} - a_{1,n}.$$

Then, $u_n \in A_0 \cap A_1$ and

$$a - \sum_{-N}^{M} u_n = a - a_{0,M} + a_{0,-N-1} = a_{1,M} + a_{0,-N-1}.$$

Hence

$$K\left(1, a - \sum_{-N}^{M} u_n; \bar{A}\right) \leqslant \|a_{0,-N-1}\|_{A_0} + \|a_{1,M}\|_{A_1}.$$

Letting $N, M \to \infty$ and using that the K-functor is a norm (Proposition 3.1.5) we can see that

$$a = \sum_{-\infty}^{\infty} u_n,$$

where the convergence is in $A_0 + A_1$.

But, by definition of u_n we also have that

$$J(2^n, u_n; \bar{A}) \leqslant \max(\|a_{0,n}\|_{A_0} + \|a_{0,n-1}\|_{A_0}, 2^n(\|a_{1,n}\|_{A_1} + \|a_{1,n-1}\|_{A_1})).$$

Using (3.6) and that $K(t, a; \bar{A})$ is increasing with respect to t, we have that

$$\begin{aligned}
&\max(\|a_{0,n}\|_{A_0} + \|a_{0,n-1}\|_{A_0}, 2^n(\|a_{1,n}\|_{A_1} + \|a_{1,n-1}\|_{A_1}))\\
\leqslant &\max(2(1+\varepsilon)K(2^n, a; \bar{A}), 2^n\|a_{1,n}\|_{A_1} + 2 \cdot 2^{n-1}\|a_{1,n-1}\|_{A_1}))\\
\leqslant &\max(2(1+\varepsilon)K(2^n, a; \bar{A}), 3((1+\varepsilon)K(2^n, a; \bar{A})))\\
= &\,3(1+\varepsilon)K(2^n, a; \bar{A}).
\end{aligned}$$

■

Theorem 3.2.2 (The equivalence theorem)**.** *If* $0 < \theta < 1$ *and* $1 \leqslant p \leqslant \infty$*, then* $(A_0, A_1)^J_{\theta,p} = (A_0, A_1)^K_{\theta,p}$ *with equivalence of norms.*

Proof. Let us verify that $\|\cdot\|_{(A_0,A_1)_{\theta,p,K}} \leqslant \|\cdot\|_{(A_0,A_1)_{\theta,p,J}}$.

Take $a \in (A_0, A_1)^J_{\theta,p}$ and

$$a = \int_0^\infty u(t)\frac{dt}{t}.$$

Then, by Proposition 1.2.16 and the Lemma 3.1.16, we have that

$$\begin{aligned}
K(t, a; \bar{A}) &\leqslant \int_0^\infty K(t, u(s); \bar{A})\frac{ds}{s} \leqslant \int_0^\infty \min(1, t/s)J(s, u(s); \bar{A})\frac{ds}{s}\\
&= \int_0^\infty \min(1, s^{-1})J(ts, u(ts); \bar{A})\frac{ds}{s}.
\end{aligned}$$

So, we have that

$$\|a\|_{(A_0,A_1)_{\theta,p,K}} \leqslant \left(\int_0^\infty t^{-p\theta}\left(\int_0^\infty \min(1, s^{-1})J(ts, u(ts); \bar{A})\frac{ds}{s}\right)^p \frac{dt}{t}\right)^{1/p}.$$

Let $r = st$, then

$$\begin{aligned}\|a\|_{(A_0,A_1)_{\theta,p,K}} &\leqslant \left(\int_0^\infty r^{-p\theta}\left(\int_0^\infty s^{-\theta}\min(1,s^{-1})J(r,u(r);\bar{A})\frac{ds}{s}\right)^p\frac{dr}{r}\right)^{1/p}\\ &= \|a\|_{(A_0,A_1)_{\theta,p,J}}\left(\int_0^\infty s^{-\theta}\min(1,s^{-1})\frac{ds}{s}\right) = C\|a\|_{(A_0,A_1)_{\theta,p,J}}.\end{aligned}$$

Then, we have that $\|\cdot\|_{(A_0,A_1)_{\theta,p,K}} \leqslant \|\cdot\|_{(A_0,A_1)_{\theta,p,J}}$. Now, let us see the converse inequality.

Let $a \in (A_0,A_1)^K_{\theta,p}$, by Properties 3.1.9 we have that

$$K(t,a;\bar{A}) \leqslant C_{\theta,p}t^\theta\|a\|_{(A_0,A_1)_{\theta,p,K}}.$$

Thus, it follows that $\min(1,1/t)K(t,a;\bar{A}) \to 0$ as $t \to 0$ or $t \to \infty$. Consequently, the Lemma 3.2.1 implies the existence of a representation $a = \sum_n u_n$ such that

$$J(2^n,u_n;\bar{A}) \leqslant (\gamma+\varepsilon)K(2^n,a;\bar{A}).$$

Thus,

$$\|J(2^n,u_n;\bar{A})\|_{\gamma^{\theta,p}} \leqslant (\gamma+\varepsilon)\|K(2^n,a;\bar{A})\|_{\gamma^{\theta,p}}.$$

But, by Lemma 3.1.12 we have that

$$2^{-\theta}\log(2)^{1/p}\|K(2^n,a;\bar{A})\|_{\gamma^{\theta,p}} \leqslant \|a\|_{(A_0,A_1)_{\theta,p,K}}.$$

Then,

$$\|K(2^n,a;\bar{A})\|_{\gamma^{\theta,p}} \leqslant \frac{\|a\|_{(A_0,A_1)_{\theta,p,K}}2^\theta}{\log(2)^{1/p}} \leqslant \frac{2\|a\|_{(A_0,A_1)_{\theta,p,K}}}{\log(2)} = C\|a\|_{(A_0,A_1)_{\theta,p,K}}.$$

And, by Proposition 3.1.21 we have that

$$\|J(2^n,u_n;\bar{A})\|_{\gamma^{\theta,p}} \sim \|a\|_{(A_0,A_1)_{\theta,p,J}}.$$

Therefore, we have that

$$\|a\|_{(A_0,A_1)_{\theta,p,J}} \leqslant C\|a\|_{(A_0,A_1)_{\theta,p,K}}.$$

And this implies that $\|\cdot\|_{(A_0,A_1)_{\theta,p,J}} \leqslant \|\cdot\|_{(A_0,A_1)_{\theta,p,K}}$. Then, we have that $\|\cdot\|_{(A_0,A_1)_{\theta,p,J}}$ and $\|\cdot\|_{(A_0,A_1)_{\theta,p,K}}$ are equivalent and, in particular, that $(A_0,A_1)^J_{\theta,p} = (A_0,A_1)^K_{\theta,p}$. ■

This theorem tells us that if $0 < \theta < 1$ and $1 \leqslant p \leqslant \infty$, then the notations $(A_0,A_1)^J_{\theta,p}$ and $(A_0,A_1)^K_{\theta,p}$ are not necessary because they are the same space, so we can call this space $(A_0,A_1)_{\theta,p}$ as in the beginning of this chapter.

As if $p = 1$ and $0 \leqslant \theta \leqslant 1$ we only have defined the space $(A_0,A_1)^J_{\theta,p}$ and if $p = \infty$ and $0 \leqslant \theta \leqslant 1$ we only have defined the $(A_0,A_1)^K_{\theta,p}$, we can also denote these spaces by $(A_0,A_1)_{\theta,p}$. And, we denote the norm on $(A_0,A_1)_{\theta,p}$ by $\|\cdot\|_{\theta,p}$.

3.3 Some Properties of $(A_0, A_1)_{\theta,p}$

In this section we will study some properties of the space $(A_0, A_1)_{\theta,p}$. We divide the properties in three theorems, the first theorem deals with inclusions between various $(A_0, A_1)_{\theta,p}$ spaces. The second deals with the inclusion of the space $A_0 \cap A_1$ and its closure between the space $(A_0, A_1)_{\theta,p}$. And the last theorem which deals with the duals of $A_0 + A_1$ and $A_0 \cap A_1$.

Theorem 3.3.1. *Let $\bar{A} = (A_0, A_1)$ be a compatible couple of quasi-Banach spaces. Then we have that*

1. $(A_0, A_1)_{\theta,p} = (A_1, A_0)_{1-\theta,p}$, *with equal norms;*
2. $(A_0, A_1)_{\theta,p} \subset (A_0, A_1)_{\theta,r}$ *if* $p \leqslant r$;
3. $(A_0, A_1)_{\theta_0,p_0} \cap (A_0, A_1)_{\theta_1,p_1} \subset (A_0, A_1)_{\theta,p}$ *if* $\theta_0 \leqslant \theta \leqslant \theta_1$;
4. *if* $\theta_0 \leqslant \theta_1$ *and* $A_1 \subset A_0$, *then* $(A_0, A_1)_{\theta_1,p} \subset (A_0, A_1)_{\theta_0,p}$;
5. $A_0 = A_1$ *with equal norms implies that* $(A_0, A_1)_{\theta,p} = A_0$ *and*
$$\|a\|_{A_0} = (p\theta(1-\theta))^{1/p}\|a\|_{\theta,p}.$$

Proof. Let us prove that $(A_0, A_1)_{\theta,p} = (A_1, A_0)_{1-\theta,p}$, with equal norms. Let $a \in (A_0, A_1)_{\theta,p}$ by the definition of the norm in $(A_0, A_1)_{\theta,p}$ we have that

$$\begin{aligned}
\|a\|_{\theta,p}^p &= \int_0^\infty \left(t^{-\theta}K(t,a;\bar{A})\right)^p \frac{dt}{t} \\
&= \int_0^\infty \left(\inf_{a=a_0+a_1}\left\{\|a_0t^{-\theta}\|_{A_0} + t^{1-\theta}\|a_1\|_{A_1}\right\}\right)^p \frac{dt}{t} \\
&= \int_0^\infty \left(t^{1-\theta}K\left(\frac{1}{t}, a; (A_1, A_0)\right)\right)^p \frac{dt}{t}.
\end{aligned}$$

Taking $r = 1/t$ we arrive at

$$\begin{aligned}
\|a\|_{\theta,p}^p &= \int_0^\infty \left(t^{1-\theta}K\left(\frac{1}{t}, a; (A_1, A_0)\right)\right)^p \frac{dt}{t} \\
&= \int_0^\infty \left(r^{-(1-\theta)}K(r, a; (A_1, A_0))\right)^p \frac{dr}{r} = \|a\|_{1-\theta,p}^p.
\end{aligned}$$

So, $(A_0, A_1)_{\theta,p} = (A_1, A_0)_{1-\theta,p}$, with equal norms.

In order to prove that $(A_0, A_1)_{\theta,p} \subset (A_0, A_1)_{\theta,r}$ if $p \leqslant r$, we will notice that as the K-functional satisfies that
$$K(s, a; \bar{A}) \leqslant \gamma_{\theta,p}s^\theta\|a\|_{\theta,p},$$
which implies $(A_0, A_1)_{\theta,p} \subset (A_0, A_1)_{\theta,\infty}$ if $p \leqslant \infty$. So, assume that $p \leqslant r < \infty$. Then

$$\begin{aligned}
\|a\|_{\theta,r}^r &= \int_0^\infty \left(t^{-\theta}K(t,a;\bar{A})\right)^r \frac{dt}{t} \\
&= \int_0^\infty \left(t^{-\theta}K(t,a;\bar{A})\right)^p \left(t^{-\theta}K(t,a;\bar{A})\right)^{r-p} \frac{dt}{t} \\
&\leqslant C\|a\|_{\theta,p}^p\|a\|_{\theta,p}^{r-p},
\end{aligned}$$

which implies that $(A_0, A_1)_{\theta,p} \subset (A_0, A_1)_{\theta,r}$ if $p \leqslant r$.

To prove that $(A_0, A_1)_{\theta_0,p_0} \cap (A_0, A_1)_{\theta_1,p_1} \subset (A_0, A_1)_{\theta,p}$ if $\theta_0 \leqslant \theta \leqslant \theta_1$ we observe the following inequalities.

$$\begin{aligned}
\|a\|_{\theta,p} &= \left(\int_0^\infty \left(t^{-\theta} K(t, a; \bar{A}) \right)^p \frac{dt}{t} \right)^{1/p} \\
&\leqslant \left(\int_0^1 \left(t^{-\theta} K(t, a; \bar{A}) \right)^p \frac{dt}{t} \right)^{1/p} + \left(\int_1^\infty \left(t^{-\theta} K(t, a; \bar{A}) \right)^p \frac{dt}{t} \right)^{1/p} \\
&\leqslant \left(\int_0^1 \left(t^{-\theta_1} K(t, a; \bar{A}) \right)^{p_1} \frac{dt}{t} \right)^{1/p_1} + \left(\int_1^\infty \left(t^{-\theta_0} K(t, a; \bar{A}) \right)^{p_0} \frac{dt}{t} \right)^{1/p_0} \\
&\leqslant \|a\|_{\theta_1,p_1} + \|a\|_{\theta_0,p_0}
\end{aligned}$$

This implies that if $a \in (A_0, A_1)_{\theta_0,p_0} \cap (A_0, A_1)_{\theta_1,p_1}$, then $a \in (A_0, A_1)_{\theta,p}$. Therefore, $(A_0, A_1)_{\theta_0,p_0} \cap (A_0, A_1)_{\theta_1,p_1} \subset (A_0, A_1)_{\theta,p}$.

Now, we are going to see that if $\theta_0 \leqslant \theta_1$ and $A_1 \subset A_0$, then $(A_0, A_1)_{\theta_1,p} \subset (A_0, A_1)_{\theta_0,p}$. First observe that if $A_1 \subset A_0$, then there exists some constant k such that $\|a\|_{A_0} \leqslant k\|a\|_{A_1}$. So, if $t > k$ then $K(t, a; \bar{A}) = \|a\|_{A_0}$ because if $a = a_0 + a_1$ is any decomposition of a in $A_0 + A_1$ then

$$\|a\|_{A_0} \leqslant \|a_0\|_{A_0} + \frac{t}{k}\|a_1\|_{A_0} \leqslant \|a_0\|_{A_0} + t\|a_1\|_{A_1},$$

taking the infimum over the decomposition of a in both sides we arrive at $\|a\|_{A_0} \leqslant K(t, a; \bar{A})$ and by definition of infimum $\|a\|_{A_0} = K(t, a; \bar{A})$. With this equality we can conclude that

$$\|a\|_{\theta,p} \sim \left(\int_0^k \left(t^{-\theta} K(t, a; \bar{A}) \right)^p \frac{dt}{t} \right)^{1/p} + \|a\|_{A_0}.$$

Thus,

$$\begin{aligned}
\|a\|_{\theta_0,p} &\sim \left(\int_0^k \left(t^{-\theta_0} K(t, a; \bar{A}) \right)^p \frac{dt}{t} \right)^{1/p} + \|a\|_{A_0} \\
&\leqslant \left(\int_0^k \left(t^{-\theta_1} K(t, a; \bar{A}) \right)^p \frac{dt}{t} \right)^{1/p} + \|a\|_{A_0} \sim \|a\|_{\theta_1,p}.
\end{aligned}$$

Then $\|a\|_{\theta_0,p} \leqslant \|a\|_{\theta_1,p}$. So, $(A_0, A_1)_{\theta_1,p} \subset (A_0, A_1)_{\theta_0,p}$.

Now it remains to see that if $A_0 = A_1$ with equal norms, then $(A_0, A_1)_{\theta,p} = A_0$ and

$$\|a\|_{A_0} = (p\theta(1-\theta))^{1/p}\|a\|_{\theta,p}.$$

As $A_0 = A_1$ we have that $(A_0, A_0)_{\theta,p} = (A_0, A_0)_{1-\theta,p}$. Even more, we have that $(A_0, A_0)_{\theta_1,p} \subset (A_0, A_0)_{\theta,p}$ is $\theta \leqslant \theta_1$. Therefore, taking θ near 0 we have that for all $\theta_1 \in (\theta, 1)$, then $(A_0, A_0)_{\theta_1,p} \subset (A_0, A_0)_{\theta,p}$. But, in particular, if $1 - \theta > \theta_1$ we have that $(A_0, A_0)_{\theta,p} = (A_0, A_0)_{1-\theta,p} \subset (A_0, A_0)_{\theta_1,p}$. This implies that the spaces $(A_0, A_0)_{\theta_1,p}$ and $(A_0, A_0)_{\theta,p}$ are equal for all $\theta_1, \theta \in (0, 1)$.

By the Properties 3.1.9 we have that $\min(1, t/s)K(s, a; \bar{A}) \leqslant K(t, a; \bar{A})$, using this with $s = 1$ we have that

$$\begin{aligned}
\|a\|_{\theta,p} &= \left(\int_0^\infty \left(t^{-\theta} K(t, a; \bar{A})\right)^p \frac{dt}{t}\right)^{1/p} \\
&\geqslant K(1, a; \bar{A}) \left(\int_0^1 \left(t^{1-\theta}\right)^p \frac{dt}{t} + \int_1^\infty \left(t^{-\theta}\right)^p \frac{dt}{t}\right)^{1/p} \\
&= K(1, a; \bar{A}) \left(\frac{1}{(1-\theta)p} + \frac{1}{\theta p}\right)^{1/p} = K(1, a; \bar{A}) \left(\frac{1}{(1-\theta)\theta p}\right)^{1/p}.
\end{aligned}$$

Since $K(1, a; \bar{A}) = \|a\|_{A_0}$ we arrive at

$$\|a\|_{\theta,p}((1-\theta)\theta p)^{1/p} \geqslant \|a\|_{A_0}.$$

So, we have to see the other inequality, but for this we will use the Lemma 3.1.16 that tells us that

$$K(t, a; \bar{A}) \leqslant \min(1, t/s) J(s, a; \bar{A}).$$

Take $s = 1$, then we have that

$$J(1, a; \bar{A}) = \max(\|a\|_{A_0}, \|a\|_{A_0}) = \|a\|_{A_0}.$$

So, we have that

$$\begin{aligned}
\|a\|_{\theta,p} = \left(\int_0^\infty \left(t^{-\theta} K(t, a; \bar{A})\right)^p \frac{dt}{t}\right)^{1/p} &\leqslant \|a\|_{A_0} \left(\int_0^\infty \left(t^{-\theta} \min(1, t)\right)^p \frac{dt}{t}\right)^{1/p} \\
&= \|a\|_{A_0} \left(\frac{1}{(1-\theta)\theta p}\right)^{1/p}.
\end{aligned}$$

Hence, we obtain that

$$\|a\|_{A_0} \geqslant \|a\|_{\theta,p}((1-\theta)\theta p)^{1/p} \geqslant \|a\|_{A_0}.$$

So we conclude that $\|a\|_{A_0} = \|a\|_{\theta,p}((1-\theta)\theta p)^{1/p}$. And this finish the proof of the theorem.

■

Theorem 3.3.2. *Let $\bar{A} = (A_0, A_1)$ be a compatible couple of quasi-Banach spaces. Then we have that*

1. *If $p < \infty$ then $A_0 \cap A_1$ is dense in $(A_0, A_1)_{\theta,p}$.*
2. *The closure of $A_0 \cap A_1$ in $(A_0, A_1)_{\theta,p}$ is the space $(A_0, A_1)^0_{\theta,p}$ of all a such that*

   $$t^{-\theta} K(t, a; \bar{A}) \to 0$$

   as $t \to 0$ or $t \to \infty$.
3. *If A_j^0 denotes the closure of $A_0 \cap A_1$ in A_j we have for $p < \infty$,*

   $$(A_0, A_1)_{\theta,p} = (A_0^0, A_1)_{\theta,p} = (A_0, A_1^0)_{\theta,p} = (A_0^0, A_1^0)_{\theta,p}.$$

Proof. Let us prove that if $p < \infty$ then $A_0 \cap A_1$ is dense in $(A_0, A_1)_{\theta,p}$. Note, that if $p < \infty$ then $0 < \theta < 1$. By Theorem 3.2.2 and Proposition 3.1.21 every $a \in (A_0, A_1)_{\theta,p}$ can be expressed as

$$a = \sum_n u_n$$

where $u_n \in A_0 \cap A_1$ and

$$\left(\sum_n (2^{-n\theta} J(2^n, u_n; \bar{A}))^p\right)^{1/p} < \infty.$$

Then

$$\left\| a - \sum_{|n| \leqslant N} u_n \right\|_{\theta,p} \leqslant \left(\sum_{|n|>N} (2^{-n\theta} J(2^n, u_n; \bar{A}))^p \right)^{1/p} \to 0,$$

as $N \uparrow \infty$. In other words, every $a \in (A_0, A_1)_{\theta,p}$ can be approximated by a sequence in $A_0 \cap A_1$, so $A_0 \cap A_1$ is dense in $(A_0, A_1)_{\theta,p}$.

Now we are going to prove that the space $(A_0, A_1)^0_{\theta,p}$ is closed. Let a_n be a convergent sequence on $(A_0, A_1)^0_{\theta,p}$, then we know that there exists $a \in (A_0, A_1)_{\theta,p}$ such that $a_n \to a$, but we want to see that $a \in (A_0, A_1)^0_{\theta,p}$. Also, we know that

$$t^{-\theta} K(t, a_n; \bar{A}) \leqslant \gamma_{\theta,p} \|a_n\|_{\theta,p} \leqslant \gamma_{\theta,p} \sup_n \|a_n\|_{\theta,p}.$$

As a_n is convergent we have that $\sup_n \|a_n\|_{\theta,p}$ is finite, and by hypothesis we have that

$$\lim_{n \uparrow \infty} \lim_{t \downarrow 0^+} t^{-\theta} K(t, a_n; \bar{A}) = 0.$$

So, applying the Dominated Convergence Theorem we arrive at

$$0 = \lim_{n \uparrow \infty} \lim_{t \downarrow 0^+} t^{-\theta} K(t, a_n; \bar{A}) = \lim_{t \downarrow 0^+} \lim_{n \uparrow \infty} t^{-\theta} K(t, a_n; \bar{A}) = \lim_{t \downarrow 0^+} t^{-\theta} K(t, a; \bar{A}).$$

Therefore, $t^{-\theta} K(t, a; \bar{A}) \to 0$ as $t \to 0^+$. And the same happens when we take $t \to \infty$ since the bound $\sup_n \|a_n\|_{\theta,p}$ does not depend on t.

We now prove that the closure of $A_0 \cap A_1$ in $(A_0, A_1)_{\theta,p}$ is the space $(A_0, A_1)^0_{\theta,p}$. Let $a \in (A_0, A_1)^0_{\theta,p}$ and assume that $\theta \in [0, 1]$. By Lemma 3.2.1 we have that $a = \sum_n u_n$, where $u_n \in A_0 \cap A_1$ and

$$J(2^n, u_n; \bar{A}) \leqslant C K(2^n, u_n; \bar{A}).$$

Then

$$\left\| a - \sum_{|n| \leqslant N} u_n \right\|_{\theta,p} \leqslant C \sup_{|n| \geqslant N} 2^{-n\theta} K(2^n, u_n; \bar{A}) \to 0, \text{ as } N \uparrow \infty.$$

Hence, $A_0 \cap A_1$ is dense in $(A_0, A_1)^0_{\theta,p}$. If we are able to see that the closure of $A_0 \cap A_1$ is in $(A_0, A_1)^0_{\theta,p}$, we will be done because $(A_0, A_1)^0_{\theta,p}$ is closed. So, take a in the closure of $A_0 \cap A_1$ in $(A_0, A_1)_{\theta,p}$ then we can find $b \in A_0 \cap A_1$ such that $\|a - b\|_{\theta,p} \leqslant \varepsilon$. By Lemma 3.1.16 and Properties 3.1.9, we obtain that

$$K(t, a; \bar{A}) \leqslant K(t, a - b; \bar{A}) + K(t, b; \bar{A}) \leqslant C t^{\theta} \|a - b\|_{\theta,p} + \min(1, t) J(1, b; \bar{A}).$$

Thus,

$$t^{-\theta}K(t,a;\bar{A}) \leqslant C\varepsilon + t^{-\theta}\min(1,t)J(1,b;\bar{A}).$$

It follows that $a \in (A_0, A_1)^0_{\theta,p}$.

