Spectral theory and dynamical systems

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Abstract. We study transfer operators defined from vector bundle maps. We will refer to these operators transfer operators or weighted composition operators. This study appears naturally, in the study of normally hyperbolic manifolds and in the study of linearization of hydrodynamic equations.

In a first part, we introduce the notation and review the main general results that appear in the literature.

In a second part, we present the main general results of this manuscript, in particular a study of the theory of Mather spectrum. We also develop a study of the relation between the spectrum in different spaces and also study which features of the spectrum can be obtained form the study of individual orbits, in particular, periodic orbits. As an application of these results, we obtain sharp versions of structural stability and shadowing theorems including smooth dependence on parameters, which appear in the last part of the manuscript.

In a third part, we concentrate on a particular case of importance in applications, namely skew products over rotations. These systems appear as models of systems subject to external forces which are quasi-periodic. These problems appear often in applications. For example in celestial mechanics. In the last part, as an application of the theory developed, we present some results on the persistence of invariant manifolds under quasi-periodic perturbations and *a posteriori* estimates which can be used to validate numerical computations.

In a fourth part, we study the influence of the preservation of geometric structures and we introduce what we call locally constrained spaces. The main phenomenon here is that the spectrum may grow when we consider restriction to invariant subspaces. As an application, in the last part we present a spectral formulation of the Bowen conjecture.

As we have mentioned above, the fifth part is devoted to several applications: structural stability and shadowing theorems; invariant tori in quasi periodic systems; the conjecture of Bowen.

Further applications are: spectral theory in Holder spaces; normally hyperbolic invariant manifolds; non-stationary normal forms; applications to hydrodynamics (spectral properties of the perfectly conducting dynamo problem and the linearization of Euler equations); etc.

Contents

Preface

The study of Dynamical Systems has always been driven by the need to understand concrete problems and hence it has incorporated a wide variety of mathematical tools.

The incorporation of the functional analysis tools in a manner similar to the one studied here started in [**Mos69, Mat68**] in the study of structural stability. In particular, it was observed in [**Mat68**] that the property that a map f was Anosov could be expressed in terms of spectral properties of f_*, acting on C^0 vector fields. This, among other things, lead to a very clean proof of structural stability for Anosov maps (see e.g., [**Mos69**]).

The use of functional analysis to study problems of structural stability is very natural since the core of the problem is the existence of a solution of a functional equation. The study of the functional analysis properties of maps was extended considerably in scope and generality by many authors. Let us note in particular [**SS74, SS76a, SS76b, Sac78**] which develop very deep relations between recurrence properties and uniformity of several functional analysis constructs. The arguments for this interplay have a strange beauty since one has often to take hypothesis in one category and obtain result from another. See also [**MS89, JPS87**].

It should also be remarked that the weighted shift operators are very natural from the point of view of ergodic theory. For example, [**LS90, LS91, Kit97**]

Very similar problems to the problems of structural stability appears in the study of invariant objects. The study of invariant manifolds near hyperbolic points was started already by Poincaré and Lyapunov, before modern functional analysis was started. Functional analysis language was incorporated gradually. See [**CFdlL05**] for some historical notes.

More general invariant manifolds serve as landmarks that organize the long term behavior of dynamical systems and are the key to understanding many interesting geometric properties both for theoretical applications and for more applied problems dealing with numerical computations of concrete trajectories. The fact that a manifold is invariant is expressed as a functional equation, somewhat similar to the equations appearing in the study of structural stability. Hence, the study of invariant manifolds leads also to very interesting functional analysis problems. Functional analysis characterizations of normally hyperbolic manifolds were studied in [**Mn78, Swa83**].

A further application of transfer operators arises in hydrodynamics. It is very well known that many quantities in hydrodynamics are just *"transported by the flow"*. For example, Helmholtz theorem asserts that, given a solution of Euler equations, vorticity is just transported. The celebrated dynamo equations assert that in a perfectly conducting fluid, the magnetic field is also transported by the fluid. These results are also expressed as *"conservation of the flux"*. Hence the same spectral problems for transfer operators appear as the problem of linearization of hydrodynamic equations.

Nevertheless, the study of transfer operators in hydrodynamics leads to surprises. In hydrodynamics, the Physics dictates that the magnetic field, or the perturbations of Euler equations satisfy that they have zero divergence. Hence, it is natural to consider the linearization of the equations in spaces of zero divergence vector fields. It was discovered in [**dlL93**] that this has very important consequences for the spectral properties. The somewhat counterintuitive result is that, when restricting to spaces of zero divergence, which is proper closed space, the spectrum grows. Indeed, in the cases considered, the spectrum becomes an annulus. Hence, the effect of growth of the spectrum when restricting to zero divergence was termed the *no-gaps phenomenon* in [**CL99**].

Understanding this no-gaps phenomenon throws some light in the study of hydrodynamics. From our point of view, we thing that the no-gaps phenomenon begs for a more systematic study of the effect of the geometric properties (symplectic, contact, volume preserving) of dynamical systems on their spectral properties. This has practical consequences for dynamical systems since all the dynamical systems appearing in mechanics preserve a symplectic structure.

Another important link between dynamical systems and functional analysis is in the study of zeta functions introduced in [**AM65, Sma67**] as counting functions for periodic orbits. The connection with functional analysis [**Rue76a, Rue76b, Rue92a**] is that, under some assumptions on the dynamical system, the operators that appear in the applications above are nuclear. That is, one can define $\zeta(s) = \det(A - s\mathrm{Id})$ in some appropriate sense. Moreover, one can also interpret $\zeta(s)$ as a counting function on periodic orbits. By using functional analysis to study the properties of $\zeta(s)$ one obtains remarkable counting results on periodic orbits. The counting functions $\zeta(s)$ share remarkable properties with the Riemann ζ function that counts primes. These studies have lead to a remarkable theory in which there are functional analysis interpretations of many important dynamical quantities such as decay of correlations and central limit theorems.

This book grew as an attempt by the authors to put together some of the above results in a more systematic way. At the time, our main interest was the study of numerical methods for the computation of invariant manifolds for some quasi-periodically forced systems. We wanted to have results that ensured that, given a numerical solution that solves the required functional equation with good accuracy (a well written program always produces that) then, there is a true solution nearby. Some of our rigorous results, numerical algorithms and empirical results have already appeared in [**HdlLb, HdlL04, HdlL05a**].

This lead us to a reexamination of the theory of transfer operators. From the point of view of numerical analysis and from the point of view of applications to hydrodynamics, it was important to have several functional spaces at our disposal. For example, the C^r spaces and the Sobolev spaces. We wondered what was the relation of the spectrum in these spaces.

We also wondered what was the relation between the spectral properties and the spectrum on individual orbits. Besides being of interest for the numerical analysis of normally hyperbolic manifolds, the issue had appeared often in the theory of normally hyperbolic manifolds. There are two basic theories of normally hyperbolic manifolds. One theory of hyperbolicity can be found in [**HPS77**]. The theory of [**HPS77**] is based on the study of bundles and their contraction properties. Another theory of hyperbolicity is found in [**Fen72, Fen74, Fen77**]. This theory is based on the study of individual orbits. This theory lends itself to generalization where the properties are not uniform [**Pes76, Pes77, Pol93, BP01**]. The study of the relation between global properties of the spectrum and the spectrum on periodic orbits leads to a unification of these theories.

We should also remark that the relation between the spectrum in different spaces and the periodic orbits had appeared naturally in the mathematical theory of hydrodynamics [**Vis86**]. (Similar relations had been found empirically in the plasma theory, when practitioners often found that *X-points* are associated with instabilities of the magnetic fields). Another different area where spectral theory in different spaces (i.e. solvability of equations in different spaces) is related to periodic orbits is the part of rigidity theory related to the study of Livsic equation etc. See [**dlLMM86**]. Of course, the relation between periodic orbits and spectral properties is the basis of the theory of ζ functions.

We also felt that, given that the geometry affects significantly the spectrum, one should study more systematically the relations between spectrum and geometry. In particular, since the systems of Hamiltonian mechanics preserve a symplectic structure, it would be important to study the influence of the symplectic structure on the functional analysis properties and therefore on stability properties etc. The issues of the relations of periodic orbits and the regularity of splittings

For the problems mentioned above we could find scattered references in the literature and some of the results (e.g. the equivalence of the hyperbolicity theory based on C^0 and bounded sections) were in the folklore, even if not explicitly written.

We undertook the task of writing a complete explanation of many of these results, which were useful for the analysis needed for the numerical analysis of invariant manifolds. We also undertook a comparison of the different theories of normal hyperbolicity of [**HPS77**] and that of [**Fen72, Fen74, Fen77**].

We also undertook a more systematic study of the influence of geometric properties on the spectrum. The arguments of [**dlL93**] could also extended to other spaces and we obtain similar no-gaps phenomena in the study of closed versus exact forms. This leads to some spectral formulations of some global problems in dynamical systems such as the Bowen conjecture. We hope that this can also lead to some spectral characterization of other non-integrability phenomena such as those discovered in in [**JPdlL95, dlL92**]. We point out that very similar phenomena no-gap phenomena happen in spaces of jets. This gives a spectral interpretation of some results of non-integrability observed in [**JPdlL95, dlL92**]. We also expect that the theory developed here leads to a spectral theory formulation of the method of *"non-stationary normal forms"* [**DeL93, GK98, KS96, Guy02**].

Unfortunately, we have not been able to cover the very deep results on ζ function theory and its relation with issues such as decay of correlations. This is a theory still being developed. Excellent surveys are [**May80, Bal00**].

Notice that, since our main motivation is to apply the spectral results to finite-dimensional dynamical systems, the transfer operators we consider are 1-1. There are generalizations of most of the results of this manuscript to the non-invertible case. The generalization of Mather spectral theory to this context is easy. We can also use the generalization of Sacker-Sell spectral theory to the non-invertible case [**SS94, CL94, CL96, CL99**], even this theory works for infinite-dimensional transfer operators, i.e. associated to bounded linear bundle maps in Banach bundles.

This book is organized as follows: In Part 1, we cover some general results of the theory, reviewing the results of Sacker and Sell, Hirsch, Pugh and Shub, Mañé, Oseledec, etc. (see the references above).

In Part 2, we refine the Mather theory of spectrum of transfer operators. This refinement leads us to consider the spectrum of the transfer operator acting in different categories (bounded, continuous, C^r, Sobolev). The results are very general, in the sense that do not depend on having dynamical systems of a special form or considering some spaces with a local constraint. As an application of the results of this first part, we present results on smooth dependence on structural stability in different spaces (in the last part).

In Part 3, we present results which assume that the dynamical system on the base is a rotation. As an application, we summarize the results on existence of invariant manifolds in [**HdlLb**], that lead to numerical algorithms [**HdlL04, HdlL05a**] (in the last part). Systems with quasi-periodic forcing are very natural in astrodynamics where the forcing terms are the other celestial bodies, which are quasi-periodic to a very good approximation. Since quasi-periodic systems are very important in applications, we have written

Part 3 in a very self-contained way, even is this produces repetitions of some arguments of Part 1 and Part 2.

In Part 4, we study the properties related to the fact that we are considering transfer operators acting on a space of functions that satisfy some differential constraint (e.g. vector fields that are zero divergence, symplectic or forms which are closed or exact). We show that in all those cases, we obtain a non-gaps phenomenon. In the last part we present a spectral formulation of the Bowen conjecture, a problem that concerns global dynamics.

Part 5 is devoted to several applications, as we have already mentioned: Structural stability, invariant manifolds of quasi periodic systems, global dynamics (Bowen conjecture).

There are many other applications that we have in mind, and are in progress: spectral theory in Holder spaces (via interpolation spaces), normally hyperbolic invariant manifolds (with the parameterization method), non-stationary normal forms, applications to magneto-hydrodynamics (dynamo theory) and to Euler equations (with the theory of spectrum in locally constrained spaces of jets), etc.

Preliminary versions of this book have been circulated before as preprints [**dlL98, HdlL03a, HdlL03b, HdlL03c**]. This book incorporates the material of these preprints and superseeds them. This has allowed us to suppress certain duplications and to develop further connections on the material. In preparing the book, we have added significant new material not included in any of the preprints. Indeed this book, besides some more pedagogical material contains some results which, to our knowledge, have never appeared in the literature.

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Part 1

Introduction and review

In this part, we start our study of the weighted shift operators. In this part, we will cover the results which do not depend on the fact that the dynamical system is a rotation or on the fact that the space of sections satisfies some local constraints.

The first Chapter 1 collects standard results on differential geometry, dynamical systems and spectral theory. The results covered here are well known, but we will use them to establish the notation.

Since spectral theory is probably less known for the experts in dynamical systems and some of the material is rather subtle, we have developed a very detailed appendix on spectral theory, which we have placed in an appendix at the end of this manuscript. A concept that we have found specially fruitful in our studies is that of Weyl spectrum or approximate point spectrum, which is the set of approximate eigenvalues.

Chapter 2 contains a review of the spectral theory of Sacker and Sell [**SS74, SS76a, SS76b, Sac78**] compared with other spectral theories. In particular, the results of [**Mat68, HPS77, Mn78, CS80, Swa81, LS90, LS91, CL99**].

The main result is that there is a relation between the existence of gaps in the spectrum of the transfer operator and the existence of invariant subbundles for the generating vector bundle automorphism.

In the study of exponential dichotomies and invariant splittings our starting point are the sets of positively and negatively bounded orbits We have followed the lines in [**SS74**]. Later on, we obtain this sets from spectral properties of the transfer operator acting on bounded sections (Theorem 2.18). Although most of the papers in the literature consider the transfer operator acting on continuous sections, study also spectrum acting on bounded sections (see [**HPS77**]) and in other spaces of different regularity.

As it is well known, the regularity of the invariant subbundles comes from the invariant section theorem [**HP70, HPS77**]. The last part the section is devoted to the study of Lyapunov multipliers (see [**SS78**]).

CHAPTER 1

Preliminaries

In this section we will collect some standard facts from differential geometry, dynamical systems and spectral theory that we will use and state precisely our results. The results that we quote are quite standard and their proofs can be found in standard textbooks (e.g. [**DFN92, DFN85, DFN90**] for differential geometry, [**KH95**] for dynamical systems and [**Kat76, DS88a, DS88b, DS88c**] for spectral theory), nevertheless, stating them will serve to establish our notation and make precise our standing assumptions. Since the spectral theory is possibly less known for the experts in dynamical systems, we have extended the explanations in Appendix A.

1.1. Differential geometry

1.1.1. Bundles, sections. Along this manuscript, E will denote a complex *vector bundle* over a compact and connected manifold $\mathcal{P}$, whose projection is $\Pi : E \to \mathcal{P}$. The rank of the vector bundle E is the dimension of its fibers, and it will be denoted by n. The zero-section of the vector bundle is denoted by E_0.

Remark 1.1. Actually, the assumption that $\mathcal{P}$ is a manifold is only needed in the section when we discuss C^r bundles. In the sections when we discuss continuous, bounded or L^p sections we will only need that $\mathcal{P}$ is a metric space which is compact, connected and, in some theorems, perfect. This generality is useful for dynamical systems since, often one needs to deal with attractors etc. which are complicated sets.

We also note that in Section 3.8 we will study the effect of restricting the base set $\mathcal{P}$ so that one possible way of obtaining the results for invariant sets in smooth dynamical systems is to consider them as subsets of a manifold.

A *section* is a map $v : \mathcal{P} \to E$ that $\Pi \circ S = \mathrm{Id}_{\mathcal{P}}$, i.e. to a point $\theta \in \mathcal{P}$ we associate the vector $v(\theta) \in E_\theta = \Pi^{-1}(\theta)$. We will denote the vector space of sections by $\Gamma(E)$.

A *subbundle* F is a closed subset of E such that every $F_\theta = \{v_\theta \in F \mid \theta \in \mathcal{P}\}$ is a linear subspace of E_θ and all of them have the same dimension n_1. So, the fibers depend continuously on θ and n_1 is the rank of the bundle F. To do so, notice that the map $\theta \to F_\theta$ is a section defined on the Grassmannian bundle $\mathcal{G}_{n_1}(E)$ (this is a bundle whose fibers $\mathcal{G}_{n_1}(E)_\theta$ are the sets of linear subspaces of E with dimensions n_1). Since the manifold $\mathcal{P}$ is compact, this section on the Grassmannian bundle has compact graphs and hence, the mapping is continuous.

To introduce topology in the space of sections we make the standing assumption that there is a Finsler metric on the bundle, i.e. a norm $|\cdot|_\theta$ on each fiber E_θ, that depends continuously with respect to the base point θ (we will suppress the subindex θ in the norm when there is no doubt of the fiber where it is acting in). Of course, a particular case of the above situation is when the norm on each fiber comes from an inner product, that is, when the Finsler metric is a Riemannian metric. In this manuscript, we will not need that the metric is Riemannian.

As it is well known, we can transfer vector space constructions into vector bundle constructions, by extending the definitions fiberwise. For instance, if E, $E_1, \dots, E_k$, F are vector bundles over the same base $\mathcal{P}$, we define:

- The Whitney sum $E_1 \oplus \cdots \oplus E_k$;
- The k-multilinear bundle, i.e., the vector bundle of k-multilinear maps $L(E_1, \dots, E_k; F)$;
- If $E_1 = \cdots = E_k = E$, we will write $L^k(E; F) = L(E_1, \dots, E_k; F)$, and we will also consider the symmetric k-multilinear bundle $L^k_s(E; F)$ and the antisymmetric k-multilinear bundle $L^k_a(E; F)$;
- The dual bundle E^*, and its subbundles of k-symmetric forms $\mathrm{Alt}^k(E) = L^k_s(E; \mathbb{C})$ (the k-symmetric bundle), and that of k-alternate forms $\mathrm{Alt}^k(E) = L^k_a(E; \mathbb{C})$ (the k-alternate bundle).