It remains to see that if A^0_j denotes the closure of $A_0 \cap A_1$ in A_j we have for $p < \infty$,

$$(A_0, A_1)_{\theta,p} = (A^0_0, A_1)_{\theta,p} = (A_0, A^0_1)_{\theta,p} = (A^0_0, A^0_1)_{\theta,p}.$$

As $p < \infty$ then $0 < \theta < 1$ and we can use the J-functor. Even more, since $(A^0_0 + A^0_1) \subset (A_0 + A^0_1) \subset (A_0 + A_1)$ and $(A^0_0 \cap A^0_1) \subset (A_0 \cap A^0_1) \subset (A_0 \cap A_1)$, we obtain that

$$(A^0_0, A^0_1)_{\theta,p} \subset (A_0, A^0_1)_{\theta,p} \subset (A_0, A_1)_{\theta,p}.$$

So, we only need to prove that $(A_0, A_1)_{\theta,p} \subset (A^0_0, A^0_1)_{\theta,p}$ (because we have the same inclusions for $(A^0_0, A_1)_{\theta,p}$). Let $a \in (A_0, A_1)_{\theta,p}$ we want to see that $a \in (A^0_0, A^0_1)_{\theta,p}$. By Proposition 3.1.19 we have that

$$\|a\|_{(A_0,A_1)_{\theta,p}} \leqslant CJ(1,a;(A_0,A_1)) < \infty.$$

Note that $(A_0 \cap A_1) \subset (A^0_0 \cap A^0_1)$ because $(A_0 \cap A_1) \subset A^0_j$. Then, if $u(t)$ takes values in $A_0 \cap A_1$ then $u(t)$ takes values in $(A^0_0 \cap A^0_1)$. Therefore, $J(t,u(t);(A_0,A_1)) \geqslant J(t,u(t);(A^0_0,A^0_1))$, so we have that

$$\left(\int_0^\infty t^{-p\theta}J(t,u(t);(A^0_0,A^0_1))^p\frac{dt}{t}\right)^{1/p} \leqslant \left(\int_0^\infty t^{-p\theta}J(t,u(t);(A_0,A_1))^p\frac{dt}{t}\right)^{1/p}$$

So, if we take $u(t)$ satisfying that

$$a = (A_0 + A_1) - \lim_{k\downarrow 0}\int_k^{1/k} u(t)\frac{dt}{t},$$

and take the infimum over those u in both sides, we can conclude that

$$\|a\|_{(A_0,A_1)_{\theta,p}} \geqslant \|a\|_{(A^0_0,A^0_1)_{\theta,p}}.$$

Therefore, $(A_0, A_1)_{\theta,p} \subset (A^0_0, A^0_1)_{\theta,p}$ and this implies that

$$(A_0, A_1)_{\theta,p} = (A^0_0, A_1)_{\theta,p} = (A_0, A^0_1)_{\theta,p} = (A^0_0, A^0_1)_{\theta,p}.$$

■

Also we have the following theorem which gives a relation between $(A_0 + A_1)'$, $A_0 + A_1$, $A_0 \cap A_1$ and $A'_0 \cap A'_1$.

Theorem 3.3.3. *Suppose that $A_0 \cap A_1$ is dense in A_0 and in A_1. Then, $(A_0 \cap A_1)' = A'_0 + A'_1$ and $(A_0 + A_1)' = A'_0 \cap A'_1$. More precisely*

$$\|a'\|_{A'_0+A'_1} = \sup_{a \in A_0 \cap A_1} \frac{|\langle a', a\rangle|}{\|a\|_{A_0\cap A_1}}$$

and

$$\|a'\|_{A'_0\cap A'_1} = \sup_{a \in A_0 + A_1} \frac{|\langle a', a\rangle|}{\|a\|_{A_0+A_1}}$$

Proof. Let us begin by proving the first formula. First, let $a' \in A_0' + A_1'$ and $a' = a_0' + a_1'$ with $a_i' \in A_i'$. Then

$$|\langle a', a\rangle| \leqslant |\langle a_0', a\rangle| + |\langle a_1', a\rangle| \leqslant (\|a_0'\|_{A_0'} + \|a_1'\|_{A_1'}) \max(\|a\|_{A_0}, \|a\|_{A_1}),$$

with $a \in A_0 \cap A_1$. Consequently, $a' \in (A_0 \cap A_1)'$ and $\|a'\|_{(A_0\cap A_1)'} \leqslant \|a'\|_{A_0'+A_1'}$.

Conversely, let $l \in (A_0 \cap A_1)'$, i.e.,

$$|\langle l, a\rangle| \leqslant \|l\|_{(A_0\cap A_1)'} \|a\|_{A_0\cap A_1}, \quad a \in A_0 \cap A_1.$$

Then the linear from

$$\lambda : (a_0, a_1) \mapsto \left\langle l, \frac{a_0 + a_1}{2} \right\rangle$$

on $E = \{(a_0, a_1) \in A_0 \oplus A_1 : a_0 = a_1\}$ is continuous in the norm $\max(\|a_0\|_{A_0}, \|a_1\|_{A_1})$ on $A_0 \oplus A_1$, E is a subspace of $A_0 \oplus A_1$. Then, by Hahn-Banach theorem, there is $(a_0', a_1') \in A_0' \oplus A_1'$ such that

$$\|a_0'\|_{A_0'} + \|a_1'\|_{A_1'} \leqslant \|l\|_{(A_0\cap A_1)'}$$

and

$$\lambda(a_0, a_1) = \langle a_0', a_0\rangle + \langle a_1', a_1\rangle, \quad (a_0, a_1) \in E.$$

Thus, taking $a_0 = a_1 = a$, we obtain

$$\langle l, a\rangle = \langle a_0', a\rangle + \langle a_1', a\rangle = \langle a_0' + a_1', a\rangle, \quad a \in A_0 \cap A_1.$$

As $A_0 \cap A_1$ is dense in A_j a_0' and a_1' are determined by their values on $A_0 \cap A_1$. Putting $l = a_0' + a_1'$, we have that

$$\|l\|_{A_0'+A_1'} \leqslant \|l\|_{(A_0\cap A_1)'}.$$

This implies that

$$\|a'\|_{A_0'+A_1'} = \sup_{a\in A_0\cap A_1} \frac{|\langle a', a\rangle|}{\|a\|_{A_0\cap A_1}}.$$

Now we are going to prove the second formula. Let $a' \in A_0' \cap A_1'$ and $a \in A_0 + A_1$. Take $\varepsilon > 0$ and let $a_{0,\varepsilon} \in A_0$ and $a_{1,\varepsilon} \in A_1$ such that $a = a_0 + a_1$ and satisfying that

$$\|a_{0,\varepsilon}\|_{A_0} + \|a_{1,\varepsilon}\|_{A_1} \leqslant \|a\|_{A_0+A_1} + \varepsilon.$$

Then, as $a' \in A_0' \cap A_1'$ we have that $a' \in A_0'$ and $a' \in A_1'$. Therefore

$$\begin{aligned} |\langle a', a\rangle| \leqslant |\langle a', a_{0,\varepsilon}\rangle| + |\langle a', a_{1,\varepsilon}\rangle| &\leqslant \|a'\|_{A_0'} \|a_{0,\varepsilon}\|_{A_0} + \|a'\|_{A_1'} \|a_{1,\varepsilon}\|_{A_1} \\ &\leqslant (\|a'\|_{A_0'} + \|a'\|_{A_1'})(\|a\|_{A_0+A_1} + \varepsilon). \end{aligned}$$

Letting $\varepsilon \to 0$ and taking the supremum over $a \in A_0 + A_1$ we can conclude that $a' \in (A_0 + A_1)'$ and that

$$\|a'\|_{A_0'\cap A_1'} \leqslant \|a'\|_{(A_0+A_1)'}.$$

For the other inclusion, let $a' \in (A_0 + A_1)'$. We will start proving that $a' \in (A_0 \cap A_1)'$ and by density of $A_0 \cap A_1$ in A_i we will see that $a' \in A_0' \cap A_1'$.

So, take $a \in A_0 \cap A_1$, then by definition of $\|a\|_{A_0+A_1}$ we have that

$$|\langle a', a\rangle| \leqslant \|a'\|_{(A_0+A_1)'} \|a\|_{A_0+A_1} \leqslant \|a'\|_{(A_0+A_1)'} \|a\|_{A_i} \quad i = 0, 1.$$

Therefore, taking supremum of $a \in A_0 \cap A_1$, we have that

$$\|a'\|_{(A_0+A_1)'} \leqslant \|a'\|_{(A_0 \cap A_1)'}.$$

Now, since $A_0 \cap A_1$ is a subspace of A_0 and A_1 for Hahn-Banach theorem we have that there exist $b \in A_0'$ and $c \in A_1'$ such that

$$b|_{A_0 \cap A_1} = c|_{A_0 \cap A_1} = a'.$$

And as $A_0 \cap A_1$ is dense in A_0 and in A_1, we have that b and c are determined by their values in $A_0 \cap A_1$, so they are $a' = b = c$. Hence, $a' \in A_0'$ and $a' \in A_1'$ which implies that $a' \in A_0' \cap A_1'$. Therefore,

$$\|a'\|_{A_0' \cap A_1'} \geqslant \|a'\|_{(A_0+A_1)'}$$

and $(A_0 + A_1)' = A_0' \cap A_1'$. ∎

3.4 The Reiteration Theorem

In this section we will study one of the most important property of the interpolation spaces, that is the Reiteration Theorem that gives us a relation between the space obtained via a couple of interpolation spaces and the original couple. Also, we will give an expression for the K-functional under some reiterations.

Let us begin by defining intermediate space and the classes $\mathscr{C}_K$ and $\mathscr{C}_J$.

Definition 3.4.1 (Intermediate spaces)**.** Let $\bar{A} = (A_0, A_1)$ be a given couple of normed spaces. We say that X is an intermediate space with respect to $\bar{A}$ if

$$A_0 \cap A_1 \subset X \subset A_0 + A_1.$$

Definition 3.4.2. Let $\bar{A} = (A_0, A_1)$ be a given couple of normed spaces. Suppose that X is an intermediate space with respect to $\bar{A}$. Then we say that

1. X is of class $\mathscr{C}_K(\theta; \bar{A})$ if $K(t, a; \bar{A}) \leqslant C t^{\theta} \|a\|_X$ for all $a \in X$;
2. X is of class $\mathscr{C}_J(\theta; \bar{A})$ if $\|a\|_X \leqslant C t^{-\theta} J(t, a; \bar{A})$ for all $a \in X$.

Here $\theta \in [0, 1]$. We also say that X is of class $\mathscr{C}(\theta; \bar{A})$ if $X \in \mathscr{C}_K(\theta; \bar{A})$ and $X \in \mathscr{C}_J(\theta; \bar{A})$.

By Properties 3.1.9 and Proposition 3.1.19 we have that if $0 < \theta < 1$ then the spaces $(A_0, A_1)_{\theta,p} \in \mathscr{C}(\theta; \bar{A})$. Moreover, we have that $A_0 \in \mathscr{C}(0; \bar{A})$ and $A_1 \in \mathscr{C}(1; \bar{A})$. This follows from

$$K(t, a; \bar{A}) \leqslant \min(\|a\|_{A_0}, t\|a\|_{A_1}) \leqslant \max(\|a\|_{A_0}, t\|a\|_{A_1}) = J(t, a; \bar{A}).$$

Proposition 3.4.3. *Let $\bar{A} = (A_0, A_1)$ be a given couple of normed spaces. Suppose that X is an intermediate space with respect to $\bar{A}$. Then,*

1. *X is of class $\mathscr{C}_K(\theta; \bar{A})$ if and only if for any $t > 0$ there exist $a_0 \in A_0$ and $a_1 \in A_1$, such that $a = a_0 + a_1$ and $\|a_0\|_{A_0} \leqslant C t^{\theta} \|a\|_X$ and $\|a_1\|_{A_1} \leqslant C t^{\theta-1} \|a\|_X$.*

2. X is of class $\mathscr{C}_J(\theta;\bar{A})$ if and only we have

$$\|a\|_X \leqslant C\|a\|_{A_0}^{1-\theta}\|a\|_{A_1}^{\theta}.$$

Proof. Let us begin by proving that X is of class $\mathscr{C}_K(\theta;\bar{A})$ if and only if for any $t>0$ there exist $a_0 \in A_0$ and $a_1 \in A_1$, such that $a = a_0 + a_1$ and $\|a_0\|_{A_0} \leqslant Ct^{\theta}\|a\|_X$ and $\|a_1\|_{A_1} \leqslant Ct^{\theta-1}\|a\|_X$.

First, assume that for all $t>0$ there exist $a_0 \in A_0$ and $a_1 \in A_1$, such that $a = a_0 + a_1$ and $\|a_0\|_{A_0} \leqslant Ct^{\theta}\|a\|_X$ and $\|a_1\|_{A_1} \leqslant Ct^{\theta-1}\|a\|_X$. Then

$$\|a_0\|_{A_0} + t\|a_1\|_{A_1} \leqslant C\|a\|_X(t^{\theta}+t^{\theta}) = 2Ct^{\theta}\|a\|_X.$$

Taking the infimum over the decompositions of a we arrive at

$$K(t,a;\bar{A}) \leqslant 2Ct^{\theta}\|a\|_X.$$

For the other implication, as we know that X is of class $\mathscr{C}_K(\theta;\bar{A})$, then we have that $K(t,a;\bar{A}) \leqslant Ct^{\theta}\|a\|_X$ for all $a \in X$. Since, the K-functor is an infimum by definition we have that there exists a decomposition $a_0 + a_1 = a$ such that

$$\|a_0\|_{A_0} + t\|a_1\|_{A_1} \leqslant Ct^{\theta}\|a\|_X$$

Therefore, $\|a_0^0\|_{A_0} \leqslant Ct^{\theta}\|a\|_X$ and $\|a_1^0\|_{A_1} \leqslant Ct^{\theta-1}\|a\|_X$.

Now we are going to see that X is of class $\mathscr{C}_J(\theta;\bar{A})$ if and only we have

$$\|a\|_X \leqslant C\|a\|_{A_0}^{1-\theta}\|a\|_{A_1}^{\theta}.$$

If $X \in \mathscr{C}_J(\theta;\bar{A})$ then

$$\|a\|_X \leqslant C\max(t^{-\theta}\|a\|_{A_0}, t^{1-\theta}\|a\|_{A_1}).$$

As it holds for any $t>0$ we can take

$$t = \frac{\|a\|_{A_1}}{\|a\|_{A_0}}$$

and obtain that

$$\|a\|_X \leqslant C\|a\|_{A_0}^{1-\theta}\|a\|_{A_1}^{\theta}.$$

For the converse implication, if $\|a\|_X \leqslant C\|a\|_{A_0}^{1-\theta}\|a\|_{A_1}^{\theta}$ then we can write this inequality as

$$\|a\|_X \leqslant Ct^{-\theta}\|a\|_{A_0}^{1-\theta}(t\|a\|_{A_1})^{\theta} \leqslant Ct^{-\theta}J(t,a;\bar{A}).$$

Therefore,

$$\|a\|_X \leqslant Ct^{-\theta}J(t,a;\bar{A}) \Rightarrow X \in \mathscr{C}_J(\theta;\bar{A}).$$

■

We can formulate the definition of those classes in another useful way given by the following theorem.

Theorem 3.4.4. *Suppose that* $0 < \theta < 1$. *Then*

1. $X \in \mathscr{C}_K(\theta; \bar{A})$ *if and only if* $A_0 \cap A_1 \subset X \subset (A_0, A_1)_{\theta,\infty}$.
2. *A Banach space* X *is of class* $\mathscr{C}_J(\theta; \bar{A})$ *if and only if* $(A_0, A_1)_{\theta,1} \subset X \subset A_0 + A_1$.

Proof. By definition of $(A_0, A_1)_{\theta,\infty}$ we have that $X \subset (A_0, A_1)_{\theta,\infty}$ if and only if

$$\sup_{t>0} t^{-\theta} K(t, a; \bar{A}) \leqslant C\|a\|_X.$$

By definition of supremum, for all $t > 0$ we have that

$$t^{-\theta} K(t, a; \bar{A}) \leqslant \sup_{t>0} t^{-\theta} K(t, a; \bar{A}) \leqslant C\|a\|_X \Rightarrow K(t, a; \bar{A}) \leqslant t^{\theta} C\|a\|_X.$$

So, we have that $X \in \mathscr{C}_K(\theta; \bar{A})$ if and only if $A_0 \cap A_1 \subset X \subset (A_0, A_1)_{\theta,\infty}$.

In order to prove that a Banach space X is of class $\mathscr{C}_J(\theta; \bar{A})$ if and only if $(A_0, A_1)_{\theta,1} \subset X \subset A_0 + A_1$, we assume that $a = \sum_n u_n$ in $A_0 + A_1$. Then if X is a Banach space of class $\mathscr{C}_J(\theta; \bar{A})$ we have that

$$\|a\|_X \leqslant \sum_{n\in\mathbb{Z}} \|u_n\|_X \leqslant C \sum_{n\in\mathbb{Z}} 2^{-n\theta} J(2^n, u_n; \bar{A}).$$

And $\sum_{n\in\mathbb{Z}} 2^{-n\theta} J(2^n, u_n; \bar{A})$ converges by Proposition 3.1.21. Therefore, $(A_0, A_1)_{\theta,1} \subset X$.

For the other implication, if we have that $(A_0, A_1)_{\theta,1} \subset X$ then we can put

$$u_n = \begin{cases} a, & \text{if } n = m, \\ 0, & \text{otherwise.} \end{cases}$$

Hence

$$\|a\|_X \leqslant C\|a\|_{\theta,1} \leqslant C 2^{-m\theta} J(2^m, a; \bar{A}),$$

which shows that $X \in \mathscr{C}_J(\theta; \bar{A})$. ∎

We now are going to see one of the most important results in interpolation theory, which is the Reiteration Theorem also called the stability theorem.

Theorem 3.4.5 (The Reiteration Theorem)**.** *Let* $\bar{A} = (A_0, A_1)$ *and* $\bar{X} = (X_0, X_1)$ *be two compatible couple of normed linear spaces, and assume that* X_i *are complete and of class* $\mathscr{C}(\theta_i; \bar{A})$, *where* $0 \leqslant \theta_i \leqslant 1$ *and* $\theta_0 \neq \theta_1$.

Put $\theta = (1 - \eta)\theta_0 + \eta\theta_1$ *with* $0 < \eta < 1$. *Then, for* $1 \leqslant p \leqslant \infty$

$$(X_0, X_1)_{\eta,p} = (A_0, A_1)_{\theta,p},$$

with equivalent norms. In particular, if $0 < \theta_i < 1$ *and* $(A_0, A_1)_{\theta_i,p_i}$ *are complete then*

$$((A_0, A_1)_{\theta_0,p_0}, (A_0, A_1)_{\theta_1,p1})_{\eta,p} = (A_0, A_1)_{\theta,p},$$

with equivalent norms.

Proof. Suppose that $a = a_0 + a_1 \in (X_0, X_1)_{\eta,p}$ with $a_i \in X_i$. Since $X_i \in \mathscr{C}(\theta_i; \bar{A})$, we have

$$K(t, a; \bar{A}) \leqslant K(t, a_0; \bar{A}) + K(t, a_1; \bar{A}) \leqslant C(t^{\theta_0}\|a_0\|_{X_0} + t^{\theta_1}\|a_1\|_{X_1}).$$

It follows that

$$K(t, a; \bar{A}) \leqslant Ct^{\theta_0} K(t^{\theta_1-\theta_0}, a; \bar{X}).$$

So, we have that

$$\left(\int_0^\infty (t^{-\theta} K(t, a; \bar{A}))^p \frac{dt}{t}\right)^{1/p} \leqslant C\left(\int_0^\infty (t^{-(\theta-\theta_0)} K(t^{\theta_1-\theta_0}, a; \bar{X}))^p \frac{dt}{t}\right)^{1/p}.$$

Changing $s = t^{\theta_1-\theta_0}$ and as $\theta = (1-\eta)\theta_0 + \eta\theta_1$, then $\eta = (\theta-\theta_0)/(\theta_1-\theta_0)$, we have that

$$\begin{aligned}\|a\|_{(A_0,A_1)_{\theta,p}} = \left(\int_0^\infty (t^{-\theta} K(t, a; \bar{A}))^p \frac{dt}{t}\right)^{1/p} &\leqslant C\left(\int_0^\infty (s^{-\eta} K(s, a; \bar{X}))^p \frac{ds}{s}\right)^{1/p} \\ &= \|a\|_{(X_0,X_1)_{\eta,p}}.\end{aligned}$$

This inequality implies that $(X_0, X_1)_{\eta,p} \subset (A_0, A_1)_{\theta,p}$. For the reverse inclusion, assume that $a \in (A_0, A_1)_{\theta,p}$ and choose a representation

$$a = \int_0^\infty u(t)\frac{dt}{t}$$

of $a \in A_0 + A_1$. If $a \in (X_0, X_1)_{\eta,p}$ we have that

$$\|a\|^p_{(X_0,X_1)_{\eta,p}} = C\int_0^\infty \left(t^{-(\theta-\theta_0)} K(t^{\theta_1-\theta_0}, a; \bar{X})\right)^p \frac{dt}{t}.$$

Using Lemma 3.1.16 and that $X_i \in \mathscr{C}(\theta_i, \bar{A})$ we obtain that

$$\begin{aligned}\left(t^{\theta_0} K(t^{\theta_1-\theta_0}, a; \bar{X})\right)^p &\leqslant \int_0^\infty \left(t^{\theta_0} K(t^{\theta_1-\theta_0}, u(s); \bar{X})\right)^p \frac{ds}{s} \\ &\leqslant \int_0^\infty \left(t^{\theta_0} \min(1, (t/s)^{\theta_1-\theta_0}) J(s^{\theta_1-\theta_0}, u(s); \bar{X})\right)^p \frac{ds}{s} \\ &\leqslant C\int_0^\infty \left(t^{\theta_0} \min((t/s)^{\theta_0}, (t/s)^{\theta_1}) J(s, u(s); \bar{A})\right)^p \frac{ds}{s}.\end{aligned}$$

Integrating this inequality with respect to dt/t, changing $t = s/r$ and using again the Lemma 3.1.16, we arrive at

$$\|a\|^p_{(X_0,X_1)_{\eta,p}} \leqslant C\left(\int_0^\infty r^\theta \min(r^{-\theta_0}, r^{-\theta_1})\frac{dr}{r}\right)\left(\int_0^\infty (s^{-\theta} J(s, u(s); \bar{A}))^p \frac{ds}{s}\right).$$

As

$$\int_0^\infty r^\theta \min(r^{-\theta_0}, r^{-\theta_1})\frac{dr}{r} = D$$

is finite, if we take infimum over u and using the Theorem 3.2.2 we arrive at

$$\|a\|^p_{(X_0,X_1)_{\eta,p}} \leqslant CD\|a\|^p_{(A_0,A_1)_{\theta,p}}.$$

So, we have that $(X_0, X_1)_{\eta,p} = (A_0, A_1)_{\theta,p}$ and that the norms $\|\cdot\|_{(X_0,X_1)_{\eta,p}}$ and $\|\cdot\|_{(A_0,A_1)_{\theta,p}}$ are equivalent.

In particular, if $0 < \theta_i < 1$ and $(A_0, A_1)_{\theta_i,p_i}$ are complete, then $(A_0, A_1)_{\theta_i,p_i} \in \mathscr{C}(\theta_i; \bar{A})$. So, taking $X_i = (A_0, A_1)_{\theta_i,p_i}$ we arrive at

$$((A_0, A_1)_{\theta_0,p_0}, (A_0, A_1)_{\theta_1,p1})_{\eta,p} = (A_0, A_1)_{\theta,p},$$

with equivalent norms. ■

Suggested by this theorem we have a formula connecting the functional $K(t, a; \bar{A})$ and $K(t, a; \bar{X})$, where $\bar{X} = ((A_0, A_1)_{\theta_0,p_0}, (A_0, A_1)$. Such formula was given by Holmstedt in [8].

Theorem 3.4.6 (T. Holmstedt)**.** *Let $\bar{A} = (A_0, A_1)$ be a given couple of normed spaces and put $X_0 = (A_0, A_1)_{\theta_0,p_0}$ and $X_1 = (A_0, A_1)_{\theta_1,p_1}$, where $0 \leqslant \theta_0 < \theta_1 \leqslant 1$ and $p_0, p_1 \in [1, \infty]$. Put $\lambda = \theta_1 - \theta_0$. Then*

$$K(t, a; \bar{X}) \sim \left(\int_0^{t^{1/\lambda}} (s^{-\theta_0} K(s, a; \bar{A}))^{p_0} \frac{ds}{s}\right)^{1/p_0} + t\left(\int_{t^{1/\lambda}}^{\infty} (s^{-\theta_1} K(s, a; \bar{A}))^{p_1} \frac{ds}{s}\right)^{1/p_1}.$$

Proof. We first prove that

$$K(t, a; \bar{X}) \gtrsim \left(\int_0^{t^{1/\lambda}} (s^{-\theta_0} K(s, a; \bar{A}))^{p_0} \frac{ds}{s}\right)^{1/p_0} + t\left(\int_{t^{1/\lambda}}^{\infty} (s^{-\theta_1} K(s, a; \bar{A}))^{p_1} \frac{ds}{s}\right)^{1/p_1}.$$

Let $a = a_0 + a_1 \in A_0 + A_1$. By Properties 3.1.9 and Minkowski's inequality it follows that

$$\begin{aligned}
\left(\int_0^{t^{1/\lambda}} (s^{-\theta_0} K(s, a; \bar{A}))^{p_0} \frac{ds}{s}\right)^{1/p_0} &\leqslant \left(\int_0^{t^{1/\lambda}} (s^{-\theta_0} K(s, a_0; \bar{A}))^{p_0} \frac{ds}{s}\right)^{1/p_0} \\
&\quad + \left(\int_0^{t^{1/\lambda}} (s^{-\theta_0} K(s, a_1; \bar{A}))^{p_0} \frac{ds}{s}\right)^{1/p_0} \\
&\leqslant \|a_0\|_{X_0} + C\left(\int_0^{t^{1/\lambda}} (s^{\lambda} \|a_1\|_{X_1})^{p_0} \frac{ds}{s}\right)^{1/p_0} \\
&\leqslant C(\|a_0\|_{X_0} + t\|a_1\|_{X_1}).
\end{aligned}$$

Using a similar argument we arrive at

$$t\left(\int_{t^{1/\lambda}}^{\infty} (s^{-\theta_1} K(s, a; \bar{A}))^{p_1} \frac{ds}{s}\right)^{1/p_1} \leqslant C'(\|a_0\|_{X_0} + t\|a_1\|_{X_1}).$$

Therefore, we have that

$$\begin{aligned}
\left(\int_0^{t^{1/\lambda}} (s^{-\theta_0} K(s, a; \bar{A}))^{p_0} \frac{ds}{s}\right)^{1/p_0} &+ t\left(\int_{t^{1/\lambda}}^{\infty} (s^{-\theta_1} K(s, a; \bar{A}))^{p_1} \frac{ds}{s}\right)^{1/p_1} \\
&\leqslant C''(\|a_0\|_{X_0} + t\|a_1\|_{X_1}).
\end{aligned}$$

Taking the infimum over the decompositions of $a \in A_0 + A_1$, we conclude that

$$K(t,a;\bar{X}) \gtrsim \left(\int_0^{t^{1/\lambda}} (s^{-\theta_0} K(s,a;\bar{A}))^{p_0} \frac{ds}{s}\right)^{1/p_0} + t\left(\int_{t^{1/\lambda}}^{\infty} (s^{-\theta_1} K(s,a;\bar{A}))^{p_1} \frac{ds}{s}\right)^{1/p_1}.$$

Let us prove the other inequality. By definition of $K(t,a;\bar{A})$, we may choose $a_0(t) \in A_0$ and $a_1(t) \in A_1$ such that $a = a_0(t) + a_1(t)$ and

$$\|a_0(t)\|_{A_0} + t\|a_1(t)\|_{A_1} \leqslant 2K(t,a;\bar{A}).$$

With this choice we have

$$\begin{aligned}
K(t,a;\bar{X}) &\leqslant \|a_0(t^{1/\lambda})\|_{X_0} + t\|a_1(t^{1/\lambda})\|_{X_1} \\
&= \left(\int_0^{\infty} (s^{-\theta_0} K(s,a_0(t^{1/\lambda});\bar{A}))^{p_0} \frac{ds}{s}\right)^{1/p_0} \\
&\quad + t\left(\int_0^{\infty} (s^{-\theta_1} K(s,a_1(t^{1/\lambda});\bar{A}))^{p_1} \frac{ds}{s}\right)^{1/p_1} \\
&\leqslant \left(\int_0^{t^{1/\lambda}} (s^{-\theta_0} K(s,a_0(t^{1/\lambda});\bar{A}))^{p_0} \frac{ds}{s}\right)^{1/p_0} \\
&\quad + \left(\int_{t^{1/\lambda}}^{\infty} (s^{-\theta_0} K(s,a_0(t^{1/\lambda});\bar{A}))^{p_0} \frac{ds}{s}\right)^{1/p_0} \\
&\quad + t\left(\int_0^{t^{1/\lambda}} (s^{-\theta_1} K(s,a_1(t^{1/\lambda});\bar{A}))^{p_1} \frac{ds}{s}\right)^{1/p_1} \\
&\quad + t\left(\int_{t^{1/\lambda}}^{\infty} (s^{-\theta_1} K(s,a_1(t^{1/\lambda});\bar{A}))^{p_1} \frac{ds}{s}\right)^{1/p_1}.
\end{aligned}$$

Call

$$\begin{aligned}
(I) &= \left(\int_0^{t^{1/\lambda}} (s^{-\theta_0} K(s,a_0(t^{1/\lambda});\bar{A}))^{p_0} \frac{ds}{s}\right)^{1/p_0}; \\
(II) &= \left(\int_{t^{1/\lambda}}^{\infty} (s^{-\theta_0} K(s,a_0(t^{1/\lambda});\bar{A}))^{p_0} \frac{ds}{s}\right)^{1/p_0}; \\
(III) &= t\left(\int_0^{t^{1/\lambda}} (s^{-\theta_1} K(s,a_1(t^{1/\lambda});\bar{A}))^{p_1} \frac{ds}{s}\right)^{1/p_1}; \\
(IV) &= t\left(\int_{t^{1/\lambda}}^{\infty} (s^{-\theta_1} K(s,a_1(t^{1/\lambda});\bar{A}))^{p_1} \frac{ds}{s}\right)^{1/p_1}.
\end{aligned}$$

We will study $(I), (II), (III)$ and (IV) separately. Since the study of (III) and (IV) is analogous to the study of (I) and (II) we only study such integrals. So, let us estimate the term (I). By the triangle inequality, we obtain that

$$(I) \leqslant \left(\int_0^{t^{1/\lambda}} (s^{-\theta_0} K(s,a;\bar{A}))^{p_0} \frac{ds}{s}\right)^{1/p_0} + \left(\int_0^{t^{1/\lambda}} (s^{-\theta_0} K(s,a_1(t^{1/\lambda});\bar{A}))^{p_0} \frac{ds}{s}\right)^{1/p_0}.$$

Since by Remark 3.1.6 $s^{-1}K(s,a;\bar{A})$ is a decreasing function with respect to s, we have that

$$\begin{aligned}\left(\int_0^{t^{1/\lambda}}(s^{-\theta_0}K(s,a_1(t^{1/\lambda});\bar{A}))^{p_0}\frac{ds}{s}\right)^{1/p_0} &\leqslant \left(\int_0^{t^{1/\lambda}}(s^{-\theta_0}s\|a_1(t^{t^{1/\lambda}})\|_{A_1}\frac{ds}{s}\right)^{1/p_0}\\ &\leqslant Ct^{-1/\lambda}K(t^{1/\lambda},a;\bar{A})t^{(1-\theta_0/\lambda}\\ &\leqslant C\left(\int_0^{t^{1/\lambda}}(s^{-\theta_0}K(s,a;\bar{A}))^{p_0}\frac{ds}{s}\right)^{1/p_0}.\end{aligned}$$

Then,

$$(I)\leqslant(1+C)\left(\int_0^{t^{1/\lambda}}(s^{-\theta_0}K(s,a;\bar{A}))^{p_0}\frac{ds}{s}\right)^{1/p_0}.$$

For (II) we have that

$$\begin{aligned}(II)\leqslant\left(\int_{t^{1/\lambda}}^{\infty}(s^{-\theta_0}\|a_0(t^{1/\lambda})\|_{A_0})^{p_0}\frac{ds}{s}\right)^{1/p_0} &\leqslant C't^{-\theta_0/\lambda}\|a_0(t^{1/\lambda})\|_{A_0}\\ &\leqslant C't^{-\theta_0/\lambda}K(t^{1/\lambda},a;\bar{A})\\ &\leqslant C'\left(\int_0^{t^{1/\lambda}}(s^{-\theta_0}K(s,a;\bar{A}))^{p_0}\frac{ds}{s}\right)^{1/p_0}.\end{aligned}$$

Therefore,

$$(I)+(II)\leqslant K\left(\int_0^{t^{1/\lambda}}(s^{-\theta_0}K(s,a;\bar{A}))^{p_0}\frac{ds}{s}\right)^{1/p_0},$$

where $K=(1+C+C')$. As we already mentioned an analogous argument holds for (III) and (IV). So, we have that

$$(III)+(IV)\leqslant K't\left(\int_{t^{1/\lambda}}^{\infty}(s^{-\theta_1}K(s,a;\bar{A}))^{p_1}\frac{ds}{s}\right)^{1/p_1}.$$

Hence, we conclude that

$$K(t,a;\bar{X})\lesssim\left(\int_0^{t^{1/\lambda}}(s^{-\theta_0}K(s,a;\bar{A}))^{p_0}\frac{ds}{s}\right)^{1/p_0}+t\left(\int_{t^{1/\lambda}}^{\infty}(s^{-\theta_1}K(s,a;\bar{A}))^{p_1}\frac{ds}{s}\right)^{1/p_1}.$$

Therefore,

$$K(t,a;\bar{X})\sim\left(\int_0^{t^{1/\lambda}}(s^{-\theta_0}K(s,a;\bar{A}))^{p_0}\frac{ds}{s}\right)^{1/p_0}+t\left(\int_{t^{1/\lambda}}^{\infty}(s^{-\theta_1}K(s,a;\bar{A}))^{p_1}\frac{ds}{s}\right)^{1/p_1}.$$

■

3.5 Duality Theorem

The last property of the real methods that we will see is the relation between interpolate a dual couple of Banach spaces and dualize a interpolation space.

Theorem 3.5.1 (The duality theorem). *Let $\bar{A}$ be a compatible couple of Banach spaces, such that $A_0 \cap A_1$ is dense in A_0 and in A_1. Assume $p \in [1, \infty)$ and $0 < \theta < 1$. Then*

$$(A_0, A_1)'_{\theta,p} = (A'_0, A'_1)_{\theta,p'}$$

with equivalent norms, where $1 = 1/p + 1/p'$.

Proof. For the proof of this theorem we will use the Equivalence Theorem 3.2.2 and the Theorem 3.3.1. Also we will use the discretization of the $K-$ and $J-$ methods (Theorem 3.1.12 and Theorem 3.1.21). In fact we will see the following inclusions

$$(A_0, A_1)'^{J}_{\theta,p} \subset (A'_1, A'_0)^{K}_{1-\theta,p'} \tag{3.7}$$

$$(A_0, A_1)'^{K}_{\theta,p} \supset (A'_1, A'_0)^{J}_{1-\theta,p'}. \tag{3.8}$$

Recall that the superindex is the method used to obtain those spaces. In order to prove (3.7), we take $a' \in (A_0, A_1)'^{J}_{\theta,p}$, and apply the first formula of the Theorem 3.3.3. Thus, given $\varepsilon > 0$, we can find $b_n \in A_0 \cap A_1$ such that $b_n \neq 0$ and, since $a' \in (A_0 \cap A_1)' = A'_0 + A'_1$,

$$K(2^{-n}, a'; (A'_0, A'_1)) - \varepsilon \min(1, 2^{-n}) \leqslant (J(2^n, b_n; \bar{A}))^{-1} \langle a', b_n \rangle.$$

Choose a sequence $(\alpha_n) \subset \lambda^{\theta,p}$, and put

$$a_\alpha = \sum_n (J(2^n, b_n; \bar{A}))^{-1} \alpha_n \cdot b_n.$$

Then since $(\alpha_n) \in \lambda^{\theta,p}$ we have that

$$\sum_n (J(2^n, b_n; \bar{A}))^{-1} \alpha_n < \infty$$

and as $b_n \in A_0 \cap A_1$ we have that $a_\alpha \in (A_0, A_1)^J_{\theta,p}$. Moreover we have that

$$\sum_n (K(2^{-n}, a'; (A'_0, A'_1)) - \varepsilon \min(1, 2^{-n})) \leqslant \langle a', a_\alpha \rangle$$

and, since $\|a\|_{\theta,p,J} \leqslant \|\alpha\|_{\lambda^{\theta,p}}$ we have that

$$\langle a', a_\alpha \rangle \leqslant \|\alpha\|_{\lambda^{\theta,p}} \|a'\|_{(A_0,A_1)'_{\theta,p,J}}.$$

Also, by Theorem 3.3.1 we have that $K(2^{-n}, a'; (A'_0, A'_1)) = 2^{-n} K(2^n, a'; (A'_1, A'_0))$, so we obtain that

$$\sum_n 2^{-n} \alpha_n (K(2^n, a'; (A'_1, A'_0)) - \varepsilon \min(1, 2^{-n})) \leqslant \|\alpha\|_{\lambda^{\theta,p}} \|a'\|_{(A_0,A_1)'_{\theta,p,J}}.$$

Now, since $\lambda^{\theta,p}$ and $\lambda^{1-\theta,p'}$ are dual via the duality

$$\sum_n 2^{-n} \alpha_n \beta_n$$

and ε is arbitrary, letting $\varepsilon \to 0$ we obtain that

$$\|\alpha\|_{\lambda^{1-\theta,p'}} \|a'\|_{(A_1',A_0')_{\theta,p',K}} \leqslant \|\alpha\|_{\lambda^{\theta,p}} \|a'\|_{(A_0,A_1)'_{\theta,p,J}}.$$

And by the duality of $\lambda^{\theta,p}$ and $\lambda^{1-\theta,p'}$, we arrive at

$$\|a'\|_{(A_1',A_0')_{\theta,p',K}} \leqslant \|a'\|_{(A_0,A_1)'_{\theta,p,J}}.$$

In order to prove (3.8), we take an element $a' \in (A_1', A_0')^J_{1-\theta,p'}$ and $a \in (A_0, A_1)_{[\theta]}$. We write a' as

$$a' = \sum_n a_n'$$

with convergence in $A_0' + A_1' = (A_0 \cap A_1)'$. Then it follows that

$$|\langle a', a\rangle| \leqslant \sum_n |\langle a_n', a\rangle| \leqslant \sum_n J(2^{-n}, a_n'; (A_0', A_1'))K(2^n, a; \bar{A}).$$

And since

$$J(2^{-n}, a'; (A_0', A_1')) = 2^{-n} J(2^n, a'; (A_1', A_0'))$$

we obtain that

$$|\langle a', a\rangle| \leqslant \sum_n 2^{-n} J(2^n, a_n'; (A_1', A_0'))K(2^n, a; \bar{A}).$$

But, using Hölder's inequality we have that

$$\sum_n 2^{-n} J(2^n, a_n'; (A_1', A_0'))K(2^n, a; \bar{A}) \leqslant \|a\|_{(A_0,A_1)_{\theta,p,K}} \|a'\|_{(A_1',A_0')_{1-\theta,p',J}}.$$

So, we have that

$$|\langle a', a\rangle| \leqslant \|a\|_{(A_0,A_1)_{\theta,p,K}} \|a'\|_{(A_1',A_0')_{1-\theta,p',J}}$$

and this implies that

$$\frac{|\langle a', a\rangle|}{\|a\|_{(A_0,A_1)_{\theta,p,K}}} \leqslant \|a'\|_{(A_1',A_0')_{1-\theta,p',J}}.$$

Taking the supremum over $a \in (A_0, A_1)^K_{\theta,p}$, we have that

$$\|a'\|_{(A_0,A_1)`_{\theta,p,K}} \leqslant \|a'\|_{(A_1',A_0')_{1-\theta,p',J}}.$$

Hence, we have that

$$\begin{aligned} (A_0, A_1)'^J_{\theta,p} &\subset (A_1', A_0')^K_{1-\theta,p'} \\ (A_0, A_1)'^K_{\theta,p} &\supset (A_1', A_0')^J_{1-\theta,p'}. \end{aligned}$$

But, by the Theorem 3.3.1 we have that

$$\begin{aligned} (A_0, A_1)'^J_{\theta,p} &\subset (A_1', A_0')^K_{1-\theta,p'} = (A_0', A_1')^K_{\theta,p'} \\ (A_0, A_1)'^K_{\theta,p} &\supset (A_1', A_0')^J_{1-\theta,p'} = (A_0', A_1')^J_{\theta,p'}. \end{aligned}$$

And by the Equivalence Theorem 3.2.2 we have that

$$(A_0, A_1)'_{\theta,p} \subset (A'_0, A'_1)_{\theta,p'}$$
$$(A_0, A_1)'_{\theta,p} \supset (A'_0, A'_1)_{\theta,p'}.$$

Therefore

$$(A_0, A_1)'_{\theta,p} = (A'_0, A'_1)_{\theta,p'}.$$

■

Remark 3.5.2. The $J-$ and $K-$ functional with $p = 1$ and $p = \infty$ respectively are extremals in the sense that if F is any interpolation functor of exponent θ then $(A_0, A_1)_{\theta,1} \subset F(\bar{A}) \subset (A_0, A_1)_{\theta,\infty}$, where $F(\bar{A})$ is the interpolation space obtained with F.

We say that F is an interpolation functor of exponent θ if for all pairs of couples $\bar{A}$ and $\bar{B}$, and for all linear and continuous operators $T : A_j \to B_j$, with norms M_0 and M_1 respectively, then we have that $T : F(\bar{A}) \to F(\bar{B})$ is linear and continuous with norm $M \leqslant CM_0^{1-\theta}M_1^{\theta}$, where C is a positive constant. If $C = 1$ we say that F is an exact interpolation functor of exponent θ, so the $J-$ and $K-$ functional are exact interpolation functors of exponent θ.

Chapter 4

Complex Interpolation

In this chapter we will study the most relevant methods in the complex interpolation theory, those techniques are based in the Calderon's theory. We will see that usually the results are analogous to those obtained in the real case, but they are more precise.

In this chapter our couple of spaces need to be Banach instead of quasi-Banach or normed spaces as in the previous chapter.

We have two interpolation functors, C_θ and C^θ, we will define them and their spaces, and we will see that unlike in the real case those spaces, in general, are not the same, but we have an inclusion of C_θ in C^θ.

We will focus to study of the properties of the spaces $\bar{A}_{[\theta]}$, letting the space $\bar{A}^{[\theta]}$ as a technical tool.

4.1 Definition of Methods

In this section we will define the C_θ and C^θ methods and see some properties of this methods. We will work with analytic functions with values in Banach spaces.

Given a couple of Banach spaces $\bar{A} = (A_0, A_1)$, we consider the space $\mathscr{F}(\bar{A})$ of all functions f with values in $A_0 + A_1$, which are bounded and continuous on the closed strip

$$S = \{z \in \mathbb{C} : 0 \leqslant \Re z \leqslant 1\},$$

and analytic in the open strip

$$\mathring{S} = \{z \in \mathbb{C} : 0 < \Re z < 1\},$$

and moreover, the functions $t \to f(j + it)$ with $j = 0, 1$ are continuous functions from the real line into A_j, which tends to 0 as $|t| \to \infty$. We provide $\mathscr{F}$ with the norm

$$\|f\|_{\mathscr{F}} = \max(\sup(\|f(it)\|_{A_0}), \sup(\|f(1 + it)\|_{A_1})). \tag{4.1}$$

Lemma 4.1.1. *The space $\mathscr{F}$ is a Banach space.*

Proof. In order to prove that $\mathscr{F}$ is a Banach space with the norm $\|\cdot\|_{\mathscr{F}}$ we have to see that

1. $\|\cdot\|_{\mathscr{F}}$ is a norm,

2. $\mathscr{F}$ is complete with this norm.

That $\|\cdot\|_{\mathscr{F}}$ is a norm follows because, by (4.1), for any $t \in \mathbb{R}$ we have that $\|f(j+it)\|_{A_j}$ is a norm, that is that for all $f, g \in \mathscr{F}$ and $\lambda \in \mathbb{C}$ we have

$$\|f(j+it)\|_{A_j} \geqslant 0;$$
$$\|f(j+it)\|_{A_j} = 0 \Leftrightarrow f(j+it) = 0;$$
$$\|f(j+it) + g(j+it)\|_{A_j} \leqslant \|f(j+it)\|_{A_j} + \|g(j+it)\|_{A_j};$$
$$\|f(j+it)\lambda\|_{A_j} = |\lambda| \|f(j+it)\|_{A_j}.$$

Then, taking supremum in those inequalities we arrive at $\sup(\|f(j+it)\|_{A_j})$ is a norm for $j = 0, 1$, so $\|\cdot\|_{\mathscr{F}}$ is a norm.

Now we are going to see the completeness of $\mathscr{F}$ with this norm. By Theorem 1.2.1 we can see the completeness via series. Assume that

$$\sum_n \|f_n\|_{\mathscr{F}} < \infty.$$

Since $f_n(z)$ is bounded in $A_0 + A_1$ and $A_j \subset A_0 + A_1$, and $f_n(z)$ are analytic in $\mathring{S}$ and continuous in S, we can apply the Hadamard Three Line Theorem 1.1.2 to each f_n and obtain that

$$\|f_n(z)\|_{A_0+A_1} \leqslant \max(\sup(\|f_n(it)\|_{A_0+A_1}), \sup(\|f_n(1+it)\|_{A_0+A_1})) \leqslant \|f_n\|_{\mathscr{F}}$$

for all $n \in \mathbb{N}$. Then, we have that

$$\sum_n \|f_n(z)\|_{A_0+A_1} < \infty,$$

but as $A_0 + A_1$ is a Banach space there exists $f \in A_0 + A_1$ such that

$$f(z) = \sum_n f_n(z)$$

and the convergence is uniformly in the closed strip S. Moreover, we have that this happens also in the boundary of S, so it happens for $z = j + it$, this means that $f(j+it) \in A_j$ and

$$f(j+it) = \sum_n f_n(j+it).$$

Also, since the convergence is uniformly in S, we have that f is bounded and continuous in S and analytic in $\mathring{S}$. Therefore, $f \in \mathscr{F}$, so $\mathscr{F}$ is a Banach space with the norm $\|\cdot\|_{\mathscr{F}}$.

■

In order to define the space generated by the C^θ–functional we need to define the space $\mathscr{G}(\bar{A})$.

Definition 4.1.2. The space $\mathscr{G}(\bar{A})$ is the space of analytic functions g defined on the strip S with values in $A_0 + A_1$, satisfying the following properties:

a) $\|g(z)\|_{A_0+A_1} \leqslant c(1+|z|)$,

b) g is continuous on S and analytic on $\mathring{S}$,

c) $g(j+it_1)-g(j+it_2)$ has values in A_j for all $t_1,t_2 \in \mathbb{R}$ and for $j=0,1$, and

$$\|g\|_{\mathscr{G}} = \max\left(\sup_{t_1,t_2}\left\|\frac{g(it_1)-g(it_2)}{t_1-t_2}\right\|_{A_0}, \sup_{t_1,t_2}\left\|\frac{g(1+it_1)-g(1+it_2)}{t_1-t_2}\right\|_{A_1}\right)$$

is finite.