REMARK 1.2. Since we will be working on spectral theory, it is natural to consider only complex vector spaces. Study of the spectrum of real operators can be reduced to the complex case using the method of complexification.

At the functional analysis level, we recall that we can complexify a Banach space and its operator as follows.

Given $\tilde{X}$ a real Banach space and $L : \tilde{X} \to \tilde{X}$ a real linear operator, set $X = \tilde{X} \oplus \tilde{X}\mathbf{i}$ and set for $(a + b\mathbf{i}) \in \mathbb{C}$, $x_1 + x_2\mathbf{i} \in X$

$$(a + b\mathbf{i})(x_1 + x_2\mathbf{i}) = (ax_1 - bx_2) + (ax_2 + bx_1)\mathbf{i} \ . \tag{1.1}$$

The complexification of $\tilde{L}$ is given by $L(x_1 + x_2\mathbf{i}) = \tilde{L}x_1 + \tilde{L}x_2\mathbf{i}$.

This construction on Banach spaces, has a counterpart in the study of bundle automorphisms.

If $\tilde{E}$ is a real vector bundle, we complexify it just complexifying the fibers: $E_\theta = \tilde{E}_\theta + \tilde{E}_\theta\mathbf{i}$. We have a complex conjugation on the fibers: $\overline{v_\theta + w_\theta\mathbf{i}} = v_\theta - w_\theta\mathbf{i}$. If we have a Finsler metric on $\tilde{E}$ we define a Finsler metric on E by $|v_\theta + w_\theta\mathbf{i}| = \sqrt{|v_\theta|^2 + |w_\theta|^2}$.

If $\tilde{M}$ is a real vector bundle automorphism on a real vector bundle $\tilde{E}$ we can consider it also as acting on the complexification $E = \tilde{E} + \tilde{E}\mathbf{i}$: $M(\theta)(v_\theta + w_\theta\mathbf{i}) = M(\theta)v_\theta + \tilde{M}(\theta)w_\theta\mathbf{i}$.

When we consider spaces of sections into the complexified bundle, the complexified bundle automorphism induces the complexified operator.

As it is usual in linear algebra and functional analysis, when we speak about the spectrum of a real operator – which are often the ones which appear naturally in geometric operations – we will mean the spectrum of the complexification.

As we will point out in remarks, after one performs the functional analysis calculations, one can often check that the results in the complex spaces (e.g spectral projections) project down to real space. In doing so, it is useful to recall that in complexified real spaces, one can define a complex conjugation $^{-}$ by $\overline{x_1 + x_2\mathbf{i}} = x_1 - x_2\mathbf{i}$. This conjugation is not a linear operation. but

$$\overline{x+y} = \overline{x} + \overline{y} \quad \forall x, y \in \tilde{X}$$
$$\overline{z \cdot x} = \overline{z} \cdot \overline{y} \quad \forall z \in \mathbb{C}, x \in \tilde{X}$$

1.1.2. Vector bundle maps, transfer operators. We are interested in the invariant objects of vector bundle automorphisms in E, also called linear extensions [**AA88**]. Recall some definitions.

DEFINITION 1.3. *Following the notation above, a* vector bundle automorphism M *over a homeomorphism* $f : \mathcal{P} \to \mathcal{P}$ *is a homeomorphism* $M : E \to E$ *such that* $\Pi \circ M = f \circ \Pi$ *and for every* $\theta \in \mathcal{P}$ *the map* $M(\theta) : E_\theta \to E_{f(\theta)}$ *is linear and* $1-1$*. We will write* M_f *to denote a vector bundle automorphism* M *over* f*.*

DEFINITION 1.4. *Given a vector bundle automorphism* M_f *in* E*, an invariant subbundle is a subbundle* F *of* E *such that for all* $\theta \in \mathcal{P}$

$$M(\theta)F_\theta = F_{f(\theta)} \ .$$

We say that a (Whitney sum) splitting

$$E = E_1 \oplus E_2$$

is invariant if E_1 *and* E_2 *are invariant.*

DEFINITION 1.5. *To a vector bundle automorphism* M_f *we associate:*

- *A* transfer operator*. That is, the linear map* $\mathcal{M}_f : \Gamma(E) \to \Gamma(E)$ *defined by*

$$(\mathcal{M}_f v)(\theta) = M(f^{-1}(\theta))v(f^{-1}(\theta)) \ . \tag{1.2}$$

- *A* cocycle*. That is, the set of maps*

$$M(\theta, m) : E_\theta \to E_{f^m(\theta)} \ ,$$

where $\theta \in \mathcal{P}$ *and* $m \in \mathbb{Z}$ *defined by*

$$\begin{aligned} &M(\theta, 0) = \mathrm{Id} \ , \\ &M(\theta, m) = M(f^{m-1}(\theta)) \dots M(f(\theta))M(\theta) \quad \text{if } m > 0 \ , \\ &M(\theta, -m) = M(f^{-m}(\theta))^{-1} \dots M(f^{-2}(\theta))^{-1}M(f^{-1}(\theta))^{-1} \ \text{if } m > 0 \ . \end{aligned} \tag{1.3}$$

Notice that for a given section $v \in \Gamma(E)$, and for any $m \in \mathbb{Z}$ and $\theta \in \mathcal{P}$,

$$\mathcal{M}_f^m v(\theta) = M(f^{-m}(\theta), m) v(f^{-m}(\theta)) \ . \tag{1.4}$$

REMARK 1.6. In general, given two vector bundles $\pi_1 : E_1 \to \mathcal{P}_1$ and $\pi_2 : E_2 \to \mathcal{P}_2$, and a vector bundle map $M_f : E_1 \to E_2$ over $f : \mathcal{P}_1 \to \mathcal{P}_2$, we associate the transfer map $\mathcal{T}(M_f) = \mathcal{M}_f : \Gamma(E_1) \to \Gamma(E_2)$ defined by

$$\mathcal{M}_f v = M \circ f^{-1} v \circ f^{-1} \ .$$

This defines a covariant functor from the category of vector bundles to the category of vector spaces. That is to say, if $M_f : E \to F$ and $N_g : F \to G$ are vector bundle maps, then

$$\mathcal{T}(N_g \circ M_f) = \mathcal{T}(N_g) \circ \mathcal{T}(M_f) \ .$$

As Mather pointed out in [**Mat68**], there is a nice connection between the functional analysis of the transfer operator $\mathcal{M}_f$ (i.e., its spectrum), the geometry of the vector bundle automorphism (i.e., invariant subbundles) and the dynamics of the cocycle (rates of growth of orbits).

Two examples that we have in mind are linear skew product and push-forward operator.

A linear skew product is a vector bundle automorphism in the trivial bundle $\mathcal{P} \times \mathbb{C}^n$. It is given by a matrix valued map $M : \mathcal{P} \to \mathrm{GL}_n(\mathbb{C})$ and a base homeomorphism $f : \mathcal{P} \to \mathcal{P}$ by

$$\begin{aligned} \bar{x} &= M(\theta) x \ , \\ \bar{\theta} &= f(\theta) \ . \end{aligned} \tag{1.5}$$

Linear skew products appear naturally in Dynamics when considering the linearization of the Dynamics around an invariant manifold of a skew product system [**HdlLb, HdlL04, HdlL05a**].

Given a smooth and discrete dynamical system on a real n-dimensional manifold $F : \mathcal{N} \to \mathcal{N}$, and an invariant manifold P, the push-forward of F, $F_* = TF$ restricted to the tangent bundle of $\mathcal{N}$ in $\mathcal{P}$, $T_{\mathcal{P}}\mathcal{N}$, is a vector bundle automorphism in $T_{\mathcal{P}}\mathcal{N}$. Notice in this case we go through complex spaces by means of complexifying the bundle $T_{\mathcal{P}}\mathcal{N}$: $E = T_{\mathcal{P}}\mathcal{N} \oplus T_{\mathcal{P}}\mathcal{N}\mathbf{i}$. The push-forward appear naturally in dynamical systems as variational equations. Heuristically, it is useful to think of the vectors in the tangent space as *infinitesimal perturbations*. Hence, the push-forward heuristically describes how infinitesimal perturbations are propagated by the dynamical system. This, of course is a fundamental tool in the analysis of stability and instability properties.

As it is well known in differential geometry, one can define the push-forward in spaces of tensors, forms, etc. These are all vector bundles and fit into our general framework. The spectral properties of the push-forward in those spaces plays a role in dynamical systems. In particular, in Part 4, we will have some occasion of studying the spectrum acting on forms.

Note that the operator $\mathcal{M}_f$ in (1.2) can be considered as acting on different spaces. If the vector bundle E is C^r and the vector bundle automorphism

(f, M) is C^r, we can consider $\mathcal{M}_f$ as acting on C^l sections, with $l \leq r$. The spectral theory of these operators in different spaces has been considered in different places (e.g. [**LS90, LS91, CL99**]). Section 3.7 of this manuscript is devoted to study some related questions, and Part 3 makes a particular study of transfer operators over rotations.