Lemma 4.1.3. *The space $\mathscr{G}(\bar{A})$, reduced modulo constant functions and provided with the norm $\|\cdot\|_{\mathscr{G}(\bar{A})}$ is a Banach space.*

Proof. In order to see that $\|\cdot\|_{\mathscr{G}}$ is a norm we only need to see that $\|g\|_{\mathscr{G}}=0$ if and only if g is constant. Because for any $t_1,t_2 \in \mathbb{R}$ we have that

$$\left\|\frac{g(j+it_1)-g(j+it_2)}{t_1-t_2}\right\|_{A_j}$$

satisfies the other properties of being a norm for $j=0,1$, so taking supremum in t_1,t_2 and taking the maximum still satisfying those properties.

So, let us check that $\|g\|_{\mathscr{G}}=0$ if and only if g is constant. Take $h \neq 0$ a real number then

$$\left\|\frac{g(z+ih)-g(z)}{ih}\right\|_{A_0+A_1} \leqslant \|g\|_{\mathscr{G}}.$$

Thus, letting $h \to 0$ we have that

$$\|g'(z)\|_{A_0+A_1} \leqslant \|g\|_{\mathscr{G}}.$$

Therefore, if $\|g\|_{\mathscr{G}}=0$ then g is constant, and then for all $t_1,t_2 \in \mathbb{R}$ we have that

$$\left\|\frac{g(j+it_1)-g(j+it_2)}{t_1-t_2}\right\|_{A_j} = 0.$$

So, $\|g\|_{\mathscr{G}}=0$ and therefore $\|\cdot\|_{\mathscr{G}}$ is a norm.

Now we are going to see the completeness. We have that on the open strip $\mathring{S}$

$$\|g(z)-g(0)\|_{A_0+A_1} \leqslant |z|\|g\|_{\mathscr{G}}. \tag{4.2}$$

By Theorem 1.2.1 we can study the completeness using series. So, take $(g_n)_n \in \mathscr{G}(\bar{A})$ such that

$$\sum_n \|g_n\|_{\mathscr{G}} < \infty.$$

By (4.2) and since A_0+A_1 is Banach we have that

$$\sum_n (g_n(z)-g_n(0))$$

converges uniformly to g on every compact subset of $\mathring{S}$. Again, by (4.2) we have that $g(z)$ satisfies the property a) of Definition 4.1.2 and as the convergence is uniformly we have that $g(z)$ also satisfies the property b) of Definition 4.1.2. So, we need to see that $g(j+it_1) - g(j+it_2)$ has values in A_j for all $t_1, t_2 \in \mathbb{R}$ because if this happens then

$$\sum_n \|g_n\|_{\mathscr{G}} = \|g\|_{\mathscr{G}} < \infty.$$

So, let us see that $g(j+it_1) - g(j+it_2)$ has values in A_j for all $t_1, t_2 \in \mathbb{R}$. Notice that by definition of supremum and maximum we have that

$$\sum_n \left\| \frac{g_n(j+it_1) - g_n(j+it_2)}{t_1 - t_2} \right\|_{A_j} \leqslant \sum_n \|g_n\|_{\mathscr{G}} < \infty.$$

And as A_j are Banach spaces then

$$\sum_n (g_n(j+it_1) - g_n(j+it_2))$$

converges to $g(j+it_1) - g(j+it_2)$ in A_j. So, $g \in \mathscr{G}(\bar{A})$ and this implies that $\mathscr{G}(\bar{A})$ is a Banach space. ∎

Now we are able to define the functors C_θ and C^θ which are based in the spaces $\mathscr{F}(\bar{A})$ and $\mathscr{G}(\bar{A})$ respectively.

4.1.1 Functional C_θ

In this section we will define the spaces generated by the functor C_θ and we will see that those spaces are Banach. Recall that those method is based in the $\mathscr{F}(\bar{A})$ space.

Definition 4.1.4. Given $\bar{A} = (A_0, A_1)$ a compatible couple of Banach spaces and $\theta \in (0,1)$, we define the space $\bar{A}_{[\theta]} = C_\theta(\bar{A})$ as

$$\bar{A}_{[\theta]} = \{a \in A_0 + A_1 : \exists f \in \mathscr{F} \text{ such that } f(\theta) = a\}.$$

We define the norm in $\bar{A}_{[\theta]}$ as

$$\|a\|_{[\theta]} = \inf\{\|f\|_{\mathscr{F}} : f(\theta) = a, f \in \mathscr{F}\}.$$

Now we are going to check that $\|\cdot\|_{[\theta]}$ is really a norm.

Proposition 4.1.5. *Given $\bar{A} = (A_0, A_1)$ a compatible couple of Banach spaces. $\|\cdot\|_{[\theta]}$ is a norm.*

Proof. In order to see that $\|\cdot\|_{[\theta]}$ is a norm we need to check that for all $a, b \in A_\theta$ and for all $\lambda \in \mathbb{C}$ we have that

(i) $\|a\|_{[\theta]} \geqslant 0$;

(ii) $\|a\|_{[\theta]} = 0 \Leftrightarrow a = 0$;

(iii) $\|a+b\|_{[\theta]} \leqslant \|a\|_{[\theta]} + \|b\|_{[\theta]}$;

(iv) $\|a\lambda\|_{[\theta]} = |\lambda| \|a\|_{[\theta]}$.

(i) follows because the infimum of positive numbers is positive. If $a = 0$ then taking $f \equiv 0$ we have that $\|a\|_{[\theta]} = 0$. Conversely, if $\|a\|_{[\theta]} = 0$ then the infimum is taken by $f \equiv 0$ and as $0 = f(\theta) = a$, this implies that $a = 0$. So this shows (ii). For (iii), Let $a, b \in \bar{A}_{[\theta]}$ and let any $f, g \in \mathscr{F}(\bar{A})$ such that $f(\theta) = a$ and $g(\theta) = b$. So, $f(\theta) + g(\theta) = a + b$, but

$$\|f + g\|_{\mathscr{F}} \leqslant \|f\|_{\mathscr{F}} + \|g\|_{\mathscr{F}}.$$

Taking infimums we have that $\|a+b\|_{[\theta]} \leqslant \|a\|_{[\theta]} + \|b\|_{[\theta]}$. In order to prove (iv), if $f(\theta) = a$ then $\lambda f(\theta) = \lambda a$, and $\|\lambda f\|_{\mathscr{F}} = |\lambda| \|f\|_{\mathscr{F}}$. Taking infimums we have that $\|a\lambda\|_{[\theta]} = |\lambda| \|a\|_{[\theta]}$. ■

Theorem 4.1.6. *The space $\bar{A}_{[\theta]}$ is a Banach space and an intermediate space with respect to $\bar{A}$.*

Proof. Let us start proving that $\bar{A}_{[\theta]}$ is an intermediate space with respect to $\bar{A}$, that is that $(A_0 \cap A_1) \subset \bar{A}_{[\theta]} \subset (A_0 + A_1)$. Since, $f(\theta) = a \in A_0 + A_1$ we have that

$$\|a\|_{A_0+A_1} = \|f(\theta)\|_{A_0+A_1} \leqslant \|f\|_{\mathscr{F}}.$$

Taking infimum we have that $\bar{A}_{[\theta]} \subset (A_0 + A_1)$. Let $\delta > 0$ and take

$$f(z) = e^{\delta(z-\theta)^2} a.$$

Note that $f(z) = a$ if and only if $z = \theta$. As $(A_0 \cap A_1) \subset A_j$ we have that

$$\|a\|_{[\theta]} \leqslant \|f\|_{\mathscr{F}} \leqslant \max(\sup(\|f(it)\|_{A_0 \cap A_1}), \sup(\|f(1 + it)\|_{A_0 \cap A_1})).$$

But, by definition of $f(z)$ we have that

$$\begin{aligned} &\max(\sup(\|f(it)\|_{A_0 \cap A_1}), \sup(\|f(1 + it)\|_{A_0 \cap A_1})) \\ =&\|a\|_{A_0 \cap A_1} \max\left(\sup\left(\left|e^{\delta(it-\theta)^2}\right|\right), \sup\left(\left|e^{\delta(1+it-\theta)^2}\right|\right)\right). \end{aligned}$$

So, letting $\delta \to 0$ we have that

$$\left|e^{\delta(j+it-\theta)^2}\right| \to 1.$$

for $j = 0, 1$, then when $\delta \to 0$ we have that

$$\max(\sup(\|f(it)\|_{A_0 \cap A_1}), \sup(\|f(1 + it)\|_{A_0 \cap A_1})) \to \|a\|_{A_0 \cap A_1}.$$

Therefore, $\|a\|_{[\theta]} \leqslant \|a\|_{A_0 \cap A_1}$. So, we have that $(A_0 \cap A_1) \subset \bar{A}_{[\theta]}$.

Take the linear mapping $f \mapsto f(\theta)$, it is continuous because

$$\|f(\theta)\|_{A_0+A_1} \leqslant \|f\|_{\mathscr{F}}.$$

Denote by $\mathcal{N}_\theta = \{f : f \in \mathscr{F}(\bar{A}), f(\theta) = 0\}$ the kernel of this map. Then, by the First Isomorphism Theorem $\bar{A}_{[\theta]}$ is isomorphic and isometric to the quotient $\mathscr{F}(\bar{A})/\mathcal{N}_\theta$. As this mapping is continuous and $\{0\}$ is closed we have that $\mathcal{N}_\theta$ is closed, so the quotient $\mathscr{F}(\bar{A})/\mathcal{N}_\theta$ is closed. Since, closed subspaces of a complete space are complete spaces and $A_0 + A_1$ is a Banach space, we have that $\bar{A}_{[\theta]}$ is a Banach space. ■

The last theorem shows that the functor C_θ is an exact interpolation functor of exponent θ (see Remark 3.5.2).

Theorem 4.1.7. *The functor C_θ is an exact interpolation functor of exponent θ.*

Proof. Let $\bar{A} = (A_0, A_1)$ and $\bar{B} = (B_0, B_1)$ be a pair of compatible couple of Banach spaces, and T is continuous with norms M_j. Let $a \in \bar{A}_{[\theta]}$ and $\varepsilon > 0$ such that there exists $f \in \mathscr{F}(\bar{A})$ with $f(\theta) = a$ and $\|f\|_{\mathscr{F}} \leqslant \|a\|_{[\theta]} + \varepsilon$. Since T is continuous in $A_0 + A_1$ then $T(f)$ is continuous on the closed strip S. Even more, as f is bounded and analytic in $\mathring{S}$ then $T(f)$ is bounded and analytic in $\mathring{S}$. Therefore if we consider

$$g(z) = M_0^{z-1} M_1^{-z} T(f(z))$$

then, $g \in \mathscr{F}(\bar{B})$. Moreover, since $\|T(f)\|_{A_j} \leqslant \|f\|_{A_j}$ we have that

$$\|g\|_{\mathscr{F}(\bar{B})} \leqslant \|f\|_{\mathscr{F}(\bar{A})} \leqslant \|a\|_{[\theta]} + \varepsilon.$$

Taking $z = \theta$ we have that

$$g(\theta) = M_0^{\theta-1} M_1^{-\theta} T(f(\theta)) = M_0^{\theta-1} M_1^{-\theta} T(a) \Leftrightarrow T(a) = M_0^{1-\theta} M_1^{\theta} g(\theta).$$

Hence,

$$\|T(a)\|_{[\theta]} \leqslant M_0^{1-\theta} M_1^{\theta} \|g\|_{\mathscr{F}(\bar{B})} \leqslant M_0^{1-\theta} M_1^{\theta} (\|a\|_{[\theta]} + \varepsilon).$$

Letting $\varepsilon \to 0$, we have that

$$\|T(a)\|_{[\theta]} \leqslant M_0^{1-\theta} M_1^{\theta} \|a\|_{[\theta]}.$$

So, $T : \bar{A}_{[\theta]} \to B_{[\theta]}$ with norm

$$M = M_0^{1-\theta} M_1^{\theta}.$$

∎

4.1.2 Functional C^θ

In this section we will define the space $\bar{A}^{[\theta]} = C^\theta(\bar{A})$ where $\bar{A} = (A_0, A_1)$ is a couple of compatible Banach spaces. Also, we will see that those spaces are Banach spaces and that the functor C^θ is an exact interpolation functor of exponent θ.

Definition 4.1.8. Let $0 < \theta < 1$ we define the space $\bar{A}^{[\theta]}$ as

$$\bar{A}^{[\theta]} = \{a \in A_0 + A_1 : a = g'(\theta),\ g \in \mathscr{G}(\bar{A})\}.$$

We define the norm on $\bar{A}^{[\theta]}$ as

$$\|a\|^{[\theta]} = \inf\{\|g\|_{\mathscr{G}} : g'(\theta) = a,\ g \in \mathscr{G}\}.$$

The proof that

$$\|a\|^{[\theta]} = \inf\{\|g\|_{\mathscr{G}} : g'(\theta) = a,\ g \in \mathscr{G}\}$$

is a norm is analogous to the proof of Proposition 4.1.5.

Theorem 4.1.9. *The space $\bar{A}^{[\theta]}$ is a Banach space and an intermediate space with respect to $\bar{A}$.*

Proof. We first will prove that $\bar{A}^{[\theta]}$ is an intermediate space with respect to $\bar{A}$. The fact that $\bar{A}^{[\theta]} \subset A_0 + A_1$ follows from the definition of $\bar{A}^{[\theta]}$. So, we have to see that $A_0 \cap A_1 \subset \bar{A}^{[\theta]}$. Let $a \in A_0 \cap A_1$ and let $g(z) = a$ then $g'(z) = a$, so

$$\|a\|_{A_0 \cap A_1} = \|g(z)\|_{A_0 \cap A_1}.$$

But, as $\|a\|_{A_0 \cap A_1} \geqslant \|a\|_{A_j}$ for $j = 0, 1$ we have that

$$\|g(z)\|_{A_0 \cap A_1} \geqslant \|g\|_{\mathscr{G}}.$$

And by definition of infimum we have that

$$\|g(z)\|_{A_0 \cap A_1} \geqslant \|g\|_{\mathscr{G}} \geqslant \|a\|^{[\theta]}.$$

Therefore, $A_0 \cap A_1 \subset \bar{A}^{[\theta]}$.

Now we are going to the that $\bar{A}^{[\theta]}$ is complete. We will use an analogous argument as in Theorem 4.1.6. Since $\|g'(\theta)\|_{A_0+A_1} \leqslant \|g\|_{\mathscr{G}}$, we see that the mapping $g \mapsto g'(\theta)$ is continuous from $\mathscr{G}$ to $A_0 + A_1$. The kernel $\mathscr{N}^\theta$ of this mapping is closed and $\bar{A}^{[\theta]}$ is isomorphic and isometric to $\mathscr{G}/\mathscr{N}^\theta$ which is closed. So, $\bar{A}^{[\theta]}$ is a closed subspace of $A_0 + A_1$ which is a Banach space, therefore, $\bar{A}^{[\theta]}$ is a Banach space. ∎

The following theorem shows that as the functor C_θ the functional C^θ is an exact interpolation functor of exponent θ.

Theorem 4.1.10. *The functional C^θ is an exact interpolation functor of exponent θ.*

Proof. Assume that $T : A_j \to B_j$ with norm M_j for $j = 0, 1$. Then we choose a function $g \in \mathscr{G}(\bar{A})$ such that $g'(\theta) = a$,

$$\|g\|_{\mathscr{G}(\bar{A})} \leqslant \|a\|^{[\theta]} + \varepsilon.$$

Consider the function

$$h(z) = M_0^{\eta-1} M_1^{-\eta} T(g(\eta))\Big|_{\eta=0}^{\eta=z} - \int_{[0,z]} \left(\log \frac{M_0}{M_1}\right) M_0^{\eta-1} M_1^{-\eta} T(g(\eta)) d\eta, \tag{4.3}$$

there $[0, z]$ means any path in the closed strip S connecting 0 and z. Notice that if we have $\eta \in \mathring{S}$ then

$$\frac{d(T(g(\eta)))}{d\eta} = T(g'(\eta))$$

and by definition $g'(\eta)$ is bounded and continuous on $\mathring{S}$. Thus $T(g'(\eta))$ is continuous on $\mathring{S}$ and bounded in $B_0 + B_1$. So, if the path $[0, z]$ has all its points except the point 0 and maybe z in $\mathring{S}$ then we can integrate (4.3) by parts. Therefore, we obtain

$$h(z) = \int_{[0,z]} M_0^{\eta-1} M_1^{-\eta} T(dg(\eta)),$$

where in general the integral is to be interpreted as a vector-valued Stieltjes integral. As $T(dg(\eta))$ is bounded in $B_0 + B_1$ we have that

$$\|h\|_{B_0+B_1} \leqslant c|z|.$$

Notice that since $g(j + it)$ takes values in A_j then $T(g(j + it))$ takes values in B_j for $j = 0, 1$, and $T(g(j + it))$ is a Lipschitz function in B_j. Thus it follows that

$$\|h(j + it_1) - h(j + it_2)\|_{B_j} \leqslant M_j^{-1} \int_{t_1}^{t_2} \|T(dg(j + it))\|_{B_j} dt$$

if $t_1 < t_2$. But

$$M_j^{-1} \int_{t_1}^{t_2} \|T(dg(j + it))\|_{B_j} dt \leqslant \int_{t_1}^{t_2} \|dg(j + it)\|_{A_j} dt \leqslant (t_2 - t_1)\|g\|_{\mathscr{G}(\bar{A})}.$$

And as $\|g\|_{\mathscr{G}(\bar{A})} \leqslant \|a\|^{[\theta]} + \varepsilon$ we arrive at

$$\|h\|_{\mathscr{G}(\bar{B})} \leqslant \|a\|^{[\theta]} + \varepsilon.$$

Moreover

$$h'(\theta) = M_0^{\theta-1} M_1^{-\theta} \left(\frac{d}{d\eta} T(g(\eta)) \right)_{\eta=\theta} = M_0^{\theta-1} M_1^{-\theta} T(a).$$

Then

$$T(a) = h'(\theta) M_0^{1-\theta} M_1^{\theta} \in \bar{B}^{[\theta]},$$

and we conclude that

$$\|T(a)\|^{[\theta]} \leqslant M_0^{1-\theta} M_1^{\theta} (\|a\|^{[\theta]} + \varepsilon).$$

Letting $\varepsilon \to 0$, we have that $\|T(a)\|^{[\theta]} \leqslant M_0^{1-\theta} M_1^{\theta} \|a\|^{[\theta]}$. So, $T : A^{[\theta]} \to B^{[\theta]}$ with norm $M = M_0^{1-\theta} M_1^{\theta}$. ■

4.2 Some properties of C_θ

In this section we will prove two theorems concerning with inclusion and density properties of the spaces $\bar{A}_{[\theta]}$. The first theorem deals with the inclusions between $\bar{A}_{[\theta_0]}$ and $\bar{A}_{[\theta_1]}$. The second theorem deals with the density and closures of the space $A_0 \cap A_1$ in these spaces.

Theorem 4.2.1. *We have*

(i) $(A_0, A_1)_{[\theta]} = (A_1, A_0)_{[1-\theta]}$ *with equal norms,*

(ii) if $0 < \theta < 1$ *then* $(A, A)_{[\theta]} = A$,

(iii) if $A_1 \subset A_0$ *and* $\theta_0 < \theta_1$ *then* $\bar{A}_{[\theta_1]} \subset \bar{A}_{[\theta_0]}$.

Proof. In order to prove that $(A_0, A_1)_{[\theta]} = (A_1, A_0)_{[1-\theta]}$ with equal norms, note that if $f \in \mathscr{F}((A_0, A_1))$ then if we define $g(z) = f(1 - z)$ we have that $g \in \mathscr{F}((A_1, A_0))$ and

$f(\theta) = a = g(1-\theta)$. Therefore, we have that $(A_0, A_1)_{[\theta]} = (A_1, A_0)_{[1-\theta]}$ with equal norms.

Now, for proving that if $0 < \theta < 1$ then $(A, A)_{[\theta]} = A$, but if $A_0 = A_1 = A$ then $A \cap A = A$ and $A + A = A$. Therefore, since $\bar{A}_{[\theta]}$ is an intermediate space we have that.

$$A \subset \bar{A}_{[\theta]} \subset A.$$

So, if $0 < \theta < 1$ then $(A, A)_{[\theta]} = A$.

Now, it remains to see that if $A_1 \subset A_0$ and $\theta_0 < \theta_1$ then $\bar{A}_{[\theta_1]} \subset \bar{A}_{[\theta_0]}$ we will use that $(A_0, A_1)_{[\theta]} = (A_1, A_0)_{[1-\theta]}$ with equal norms. We are going to see that $A_0 \subset A_1$ implies $(A_0, A_1)_{[\theta]} \subset (A_0, A_1)_{[\tilde{\theta}]}$ when $\theta < \tilde{\theta}$. Let $a \in (A_0, A_1)[\theta]$ we can choose $f \in \mathscr{F}(\bar{A})$ such that $f(\theta) = a$ and $\|f\|_{\mathscr{F}} \leqslant \|a\|_{[\theta]} + \varepsilon$. Put $\theta = \lambda\tilde{\theta}$ where $0 \leqslant \lambda < 1$ and

$$\varphi(z) = f(\tilde{\theta} z) \exp(\varepsilon(z^2 - \lambda^2)).$$

Writing $B_1 = (A_0, A_1)_{[\tilde{\theta}]}$ we have that

$$\|f(\tilde{\theta} + it)\|_{B_1} \leqslant \|f\|_{\mathscr{F}(\bar{A})}.$$

It follows that

$$\|\varphi\|_{\mathscr{F}(A_0, B_1)} \leqslant (\|a\|_{[\theta]} + \varepsilon)e^{\varepsilon}.$$

But $\varphi(\lambda) = a$ and $(A_0, B_1)_{[\lambda]} \subset (B_1, B_1)_{[\lambda]} = B_1$, the equality follows from (ii), and that $(A_0, B_1)_{[\lambda]} \subset (B_1, B_1)_{[\lambda]}$ follows since $A_0 \subset B_1$. And this holds since if we take $a \in A_0$ and $f \in \mathscr{F}(\bar{A})$ then $f(\tilde{\theta} - z) \in \mathscr{F}(\bar{A})$, if we call $g(z) = f(\tilde{\theta} - z)$ then $g(\tilde{\theta}) = f(0)$. So, $a \in B_1$, as this happens for all $a \in A_0$ we have that $A_0 \subset B_1$. Thus

$$\|a\|_{[\tilde{\theta}]} \leqslant c\|\varphi(\lambda)\|_{(A_0, B_1)_{[\lambda]}} \leqslant c\|\varphi\|_{\mathscr{F}(A_0, B_1)}.$$

Then, $\|a\|_{[\tilde{\theta}]} \leqslant c\|a\|_{[\theta]}$. ∎

As we said the second theorem deals withe the closure and density of $A_0 \cap A_1$ in the space $\bar{A}_{[\theta]}$. But, the proof of this theorem requires the following lemma.

Lemma 4.2.2. *Let $\mathscr{F}_0(\bar{A})$ be the space of all linear combination of functions of the form*

$$e^{\delta z^2} \sum_{n=1}^{N} a_n e^{\lambda_n z}$$

where $a_n \in A_0 \cap A_1$, $\lambda_n \in \mathbb{R}$ and $\delta > 0$, then $\mathscr{F}_0(\bar{A})$ is dense in $\mathscr{F}(\bar{A})$.