We are interested also in spectral theory of transfer operators acting in other bundles. For instance, given k bundles over the manifold $\mathcal{P}$, $E_1, \dots, E_k$, and k vector bundle automorphisms over the homeomorphism f, $M_f^1 : E_1 \to E_1, \dots, M_f^k : E_k \to E_k$, we can consider the vector bundle automorphism on $L(E_1, \dots, E_k; F)$ over f^{-1} given by:

$$M(\theta)^{*k} v^{*k}_{f(\theta)}(v^1_\theta, \dots, v^k_\theta) = v^{*k}_\theta(M^1(\theta)v^1_\theta, \dots, M^k(\theta)v^k_\theta) \ ,$$

for any $\theta \in \mathcal{P}$, $v^{*k}_{f(\theta)} \in L(E^1_{f(\theta)}, \dots E^k_{f(\theta)}; F_{f(\theta)})$ and $v^1_\theta \in E^1_\theta, \dots, v^k_\theta \in E^k_\theta$.

Similar definitions give vector bundle automorphisms on the Whitney sum (using push forward) or the dual bundle (using, as before, pull back), etc.

1.2. Function spaces

1.2.1. Some standard functions spaces. We collect some standard definitions on spaces of (real or complex) functions defined in a compact manifold $\mathcal{P}$. We consider also a Borel measure μ on $\mathcal{P}$.

DEFINITION 1.7. *In the following, $\varphi : \mathcal{P} \to \mathbb{C}$ denotes a complex valued function.*

We denote by $B(\mathcal{P})$, the Banach space of bounded functions with the sup norm:

$$\|\varphi\|_B = \|\varphi\|_\infty = \sup_{\theta \in \mathcal{P}} |\varphi(\theta)| \ . \tag{1.6}$$

For $r \in \mathbb{N}$, we denote $C^r(\mathcal{P})$, the Banach space of r-times continuously differentiable functions equipped with the norm

$$\|\varphi\|_{C^r} = \max_{i \leq r} \sup_{\theta \in \mathcal{P}} |\mathrm{D}^i \varphi(\theta)| \ . \tag{1.7}$$

Recall that $|\mathrm{D}^i\varphi(\theta)|$ is computed according to the usual norm of i-multilinear operators. Notice that $C^0(\mathcal{P})$ is a closed subspace of $B(\mathcal{P})$.

For $p \in [1, \infty[$, we consider the space L^p-functions, (or, more properly equivalence classes of functions defined modulo equivalence of zero measure) $L^p(\mathcal{P}, \mu)$, with the norm given, for $p < \infty$ by

$$\|\varphi\|_{L^p} = \left(\int_P |\varphi|^p \, d\mu \right)^{\frac{1}{p}} \ . \tag{1.8}$$

For a L^∞-function φ there exists a finite constant K such that $|\varphi(\theta)| \leq K$ for $\mu-$a.e. $\theta \in \mathcal{P}$. The norm in $L^\infty(\mathcal{P}, \mu)$ is given by the essential supremum

$$\|\varphi\|_{L^\infty} = \inf\{K \in \mathbb{R} \mid |\varphi(\theta)| \leq K \text{ for } \mu - a.e.\ \theta \in \mathcal{P}\} \ . \tag{1.9}$$

When $r \in \mathbb{N}, p \in [1, \infty]$, *we say* $\varphi \in W^{r,p}(\mathcal{P}, \mu)$ *when* $\mathrm{D}^i\varphi \in L^p,\ i = 0, \dots, r$ *(we understand the derivative in the sense of distributions). At it is also well known, the Sobolev space* $W^{r,p}(\mathcal{P}, \mu)$ *is a Banach space under the norm*

$$\|\varphi\|_{W^{r,p}} = \max_{i \le r} \|\mathrm{D}^i\varphi\|_{L^p} \ . \tag{1.10}$$

An important characterization of Sobolev spaces is given by the fact that, for $r \in \mathbb{N} \setminus \{0\}$, $1 < p < \infty$,

$$\varphi \in W^{r,p} \ \Leftrightarrow\ (-\Delta + \mathrm{Id})^{r/2} \in L^p \ ,$$

where Δ denotes the Laplacian. This let us extend the definition of the Sobolev spaces to real indices r.

Definition 1.8. *For* $r \in \mathbb{R}$ *and* $p \in]1, \infty[$, *we define the Sobolev space*

$$W^{r,p} = \{\varphi : \mathcal{P} \to \mathbb{C} \mid (-\Delta + \mathrm{Id})^{r/2} \in L^p\} \ . \tag{1.11}$$

The norm

$$\|\varphi\|_{W^{r,p}} = \|(-\Delta + \mathrm{Id})^{r/2}\|_{L^p} \ , \tag{1.12}$$

makes $W^{r,p}$ *a Banach space.*

Remark 1.9. Definitions if $W^{r,p}$ in 1.7 and 1.8 are equivalent for $r \in \mathbb{N} \setminus \{0\}$ and $1 < p < \infty$. The equivalence of the two norms (1.10) and (1.12) is contained in Theorem 3, p. 303 of [**Ste70**]. The proof of this result uses the theory of Bessel potentials. It depends very heavily in the assumption – which we will be making in this chapter – that $p \in]1, \infty[$. In the cases, $p = 1, \infty$, the two norms mentioned above are not equivalent and the corresponding spaces are not equivalent. We refer to [**Ada75**] for variants of Sobolev spaces.

Lemma 1.10. *Assume that*

$$\frac{1}{p} - \frac{r}{d} < 0 \ . \tag{1.13}$$

Then,

a) *The space* $W^{r,p}$ *is a Banach algebra under multiplication. That is, for* $\varphi, \psi \in W^{r,p}$: $\|\varphi \cdot \psi\|_{W^{r,p}} \le K\|u\|_{W^{r,p}}\|v\|_{W^{r,p}}$.

b) *If* $s \ge r$ *and* $\frac{1}{q} - \frac{s}{d} \le \frac{1}{p} - \frac{r}{d}$, *then* $W^{s,q} \subset W^{r,p}$ *and the imbedding is continuous. Moreover, for* $\psi \in W^{s,q}$ *given, the multiplication operator* $\varphi \longrightarrow \psi\varphi$ *is continuous in* $W^{r,p}$.

c) *We have* $W^{r,p} \subset C^0$ *and the imbedding is continuous.*

Proof: Part a) of Lemma 1.10 is very standard. For integer k we just use Leibniz rule for the derivative of the product and the Sobolev imbedding $W^{r,p} \subset L^q$ with $1/q = (1/p - r/d)_+$ (allowing $q = \infty$ if $1/p - r/d < 0$), to control the factors. Then, the case for $r \in \mathbb{R}^+$ follows by interpolation. (See [**dlL01a**, p. 1143] or [**Tay97**, Proposition 3.7].

The proof of b) is again a direct application of the Sobolev imbedding.

Part c) of Lemma 1.10 is again part of Sobolev embedding Theorem. See e.g. [**Tay97**, Proposition 8.5 p. 39]. □

REMARK 1.11. We note that part a) of Lemma 1.10 is valid also for

$$\frac{1}{p} - \frac{r}{d} = 0$$

nevertheless, we will have few applications for this borderline case.

REMARK 1.12. Notice that in the definition of $B(\mathcal{P})$, contrary to that of $L^\infty(\mathcal{P}, \mu)$, we do not identify functions differing in a set of measure zero.
This difference will be extremely important in our applications.

1.2.2. Interpolation spaces. In this section, we discuss the technique of interpolation of operators. This is a standard technique in harmonic analysis, but does not seem to be so well known for dynamicists.

It has some advantages since it unifies some of the results for different spaces. One can often obtain estimates for intermediate spaces by obtaining estimates for spaces of very smooth function as and in spaces of very rough functions and then, interpolating. Since we are concerned with comparing spectrum in different spaces, the theory of interpolation will give us some convexity properties of spectral radius when considered as acting on different spaces. Since this section is not much used yet, it could perhaps be skipped in a first reading. Good surveys on interpolation spaces, which gives particular attention to Hölder spaces, which are our main concern is [**KP66, BL76, KPS82**]. We have also found useful the expositions in [**Tay96, Tay97, SW71**]. We refer to these references for proofs and for accounts of the primary literature. We will mainly follow [**Tay96**, p. 276 ff.].

DEFINITION 1.13. *Let E, F be Banach spaces such that $F \subset E$ and the inclusion is continuous. Let $\Omega = \{z \in \mathbb{C} \mid 0 < \Re z < 1\}$. Denote $\mathcal{G}H_{EF}$ the set of functions $\mu : \Omega \to E$ such that*

a) *μ is analytic in Ω,*
b) *μ is bounded and continuous in $\bar{\Omega}$*
c) *$\|\mu(1+iy)\|_F$ bounded uniformly for $y \in \mathbb{R}$.*

$\mathcal{G}H_{EF}$ is a Banach space endowed with the norm

$$\|\mu\|_{\mathcal{G}H_{EF}} = \max\left(\sup_{z\in\bar{\Omega}} \|\mu(z)\|_E, \sup_{y\in\mathbb{R}} \|\mu(1+iy)\|_F\right)$$

For $\theta \in [0, 1]$ we define the interpolation spaces

$$[E, F]_\theta = \{\mu(\theta) \mid \mu \in \mathcal{G}H_{EF}\}$$

Endowed with the norm

$$\|f\|_{[E,F]_\theta} = \inf_{\mu(\theta)=f} \|\mu\|_{\mathcal{G}H_{E,F}}$$

We note that $F \subset [E,F]_\theta \subset [E,F]_\theta \subset E$ for $0 \le \theta \le \theta' \le 1$.