Proof. Since $\|\exp(\delta z^2) f(z) - f(z)\|_{\mathscr{F}} \to 0$ as $\delta \to 0$ for all $f \in \mathscr{F}(\bar{A})$, it is enough to show that all functions $g(z) = \exp(\delta z^2) f(z)$ with $f \in \mathscr{F}(\bar{A})$ can be approximated by functions in $\mathscr{F}_0(\bar{A})$. Take

$$g_n(z) = \sum_k g(z + 2\pi i k n),$$

where $n \geqslant 1$. Then g_n is analytic in the open strip $\mathring{S}$ and continuous on the closed strip S with values in $A_0 + A_1$. Moreover, g_n is periodic with period $2\pi in$, and $g_n(j+it) \in A_j$ for $j = 0, 1$. Even more,

$$\|g_n(j+it) - g(j+it)\|_{A_j} \to 0$$

as $n \to \infty$ uniformly in compact sets of t-values and $\|g_n(j+it)\|_{A_j}$ is bounded as a function of n and t. It follows that, for all $s > 0$, we have

$$e^{sz^2} g_n(z) \in \mathscr{F}(\bar{A}).$$

Therefore, we can find s and n so that

$$\|e^{sz^2} g_n(z) - g(z)\|_{\mathscr{F}} < \varepsilon.$$

But $g_n(z)$ can be represented by a Fourier series

$$g_n(z) = \sum_k a_{kn} e^{kz/n}, \quad z = s + it, \tag{4.4}$$

where

$$a_{kn} = \frac{1}{2\pi nm} \int_{-\pi nm}^{\pi nm} g_n(s+it) e^{-k(s+it)/n} dt.$$

As g_n are periodic we have that this integral is independent of m. Let s_1, s_2 two values of s then

$$\frac{1}{2\pi nm} \left| \int_{-\pi nm}^{\pi nm} g_n(s_1+it) e^{-k(s_1+it)/n} dt - \int_{-\pi nm}^{\pi nm} g_n(s_2+it) e^{-k(s_2+it)/n} dt \right| \to 0$$

as $m \to \infty$, and these integrals do not depend of m we have that

$$a_{kn} = \frac{1}{2\pi nm} \int_{-\pi nm}^{\pi nm} g_n(s+it) e^{-k(s+it)/n} dt$$

is also independent of s, so we can take $m = 1$ and $s = j$ for $j = 0, 1$. Then we have that $a_{kn} \in A_0 \cap A_1$. Now we consider the $(C, 1)$-means of the sum (4.4), i.e. we consider

$$\sigma_m g_n(z) = \sum_{|k| \leqslant m} \left(1 - \frac{|k|}{m+1}\right) a_{kn} e^{kz/n}.$$

Then

$$\|\sigma_m g_n(j+it) - g_n(j+it)\|_{A_j} \to 0, \quad \text{as} \quad m \to \infty,$$

uniformly in n. Thus

$$\|e^{sz^2}(\sigma_m g_n - g)\|_{\mathscr{F}} < 2\varepsilon.$$

But $e^{sz^2}\sigma_m g_n \in \mathscr{F}_0(\bar{A})$. So, $\mathscr{F}_0(\bar{A})$ is dense in $\mathscr{F}(\bar{A})$. ■

Theorem 4.2.3. *Let $0 \leqslant \theta \leqslant 1$. Then*

(i) $A_0 \cap A_1$ is dense in $\bar{A}_{[\theta]}$;

(ii) let A_j^0 denote the closure of $A_0 \cap A_1$ in A_j, then

$$(A_0, A_1)_{[\theta]} = (A_0^0, A_1)_{[\theta]} = (A_0, A_1^0)_{[\theta]} = (A_0^0, A_1^0)_{[\theta]};$$

(iii) the space $B_j = \bar{A}_{[\theta]}$ *with* $j = 0,1$ *is a closed subspace of* A_j *and the norms coincide in* B_j*;*

(iv) $(A_0, A_1)_{[\theta]} = (B_0, B_1)_{[\theta]}$*, with* B_j *as in (iii).*

Proof. Let us prove (i), if $a \in \bar{A}_{[\theta]}$ there exists a function $f \in \mathscr{F}(\bar{A})$, such that $f(\theta) = a$. Then, by Lemma 4.2.2, there exists $g \in \mathscr{F}_0(\bar{A})$, such that $\|f - g\|_{\mathscr{F}} < \varepsilon$. Therefore $\|a - g(\theta)\|_{[\theta]} < \varepsilon$ and since $g(\theta) \in A_0 \cap A_1$ we have that $A_0 \cap A_1$ is dense in $\bar{A}_{[\theta]}$.

In order to prove (ii) notice that

$$(A_0, A_1)_{[\theta]} \supset (A_0^0, A_1)_{[\theta]} \supset (A_0^0, A_1^0)_{[\theta]}.$$

Then, it is enough to prove that $(A_0, A_1)_{[\theta]} = (A_0^0, A_1^0)_{[\theta]}$. Note that A_j^0 are closed in A_j and since A_j are Banach spaces then A_j^0 are also Banach spaces so, by Theorem 4.1.6, the space $(A_0^0, A_1^0)_{[\theta]}$ is also Banach.

Now we want to see that $(A_0^0, A_1^0)_{[\theta]}$ is closed in $(A_0, A_1)_{[\theta]}$. So, let $(a_n)_n \subset (A_0^0, A_1^0)_{[\theta]}$ be a convergent sequence with the norm in $(A_0, A_1)_{[\theta]}$, call a the limit in this norm. Then, we want to see that $a \in (A_0^0, A_1^0)_{[\theta]}$. Since

$$\|a_n\|_{(A_0,A_1)_{[\theta]}} \to \|a\|_{(A_0,A_1)_{[\theta]}}, \text{ as } n \uparrow \infty$$

we have that for $j = 0,1$

$$\sup \|f_n(j + it)\|_{A_j} \to \sup \|f(j + it)\|_{A_j},$$

where $f_n, f \in \mathscr{F}((A_0, A_1))$, $f_n(\theta) = a_n$ and $f(\theta) = a$. But, since by definition A_j^0 are closed in A_j and $(a_n)_n \subset (A_0^0, A_1^0)_{[\theta]}$, we have that $f_n, f \in \mathscr{F}((A_0^0, A_1^0))$ and that

$$\sup \|f_n(j + it)\|_{A_j^0} \to \sup \|f(j + it)\|_{A_j^0}.$$

As this happens for any f_n and f such that $f_n(\theta) = a_n$ and $f(\theta) = a$, we can take the infimum and obtain that

$$\|a_n\|_{(A_0^0,A_1^0)_{[\theta]}} \to \|a\|_{(A_0^0,A_1^0)_{[\theta]}}, \text{ as } n \uparrow \infty.$$

Therefore, $(A_0^0, A_1^0)_{[\theta]}$ is closed in $(A_0, A_1)_{[\theta]}$. Moreover, we have that $(A_0^0, A_1^0)_{[\theta]}$ is an intermediate space, so we have that

$$A_0 \cap A_1 \subset A_0^0 \cap A_1^0 \subset (A_0^0, A_1^0)_{[\theta]}.$$

Taking closures with respect to $(A_0, A_1)_{[\theta]}$ and using (i) we have that

$$(A_0, A_1)_{[\theta]} \overset{(i)}{=} \overline{A_0 \cap A_1}^{(A_0,A_1)_{[\theta]}} \subset \overline{(A_0^0, A_1^0)_{[\theta]}}^{(A_0,A_1)_{[\theta]}} = (A_0^0, A_1^0)_{[\theta]}.$$

Therefore, we have that

$$(A_0, A_1)_{[\theta]} = (A_0^0, A_1)_{[\theta]} = (A_0, A_1^0)_{[\theta]} = (A_0^0, A_1^0)_{[\theta]}.$$

For (iii) we have that $B_j \subset A_j$ for $j = 0,1$. Let us prove that the norm in B_0 coincides with the norm on A_0 (for B_1 it is the same argument). Take $a \in B_0$. Then, by Lemma 4.2.2, we can find $a_1 \in A_0 \cap A_1$ such that $\|a - a_1\|_{B_0} < \varepsilon$. Consider $f_n(z) = a_1 \exp(z^2 - nz) \in \mathscr{F}(\bar{A})$. Then $f_n(0) = a_1$ and $\|f_n\|_{\mathscr{F}} \leqslant \|a_1\|_{A_0} + \exp(1-n)\|a_1\|_{A_1}$. Since $\|a_1\|_{B_0} \leqslant \|f_n\|_{\mathscr{F}}$ for all n, we conclude that $\|a_1\|_{B_0} \leqslant \|a_1\|_{A_0}$. But $\|a\|_{A_0} \leqslant \|a\|_{B_0}$ and so

$$\|a - a_1\|_{A_0} \leqslant \|a - a_1\|_{B_0} \leqslant \varepsilon.$$

Therefore we have that

$$\|a\|_{B_0} \leqslant \varepsilon + \|a_1\|_{B_0} \leqslant 2\varepsilon + \|a\|_{A_0}.$$

So, $\|a\|_{B_0} \leqslant \|a\|_{A_0}$, and this proves that $\|a\|_{B_0} = \|a\|_{A_0}$. Also, since B_j are Banach spaces, they are closed spaces with their own norm, but as their norm coincides with the norm of A_j we have that B_j are closed subspaces of A_j.

(iv) follows if we are able to see that $\mathscr{F}(\bar{A}) = \mathscr{F}(\bar{B})$. As $B_j \subset A_j$ we have that $\mathscr{F}(\bar{A}) \supset \mathscr{F}(\bar{B})$. Let $f(z) \in \mathscr{F}(\bar{A})$ then $f(j+it) \in B_j$. So, $f(z) \in \mathscr{F}(\bar{B})$, this means that $\mathscr{F}(\bar{A}) \subset \mathscr{F}(\bar{B})$. Therefore, $\mathscr{F}(\bar{A}) = \mathscr{F}(\bar{B})$. ∎

4.3 The Equivalence Theorem

In this section we will study the relation between the functional C_θ and C^θ. For instance, we will prove that in general we have that $\bar{A}_{[\theta]} \subset \bar{A}^{[\theta]}$ and that if A_0 or A_1 are reflexive then $\bar{A}_{[\theta]} = \bar{A}^{[\theta]}$ with equality of norms. In order to see this we need two previous lemmas.

Let us denote by P_j with $j = 0, 1$ the Poisson kernels for the strip S. They can be obtained from the Poisson kernel for the half-plane by means of a conformal mapping. Explicitly, we have that

$$P_j(s+it,\tau) = \frac{e^{-\pi(\tau-t)}\sin \pi s}{\sin^2 \pi s + (\cos \pi s - e^{ij\pi - \pi(\tau-t)})^2}, \quad j = 0, 1.$$

Lemma 4.3.1. *If $f \in \mathscr{F}(\bar{A})$ we have that*

(i) $\log \|f(\theta)\|_{[\theta]} \leqslant \sum_{j=0}^{1} \left(\int_{\mathbb{R}} \log \|f(j+i\tau)\|_{A_j} P_j(\theta,\tau)d\tau\right)$.

(ii)

$$\|f(\theta)\|_{[\theta]} \leqslant \left(\frac{1}{1-\theta}\int_{\mathbb{R}} \|f(i\tau)\|_{A_0} P_0(\theta,\tau)d\tau\right)^{1-\theta} \left(\frac{1}{\theta}\int_{\mathbb{R}} \|f(1+i\tau)\|_{A_1} P_1(\theta,\tau)d\tau\right)^{\theta}.$$

(iii) $\|f(\theta)\|_{[\theta]} \leqslant \sum_{j=0}^{1} \left(\int_{\mathbb{R}} \|f(j+i\tau)\|_{A_j} P_j(\theta,\tau)d\tau\right)$.

Proof. The most difficult part of this proof resides in (i), since (ii) follows applying Jensen's inequality to the exponential to (i) and using that

$$\int_{\mathbb{R}} P_0(\theta,\tau)d\tau = 1 - \theta$$

and that

$$\int_{\mathbb{R}} P_1(\theta,\tau)d\tau = \theta.$$

In fact, applying the exponential to (i) we have that

$$\|f(\theta)\|_{[\theta]} \leqslant \exp\left(\int_{\mathbb{R}} \log\|f(i\tau)\|_{A_0} P_0(\theta,\tau)d\tau\right)\exp\left(\int_{\mathbb{R}} \log\|f(1+i\tau)\|_{A_1} P_1(\theta,\tau)d\tau\right).$$

Now, in order to apply Jensen's inequality we will multiply and divide the first integral by $1-\theta$ and the second by θ. Then,

$$\frac{P_0(\theta,\tau)d\tau}{1-\theta} \quad \text{and} \quad \frac{P_1(\theta,\tau)d\tau}{\theta}$$

are probabilistic measures (this means that the measure of the full space is 1), and using that $\exp(ab) = (\exp(b))^a$, we have that

$$\|f(\theta)\|_{[\theta]} \leqslant \exp\left(\int_{\mathbb{R}} \log\|f(i\tau)\|_{A_0} \frac{P_0(\theta,\tau)d\tau}{1-\theta}\right)^{1-\theta}\exp\left(\int_{\mathbb{R}} \log\|f(1+i\tau)\|_{A_1} \frac{P_1(\theta,\tau)d\tau}{\theta}\right)^{\theta}.$$

Now, we can apply Jensen's inequality and obtain that

$$\|f(\theta)\|_{[\theta]} \leqslant \left(\frac{1}{1-\theta}\int_{\mathbb{R}} \|f(i\tau)\|_{A_0} P_0(\theta,\tau)d\tau\right)^{1-\theta}\left(\frac{1}{\theta}\int_{\mathbb{R}} \|f(1+i\tau)\|_{A_1} P_1(\theta,\tau)d\tau\right)^{\theta}.$$

Finally (iii) follows since

$$\left(\frac{1}{1-\theta}\int_{\mathbb{R}} \|f(i\tau)\|_{A_0} P_0(\theta,\tau)d\tau\right)^{1-\theta}\left(\frac{1}{\theta}\int_{\mathbb{R}} \|f(1+i\tau)\|_{A_1} P_1(\theta,\tau)d\tau\right)^{\theta}$$

is the geometric mean and

$$\sum_{j=0}^{1}\left(\int_{\mathbb{R}} \|f(j+i\tau)\|_{A_j} P_j(\theta,\tau)d\tau\right)$$

is the arithmetic mean of these integrals, and the geometric mean is less than or equal to the arithmetic mean. So, proving (i) we will have the lemma proved. Then, let us prove (i). Since $f \in \mathscr{F}(\bar{A})$ we have that $\log\|f(j+it)\|_{A_j}$ is upper bounded we have that there exists φ_j infinitely differentiable bounded function such that

$$\log\|f(j+it)\|_{A_j} \leqslant \varphi_j(t), \quad j = 0, 1.$$

Let $\Phi(z)$ be an analytic function such that

$$\Re\Phi(z) = \int_{\mathbb{R}} \varphi_0(\tau)P_0(z,\tau)d\tau + \int_{\mathbb{R}} \varphi_1(\tau)P_1(z,\tau)d\tau.$$

Therefore, $\Re\Phi(j+it) = \varphi_j(it)$ for $j = 0, 1$ and Φ is continuous and bounded on S. Since

$$\|e^{-\Phi(j+it)}f(j+it)\|_{A_j} \leqslant e^{-\varphi_j(t)}\|f(j+it)\|_{A_j} \leqslant 1$$

it follows that $\|e^{-\Phi}f\|_{\mathscr{F}} \leqslant 1$, thus $e^{-\Phi}f \in \mathscr{F}(\bar{A})$ and

$$\|e^{-\Phi}f\|_{[\theta]} \leqslant 1.$$

So,

$$\|f\|_{[\theta]} \leqslant \|e^{\Phi}\|_{[\theta]}.$$

Therefore we conclude that

$$\log \|f\|_{[\theta]} \leqslant \Re\Phi(\theta) = \int_{\mathbb{R}} \varphi_0(\tau)P_0(\theta,\tau)d\tau + \int_{\mathbb{R}} \varphi_1(\tau)P_1(\theta,\tau)d\tau.$$

Since $f \in \mathscr{F}(\bar{A})$ we have that f is continuous in the boundary of S. Then, we can take a decreasing sequence of functions φ_j converging to $\log \|f(j+it)\|_{A_j}$, we get

$$\log \|f(\theta)\|_{[\theta]} \leqslant \sum_{j=0}^{1} \left(\int_{\mathbb{R}} \log \|f(j+i\tau)\|_{A_j} P_j(\theta,\tau)d\tau \right).$$

■

Lemma 4.3.2. *If $f \in \mathscr{G}(\bar{A})$ satisfies that*

$$\frac{f(it+ih)-f(it)}{h}$$

converges in A_0 on a set E of positive measure as $h \to 0$ with $h \in \mathbb{R}$, then $f'(\theta) \in \bar{A}_{[\theta]}$ for $0 < \theta < 1$.

Proof. Take

$$f_n(z) = \left(\frac{i}{n}\right)^{-1} \left(f\left(z+\frac{i}{n}\right) - f(z) \right).$$

Then $\|f_n(it) - f_m(it)\|_{A_0} \to 0$ as $n, m \to \infty$ for all t on a set E of positive measure. Even more, we have that

$$e^{\varepsilon z^2} f_n(z) \in \mathscr{F}(\bar{A})$$

for all $\varepsilon > 0$. From Lemma 4.3.1 we obtain that

$$\begin{aligned} &\log\|e^{\varepsilon\theta^2}(f_n(\theta) - f_m(\theta)\|_{[\theta]} \\ &\quad \leqslant \sum_{j=0}^{1} \left(\int_{\mathbb{R}} \log \|e^{\varepsilon(j+i\tau)^2}(f_n(j+i\tau) - f_m(j+i\tau)\|_{A_j} P_j(\theta,\tau)d\tau \right). \end{aligned}$$

Since $\|f_n(j+it) - f_m(j+it)\|_{A_j} \leqslant 2\|f\|_{\mathscr{G}}$ and since $\|f_n(it) - f_m(it)\|_{A_0} \to 0$ for all $t \in E$, we obtain that $\|f\|_{\mathscr{G}} \to -\infty$ as $n, m \to \infty$. Thus

$$\log \|e^{\varepsilon\theta^2}(f_n(\theta) - f_m(\theta))\|_{[\theta]} \to -\infty$$

as $n, m \to \infty$. Therefore $\|(f_n(\theta) - f_m(\theta))\|_{[\theta]} \to 0$. So, $f_n(\theta)$ converges in $\bar{A}_{[\theta]}$. But, we have that $f_n(\theta) \to f'(\theta)$ in $A_0 + A_1$. Since, by Theorem 4.1.6 we have that $\bar{A}_{[\theta]}$ is Banach and then it is closed, we can conclude that $f_n(\theta) \to f'(\theta)$ in $\bar{A}_{[\theta]}$. ■

Theorem 4.3.3 (The complex equivalence theorem)**.** *For any couple $\bar{A} = (A_0, A_1)$ we have that*

$$\bar{A}_{[\theta]} \subset \bar{A}^{[\theta]}$$

and $\|a\|^{[\theta]} \leqslant \|a\|_{[\theta]}$.

Also, if at least one of the two spaces A_0 or A_1 is reflexive and $0 < \theta < 1$, then

$$\bar{A}_{[\theta]} = \bar{A}^{[\theta]}$$

with equality of norms.

Proof. We begin by proving that for any couple $\bar{A} = (A_0, A_1)$ we have that

$$\bar{A}_{[\theta]} \subset \bar{A}^{[\theta]}$$

and $\|a\|^{[\theta]} \leqslant \|a\|_{[\theta]}$. Take $a \in \bar{A}_{[\theta]}$, and choose $f \in \mathscr{F}(\bar{A})$ so that $f(\theta) = a$ and $\|f\|_{\mathscr{F}} \leqslant \|a\|_{[\theta]} + \varepsilon$. Then put

$$g(z) = \int_{[0,z]} f(\zeta)d\zeta,$$

where $[0, z]$ is a path connecting 0 and z. Then $g'(z) = f(z)$ and

$$\|g(z)\|_{A_0+A_1} \leqslant \int_{[0,z]} \|f(\zeta)\|_{A_0+A_1} d\zeta \leqslant \|f\|_{\mathscr{F}} \int_{[0,z]} d\zeta \leqslant (1 + |z|)\|f\|_{\mathscr{F}}.$$

So, $g \in \mathscr{G}(\bar{A})$ and

$$\|g\|_{\mathscr{G}} \leqslant \|f\|_{\mathscr{F}}.$$

Since $g'(z) = f(z)$, we have that $g'(\theta) = f(\theta) = a$. So,

$$\|a\|^{[\theta]} \leqslant \|g\|_{\mathscr{G}} \leqslant \|f\|_{\mathscr{F}} \leqslant \|a\|_{[\theta]}.$$

Therefore we have that

$$\bar{A}_{[\theta]} \subset \bar{A}^{[\theta]}.$$

Now we are going to see the second part of the theorem. By Theorem 4.2.1 it is enough to prove that if A_0 is reflexive then $\bar{A}_{[\theta]} = \bar{A}^{[\theta]}$ with equality of norms.

If $f \in \mathscr{G}(\bar{A})$ then $f(it)$ is continuous and therefore its range lies in a separable subspace V of A_0. Take

$$f_n(z) = \left(f\left(z + \frac{i}{n}\right) - f(z)\right)\left(\frac{n}{i}\right)$$

and let $R_m(t)$ be the weak closure of the set $\{f_n(it) : n \geqslant m\}$. Put $R(t) = \bigcap_m R_m(t)$. Then $R_m(t)$ and $R(t)$ are uniformly bounded subsets of A_0 wit respect to t and m. Since $R_m(t)$ is bounded and weakly closed, and since by Corollary 1.2.11 we have that the unit sphere of A_0 is weakly compact (because A_0 is reflexive), we can deduce that $R_m(t)$ is weakly compact. Therefore $R(t)$ is non-empty. Let $g(t)$ be a function such that $g(t) \in R(t)$ for each t. Since $R(t) \subset V$, the range of g is separable.

Now we want to prove that $f(it) = f(0) + i\int_0^t g(\tau)d\tau$, because if this happens then $f(it)$ is derivable almost everywhere and we can use Lemma 4.3.2 to finish the proof of this theorem. So, let us prove that $f(it) = f(0) + i\int_0^t g(\tau)d\tau$. Let L be a continuous linear functional on A_0, and put $\varphi(t) = -iL(f(it))$. Since $f \in \mathscr{G}(\bar{A})$ we have that φ is Lipschitz continuous. Even more,

$$L(f_n(it)) = n\left(\varphi\left(t + \frac{1}{n}\right) - \varphi(t)\right).$$

The image of $R_m(t)$ under L is the closure of the set $\{n(\varphi(t+1/n)-\varphi(t)) : n \geqslant m\}$. The image of $R(t)$ is contained in the intersection of these sets. If φ is differentiable at the point t, then $L(R(t)) = \{\varphi'(t)\}$ and $L(g(t)) = \varphi'(t)$. But φ is Lipschitz continuous, then by Rademacher's theorem (see [6, Theorem 3.1.6]), $\varphi'(t)$ exists almost everywhere and is measurable . It follows that $L(g(t))$ exists almost everywhere and is measurable. Since the range of g is separable, it follows that g is strongly measurable. Since the sets $R(t)$ are all contained in a bounded set, $g(t)$ is also bounded. Then

$$L(f(it)) = i\varphi(t) = i\varphi(0) + i\int_0^t \varphi'(\tau)d\tau = L(f(0)) + i\int_0^t L(g(\tau))d\tau.$$

And as L is continuous and linear we have that

$$L(f(it)) = L(f(0)) + i\int_0^t L(g(\tau))d\tau = L\left(f(0) + i\int_0^t g(\tau)d\tau\right).$$

And we have that $f(it) = f(0) + i\int_0^t g(\tau)d\tau$. Then $f(it)$ has a strong derivative almost everywhere. Thus Lemma 4.3.2 implies that $f'(\theta) \in \bar{A}_{[\theta]}$. But $f'(\theta) \in \bar{A}^{[\theta]}$, so $\bar{A}_{[\theta]} = \bar{A}^{[\theta]}$.

It remains to see that $\|a\|_{[\theta]} \leqslant \|a\|^{[\theta]}$. Let $a \in \bar{A}^{[\theta]}$ we can choose $f \in \mathscr{G}(\bar{A})$ such that $f'(\theta) = a$ and

$$\|f\|_{\mathscr{F}} \leqslant \|a\|^{[\theta]} + \varepsilon.$$

Consider the function

$$h_n(z) = e^{\varepsilon z^2} f_n(z).$$

Then $h_n \in \mathscr{F}(\bar{A})$ and $\|h_n\|_{\mathscr{F}} \leqslant e^{\varepsilon}\|f\|_{\mathscr{G}}$. Thus $\|h_n(\theta)\|_{[\theta]} \leqslant e^{\varepsilon}(\|a\|^{[\theta]} + \varepsilon)$. But

$$\|h_n(\theta) - e^{\varepsilon\theta^2} a\|_{[\theta]} \to 0 \quad \text{as} \quad n \to \infty.$$

So, letting $n \to \infty$ in the inequality $\|h_n(\theta)\|_{[\theta]} \leqslant e^{\varepsilon}(\|a\|^{[\theta]} + \varepsilon)$ we have that

$$e^{\varepsilon\theta^2}\|a\|_{[\theta]} \leqslant e^{\varepsilon}(\|a\|^{[\theta]} + \varepsilon).$$

Now let $\varepsilon \to 0$ and we arrive at $\|a\|_{[\theta]} \leqslant \|a\|^{[\theta]}$. So $\|a\|_{[\theta]} = \|a\|^{[\theta]}$. ■

4.4 The Reiteration Theorem

In this section we will see that the complex method C_θ is stable under iterations in the same sense that in the real case. In order to prove this theorem we will use the complex equivalence Theorem 4.3.3 and the duality theorem (see [4, Chapter 4, Section 5]).

Theorem 4.4.1 (The Reiteration Theorem). *Let $\bar{A}$ be a compatible couple of Banach spaces and put*

$$X_j = \bar{A}_{[\theta_j]} \quad (0 \leqslant \theta_j \leqslant 1; j = 0, 1).$$

Assume that $A_0 \cap A_1$ is dense in the spaces A_0, A_1 and $X_0 \cap X_1$. Then

$$\bar{X}_{[\eta]} = \bar{A}_{[\theta]} \quad (0 \leqslant \eta \leqslant 1),$$

with equality of norms, where $\theta = (1-\eta)\theta_0 + \eta\theta_1$.

Proof. We will begin by proving that $\|a\|_{\bar{X}_{[\eta]}} \leqslant \|a\|_{\bar{A}_{[\theta]}}$ if $a \in \bar{A}_{[\theta]}$. Take $a \in \bar{A}_{[\theta]}$, then there exists $f \in \mathscr{F}(\bar{A})$ such that $f(\theta) = a$ and $\|f\|_{\mathscr{F}} \leqslant \|a\|_{\bar{A}_{[\theta]}} + \varepsilon$. Put $f_1(z) = f((1-z)\theta_0 + z\theta_1)$. Then $f_1(\eta) = a$ and

$$\begin{aligned} &f_1(j+it) = f((1-j)\theta_0 + it(\theta_1 - \theta_0)) \in X_j, \quad j = 0,1; \\ &f_1(j+it) \to 0, \quad |t| \to \infty. \end{aligned}$$

Also, by the Hadamard Three Line Theorem 1.1.2 we have that the maximum of f is given in the boundary of S. So we arrive at

$$\|a\|_{\bar{X}_{[\eta]}} = \|f_1\|_{\mathscr{F}(\bar{X})} \leqslant \|f\|_{\mathscr{F}(\bar{A})} \leqslant \|a\|_{\bar{A}_{[\theta]}} + \varepsilon.$$

Using a similar argument with $g \in \mathscr{G}(\bar{A})$ we can see that $\|a\|_{\bar{Y}^{[\eta]}} \leqslant \|a\|^{\bar{A}^{[\theta]}}$ where $Y_j = \bar{A}^{[\theta_j]}$ and $a \in \bar{A}^{[\theta]}$.

Let us prove that $\|a\|_{\bar{X}_{[\eta]}} \geqslant \|a\|_{\bar{A}_{[\theta]}}$ if $a \in \bar{X}_{[\eta]}$. By Theorem 4.2.3 we know that $X_0 \cap X_1$ is dense in X_0 and in X_1, and also by Theorem 4.2.3 we have that $X_0 \cap X_1$ is dense in $\bar{X}_{[\eta]}$ and $A_0 \cap A_1$ is dense in $\bar{A}_{[\theta]}$. But as $A_0 \cap A_1$ is dense in $X_0 \cap X_1$ we have that $A_0 \cap A_1$ is dense in $\bar{X}_{[\eta]}$. Also, if we are able to see that $\|l\|_{\bar{A}'_{[\theta]}} \geqslant \|l\|_{\bar{X}'_{[\eta]}}$ for $l \in \bar{A}'_{[\theta]}$, then by Duality Theorem we will have that

$$\|l\|_{\bar{A}'_{[\theta]}} = \|l\|_{\bar{A}'^{[\theta]}} \geqslant \|l\|_{(\bar{A}'^{[\theta_0]}, \bar{A}'^{[\theta_1]})^{[\eta]}} = \|l\|_{\bar{X}'^{[\eta]}} = \|l\|_{\bar{X}'_{[\eta]}}.$$

And this is telling us that the norms on $\bar{X}_{[\eta]}$ and $\bar{A}_{[\theta]}$ coincide, and as $A_0 \cap A_1$ is dense in both spaces we have that $\bar{X}_{[\eta]} = \bar{A}_{[\theta]}$ with equal norms. ∎

Remark 4.4.2. If $A_0 \subset A_1$ then $A_0 \cap A_1$ is dense in $X_0 \cap X_1$, since by Theorem 4.2.1 $X_0 \subset X_1$ (if $\theta_1 < \theta_0$) and by Theorem 4.2.3 $A_0 \cap A_1$ is dense in X_0 and in X_1.

Chapter 5

Interpolation Spaces

In this chapter we will study several spaces and apply the interpolation methods studied in the last chapters to these spaces. In particular, we will see that we obtain when we interpolate the L^p and the Hardy spaces.

5.1 Lorentz Spaces

In this section we will interpolate the L^p and the Lorentz spaces, one example of what we obtain are the weak L^p spaces, $L^{p,\infty}$, studied in the Section 1.3. For instance, we will use the real methods and we will use strongly the distribution function of f and the non-decreasing rearrangement of f.

5.1.1 Definition

In this section we will define the Lorentz space.

Definition 5.1.1. We say that $f \in L^{p,q}$ with $1 \leqslant p \leqslant \infty$ if and only if

$$\|f\|_{p,q} = \left(\int_0^\infty (t^{1/p} f^*(t))^q \frac{dt}{t}\right)^{1/q} < \infty, \quad \text{if } 1 \leqslant q < \infty,$$
$$\|f\|_{p,\infty} = \sup_t t^{1/p} f^*(t) < \infty, \quad \text{if } q = \infty.$$

Here f^* is the non-decreasing rearrangement of f (see Definition 1.3.5).

5.1.2 Interpolation

In this section we will see the results obtained when we interpolate the L^p and the $L^{p,q}$ spaces. In particular, we will see the general Marcinkiewicz interpolation theorem and the Calderón's interpolation theorem.

The first result that we find gives us a formula for the $K-$ functional for the couple (L^p, L^∞), and also says us that the interpolation space of the couple (L_{p_0}, L_{p_1}) is a Lorentz space.

Theorem 5.1.2. *Assume that $f \in L^p + L^\infty$ with $0 < p < \infty$. Then*

$$K(t, f; L^p, L^\infty) \sim \left(\int_0^{t^p} (f^*(s))^p ds \right)^{1/p}. \tag{5.1}$$

If $p = 1$ then it is an equality. Moreover, if $0 < p_0 < p_1 \leqslant \infty$, $p_0 < q \leqslant \infty$ and $1/p = (1-\theta)/p_0 + \theta/p_1$ with $0 < \theta < 1$, then

$$(L_{p_0}, L_{p_1})_{\theta,q} = L^{p,q} \quad \text{with equivalent norms.} \tag{5.2}$$

Proof. Fix a measure space $(A, d\mu)$. We will begin by proving (5.1), first we will prove that

$$K(t, f; L^p, L^\infty) \leqslant C \left(\int_0^{t^p} (f^*(s))^p ds \right)^{1/p}$$

for some constant C. Take

$$f_0(x) = \begin{cases} f(x) - f^*(x)\frac{f(x)}{|f(x)|} & \text{if } |f(x)| > f^*(t^p) \\ 0 & \text{otherwise} \end{cases}$$

and $f_1(x) = f(x) - f_0(x)$. Let $E = \{x \in A : f_0(x) \neq 0\}$, then $\mu(E) = a \leqslant t^p$, that is $\lambda_f(f^*(t^p)) \leqslant t^p$ where $\lambda_f(s)$ is the distribution function of f. Also, we have that

$$f^*(a) = \inf\{s : \lambda_f(s) \leqslant a\} = \inf\{s : \lambda_f(s) \leqslant \lambda_f(f^*(t^p))\}$$

and, as we see in Section 1.3, $\lambda_f(s)$ is decreasing with respect to s we have that $f^*(t^p) \leqslant f^*(a)$. But, since $f^*(t)$ is increasing with respect to t we have that $f^*(t^p) \geqslant f^*(a)$, then we obtain that $f^*(s)$ is constant in the interval $[a, t^p]$. Also, since $f_1 \in L^\infty$ because $|f_1(x)| = f^*(t^p)$, so $\|f_1\|_\infty = f^*(t^p) < \infty$. Therefore, we have that

$$\begin{aligned}
K(t, f; L^p, L^\infty) \leqslant \|f_0\|_p + t\|f_1\|_\infty &= \left(\int_E (|f(x)| - f^*(t^p))^p d\mu \right)^{1/p} + t f^*(t^p) \\
&= \left(\int_0^{\mu(E)} (|f^*(s)| - f^*(t^p))^p ds \right)^{1/p} + \left(\int_0^{t^p} (f^*(s))^p ds \right)^{1/p} \\
&= \left(\int_0^{t^p} (|f^*(s)| - f^*(t^p))^p ds \right)^{1/p} + \left(\int_0^{t^p} (f^*(s))^p ds \right)^{1/p} \\
&\leqslant C \left(\int_0^{t^p} (f^*(s))^p ds \right)^{1/p}.
\end{aligned}$$

Note that if $p = 1$, then $C = 1$. In order to see that

$$K(t, f; L^p, L^\infty) \geqslant C \left(\int_0^{t^p} (f^*(s))^p ds \right)^{1/p}$$

we assume that $f = f_0 + f_1$, with $f_0 \in L^p$ and $f_1 \in L^\infty$. Using that $\lambda_f(t+s) \leqslant \lambda_{f_0}(t) + \lambda_{f_1}(s)$, we obtain that

$$f^*(s) \leqslant f_0^*((1-\varepsilon)s) + f_1^*(\varepsilon s), \quad 0 < \varepsilon < 1.$$

Then

$$\begin{aligned}\left(\int_0^{t^p} (f^*(s))^p ds\right)^{1/p} &\leqslant C\left[\left(\int_0^{t^p} (f_0^*((1-\varepsilon)s))^p ds\right)^{1/p} + \left(\int_0^{t^p} (f_1^*(\varepsilon s))^p ds\right)^{1/p}\right] \\ &\leqslant C\left[\left(\int_0^{t^p} (f_0^*((1-\varepsilon)s))^p ds\right)^{1/p} + t f_1^*(0)\right] \\ &= C\left[(1-\varepsilon)^{-1/p}\|f_0\|_p + t\|f_1\|_\infty\right].\end{aligned}$$

Taking the infimum over the decomposition of f and letting $\varepsilon \to 0$ we obtain that

$$K(t,f;L^p,L^\infty) \geqslant C\left(\int_0^{t^p} (f^*(s))^p ds\right)^{1/p}.$$

Notice that if $p=1$ then $C=1$.

Now we are going to see (5.2). In order to prove this we will use the Reiteration Theorem 3.4.5 with the couple (L^{p_0}, L^∞). Let $p_1 = \infty$, by (5.1), we have that

$$\begin{aligned}\|f\|_{\theta,q} &= \left(\int_0^\infty (t^{-\theta}K(t,f;L^{p_0},L^\infty))^p \frac{dt}{t}\right)^{1/q} \\ &\sim \left(\int_0^\infty \left(t^{-\theta p_0}\int_0^{t^{p_0}} (f^*(s))^{p_0} ds\right)^{q/p_0} \frac{dt}{t}\right)^{1/q} \\ &= \left(\int_0^\infty \left(t^{p_0-\theta p_0}\int_0^1 (f^*(t^{p_0}s))^{p_0} s\frac{ds}{s}\right)^{q/p_0} \frac{dt}{t}\right)^{1/q}.\end{aligned}$$

Since $q > p_0$ we have that $q/p_0 > 1$, so we can apply Minkowski's inequalities 1.3.17 and we arrive at

$$\|f\|_{\theta,q} \leqslant C\int_0^1 \left(s^{q/p_0}\int_0^\infty t^{(1-\theta)q}(f^*(st^{p_0}))^q \frac{dt}{t}\right)\frac{ds}{s} \leqslant C\|f\|_{p,q},$$

because $p = p_0/(1-\theta)$. Conversely, since f^* is nonnegative and decreasing, it follows that

$$C\|f\|_{p,q} \leqslant C\left(\int_0^\infty (t^{(1-\theta)p_0}(f^*(t^{p_0}))^{p_0})^{q/p_0}\frac{dt}{t}\right)^{1/q} \leqslant \|f\|_{\theta,q}.$$

So, we have proved (5.2) for $p_1 = \infty$, using the Reiteration Theorem 3.4.5 we obtain that

$$(L^{p_0}, L^{p_1})_{\eta,q} = ((L^r,L^\infty)_{\theta_0,q_0}, (L^r,L^\infty)_{\theta_1,q_1})_{\eta,q} = (L^r,L^\infty)_{\theta,q} = L^{p,q},$$

where $r < p_0$ and $\theta = (1-\eta)\theta_0 + \eta\theta_1$. ∎

The following theorem identifies the space $(L^{p_0,q_0}, L^{p_1,q_1})_{\theta,q}$.

Theorem 5.1.3. *Suppose that p_0, p_1, q_0, q_1 and q are positive, possibly infinite numbers and take*

$$\frac{1}{p} = \frac{1-\eta}{p_0} + \frac{\eta}{p_1}$$

where $0<\eta<1$. Then, if $p_0 \neq p_1$, we have that

$$(L^{p_0,q_0}, L^{p_1,q_1})_{\theta,q} = L^{p,q}.$$

Proof. By the reiteration Theorem 3.4.5 and the theorem 5.1.2 we have that taking $0 < r < \min(p_0, p_1)$ and

$$\frac{1}{p_i} = \frac{1-\theta_i}{r}, \quad \theta = (1-\eta)\theta_0 + \eta\theta_1$$

we obtain that

$$\frac{1}{p} = \frac{1-\theta}{r}$$

and that

$$(L^{p_0,q_0}, L^{p_1,q_1})_{\theta,q} = ((L^r, L^\infty)_{\theta_0,q_0}, (L^r, L^\infty)_{\theta_1,q_1})_{\theta,q} = (L^r, L^\infty)_{\theta,q} = L^{p,q}.$$

■

Remarks 5.1.4.

(a) If $p_0 = p_1 = p$ then the Theorem 5.1.3 holds, using that $\theta_i = \theta$ and for any $0 < \eta < 1$ the conditions of the theorem hold.

(b) From Theorem 3.3.1 we have that if $1 \leqslant s_1 \leqslant s_2 \leqslant \infty$ then

$$(L^{p_0,q_0}, L^{p_1,q_1})_{\theta,s_1} \subset (L^{p_0,q_0}, L^{p_1,q_1})_{\theta,s2}.$$

But, by Theorem 5.1.3 we have that $(L^{p_0,q_0}, L^{p_1,q_1})_{\theta,s_i} = L^{p,s_i}$. So, we have that

$$L^{p,s_1} \subset L^{p,s_2}.$$

In other words, the spaces $L^{p,q}$ are increasing with respect to q.

As a consequence of the Theorem 5.1.3 we have the Generalization of the Marcinkiewicz Theorem 2.2.1.

Theorem 5.1.5 (The general Marcinkiewicz Interpolation Theorem)**.** *Take two measurable spaces $(U, d\mu)$ and $(V, d\nu)$, assume that*

$$T : L^{p_0,r_0}(U, d\mu) \to L^{q_0,s_0}(V, d\nu),$$
$$T : L^{p_1,r_1}(U, d\mu) \to L^{q_1,s_1}(V, d\nu),$$

where $p_0 \neq p_1$ and $q_0 \neq q_1$. Take

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Then

$$T : L^{p,r}(U, d\mu) \to L^{q,r}(V, d\nu), \quad 0 < r \leqslant \infty. \tag{5.3}$$

In particular, we have that

$$T : L^p(U, d\mu) \to L^q(V, d\nu), \quad \text{if } p \leqslant q. \tag{5.4}$$

Proof. Notice that if $r = p$ we have that $L^{p,r}(U, d\mu) = L^p(U, d\mu)$ and since $L^{q,r}(V, d\nu) = L^{q,p}(V, d\nu) \subset L^p(V, d\nu)$ we have that (5.4) follows from (5.3). But, (5.3) follows from Theorem 5.1.3 and the fact that the real methods are exacts (Theorem 3.1.11 and Theorem 3.1.20). Since, by Theorem 5.1.3 we have that

$$L^{p,r}(U, d\mu) = (L^{p_0,r_0}(U, d\mu), L^{p_1,r_1}(U, d\mu))_{\theta,r}$$

and the same for $L^{q,r}(V, d\nu)$.

■

The most general consequence of Theorem 5.1.5 is that if $r \leqslant s \leqslant \infty$ then we can write (5.3) as

$$T : L^{p,r}(U, d\mu) \to L^{q,s}(V, d\nu), \quad 0 < r \leqslant \infty. \tag{5.5}$$

As a particular case we have the Calderón's interpolation theorem.

Theorem 5.1.6 (Calderón's interpolation theorem)**.** *Suppose that $\rho > 0$ and that*

$$T : L^{p_0,r_0} \to L^{q_0,s_0},$$
$$T : L^{p_1,r_1} \to L^{q_1,s_1},$$

where $p_0 \neq p_1$ and $q_0 \neq q_1$. Put

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Then

$$T : L^{p,r} \to L^{q,s}$$

if $r \leqslant s$.

Proof. By Theorem 5.1.3 and Theorem 3.3.1 we have that if $r \leqslant s$ then

$$L^{q,r} \subset L^{q,s}.$$

And by Theorem 5.1.5 we have that

$$T : L^{p,r} \to L^{q,r} \subset L^{q,s}, \quad 0 < r \leqslant s \leqslant \infty.$$

Therefore,

$$T : L^{p,r} \to L^{q,s}.$$

■

One of the most important motivations of the Lorentz spaces is the following theorem, that is an Improvement of Hausdorff-Young inequality.

Theorem 5.1.7. *The Fourier transform is a continuous operator from L^p to $L^{p',p}$ where $1 \leqslant p \leqslant 2$ and $1 = 1/p + 1/p'$.*

Proof. Since, by Section 1.3.3, we know that

$$L^1 \xrightarrow{\wedge} L^\infty,$$
$$L^2 \xrightarrow{\wedge} L^2,$$

we have that

$$(L^1, L^2)_{\theta,p} \xrightarrow{\wedge} (L^\infty, L^2)_{\theta,p}.$$

Taking $\theta = 1/p$ with $1 \leqslant p \leqslant 2$ we have that, by Theorem 5.1.2, $(L^1, L^2)_{\theta,p} = L^{p,p}$ and that $(L^\infty, L^2)_{\theta,p} = L^{p',p}$. And since $L^p = L^{p,p}$, we have that

$$L^p \xrightarrow{\wedge} L^{p',p}.$$

■

5.2 Hardy Spaces

In this section we will study the Hardy spaces and we will apply the complex interpolation method to these spaces.

5.2.1 Definition

In this section we will introduce the Hardy Spaces and we will see some properties, as for example, that they are Banach spaces. Here $\mathbb{D}$ will denote the unit ball in $\mathbb{C}$.

Let us begin by defining the smallest of the Hardy spaces, the space H^∞.

Definition 5.2.1. We define the space H^∞ as

$$H^\infty = H^\infty(\mathbb{D}) = \mathrm{Hol}(\mathbb{D}) \cap L^\infty(\mathbb{D}).$$

where $\mathbb{D}$ is the unit disk, $\mathrm{Hol}(\mathbb{D})$ is the space of Holomorphic functions in $\mathbb{D}$ and $L^\infty(\mathbb{D})$ is the space of bounded functions in $\mathbb{D}$.

We define the norm in H^∞ as $\|f\|_{H^\infty} = \sup_{z\in\mathbb{D}} |f(z)|$.

Proposition 5.2.2. *The space H^∞ is a Banach space.*

Proof. We need to prove that H^∞ is closed on $C(\mathbb{D}) \cap L^\infty(\mathbb{D})$ where $C(\mathbb{D})$ is the space of continuous functions. Let $(f_n)_n \in H^\infty$ such that $f_n \to f$ with the norm $\|\cdot\|_\infty$. Since $C(\mathbb{D}) \cap L^\infty(\mathbb{D})$ is Banach we know that $f \in C(\mathbb{D}) \cap L^\infty(\mathbb{D})$, so we only need to check that $f \in \mathrm{Hol}(\mathbb{D})$. But, as

$$\|f_n - f\|_{H^\infty} \to 0 \Rightarrow f_n \to f$$

uniformly on compact subsets of $\mathbb{D}$. Then, $f \in \mathrm{Hol}(\mathbb{D})$. ■

Now we are going to define the Hardy spaces H^p, but first we will define what is a subharmonic function and for that we need to define what is an upper semicontinuous function.

Definition 5.2.3. Let X be a topological space. We say that a function $u : X \to [-\infty, \infty]$ is upper semicontinuous if the set

$$\{x \in X : u(x) < \alpha\}$$

is open in X for each $\alpha \in \mathbb{R}$.

Remark 5.2.4.

(a) We say that u is lower semicontinuous if $-u$ is upper semicontinuous.

(b) It can be proved that u is upper semicontinuous if and only if

$$\limsup_{x \to y} u(x) \leqslant u(y).$$

Now we can define what is a subharmonic function.

Definition 5.2.5. Let Ω be a domain in $\mathbb{C}$ and let $u : \Omega \to [-\infty, \infty)$, then we say that u is subharmonic in Ω if satisfies

1. u is upper semicontinuous,
2. $u \not\equiv -\infty$ on each component of Ω,
3. u satisfies the submean value property, i.e. if $\overline{B(a,r)} \subset \Omega$, then

$$u(a) \leqslant \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{it}) dt.$$

Definition 5.2.6. Let $0 < p < \infty$ and

$$M_p(f, r) = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{1/p}.$$

Then $f \in H^p$ if $f \in \mathrm{Hol}(\mathbb{D})$ and

$$\|f\|_{H^p} = \sup_{0<r<1} M_p(f, r) < \infty.$$

Remark 5.2.7. Since $|f|^p$ is subharmonic we have that $M_p(f, r)$ is increasing with respect to r. So, $\|f\|_{H^p} = \lim_{r \to 1^-} M_p(f, r)$.

Lemma 5.2.8. *Let $0 < p < \infty$. If $f \in H^p$, then*

$$|f(z)| \leqslant \left(\frac{2}{1 - |z|} \right)^{1/p} \|f\|_{H^p}.$$

Proof. Let $0 < r < 1$ since f is analytic we have that

$$|f(z)|^p = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p \frac{r^2 - |z|^2}{|re^{it} - z|^2} dt$$

(see [1, Theorem 24]). Using that $|re^{it} - z| \geqslant r - |z|$, we arrive at

$$\frac{1}{2\pi}\int_0^{2\pi}|f(re^{it})|^p\frac{r^2-|z|^2}{|re^{it}-z|^2}dt \leqslant \frac{r^2-|z|^2}{(r-|z|)^2}M_p(f,r)^p = \frac{r+|z|}{r-|z|}M_p(f,r)^p$$
$$\leqslant \frac{2}{r-|z|}M_p(f,r)^p.$$

Letting $r \uparrow 1^-$ we have that

$$|f(z)| \leqslant \left(\frac{2}{1-|z|}\right)^{1/p}\|f\|_{H^p}.$$

■

Corollary 5.2.9. *For $p \geqslant 1$ we have that H^p is a Banach space.*

Proof. Let $(f_n)_n \subset H^p$ be a Cauchy sequence, by Lemma 5.2.8 we have that $(f_n)_n$ is uniformly Cauchy on compact subsets of $\mathbb{D}$. Then there exists $f \in \mathrm{Hol}(\mathbb{D})$ such that $f_n \to f$ uniformly on compact subsets of $\mathbb{D}$. From the Cauchy condition we have that

$$\|f_n - f\|_{H^p} \to 0 \Rightarrow \|f\|_{H^p} \leqslant \|f_n - f\|_{H^p} + \|f_n\|_{H^p} < \infty.$$

■

Since if $f \in H^p$, then we have that $f_r = f(re^{i\theta}) \in L^p(\mathbb{D})$ for all $0 < r < 1$, we have that $f \in L^p(\mathbb{D})$. Moreover, since $\mathbb{D}$ is a compact subset of $\mathbb{C}$ we have that $L^p(\mathbb{D}) \subset L^q(\mathbb{D})$ if $p \geqslant q$. Hence, we have that $H^p \subset H^q$ if $p \geqslant q$.