The main result from abstract interpolation theory is the following result of [**Cal64**]. See also Proposition 2.1 in [**Tay96**, p.276] and [**SW71**, p.211].

THEOREM 1.14. *Let E, $\tilde{E}$, F, $\tilde{F}$ be Banach spaces, $F \subset E$, $\tilde{F} \subset \tilde{E}$ where the inclusions are continuous. Let A be a bounded, linear operator $A : E \to \tilde{E}$ such that $A(F) \subset \tilde{F}$ and $A : F \to \tilde{F}$ is also bounded. Then,*

a) *$A([E,F]_\theta) \subset [\tilde{E},\tilde{F}]_\theta$ and the mappings are continuous*
b) $\|A\|_{[E,F]_\theta \to [\tilde{E},\tilde{F}]_\theta} \le \|A\|^\theta_{E\to\tilde{E}} \|A\|^{1-\theta}_{F\to\tilde{E}}$

Of course the equalities for the norm in b) are only true when we use the interpolation norm.

Often, as it will be the case in the next section, one considers the interpolation spaces equipped with an equivalent norm, which sometimes arises more naturally.

In such cases, we do not have the inequality b). Nevertheless, we observe that in the case that $E = \tilde{E}$, $F = \tilde{F}$, (and, hence, $[E,F]_\theta = [\tilde{E},\tilde{F}]_\theta$) the spectral radius $\rho(A) = \lim_{n\to\infty} \|A^n\|^{1/n}$ is independent of the norm considered. Hence, from b) we have:

PROPOSITION 1.15. *Under the conditions above,*

$$\rho(A|_{[E,F]_\theta}) \le \rho(A|_E)^\theta \rho(A|_F)^{1-\theta} \tag{1.14}$$

The inequality (1.14) is clearly independent of the equivalent norm used.

As it turns out, many of the spaces that we have already considered are interpolation spaces.

THEOREM 1.16. *Let $r, s \in \mathbb{N} + (0,1)$, $r > s$. If $\theta \in [0,1]$ and $s\theta + (1-\theta)r \notin \mathbb{N}$, then the interpolation space $[C^s(\mathbb{R}^d), C^r(\mathbb{R}^d)]_\theta$ is isomorphic to $C^{s\theta+(1-\theta)r}$.*

That is, the functions in one space are in the other and the norms in both spaces are equivalent.

The result is Proposition 8.4 of [**Tay97**, p.39]. The proof is based on the Littlewood-Paley theory characterization of C^r spaces

$$\|f\|_{C^r} \approx \sup_k 2^{kr} \|\psi_k(D) f\|_{L^\infty} \tag{1.15}$$

where ψ_k is a Payley-Littlewood decomposition.

The equivalence of the norm in (1.7) and the interpolation norm is false when $r \in \mathbb{N}$ and this is why Theorem 1.16 includes hypothesis that the regularities involved are not integers.

Indeed, when When $s\theta + (1-\theta)r$ is an integer, the interpolation space is different from the C^r space. It is the space which in [**Ste70**] is called $\Lambda_{s\theta+(1-\theta)r}$.

It is rather straightforward to extend Theorem 1.16 to compact — hence finite dimensional — manifolds without boundary. It suffices to use partitions of unity.

One application that will be very important for us is the following

LEMMA 1.17. *Let M be a manifold, $f : M \to M$ a diffeomorphism. Let X be a Banach space. Let $A : M \to Ł(X, X)$. Consider the operator acting on functions $\sigma : M \to X$ defined by*

$$(\mathcal{T}\sigma)(x) = Af(x)\sigma f(x) \tag{1.16}$$

Assume that $f \in C^r(M, M)$, $A \in C^r(M, Ł(X, X))$, $r \geq 0$. Then, for $s \leq r$, $\mathcal{T}(C^s(M, X)) \subset C^s(M, X)$ and $\mathcal{T} : C^s(M, X) \to C^s(M, x)$ is linear.

Then, for $0 \leq s \leq r$ such that $s \notin \mathbb{N}$, we have $\rho(A|_{C^s})$ is log-convex as a function on s.

The log convexity for non-integer values of s is a direct consequence of Proposition 1.15 and Theorem 1.16.

We also recall the well known results

PROPOSITION 1.18. *The spaces L^p, $p \in (1, \infty)$ are interpolation spaces.*
For fixed p, the spaces $W^{r,p}$, $r \in (0, \infty)$ are interpolation spaces.
For fixed r, the spaces $W^{r,p}$, $p \in (1, \infty)$ are interpolation spaces.

1.3. Spaces of sections

We can also consider different norms of the sections of a Finsler vector bundle E over $\mathcal{P}$.

DEFINITION 1.19. *We will denote by $\Gamma_B(E)$ the space of bounded sections, $\Gamma_{C^0}(E)$ the space of continuous sections, and $\Gamma_{L^p}(E)$ the space of L^p sections. When assuming enough regularity on the objects, we also can consider the spaces of C^r sections, $\Gamma_{C^r}(E)$, and that of Sobolev sections, $\Gamma_{W^{r,p}}(E)$.*

All of these spaces are Banach spaces when topologyzing them with the corresponding norms above.

REMARK 1.20. Given a continuous vector bundle automorphism $M_f : E \to E$, we can consider the corresponding transfer operator $\mathcal{M}_f$ as acting on different spaces of sections (bounded, continuous and L^p).

We have found very useful to study first the case in which $\mathcal{M}_f$ acts on $\Gamma_B(E)$, for which

$$\|\mathcal{M}_f\|_{\Gamma_B(E)} = \sup_{\theta \in \mathcal{P}} |M(\theta)| \ .$$

REMARK 1.21. Notice that all the spaces of sections mentioned above for the complex vector bundle E can be stated for a real vector bundle $\tilde{E}$. Notice also that $\Gamma(\tilde{E} + \tilde{E}\mathbf{i}) = \Gamma(\tilde{E}) + \Gamma(\tilde{E})\mathbf{i}$, where Γ denotes Γ_B, Γ_{C^r}, etc. This means that a space of complex sections is the complexification of the corresponding space of real sections.

1.4. Local Trivializations

In the functional analysis of a transfer operator $\mathcal{M}_f$ we can use any norm on say $\Gamma_B(E)$ equivalent to that induced by the Finsler metric in E. The following constructions are useful when we need to do computations and estimates.

Recall we have defined a norm $|\cdot|_\theta$ on each fiber E_θ of the bundle E, that depends continuously on $\theta \in \mathcal{P}$. Since $\mathcal{P}$ is compact, the topologies in $\Gamma_B(E)$, $\Gamma_{C^0}(E)$, etc. do not depend on the continuous Finsler metric we have chosen. If the bundle is C^r the Finsler metric is supposed to be C^r.

Moreover, since $\mathcal{P}$ is compact we can cover it by a finite number of trivializing neighborhoods $\{U_i\}_{i=1,\dots,p}$. That is, for each $i = 1, \dots, p$, $U_i \subset \mathcal{P}$ is open and there exists a homeomorphism (C^r diffeomorphism if the objects involved are C^r)

$$\varphi_i : \Pi^{-1}(U_i) \longrightarrow U_i \times \mathbb{C}^n \ ,$$

such that for each $\theta \in U_i$

$$\varphi_{i,\theta} : E_\theta \longrightarrow \{\theta\} \times \mathbb{C}^n$$

is an isomorphism. Notice that, say, the sup-norm in $\{\theta\} \times \mathbb{C}^n \simeq \mathbb{C}^n$ is equivalent to the norm in E_θ. Using again the compactness of $\mathcal{P}$, we can choose the neighborhoods $\{U_i\}_{i=1,\dots,p}$ in such a way that there exists positive constants $0 < c < C$ such that for all $\theta \in P$, if $\theta \in U_i$ then for all $v_\theta \in E_\theta$,

$$c|v_\theta|_\theta \le |\varphi_i(v_\theta)|_\infty \le C|v_\theta|_\theta \ . \tag{1.17}$$

Let $R > 0$ a Lebesgue radius of the finite covering $\{U_i\}_{i=1,\dots,p}$, that is to say, for each $\theta \in \mathcal{P}$ there exists a chart U_i such that $\bar{B}(\theta, R) \subset U_i$. We define a new norm on E by

$$|v_\theta|_\infty = |\varphi_\nu v_\theta|_\infty \ \text{ where } \nu = \nu(\theta) = \min\{i = 1, \dots, p \mid \bar{B}(\theta, R) \subset U_i\} \ .$$

We point out that this Finsler metric on E does not depend continuously on θ, but the norms induced in the spaces of sections are equivalent to the corresponding norms induced by the original continuous Finsler. We can then represent a vector bundle automorphism M_f by $n \times n$ matrices.