5.2.2 Interpolation

In this section we will see that if we have two couples of Hardy spaces, (H^{p_1}, H^{q_1}) and (H^{p_2}, H^{q_2}), and a linear and continuous operator T such that

$$T : H^{p_1} \to H^{q_1}, \quad \text{with norm } M_1,$$
$$T : H^{p_2} \to H^{q_2}, \quad \text{with norm } M_2,$$

then, T is a linear and continuous operator from H^p to H^q with $p \in [p_1, p_2]$ and $q \in [q_1, q_2]$ as in Theorem 5.2.13. In order to see this we need to introduce some tools. We will begin by defining the Blaschke condition and the Blaschke product.

Definition 5.2.10. Let $\{z_k\} \in \mathbb{D}$, the condition $\sum_k(1-|z_k|) < \infty$ is called the Blaschke condition.

Now we will define the Blaschke product and we will see that it converges uniformly in compact sets.

Theorem 5.2.11. *Let $\{z_k\} \subset \mathbb{D}$ satisfying the Blaschke condition. Then, the Blaschke product defined as*

$$B(z) = \prod_k \frac{\overline{z_k}}{|z_k|}\left(\frac{z_k - z}{1 - z\overline{z_k}}\right)$$

converges uniformly on compact subset of $\mathbb{D}$, and therefore defines an analytic function on $\mathbb{D}$ vanishing exactly at the points $\{z_k\}$. Moreover, $|B(z)| \leqslant 1$ for all $z \in \mathbb{D}$.

Proof. We need to show that for $|z| \leqslant R < 1$, $\sum_k |1 - b_k(z)| \leqslant C_R < \infty$, where $b_k(z)$ is the k-th term of $B(z)$.

$$\begin{aligned} 1 - b_k(z) &= \frac{|z_k|(1 - \overline{z_k} z) - \overline{z_k}(z_k - z)}{|z_k|(1 - \overline{z_k} z)} \\ &= \frac{|z_k| - |z_k|^2 + \overline{z_k} z(1 - |z_k|)}{|z_k|(1 - \overline{z_k} z)} = \frac{(1 - |z_k|)(|z_k| + \overline{z_k} z)}{|z_k|(1 - \overline{z_k} z)}. \end{aligned}$$

Therefore,

$$|1 - b_k(z)| \leqslant \frac{(1 - |z_k|)|z_k|(1 + |z|)}{|z_k||1 - \overline{z_k} z|} \leqslant \frac{2(1 - |z_k|)}{|1 - \overline{z_k} z|} \leqslant \frac{2(1 - |z_k|)}{1 - |z|} \leqslant \frac{2}{1 + R}.$$

■

The following theorem tells us that given any function in H^p we can take out its zeros and this does not affect the norm of the function.

Theorem 5.2.12 (Riesz Factorization Theorem). *Any function $f \in H^p$, $p > 0$ such that $f \not\equiv 0$ can be factored on the form $f = g \cdot B$, where B is a Blaschke product and g is an H^p function without zeros on $\mathbb{D}$. Moreover*

$$\|f\|_{H^p} = \|g\|_{H^p}.$$

Proof. Let $\{z_n\}$ the zeros of f in $\mathbb{D}$. Then $\{z_n\}$ satisfies the Blaschke condition (see [2, Chapter 5, Theorem 2.2]). Let B be the Blaschke product with these zeros and consider the function

$$g(z) = \frac{f(z)}{B(z)}$$

that has no zeros and is analytic on $\mathbb{D}$. So, we have to see that $g \in H^p$ and that

$$\|f\|_{H^p} = \|g\|_{H^p}.$$

Consider the finite Blaschke product B_n with zeros $z_1, \cdots, z_n$ and let

$$g_n(z) = \frac{f(z)}{B_n(z)}.$$

For fixed n and $\varepsilon > 0$, and for $|z|$ near 1 we have that $|B_n(z)| > 1 - \varepsilon$. Then

$$\frac{1}{2\pi} \int_0^{2\pi} |g_n(re^{i\theta})|^p d\theta < (1 - \varepsilon)^{-p} M_p(f, r)^p \leqslant (1 - \varepsilon)^{-p} \|f\|_{H^p}^p.$$

Since the integral is monotone we have that this holds for all $r < 1$. Letting $\varepsilon \to 0$ we get

$$\sup_{r<1} \frac{1}{2\pi} \int_0^{2\pi} |g_n(re^{i\theta})|^p d\theta \leqslant \|f\|_{H^p}^p.$$

Now, using the Monotone Convergence Theorem we have that $g \in H^p$ with $\|g\|_{H^p} \leqslant \|f\|_{H^p}$. And since $|f(z)| \leqslant |f(z)||B(z)| = |g(z)|$, we have that $\|g\|_{H^p} = \|f\|_{H^p}$. ■

Now we can give the statement and the proof of the main theorem of this section.

Theorem 5.2.13. *Let $0 < \theta < 1$, $1 < p_1, p_2 < \infty$ and $1 \leqslant q_1, q_2 \leqslant \infty$, consider the spaces H^{p_i} and $L^{q_i}(\mathbb{D})$, and let T be a continuous operator from H^{p_j} to $L^{q_j}(\mathbb{D})$ such that has norm M_1 and M_2 respectively. Then,*

$$T : H^p \to L^q(\mathbb{D})$$

where

$$\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2},$$
$$\frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2},$$

with norm less than or equal to $KM_1^{1-\theta}M_2^{\theta}$ for some constant K depending only of p_1 and p_2.

Proof. Assume that $p_1 \geqslant p_2$, then $H^{p_1} \subset H^{p_2}$, and let $n \in \mathbb{N}$ such that $p_1 < n$. For any system of n complex-valued simple functions $\{g_1, \cdots, g_n\}$ (i.e. g_j takes values in $\mathbb{C}$ and are of the form $g_j(z) = \chi_K(z)$ for some compact $K \subset \mathbb{D}$), we define an operation T^* as

$$T^*(g_1, \cdots, g_n) = T(F_1 \cdot F_2 \cdots F_n), \tag{5.6}$$

where

$$F_j(z) = \frac{1}{2\pi}\int_0^{2\pi} g_j(t)\frac{e^{it}+z}{e^{it}-z}dt.$$

By Lemma 5.2.8 we have that $\|F_j\|_{H^r} \leqslant A_r\|g_j\|_{H^r}$ for $1 < r < \infty$. So, $F_j \in H^{n/p_1}$. By Hölder's inequality we have that $\prod_j F_j \in H^{p_1}$, and so the left-hand side of (5.6) is defined. T^* is additive in each g_j because

$$F_1 \cdot F_2 \cdots (F_j + F_j') \cdot F_{j+1} \cdots F_n = \prod_{k=1}^{n} F_k + \left(\prod_{\substack{k=1\\k\neq j}}^{n} F_k\right) \cdot F_j',$$

and the linearity of T. Moreover, by Hölder's inequality we have that

$$\|T^*(g_1, \cdots, g_n)\|_{L^{q_k}(\mathbb{D})} \leqslant M_k\|F_1 \cdots F_n\|_{H^{p_k}} \leqslant M_k\|F_1\|_{H^{p_k}} \cdots \|F_n\|_{H^{p_k}},$$

for $k = 1, 2$. Using that $\|F_j\|_{H^r} \leqslant A_r\|g_j\|_{H^r}$ for $1 < r < \infty$, we arrive at

$$\|T^*(g_1, \cdots, g_n)\|_{L^{q_k}(\mathbb{D})} \leqslant M_k\|F_1 \cdots F_n\|_{H^{p_k}} \leqslant M_k(A_{np_k}^n)\prod_{j=1}^{n}\|g_j\|_{H^{np_k}}.$$

Then, T^* is a multilinear operation defined for all simple functions $g_1, \cdots, g_n$. Then, it can be proved (see [13, Chapter XII, Theorem 3.3]) that

$$\|T^*(g_1, \cdots, g_n)\|_{L^q(\mathbb{D})} \leqslant (A_{np_1}^{1-\theta}A_{np_2}^{\theta})^n(M_1^{1-\theta}M_2^{\theta})\prod_{j=1}^{n}\|g_j\|_{H^{np}}, \tag{5.7}$$

where $0 < \theta < 1$ and

$$\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2},$$
$$\frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}.$$

As it holds for g_j simple functions, we can extend T^* to $L^{nq_2}(\mathbb{D}) \times \cdots \times L^{nq_2}(\mathbb{D})$, preserving the inequality (5.7). But, if $g_j \in L^{np_2}(\mathbb{D})$ then $F_j \in H^{np_2}$ and so $F_1 \cdot F_2 \cdots F_n \in H^{p_2}$. Therefore, the right-hand of (5.6) makes sense. Now we are going to show that (5.6) still holds in the case that $g_j \in L^{np_2}(\mathbb{D})$. Let $g_j \in H^{np_2}$ and g_j^m be simple functions such that

$$\|g_j^m - g_j\|_{H^{np_2}} \to 0 \quad \text{as } m \to \infty,$$

then

$$\|T^*(g_1^m, \cdots, g_n^m) - T^*(g_1, \cdots, g_n)\|_{q_2} \to 0$$

as $m \to \infty$. But, by definition of F_j we have that

$$\|F_j^m - F_j\|_{H^{np_2}} \to 0, \quad \|F_j^m\|_{H^{np_2}} \leqslant A_{np_2}\|g_j^m\|_{H^{np_2}}.$$

Therefore, we have that

$$\left\| T\left(\prod_{j=1}^n F_j^m\right) - T\left(\prod_{j=1}^n F_j\right)\right\|_{L^{q_2}(\mathbb{D})} \to 0.$$

Hence, (5.6) still holds in the case that $g_j \in L^{np_2}(\mathbb{D})$. Even more, since $p_1 \geqslant p \geqslant p_2$ we have that (5.6) holds if $g_i \in L^{np}(\mathbb{D})$. Now we are going to see that if P is any polynomial then $\|TP\|_{L^q(\mathbb{D})} \leqslant \|KM_1^{1-\theta}M_2^{\theta}\|P\|_{H^p}$. If we can see this then we can extend T to the whole H^p.

Given P any polynomial, by the Riesz Factorization Theorem 5.2.12, we can write $P(z) = B(z)G(z)$ where G is a polynomial without zeros and B is the Blaschke product of P. Hence, define

$$F_1 = BG^{1/n}, \quad F_2 = F_3 = \cdots = F_n = G^{1/n}.$$

Multiplying P by a number of modulus 1, we can assume that $P(0)$ is real and since $G(0) > 0$ then $B(0)$ is also real. Taking the main branch of $G^{1/n}$, we have that $F_j(0)$ is real for all j. Then, since F_j are bounded and $F_j(0)$ are real we have that

$$F_j(z) = \frac{1}{2\pi}\int_0^{2\pi} g_j(t)\frac{e^{it}+z}{e^{it}-z}dt,$$

with $g_j \in L^{np}(\mathbb{D})$ and g_j are real valued. Hence, (5.6) holds and,

$$\|TP\|_{L^q(\mathbb{D})} = \left\|T\left(\prod_{j=1}^n F_j\right)\right\|_{L^q(\mathbb{D})} \leqslant (A_{np_1}^{1-\theta}A_{np_2}^{\theta})^n(M_1^{1-\theta}M_2^{\theta})\prod_{j=1}^n \|g_j\|_{H^{np}}.$$

But

$$\prod_{j=1}^n\left(\int_0^{2\pi}|F_j(e^{it})|^{np}dt\right)^{p/n} = \prod_{j=1}^n\left(\int_0^{2\pi}|G_j(e^{it})|^{p}dt\right)^{p/n} = (2\pi)^p\|P\|_{H^p}.$$

Therefore,

$$\|TP\|_{L^q(\mathbb{D})} \leqslant (A_{np_1}^{1-\theta} A_{np_2}^{\theta})^n (M_1^{1-\theta} M_2^{\theta})(2\pi)^p \|P\|_{H^p}.$$

Taking $K = \max(A_{np_1}^n A_{np_2}^n)(2\pi)^p$, we arrive at

$$\|TP\|_{L^q(\mathbb{D})} \leqslant K(M_1^{1-\theta} M_2^{\theta}) \|P\|_{H^p}.$$

Then, as it holds for any polynomial P, it still holds for any $f \in H^p$. ■

Chapter 6

Boundedness of Operators

In this chapter we will see some examples of operators and we will show how the interpolation methods affect to these operators. In particular, we will study the Fourier multipliers and, as a particular case, the Hilbert transform.

The results and definitions of those operators are in [11, Chapter II and Chapter IV.3].

6.1 Fourier Multipliers

In this section we will apply the theory of interpolation to the Fourier multipliers. These type of operators are called "multipliers" because its Fourier transform acts by multiplication. We will define and give some examples of them. We will begin by defining what is a Fourier multiplier.

Definition 6.1.1. Let m be a measurable function on $\mathbb{R}^n$ and define T_m with domain in $L^2 \cap L^p$ by the following relation

$$\widehat{T_m(f)}(x) = m(x)\hat{f}(x), \quad f \in L^2 \cap L^p.$$

We say that m is a multiplier for L^p with $1 \leqslant p \leqslant \infty$ if $T_m f \in L^p$ and satisfies that

$$\|T_m f\|_p \leqslant C\|f\|_p$$

where C is a constant independent of f.

Since $L^2 \cap L^p$ is dense in L^p we have that T_m extends uniquely in L^p. By simplicity we shall denote by T_m this extension. And we denote by $\mathcal{M}_p$ the class of multipliers such that $T_m : L^p \to L^p$ continuously.

The goal of this section is to see that $\mathcal{M}_p = \mathcal{M}_{p'}$ if $1 = 1/p + 1/p'$ and that $\mathcal{M}_1 \subset \mathcal{M}_2$. Because if we see these things, then we will have that

$$\mathcal{M}_1 \subset \mathcal{M}_p \subset \mathcal{M}_2 \quad 1 \leqslant p \leqslant 2$$

and that

$$\mathcal{M}_\infty \subset \mathcal{M}_p \subset \mathcal{M}_2 \quad 2 \leqslant p \leqslant \infty.$$

Now we are going to see which are the $T_m \in \mathcal{M}_2$ and $T_m \in \mathcal{M}_1$.

Proposition 6.1.2. *The $\mathcal{M}_2$ class is the class of all bounded measurable functions and the multipliers norm is identical with the L^∞ norm.*

Proof. Let $f \in L^2(\mathbb{R}^n)$ and $m \in \mathcal{M}_2$, then by Parseval's Theorem 1.3.39 we have that

$$\|T_m f\|_2^2 = \|\widehat{T_m f}\|_2^2 = \|m\hat{f}\|_2^2 = \int_{\mathbb{R}^n} |m(\xi)\hat{f}(\xi)|^2 d\xi.$$

But, since T_m is continuous we have that $\|T_m f\|^2 \leqslant A^2\|f\|_2^2$. Take f such that

$$\hat{f}_{r,x}(\xi) = \frac{1}{|B(x,r)|^{1/2}}\chi_{B(x,r)}(\xi) \in L^2(\mathbb{R}^n).$$

Then, we have that $\|f_{r,x}\|_2^2 = 1$ and that

$$\int_{\mathbb{R}^n} |m(\xi)\hat{f}_{r,x}(\xi)|^2 d\xi = \frac{1}{|B(x,r)|}\int_{B(x,r)} |m(\xi)|^2 d\xi \leqslant A^2.$$

By the Lebesgue Differentiation Theorem 1.3.22 we arrive at

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} |m(\xi)|^2 d\xi \to |m(x)|^2 \quad \text{a.e. } x$$

as $r \to 0$. Therefore, $\|m\|_\infty \leqslant A$, but since A is the infimum over all the constants such that $\|T_m f\|_2 \leqslant A\|f\|_2$, we have that $A \leqslant \|m\|_\infty$, then $A = \|m\|_\infty$. ■

Proposition 6.1.3. *The $\mathcal{M}_1$ class is the class of Fourier transforms of elements of $\mathscr{B}(\mathbb{R}^n)$, (the finite Borel measures), and the norm of $\mathcal{M}_1$ is identical to the norm of $\mathscr{B}(\mathbb{R}^n)$.*

Proof. Let $f \in L^1$ and $\mu \in \mathscr{B}(\mathbb{R}^n)$, then the Fourier transform of μ is

$$\hat{\mu}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} d\mu(x).$$

So, we have that

$$|\hat{\mu}(\xi)| = \left|\int_{\mathbb{R}^n} e^{-ix\cdot\xi} d\mu(x)\right| \leqslant \int_{\mathbb{R}^n} |e^{-ix\cdot\xi}||d\mu(x)| = \int_{\mathbb{R}^n} |d\mu(x)| = \|\mu\|_{\mathscr{B}(\mathbb{R}^n)}.$$

So, $\|\hat{\mu}\|_\infty \leqslant \|\mu\|_{\mathscr{B}(\mathbb{R}^n)}$. Now, define

$$Tf(x) := \int_{\mathbb{R}^n} f(x-y)d\mu(y) = (f * \mu)(x).$$

Then, by Fubini, we have that

$$\begin{aligned}\|Tf\|_1 &= \int_{\mathbb{R}^n}\left|\int_{\mathbb{R}^n} f(x-y)d\mu(y)\right| dx \leqslant \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(x-y)|dx d\mu(y) = \|f\|_1 \int_{\mathbb{R}^n} d\mu(y) \\ &= \|\mu\|_{\mathscr{B}(\mathbb{R}^n)}\|f\|_1\end{aligned}$$

and $\|\widehat{Tf}\|_\infty \leqslant \|\hat{\mu}\|_\infty\|\hat{f}\|_\infty$. So, we have that

$$\hat{\mathscr{B}}(\mathbb{R}^n) \subset \mathcal{M}_1.$$

For the other inclusion see [7, Theorem 3.6.4]. ■

The following theorem shows that $\mathcal{M}_p = \mathcal{M}_{p'}$ with $1 = 1/p + 1/p'$.

Theorem 6.1.4. *Assume that $1 = 1/p + 1/p'$, $1 \leqslant p \leqslant \infty$, then $\mathcal{M}_p = \mathcal{M}_{p'}$ with equal norms.*

Proof. Let $m \in \mathcal{M}_p$ and let us denote $\sigma(f)(x) = \tilde{\bar{f}}(x) = \bar{f}(-x)$, then we have that $\sigma^{-1}(f)(x) = \bar{\bar{f}}(x)$. Moreover,

$$\widehat{(\sigma(f)(x))}(y) = \int_{\mathbb{R}^n} \bar{f}(-x)e^{-ix\cdot y}dx = -\int_{\mathbb{R}^n} \bar{f}(x)e^{ix\cdot y}dx = -\bar{\hat{f}}(y).$$

Then,

$$\widehat{(T_m\sigma(f)(x))}(y) = -m(y)\bar{\hat{f}}(y).$$

But, also we have that

$$\widehat{(\sigma^{-1}(f)(x))}(y) = \int_{\mathbb{R}^n} \overline{f(-x)}e^{-ix\cdot y}dx = -\int_{\mathbb{R}^n} \bar{f}(x)e^{ix\cdot y}dx = -\bar{\hat{f}}(y).$$

So, we arrive at

$$\widehat{(\sigma^{-1}T_m\sigma f(x))}(y) = -\overline{\widehat{(T_m\sigma(f)(x))}}(y) = \bar{m}(y)\hat{f}(y).$$

Therefore, $\sigma^{-1}T_m\sigma = T_{\bar{m}}$. Even more, if $m \in \mathcal{M}_p$ then $\bar{m} \in \mathcal{M}_p$ with the same norm. Now, since $L^2 \cap L^p$ is dense in L^p we can consider $f \in L^2 \cap L^p$ and $g \in L^2 \cap L^{p'}$, and use Plancherel's Theorem 1.3.35 to obtain

$$\langle T_m f, g\rangle = \int_{\mathbb{R}^n} T_m f(x)\bar{g}(x)dx = \int_{\mathbb{R}^n} m(x)\hat{f}(x)\bar{\hat{g}}(x)dx = \int_{\mathbb{R}^n} \hat{f}(x)\bar{\bar{m}}(x)\bar{\hat{g}}(x)dx = \langle f, T_{\bar{m}}g\rangle.$$

Assume also that $\|f\|_p = 1$, then

$$\left|\int_{\mathbb{R}^n} f(x)T_{\bar{m}}\bar{g}(x)dx\right| = \left|\int_{\mathbb{R}^n} T_m f(x)\bar{g}(x)dx\right| \leqslant \|T_m f\|_p\|g\|_{p'} \leqslant \|f\|_p A\|g\|_{p'} = A\|g\|_{p'},$$

where A is the norm of m in $\mathcal{M}_p$. Now, taking the supremum over f we obtain that

$$\|T_{\bar{m}}\bar{g}\|_{p'} \leqslant A\|g\|_{p'}.$$

Then $m \in \mathcal{M}_{p'}$ and $\|m\|_{\mathcal{M}_p} \geqslant \|m\|_{\mathcal{M}_{p'}}$. Using the same argument but assuming that $m \in \mathcal{M}_{p'}$ we obtain that $\mathcal{M}_{p'} = \mathcal{M}_p$ with equal norms. ∎

Note that by Proposition 6.1.3, we have that $\mathcal{M}_1 \subset \mathcal{M}_2$. Then, by Theorem 6.1.4 we have that $\mathcal{M}_\infty \subset \mathcal{M}_2$. So, it remains to see that for any $1 \leqslant p \leqslant q \leqslant 2$ we have that

$$\mathcal{M}_1 \subset \mathcal{M}_p \subset \mathcal{M}_q \subset \mathcal{M}_2.$$

Theorem 6.1.5. *Let $1 \leqslant p \leqslant 2$ and take $m \in \mathcal{M}_p$,then, $m \in \mathcal{M}_q$ for $p \leqslant q \leqslant 2$.*

Proof. Let $1 \leqslant p \leqslant 2$ and take $m \in \mathcal{M}_p$ and consider T_m the operator associated to m. Then,

$$T_m : L^p \to L^p.$$

But, by Theorem 6.1.4 we have that $m \in \mathcal{M}_{p'}$ so

$$T_m : L^{p'} \to L^{p'}.$$

Take $p \leqslant q \leqslant 2$, therefore $p \leqslant q \leqslant p'$. By The General Marcinkiewicz Interpolation Theorem 5.1.5 we have that

$$T_m : L^{r,q} \to L^{r,q}$$

where

$$\frac{1}{r} = \frac{1-\theta}{p} + \frac{\theta}{p'}$$

for $0 < \theta < 1$. So, we have to compute θ such that $r = q$, since

$$1 = \frac{1}{p} + \frac{1}{p'}$$

we obtain that

$$\frac{1}{q} = \frac{1-\theta}{p} + \frac{\theta}{p'} = \frac{1-\theta}{p} + \frac{\theta(p-1)}{p} = \frac{1-\theta+\theta p-\theta}{p} = \frac{1+\theta(p-2)}{p}.$$

Then, we have that

$$\frac{1}{q} = \frac{1+\theta(p-2)}{p} \Leftrightarrow \theta = \frac{\frac{p}{q}-1}{p-2} = \frac{q-p}{q(2-p)}.$$

Notice that

$$\frac{q-p}{q(2-p)} = \frac{q-p}{q+(q-p)} \in (0,1).$$

So, taking this θ we will arrive at

$$T_m : L^{q,q} \to L^{q,q}$$

but $L^{q,q} = L^q$. Hence, $m \in \mathcal{M}_q$. ∎

Now, we have that for any $1 \leqslant p \leqslant q \leqslant 2$

$$\mathcal{M}_1 \subset \mathcal{M}_p \subset \mathcal{M}_q \subset \mathcal{M}_2,$$

but, by Theorem 6.1.4 we also have that

$$\mathcal{M}_\infty \subset \mathcal{M}_{p'} \subset \mathcal{M}_{q'} \subset \mathcal{M}_2$$

and if $p \leqslant q \leqslant 2$, then $p' \geqslant q' \geqslant 2$. This means that the $\mathcal{M}_p$ is increasing with respect to p if $1 \leqslant p \leqslant 2$ and it is decreasing with respect to p if $2 \leqslant p \leqslant \infty$.

6.2 Hilbert Transform

In this section we will define the Hilbert transform and we will apply the interpolation methods to this operator. We will begin by giving the expression of this transformation.

Definition 6.2.1. Let $f \in L^p(\mathbb{R})$ we define the Hilbert transform of f as

$$H(f)(x) = \frac{1}{\pi} \,\mathrm{p.\,v.} \int_{\mathbb{R}} \frac{f(x-y)}{y} dy.$$

Note that this integral has to be interpreted as

$$\lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{|y| \geqslant \varepsilon} \frac{f(x-y)}{y} dy = \left(\mathrm{p.\,v.} \frac{1}{\pi y} * f \right)(x).$$

The purpose of this section is to show that $H : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ continuously for $1 < p < \infty$ but that for $p = \infty$ it does not hold.

In the following proposition we will see that the Hilbert transform is a tempered distribution.

Proposition 6.2.2. *The Hilbert transform applied to some point x is a tempered distribution.*

Proof. In order to see that H is a tempered distribution we will begin by proving that it is a distribution. That H is linear follows from the linearity of the integral and the limit. Let $\varphi \in D(\mathbb{R})$ and consider $|H(\varphi)(x)|$ for any $x \in R^n$, then we have that

$$\pi |H(\varphi)(x)| \leqslant \lim_{\varepsilon \to 0} \left| \int_{\varepsilon < |y| < 1} \frac{\varphi(x-y)}{y} dy \right| + \left| \int_{|y| \geqslant 1} \frac{\varphi(x-y)}{y} dy \right|$$

Without lost of generality we can consider $x = 0$, and since if $\varphi \in D(\mathbb{R})$ then $\tilde{\varphi} \in D(\mathbb{R})$ we can take $\varphi(y)$ instead of $\varphi(-y)$ and we obtain that

$$\pi |H(\varphi)(0)| \leqslant \lim_{\varepsilon \to 0} \left| \int_{\varepsilon < |y| < 1} \frac{\varphi(y)}{y} dy \right| + \left| \int_{|y| \geqslant 1} \frac{\varphi(y)}{y} dy \right|$$

Since $1/y \in L^1_{\mathrm{loc}}(\mathbb{R} \backslash \{0\})$ we have that the second integral is bounded by $C_1 \|\varphi\|_m$ where C_1 is constant and independent of φ, $\|\cdot\|_m$ is the seminorm of a test function defined in Section 1.4 and $m \in \mathbb{N}$. So, we only need to check the first integral. Since $1/y$ is odd we have that

$$\int_{\varepsilon < |y| \leqslant 1} \frac{dy}{y} = 0.$$

Hence, we can add $\varphi(0)/y$ since $\varphi(0)$ is constant and it does not modify the integral. So, we get

$$\lim_{\varepsilon \to 0} \left| \int_{\varepsilon < |y| < 1} \frac{\varphi(y)}{y} dy \right| = \lim_{\varepsilon \to 0} \left| \int_{\varepsilon < |y| < 1} \frac{\varphi(y) - \varphi(0)}{y} dy \right|.$$

Now, since φ is infinitely differentiable, in particular it is Lipschitz, so we have that

$$\lim_{\varepsilon\to 0}\left|\int_{\varepsilon<|y|<1}\frac{\varphi(y)-\varphi(0)}{y}dy\right| \leqslant \lim_{\varepsilon\to 0}\int_{\varepsilon<|y|<1}\frac{|\varphi(y)-\varphi(0)|}{|y|}dy \leqslant \lim_{\varepsilon\to 0}\int_{\varepsilon<|y|<1}\frac{|D\varphi(z)||y|}{|y|}dy$$
$$\leqslant \|D\varphi\|_{L^\infty}|B(0,1)|,$$

where $|B(0,1)|$ is the measure of the ball of center 0 and radius 1. But, $\|D\varphi\|_{L^\infty} \leqslant \|\varphi\|_1$ where $\|\varphi\|_1$ is the seminorm of φ, therefore, we have that $\|p\|_1 \leqslant \|p\|_m$ and we obtain that

$$\pi|H(\varphi)(0)| \leqslant \|\varphi\|_m(C_1 + |B(0,1)|).$$

So, H is a distribution, now we have to see that for all $f \in S(\mathbb{R})$ we have that

$$\pi|H(f)(0)| \leqslant K\|f\|_N$$

for some $N \in \mathbb{N}$ and some constant $K > 0$ independent of f. But notice that for the first integral we only have used that $\varphi \in C^\infty(\mathbb{R})$, then it holds for $f \in S(\mathbb{R})$ because if $f \in S(\mathbb{R})$ then $\tilde{f} \in S(\mathbb{R})$. Then, we have to check the second integral, then we have that

$$\left|\int_{|y|\geqslant 1}\frac{f(-y)}{y}dy\right| = \left|\int_{|y|\geqslant 1}\frac{f(y)}{y}dy\right| \leqslant \int_{|y|\geqslant 1}\frac{|f(y)|}{|y|}dy.$$

Multiplying and dividing by $|y|^2$ and using that $1 \leqslant |y|$ we have that

$$\int_{|y|\geqslant 1}\frac{|y|^2|f(y)|}{|y|^2|y|}dy \leqslant \int_{|y|\geqslant 1}\frac{\|f\|_1}{|y|^2}dy = K\|f\|_1$$

where K is a constant independent of f. Therefore, we have that

$$\pi|H(f)(0)| \leqslant \|f\|_1(K + |B(0,1)|).$$

So, $H \in S'(\mathbb{R})$. ∎

Now, let us compute the Fourier transform of H, we had seen in the Section 1.4.1 that $\hat{H}(\varphi) = H(\hat{\varphi})$ for $\varphi \in S(\mathbb{R})$. In order to simplify the notation consider $x = 0$. Then,

$$\hat{H}(\varphi)(0) = \frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{|\xi|\geqslant\varepsilon}\frac{\hat{\varphi}(\xi)}{\xi}d\xi = \frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{1/\varepsilon\geqslant|\xi|\geqslant\varepsilon}\left(\int_{\mathbb{R}}\varphi(y)e^{-iy\cdot\xi}dy\right)\frac{d\xi}{\xi}.$$

Applying Fubini and using that

$$e^{ix} = \cos(x) + i\sin(x)$$

we arrive at

$$\hat{H}(\varphi)(0) = \frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{\mathbb{R}}\varphi(y)\left(\int_{1/\varepsilon\geqslant|\xi|\geqslant\varepsilon}(\cos(-y\cdot\xi) + i\sin(-y\cdot\xi))\frac{d\xi}{\xi}\right)dy.$$

Since, cos is even and $1/\xi$ is odd we have that

$$\int_{1/\varepsilon\geqslant|\xi|\geqslant\varepsilon}\cos(-y\cdot\xi)\frac{d\xi}{\xi} = 0.$$

And since, $\sin(-x) = -\sin(x)$. we arrive at

$$\hat{H}(\varphi)(0) = \frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{\mathbb{R}}\varphi(y)\left(-i\int_{1/\varepsilon\geqslant|\xi|\geqslant\varepsilon}\sin(y\cdot\xi)\frac{d\xi}{\xi}\right)dy.$$

Now, since for all $0 < a < b < \infty$ we have that

$$\left|\int_a^b \frac{\sin(x)}{x}dx\right| \leqslant 4$$

we obtain that

$$\int_{1/\varepsilon\geqslant|\xi|\geqslant\varepsilon}\sin(y\cdot\xi)\frac{d\xi}{\xi}$$

is uniformly bounded by 8. Even more, we have that

$$\lim_{\varepsilon\to 0}\int_{1/\varepsilon\geqslant|\xi|\geqslant\varepsilon}\sin(y\cdot\xi)\frac{d\xi}{\xi} = \pi\operatorname{sgn}(y).$$

Therefore, we can apply the Dominated Convergence theorem and enter the limit inside the integral, and we get that

$$\begin{aligned}\hat{H}(\varphi)(0) = \frac{1}{\pi}\int_{\mathbb{R}}\varphi(y)\lim_{\varepsilon\to 0}\left(-i\int_{1/\varepsilon\geqslant|\xi|\geqslant\varepsilon}\sin(y\cdot\xi)\frac{d\xi}{\xi}\right)dy &= \frac{1}{\pi}\int_{\mathbb{R}}\varphi(y)(-i\pi\operatorname{sgn}(y))dy \\ &= \int_{\mathbb{R}}\varphi(y)(-i\operatorname{sgn}(y))dy.\end{aligned}$$

Hence, $\hat{H}(x) = -i\operatorname{sgn}(x)$ in the sense of distributions.

Now, since $S(\mathbb{R})$ is dense in $L^p(\mathbb{R})$ for all $1 \leqslant p < \infty$ we can compute $H(f)(x)$ for $f \in L^p(\mathbb{R})$ as the limit in $L^p(\mathbb{R})$ of Schwarz functions. Thus, we can reduce to prove that

$$\|H(f)\|_{L^p} \leqslant C\|f\|_{L^p}$$

for some constant C independent of f and for $f \in S(\mathbb{R})$.

Proposition 6.2.3. *The Hilbert transform is a continuous operator form $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$.*

Proof. Let $f \in S(\mathbb{R})$ since $S(\mathbb{R})$ is dense in $L^2(\mathbb{R})$ we have that $f \in L^2(\mathbb{R})$, and then we can apply Parseval Theorem 1.3.39 to f, and we obtain that $\|\hat{f}\|_2^2 = \|f\|_2^2$. But, using the definition of the Fourier transform for tempered distributions we have that

$$\|\hat{H}(f)\|_2^2 = \|H(\hat{f})\|_2^2,$$

and, since we saw before $\hat{H}(x) = -i\operatorname{sgn}(x)$ in the sense of distributions, we obtain that

$$\|\hat{H}(f)\|_2^2 = \|f\|_2^2.$$

Moreover, since the Fourier transform goes from $S(\mathbb{R})$ to $S(\mathbb{R})$ we have that there exists $g \in S(\mathbb{R})$ such that $\hat{f} = g$ and, then, by Parseval Theorem 1.3.39 we arrive at

$$\|g\|_2^2 = \|f\|_2^2 = \|\hat{H}(f)\|_2^2 = \|H(\hat{f})\|_2^2 = \|H(g)\|_2^2.$$

Therefore, $H : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ continuously with norm 1. ∎

In order to prove that $H : L^{2^k}(\mathbb{R}) \to L^{2^k}(\mathbb{R})$ we need the following lemma which gives us a formula for the square of the Hilbert transform.

Lemma 6.2.4. *Let $f \in S(\mathbb{R})$, then for all $x \in \mathbb{R}$ we have that*

$$(H(f)^2(x) = f^2(x) + 2H(fH(f))(x).$$

Proof. Notice that since H acts as a convolution it is a Fourier multiplier with $m(\xi) = \hat{H}(\xi) = -i\,\mathrm{sgn}(\xi)$. So, we have that

$$\hat{f}^2(\xi) + 2(\widehat{H(fH(f))}(\xi) = (\hat{f} * \hat{f})(\xi) + 2m(\xi)(\hat{f} * \widehat{H(f)})(\xi).$$

Applying that $H\hat{(}f)(x) = m(x)\hat{f}(x)$ and the definition of convolution we arrive at

$$(\hat{f} * \hat{f})(\xi) + 2m(\xi)(\hat{f} * \widehat{H(f)})(\xi) = \int_{\mathbb{R}} \hat{f}(\eta)\hat{f}(\xi-\eta)d\eta + 2m(\xi)\int_{\mathbb{R}} \hat{f}(\eta)\hat{f}(\xi-\eta)m(\eta)d\eta. \tag{6.1}$$

But, since the convolution is associative we also have that

$$(\hat{f} * \hat{f})(\xi) + 2m(\xi)(\hat{f} * \widehat{H(f)})(\xi) = \int_{\mathbb{R}} \hat{f}(\eta)\hat{f}(\xi-\eta)d\eta \tag{6.2}$$

$$+ 2m(\xi)\int_{\mathbb{R}} \hat{f}(\eta)\hat{f}(\xi-\eta)m(\xi-\eta)d\eta. \tag{6.3}$$

So, averaging (6.1) and (6.2) we get

$$\hat{f}^2(\xi) + 2(\widehat{H(fH(f))}(\xi) = \int_{\mathbb{R}} \hat{f}(\eta)\hat{f}(\xi-\eta)(1 + m(\xi)(m(\eta) + m(\xi-\eta)))d\eta.$$

Now, we are going to prove that

$$m(\eta)m(\xi-\eta) = 1 + m(\xi)m(\eta) + m(\xi)m(\xi-\eta)$$

because if this happens then

$$\begin{aligned}\int_{\mathbb{R}} \hat{f}(\eta)\hat{f}(\xi-\eta)(1 + m(\xi)(m(\eta) + m(\xi-\eta)))d\eta &= \int_{\mathbb{R}} \hat{f}(\eta)\hat{f}(\xi-\eta)m(\eta)m(\xi-\eta)d\eta \\ &= (\widehat{H(f)} * \widehat{H(f)})(\xi).\end{aligned}$$

Hence,

$$\hat{f}^2(\xi) + 2(\widehat{H(fH(f))}(\xi) = (\widehat{H(f)} * \widehat{H(f)})(\xi)$$

what implies that $(H(f)^2(x) = f^2(x) + 2H(fH(f))(x)$. So, let us prove that

$$m(\eta)m(\xi-\eta) = 1 + m(\xi)m(\eta) + m(\xi)m(\xi-\eta)$$

or equivalently,

$$0 = 1 + m(\xi)m(\eta) + m(\xi-\eta)(m(\xi) - m(\eta)).$$

Since $m(x) = -i\,\mathrm{sgn}(x)$ we have that

$$1 + m(\xi)m(\eta) + m(\xi-\eta)(m(\xi) - m(\eta)) = 1 - \mathrm{sgn}(\xi)\,\mathrm{sgn}(\eta) - \mathrm{sgn}(\xi-\eta)(\mathrm{sgn}(\xi) - \mathrm{sgn}(\eta)).$$

Assume that $\operatorname{sgn}(\eta) = \operatorname{sgn}(\xi)$, then

$$1 - \operatorname{sgn}(\xi)\operatorname{sgn}(\eta) - \operatorname{sgn}(\xi-\eta)(\operatorname{sgn}(\xi) - \operatorname{sgn}(\eta)) = 1 - \operatorname{sgn}(\xi)^2 = 1 - (\pm 1)^2 = 0.$$

Assume now that $\operatorname{sgn}(\eta) = -1$ and that $\operatorname{sgn}(\xi) = 1$, then

$$\begin{aligned} 1 - \operatorname{sgn}(\xi)\operatorname{sgn}(\eta) - \operatorname{sgn}(\xi-\eta)(\operatorname{sgn}(\xi) - \operatorname{sgn}(\eta)) &= 1 + 1 - \operatorname{sgn}(\xi-\eta)(1-(-1)) \\ &= 2 - 2\operatorname{sgn}(\xi-\eta). \end{aligned}$$

But, since $\operatorname{sgn}(\eta) = -1$ and $\operatorname{sgn}(\xi) = 1$ we have that $\xi - \eta > 0$, so $\operatorname{sgn}(\xi-\eta) = 1$ and we obtain that

$$1 - \operatorname{sgn}(\xi)\operatorname{sgn}(\eta) - \operatorname{sgn}(\xi-\eta)(\operatorname{sgn}(\xi) - \operatorname{sgn}(\eta)) = 0.$$

Finally assume that $\operatorname{sgn}(\eta) = 1$ and that $\operatorname{sgn}(\xi) = -1$, then

$$\begin{aligned} 1 - \operatorname{sgn}(\xi)\operatorname{sgn}(\eta) - \operatorname{sgn}(\xi-\eta)(\operatorname{sgn}(\xi) - \operatorname{sgn}(\eta)) &= 1 + 1 - \operatorname{sgn}(\xi-\eta)(-1-1) \\ &= 2 + 2\operatorname{sgn}(\xi-\eta). \end{aligned}$$

But, since $\operatorname{sgn}(\eta) = 1$ and $\operatorname{sgn}(\xi) = -1$ we have that $\xi - \eta < 0$, so $\operatorname{sgn}(\xi-\eta) = -1$. Therefore, we get that

$$1 - \operatorname{sgn}(\xi)\operatorname{sgn}(\eta) - \operatorname{sgn}(\xi-\eta)(\operatorname{sgn}(\xi) - \operatorname{sgn}(\eta)) = 0.$$

Hence,

$$m(\eta)m(\xi-\eta) = 1 + m(\xi)m(\eta) + m(\xi)m(\xi-\eta).$$

■

Now we are going to see that the adjoint operator of H, H^* is $-H$, if we see this, since $(\hat{H}(\xi))^2 = (-i\operatorname{sgn}(\xi))^2 = -1$, we will have that $H^2 = -I$, where I is the identity operator. But, since

$$\hat{H^2}(f)(x) = \hat{H}(H(f))(x) = (-i\operatorname{sgn}(x))\hat{H}(f)(x) = (-i\operatorname{sgn}(x))^2\hat{f} = -1,$$

we have that $\hat{H^2} = -1$ (in the sense of distributions). Denote by $m_H(x) = \hat{H}(x)$ and $m_{H^*}(x) = \hat{H}^*(x)$, then by definition of adjoint operator we get that, for all $f, g \in S(\mathbb{R})$

$$\int_{\mathbb{R}} m_H(x)\hat{f}(x)\overline{g(x)}dx = \int_{\mathbb{R}} H(\hat{f})(x)\overline{g(x)}dx = \int_{\mathbb{R}} \hat{f}(x)\overline{H^*(g(x))}dx.$$

Now using the Hat Theorem 1.3.25 we obtain that

$$\int_{\mathbb{R}} m_H(x)\hat{f}(x)\overline{g(x)} \int_{\mathbb{R}} f(x)\overline{m_{H^*}(x)\hat{g}(x)}dx.$$

Using again the Hat Theorem 1.3.25 we arrive at

$$\int_{\mathbb{R}} m_H(x)\hat{f}(x)\overline{g(x)} \int_{\mathbb{R}} \hat{f}(x)\overline{m_{H^*}(x)g(x)}dx.$$

Hence, $m_H(x) = \overline{m_{H^*}(x)}$ but as $m_{H^*}(x) = \overline{m_H(x)} = -m_H(x) = i\operatorname{sgn}(x)$ we can conclude that $H^* = -H$. So, we have that $H^2 = -I$.

The following theorem shows us that $H : L^{2^k}(\mathbb{R}) \to L^{2^k}(\mathbb{R})$ continuously.

Theorem 6.2.5. *Let $k \in \mathbb{N}\backslash\{0\}$ and let $p = 2^k$, then $H : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ with norm C_p is a constant which only depends of p.*

Proof. By Proposition 6.2.3 we have that $H : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ with norm 1. So, we have proved the case $k = 1$. Now we are going to apply induction with respect to k, assume that $H : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ is bounded with norm C_p and $p = 2^k$, since $S(\mathbb{R})$ is dense in $L^{2p}(\mathbb{R})$ it suffices to consider $f \in S(\mathbb{R})$. Then,

$$\|H(f)\|_{2p} = \|H(f)\|_p^{1/2}.$$

But, by Lemma 6.2.4 we have that

$$\|H(f)\|_p^{1/2} \leqslant (\|f\|_p^2 + 2\|H(fH(f))\|_p)^{1/2}.$$

Applying that $\|H(fH(f))\|_p$ is bounded by $C_p\|fH(f)\|_p$ and that by Hölder's inequality $\|fH(f)\|_p \leqslant \|f\|_{2p}\|H(f)\|_{2p}$, we arrive at

$$\|H(f)\|_p^{1/2} \leqslant (\|f\|_p^2 + 2\|H(fH(f))\|_p)^{1/2} \leqslant (\|f\|_p^2 + 2C_p\|f\|_{2p}\|H(f)\|_{2p})^{1/2}.$$

So, we arrive at

$$\left(\frac{\|H(f)\|_{2p}}{\|f\|_{2p}}\right)^2 - 2C_p\frac{\|H(f)\|_{2p}}{\|f\|_{2p}} - 1 \leqslant 0.$$

So, taking

$$t = \frac{\|H(f)\|_{2p}}{\|f\|_{2p}}$$

we have to solve the inequality $t^2 - 2C_pt - 1 \leqslant 0$, but this gives us that

$$\frac{\|H(f)\|_{2p}}{\|f\|_{2p}} = t \leqslant C_p + \sqrt{C_p^2 + 1}.$$

Now, taking supremum over $\|f\|_{2p} = 1$ we obtain that $H : L^{2p}(\mathbb{R}) \to L^{2p}(\mathbb{R})$ with norm $C_{2p} \leqslant C_p + \sqrt{C_p^2 + 1}$. Then, we have that $H : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ with norm C_p for all $p = 2^k$ with $k \in \mathbb{N}\backslash\{0\}$. ∎

Since we have that

$$H : L^{2^k}(\mathbb{R}) \to L^{2^k}(\mathbb{R})$$
$$H : L^{2^{k+1}}(\mathbb{R}) \to L^{2^{k+1}}(\mathbb{R})$$

continuously, by the General Marcinkiewicz Interpolation Theorem 5.1.5, we have that $H : L^q(\mathbb{R}) \to L^q(\mathbb{R})$ continuously, for all $q \in [2^k, 2^{k+1}]$. And as this happens for all $k \in \mathbb{N}\backslash\{0\}$ we can conclude that $H : L^q(\mathbb{R}) \to L^q(\mathbb{R})$ continuously, for all $q \in [2, \infty)$. But, since the Hilbert transform is a Fourier multiplier for $2 \leqslant p < \infty$, by Theorem 6.1.4, we have that that $H : L^q(\mathbb{R}) \to L^q(\mathbb{R})$ continuously, for all $q \in (1, \infty)$. So, it remains to see that for $p = \infty$ it does not holds.

Proposition 6.2.6. *Let $g(x) = \chi_{(0,1)}(x) \in L^\infty(\mathbb{R})$, then $H(g)(x) \notin L^\infty(\mathbb{R})$.*

Proof. Let $g(x) = \chi_{(0,1)}(x) \in L^\infty(\mathbb{R})$ and take $x > 1$, then

$$\pi H(g)(x) = \lim_{\varepsilon \to 0} \int_{|y| \geqslant \varepsilon} \frac{g(x-y)}{y} dy.$$

But $\chi_{(0,1)}(x-y) = 1$ if $0 < x - y < 1$, hence

$$\lim_{\varepsilon \to 0} \int_{|y| \geqslant \varepsilon} \frac{g(x-y)}{y} dy = \int_{0<x-y<1} \frac{dy}{y} = \int_{x-1}^{x} \frac{dy}{y} = \log\left(\frac{x}{x-1}\right).$$

Note that

$$\lim_{x \to 1^+} H(g)(x) = \lim_{x \to 1^+} \log\left(\frac{x}{x-1}\right) = \infty.$$

Therefore, $H(g) \notin L^\infty(\mathbb{R})$. ■

Again, using the Theorem 6.1.4 we arrive at $H : L^1(\mathbb{R}) \nrightarrow L^1(\mathbb{R})$. Hence, we conclude that $H : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ continuously, for $1 \lneqq p \lneqq \infty$.

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