Another device to produce trivializations is the embedding of the Finslered vector bundle E in a trivial one (see, for instance [**HPPS70**]).

Lemma 1.22. *Any C^r Finslered vector bundle E of rank n, can be isometrically C^r embedded in a trivial Finslered C^r vector bundle $\tilde{E} \simeq \mathcal{P} \times \mathbb{C}^{n+k}$, where k is large enough.*

To control the distance between vectors of E set in different (but close) points, we can use the previous trivializations. We can also construct of a *connector* [**HPPS70**]. It is like a 'local' parallel transport.

Definition 1.23. *A* connector T *in a vector bundle E is a continuous (we will often assume that they are more differentiable) family of isomorphisms $\mathrm{T}_{\theta,\theta'} : E_\theta \to E_{\theta'}$ defined in neighborhood of the diagonal Δ in $\mathcal{P} \times \mathcal{P}$ such that $\mathrm{T}_{\theta,\theta} = \mathrm{Id}_{E_\theta}$.*

LEMMA 1.24. *Every vector bundle E admits a connector. If the bundle is C^r, the connector is C^r.*

Proof: For the sake of completeness, we prove the lemma. Lemma 1.22 asserts that the vector bundle E of rank n has a complementary vector bundle F of large enough rank k, such that $E \oplus F \simeq \mathcal{P} \times \mathbb{C}^{n+k}$. Let $i_E : E \to E \oplus F$ and $p_E : E \oplus F \to E$ the corresponding inclusion and projection maps, respectively.

Consider the diagram:

$$\begin{array}{ccccccc} E \times \mathcal{P} & \stackrel{i_E \times id_{\mathcal{P}}}{\longrightarrow} & \mathcal{P} \times \mathbb{C}^{n+k} \times \mathcal{P} & \longrightarrow & \mathcal{P} \times \mathbb{C}^{n+k} & \stackrel{p_E}{\longrightarrow} & E \\ (v_\theta, \theta') & \longrightarrow & (\theta, \alpha, \theta') & \longrightarrow & (\theta', \alpha) & \longrightarrow & v_{\theta'} \end{array}$$

Notice that the maps $T_{\theta,\theta'} : E_\theta \to E_{\theta'}$ induced by the previous diagram are lineal, and depend continuously on θ, θ'. Obviously, $T_{\theta,\theta'} = \mathrm{Id}_{E_\theta}$, and this also implies that there exist a neighborhood of the diagonal in $\mathcal{P} \times \mathcal{P}$ for which the connectors are isomorphisms.

If the objects are C^r, the whole construction is C^r. □

REMARK 1.25. When the bundle is the tangent bundle and there is a Riemannian metric, a more geometric way of constructing connectors is to observe that of we fix a point, for all the points in a small neighborhood of it, there is a unique shortest geodesic (Hopf-Rinow theorem) connecting it to the fixed point. We can define a connector from the fixed point to the points in the neighborhood by transporting along this shortest geodesic using the Levi-Civita connection.

1.5. Recurrence in dynamical systems

In the analysis of the dynamics of a vector bundle automorphism, the motion on the base manifold $\mathcal{P}$ and, specially its recurrence properties plays an important role. In this introductory subsection we will state the main definitions that we will use in this manuscript. This should be considered as a reference section.

In the following, $f : \mathcal{P} \to \mathcal{P}$ will be a homeomorphism in a compact metric space $\mathcal{P}$ (although in most cases it will be a compact manifold). It defines a discrete dynamical system.

Given $\theta \in \mathcal{P}$, we say that θ is a *periodic point* iff there exists $N > 0$ such that $f^N(\theta) = \theta$, otherwise we will say that θ is an *aperiodic point.* The *minimal period function* is $p : \mathcal{P} \to \mathbb{N}^* \cup \{\infty\}$, defined by

$$p(\theta) = \begin{cases} \infty \text{ if } \theta \text{ is aperiodic,} \\ \min\{N \in \mathbb{N}^* \mid f^N(\theta) = \theta\} \text{ if } \theta \text{ is periodic.} \end{cases}$$

DEFINITION 1.26. *We will say that f is NPO (from No Periodic Orbits) iff f does not have periodic points. We will say that f is APD (from APeriodic orbits Dense) iff the set of aperiodic point is dense. We will also say that f is PD (from Periodic orbits Dense) iff the set of periodic orbits in dense.*

Of course, a map may be simultaneously APD and PD. Notice also that, using index theory, the NPO condition imposes strong restrictions on the topology of $\mathcal{P}$. One example of NPO map is an irrational rotation on the torus, the case of study in Part 3. A characterization of the APD property is given by the following.

PROPOSITION 1.27. *f is APD $\Leftrightarrow$ for all open set $U \subset \mathcal{P}$ the restriction $p_{|U}$ is not bounded.*

Proof: The implication $\Rightarrow$ is obvious. Suppose that for all open set $U \subset \mathcal{P}$ the restriction $p_{|U}$ is not bounded. For each $N > 0$, let P_N be the set of periodic points whose period is less or equal than N. Since P_N is closed, the set $A_N = \mathcal{P} \setminus P_N$ is open. Notice also that A_N is dense in $\mathcal{P}$, otherwise it would exist a non empty open set $U \subset \mathcal{P}$ such that $A_N \cap U = \emptyset$, that is $U \subset P_N$ and so $p_{|U} \le N$, in contradiction with the hypotheses. Notice that the set of aperiodic points is

$$A = \bigcap_{N=1}^{\infty} A_N \ .$$

Since A is a countable intersection of dense open sets, it is dense in $\mathcal{P}$ (by Baire Category Theorem). □

REMARK 1.28. Note that the negation of the hypothesis that aperiodic points are dense implies that for some N the set of periodic points of period N has non empty interior. Therefore, we can find an open set such that $f^N(\theta) = \theta$.

We will also use the following standard definitions relative to properties of recurrence.

DEFINITION 1.29. *We will say that f is* topologically transitive *iff there exists one orbit $\{f^k(\theta) \mid k \in \mathbb{Z}\}$ that is dense. If $\mathcal{P}$ is separable, this property is equivalent to for all non empty open sets $U, V \subset \mathcal{P}$ there exists $N > 0$ such that $f^N(U) \cap V \neq \emptyset$.*

We will say that f is minimal *iff all the orbits $\{f^k(\theta) \mid k \in \mathbb{Z}\}$ are dense, that is f has no proper closed invariant sets.*

A point $\theta \in \mathcal{P}$ is nonwandering *iff for all open set $U \subset \mathcal{P}$ with $\theta \in \mathcal{P}$, there exists $N > 0$ such that $f^N(U) \cap U \neq \emptyset$. The set of nonwandering points is the nonwandering set, denoted by $\Omega(f)$ (also $NW(f)$), and it is a non empty closed invariant set. We will say that f is* regionally recurrent *iff all the orbits are nonwandering, i.e. $\mathcal{P} = \Omega(f)$.*

A point $\theta \in \mathcal{P}$ is chain recurrent *iff for all $N > 0$ and $\varepsilon > 0$ there exist a chain $\theta = \theta_0, \theta_1, \dots \theta_k = \theta \in \mathcal{P}$ with $k > 0$ and $n_0, \dots, n_{k-1} \geq N$ such that*

$$d(f^{n_i}(\theta_i), \theta_{i+1}) < \varepsilon \ ,$$

for $i = 0, \dots, k-1$. The set of chain-recurrent points is the chain-recurrent set, denoted by $\mathcal{R}(f)$. We will say that f is chain-recurrent *iff all the orbits are chain-recurrent, i.e. $\mathcal{P} = \mathcal{R}(f)$.*

Notice that chain recurrence is a very general concept of "repetitiveness". For instance, PD property implies chain recurrence, and also topological transitivity implies such property. A minimal system is both chain-recurrent and NPO. Since the set $\Omega(f)$ is a subset of $\mathcal{R}(f)$, and the inclusion is in general proper, regionally-recurrent dynamical systems are also chain-recurrent. In the words of [**AA88**], chain recurrence does not mean that a motion "repeats", rather that the dynamical system "does not interfere with" the repetition of "approximate motions".

Note that Poincaré recurrence theorem implies, in particular that if the system preserves a probability measure which is positive on open sets, then, it is chain recurrent.

Since $\mathcal{P}$ is connected, there are different properties that are equivalent to chain recurrence, such that

- Non-existence of proper stable attractors;
- Weak incompressibility, that is for all non empty open set $U \subset \mathcal{P}$ and for all $m > 0$, $f^m(\overline{U}) \not\subseteq U$.

We also recall the deep theorem of [**Con88**] that any dynamical system admits a Lyapunov function outside of its chain recurrent set, so that in some sense, chain recurrence is the mildest notion of recurrence.

Another important notion that will play a role in the theory of characterization of the spectrum by orbits is *specification.*

Definition 1.30. *We say that a dynamical system f on a manifold M satisfies the specification property if for all $\varepsilon > 0$, there exists a $K \in \mathbb{N}$ such that: Given a collection of orbit segments $\{f^i(x_j)\}_{i=a_j^-}^{a_j^+}$, $j = 1, \dots, J$ satisfying that $a_j^+ + K < a_{j+1}^-$, $j = 1, \dots J-1$*
i) there exists an orbit $\{f^i(y)\}$ such that

$$d(f^i(y), f^i(x_j)) < \varepsilon \quad \text{where} \quad i \in \bigcup_{j=1,\dots,J} [a_j^-, a_j^+]$$

ii For any $T > K + \sum_{j=1}^N a_j^+ - a_j^-$ there exists a T-periodic orbit p such that 1.30 holds.

A sequence of orbit segments is usually called a specification, the property $a_i^+ + K \leq a_{i+1}^-$ is usually expressed by saying the specification is K-separated and the property $d(f^i(y)f^i(x)) \leq \varepsilon$ for i in the indices is expressed by saying that the orbit is ε-shadowed.

Hence, 1.30 is described as saying that, for the system f, sufficiently separated specifications can be ε-shadowed.

Remark 1.31. We note that, in the literature, sometimes the specification property is stated taking $J = \infty$. We note that, for compact manifolds, as we are considering in this manuscript, this follows from the finite J version we stated. If y^J is the sequence of points produced by considering just J orbit segments we can find a convergent subsequence. Its limit will satisfy 1.30 for all points.

REMARK 1.32. We also note that if the map is expansive the points satisfying 1.30 for an infinite sequence of intervals are unique. This shows that for expanding maps i) implies ii). (We consider an infinite sequence defined by $a_{i+J}^{+/-} = a_i^{+/-} + T \ x_{i+j} = x_i$. The y for this orbit has to be periodic.)

REMARK 1.33. We note that the specification property implies that periodic points are dense (we can take in the hypothesis to be any point that we want, the periodic orbit produced is, in particular arbitrarily close to x_1).

REMARK 1.34. It is a deep theorem [**Bow75, KH95**] that a transitive Axiom A system (in particular, transitive Anosov system) satisfies the specification property. Nevertheless, there are other systems that are not Anosov that satisfy specification.

We also have the elementary

PROPOSITION 1.35. *A system satisfying specification also satisfies APD.*

Proof: If APD fails we can find $n \in \mathbb{N}$ and an open set U such that $f^n(x) = a$ for all $x \in U$. If $\mathrm{Fix}(f^n)$ had empty interior for all n, since it is a closed set, by Baire category theorem $\bigcap_{n=0}^{\infty} M - \ \mathrm{Fix}(f^n)$ would be dense, hence APD would hold.

The existence of this open set violates the specification property. If $x \in U$, any orbit ε-close to it (ε sufficiently small) has to be periodic of period n. In particular, it cannot ε-approximate any orbit that goes through x_2 where x_2 is more than 2ε away from its orbit. □

Since a system satisfying specification is also APD, the orbit produced in i) can be assume to be aperiodic.

REMARK 1.36. We refer to [**Bow75, KH95**], for more details about the specification property. There are generalizations to continuous maps on metric spaces and for actions of $\mathbb{Z}^d$ instead of just dynamical systems. Large parts of the ergodic theory of Anosov systems and of thermodynamics can be generalized to systems satisfying specification and expansiveness [**Rue73, Bow75, DGS76, Rue92b**].

To study a dynamical system from a statistical point of view, one uses invariant measures. Given a continuous map $f : \mathcal{P} \to \mathcal{P}$, an invariant measure μ is a probability Borel measure such that $\mu(f^{-1}(A)) = \mu(A)$, for all Borel set $A \subset \mathcal{P}$. As it is well known, the statistical properties translate into recurrence properties on the support of the measure. Recall that this is

$$\mathrm{supp}\,\mu = \{\theta \in \mathcal{P} \mid \forall U \subset \mathcal{P}, \text{ open }, \theta \in U \Rightarrow \mu(U) > 0\}$$

and it is an invariant closed set of f. For instance, suppose that f preserves a topological measure μ (i.e. non empty open sets have positive measure). Then:

- f is chain-recurrent;
- if μ is ergodic (for all invariant Borel set A, $\mu(A) = 0$ or $\mu(\mathcal{P} \setminus A) = 0$), then f is topologically transitive;
- if μ is the only invariant measure (that is f is uniquely ergodic), then f is minimal.

(For these properties, one just needs that $\mathcal{P}$ is a complete separable metric space. For the last one, we need also $\mathcal{P}$ compact. Notice that the preservation of a topological measure in the compact case also implies that f is regionally-recurrent).

Notice that if f is uniquely ergodic then it is APD. In fact, f can not have more than one periodic orbit. Indeed, it the map had two different periodic orbits, we could define invariant measures supported in different periodic orbits. Moreover, if the unique invariant measure is topological, then f is also minimal (and NPO). If the uniquely ergodic map f has a periodic orbit, the phase space has to be just the periodic orbit.

These definitions and well known facts will be important along the manuscript.

In Part 3 we will consider $\mathcal{P} = \mathbb{T}^d$ and the motion f will be a rotation $\omega \in \mathbb{R}^d$: $f(\theta) = \theta + \omega$. Notice that rotations are chain-recurrent. Rational rotations ($\omega \in \mathbb{Q}^d$) are PD (all the orbits are periodic), while irrational rotations ($\omega \notin \mathbb{Q}^d$) are NPO (all the orbits are aperiodic). Notice also that ergodic or non-resonant rotations ($k \cdot \omega \notin \mathbb{Z}$ for all $k \in \mathbb{Z}^d \setminus \{0\}$) are minimal.

An important example of these situation is Anosov diffeomorphisms. See the definition in We recall that it is well known that Anosov diffeomorphisms satisfy APD. For an Anosov diffeomorphism, PD is equivalent to topological transitivity, hence, we will use often the name transitive Anosov diffeomorphisms instead of PD Anosov diffeomorphisms even in sections when we are discussion PD systems.

1.6. Spectral theory

In this section we recall some standard definitions and results in general spectral theory. This review is completed in Appendix A.

In the following, $L : X \to X$ will denote a bounded linear operator in a complex Banach space X.

DEFINITION 1.37. *We say that $z \in \mathbb{C}$ is in the* resolvent set *when $(L-z)$ is bijective. In such a case, the Banach isomorphism theorem implies that $(L-z)^{-1}$ is bounded. The complement of the resolvent set is the* spectrum *. We will denote the spectrum of the operator L acting on the space X by* $\mathrm{Spec}(L, X)$*, and the resolvent set by* $\mathrm{Res}(L, X)$*.*

An important subset of the spectrum is the *Weyl spectrum* of approximate eigenvalues. Sometimes it is also called the *Approximate point spectrum.*

DEFINITION 1.38. *An approximate eigenvalue is a complex number z such that*

$$z \in \mathrm{Spec}_W(L, X) \Leftrightarrow \exists \ \{v_n\}_{n=0}^{\infty} \subset X \ \Big| \ \|v_n\| = 1 \ , \ \|(L-z)v_n\| \to 0 \tag{1.18}$$

The v_n's are usually called approximate eigenvectors *of L for z. The set of approximate eigenvalues is known as* approximate point spectrum *or* Weyl spectrum *and we will denote it by* $\mathrm{Spec}_W(L, X)$*.*

Obviously, the Weyl spectrum contains the point spectrum*, denoted by* $\mathrm{Spec}_P(X, L)$*, that is the set of eigenvalues.*

Note that the definition of Weyl spectrum makes it obvious that if Y is a closed invariant subspace of X, i.e. $LY \subset Y$, then the approximate eigenvectors in Y can be considered as approximate eigenvectors in X and therefore,

$$\mathrm{Spec}_W(L, Y) \subset \mathrm{Spec}_W(L, X) \tag{1.19}$$

On the other hand, we emphasize that there is no general such inclusion for the full spectrum and indeed, we will find in Part 4 examples in which the spectrum in a proper subspace is strictly greater.

Since in many situations, the spectrum agrees with the Weyl spectrum (e.g. finite dimensions, self-adjoint and many other operators in applied mathematics) (1.19) implies the corresponding result for the spectrum, but this is not true in general. The characterization of the spectrum of (even unbounded) self-adjoint operators by (1.18) is called the *Weyl criterion*, whose proof consists in showing that self-adjoint operators (possibly unbounded) do not have residual spectrum. The non-residual spectrum is included in the Weyl spectrum (see Appendix A). Hence, the Weyl criterion asserts that for normal operators the Weyl spectrum is the whole spectrum. Unfortunately, transfer operators may fail to be normal. Indeed, specially in Part 4, we will find transfer operators in spaces for which there is residual spectrum.

From the point of view of numerical analysis, the Weyl spectrum is the most immediate to compute. Typically, in numerical analysis, one produces a sequence of discretizations of the problem, which are, in turn, approximately diagonalized or some approximate eigenvectors are produced. (e.g using Krylov, Arnoldi, Lanczos, etc. methods). In the customary idealization of numerical analysis, the sequence of discretizations is taken to be infinite even if in practice only a finite number of them are run. This procedure will locate the Weyl spectrum but will fail to locate its complementary. At the moment, it does not seem clear to us how to go about computing the non-Weyl spectrum through discretizations. The presence of this spectrum in the dynamo problems may well be one of the reasons why the survey [**Bay92**] describes the problem as “horrible”.

We call attention to the fact the boundary of the spectrum is always Weyl spectrum (see Proposition A.26). Hence, the boundary of the spectrum – in particular the spectral radius – can always be computed using approximate eigenvalues as in (1.18). This fact has the amusing corollary that if the Weyl spectrum is rotationally invariant then the full spectrum is also rotationally invariant (see Corollary A.27).

We also recall that when X is complex Banach space given an isolated part of the spectrum, That is, given $\Sigma \subset \mathrm{Spec}(L, X)$, that satisfies $\mathrm{dist}(\Sigma, \Sigma^c) > 0$ – where we denote $\Sigma^c = \mathrm{Spec}(L, X) \setminus \Sigma)$ – we can find a decomposition $X = X_\Sigma \oplus X_{\Sigma^c}$ invariant under L and

$$\mathrm{Spec}(L, X_\Sigma) = \Sigma$$
$$\mathrm{Spec}(L, X_{\Sigma^c}) = \Sigma^c \ .$$

We denote by P_Σ the projection over X_Σ and we note that $LP_\Sigma = P_\Sigma L$, hence, $LX_\Sigma \subset X_\Sigma$, and similarly for Σ^c.

When we apply these results to the complexification X of a real Banach space $\tilde{X}$ and L the complexification of $\tilde{L}$, if $\Sigma^* = \Sigma$ (where * denotes the complex conjugate of a complex number: $(a+b\mathbf{i})^* = a-b\mathbf{i}$), it can be shown that X_Σ, X_{Σ^c} are the complexification of real subspaces $\tilde{X}_\Sigma$, $\tilde{X}_{\Sigma^c}$ which give a direct decomposition of $\tilde{X}$ which is invariant under $\tilde{L}$. Therefore, provided that we consider spectral subsets that are invariant under complex conjugation, the complex constructions that we perform have a real counterpart. In the applications we will use, the spectral subsets will be annuli or complement of annuli centered at the origin, which certainly satisfy that.

We also recall that the spectral radius formula states that $\mathrm{Spec}(L, X) \subset \{z \in \mathbb{C} \mid |z| \le \rho\}$ is equivalent to

$$\forall\, \varepsilon > 0\ \exists\, C_\varepsilon \mid \forall\, m \ge 0\ \|L^m\| \le C_\varepsilon(\rho+\varepsilon)^m \ . \tag{1.20}$$

By the uniform boundedness principle this is equivalent to

$$\forall\, v \in X\ ,\ \forall\, \varepsilon > 0\ \exists\, C_{\varepsilon,v} \mid \forall\, m \ge 0\ \|L^m v\| \le C_{\varepsilon,v}(\rho+\varepsilon)^m \ . \tag{1.21}$$

This result is somewhat surprising since the spectrum does not depend on any norm – just the topology enters –. This suggests that there could be norms which are particularly useful when considering the spectral properties of an operator. These are called the *adapted norms*.

Given an operator L with spectral radius ρ, we define

$$\|v\|_\varepsilon = \sum_{m=0} \|L^m v\|(\rho+\varepsilon)^{-m} \tag{1.22}$$

In this adapted norm, we have easily $\|L^m v\|_\varepsilon \le (\rho+\varepsilon)^m \|v\|_\varepsilon$. That is, we can take the constants in (1.20) and (1.21) equal to 1. This becomes particularly useful when we consider bundles since, in this metric, certain estimates become uniform. In the case of applications to bundles, this will lead later to the adapted metric.

Since we are going to be interested in rotational properties of the spectrum, we introduce the following notation.

Definition 1.39. *Given a set* $\Sigma \subset \mathbb{C}$*, we define the* annular hull *of* Σ*,* $\mathcal{A}\Sigma$*, as the union set of the circles that intersect* Σ*:*

$$\mathcal{A}\Sigma = \{e^{\alpha \mathbf{i}} z \mid z \in \Sigma, \alpha \in \mathbb{R}\} \ .$$

Given $N \in \mathbb{N}$*, we define also the set*

$$\mathcal{A}_N\Sigma = \{e^{2\pi \frac{k}{N} \mathbf{i}} z \mid z \in \Sigma, k = 0, \cdots, N-1\} \ .$$

We will denote

$$\mathcal{A}_{\lambda,\mu} = \mathcal{A}[\lambda, \mu] = \{z \in \mathbb{C} \mid \lambda \leq |z| \leq \mu\}$$

as the annulus of radii $0 < \lambda \leq \mu$*. For* $\rho > 0$*, we will write*

$$\mathcal{S}_\rho = \{z \in \mathbb{C} | |z| = \rho\},$$

the circle of radius ρ*.*

CHAPTER 2

Dichotomies, invariant splittings and spectral gaps

In this section we review some known results on transfer operators and asymptotic properties of cocycles. The main ideas of the results we present go back to [**Mat68**]. Some further developments and variants can also be found in [**HPS77, SS74, SS76a, SS76b, Sac78, SS78, Mn78, CS80, Swa81, LS90, LS91, CL99**].

We will start by reviewing the theory of Sacker and Sell [**SS74, SS76a, SS76b, Sac78**], on dichotomies and quasi-dichotomies (or weak-dichotomies). We then relate both dynamical properties with the properties of the transfer operator associated to the cocycle. So, the relation of quasi-dichotomy and point spectrum is summarized in the Mañé lemma [**Mn78**] (see Section 4.2 for another version). Then we will review a result by [**Mat68**] relating the existence of an invariant splitting with the existence of a gap in the full spectrum of the transfer operator. This is the starting point of the Mather spectrum theory [**Mat68**], which will be developed in Section 3. We mention that the study of the regularity of the invariant splitting follows from the celebrated invariant section theorem [**HP70, HPS77**].

Notice that these transfer operators can be considered as acting on different function spaces of sections, but the first general result corresponds to $\Gamma_B(E)$, the space of bounded sections with the sup-norm. The continuous case is a corollary.

2.1. Dichotomies and invariant splittings, Sacker-Sell theory

In a series of papers [**SS74, SS76a, SS76b, Sac78**], Sacker and Sell studied the existence of invariant splittings of vector bundle maps (and flows) under some standing hypotheses on the dynamics of the map. In particular, the study the existence of the stable and unstable invariant bundles under the assumption of non existence of non-trivial bounded orbits. This section is a brief review and simplified version of a part of their work that we will use later.

Given a vector bundle automorphism $M_f : E \to E$ over $f : \mathcal{P} \to \mathcal{P}$, we consider the subsets

$$B^+ = \{v_\theta \in E \mid \sup_{m\geq 0} |M(\theta, m)v_\theta| < \infty\} \,, \tag{2.1}$$

$$B^- = \{v_\theta \in E \mid \sup_{m\leq 0} |M(\theta, m)v_\theta| < \infty\} \tag{2.2}$$

and

$$B = B^+ \cap B^- \ . \tag{2.3}$$

These are the sets of positively bounded orbits, negatively bounded orbits and bounded orbits, respectively. Other related sets are the stable and unstable sets, defined by:

$$E^s = \{v_\theta \in E \mid \lim_{m\to+\infty} |M(\theta,m)v_\theta| = 0\} \ , \tag{2.4}$$

$$E^u = \{v_\theta \in E \mid \lim_{m\to-\infty} |M(\theta,m)v_\theta| = 0\} \tag{2.5}$$

If we want to make explicit the dependence on the cocycle, we will write $B^+(M_f), B^-(M_f), etc.$

These subsets are invariant linear subspaces, but from the definitions we do not know if they are continuous, and not even if they have constant rank (that is, we do not know if they are vector subbundles). This is what the papers [**SS74, SS76a, SS76b**] studied, under the standing hypothesis $B = E_0$.

To bound the rates of growth of the orbits we will use the following definitions, which generalize the previous ones. Later on, we will relate these dynamical properties with spectral properties of transfer operators.

DEFINITION 2.1. *Given a positive constant ρ, for each $\theta \in \mathcal{P}$, we define the* growth spaces

$$W^{\le\rho} = \{v_\theta \in E \mid \exists\, C^{\le\rho}_{v_\theta} > 0 : \ \forall m \ge 0\ |M(\theta,m)v_\theta| \le C^{\le\rho}_{v_\theta}\rho^m\} \ , \tag{2.6}$$

$$W^{\ge\rho} = \{v_\theta \in E \mid \exists\, C^{\ge\rho}_{v_\theta} > 0 : \ \forall m \le 0\ |M(\theta,m)v_\theta| \le C^{\ge\rho}_{v_\theta}\rho^m\} \tag{2.7}$$

and

$$W^\rho = W^{\le\rho} \cap W^{\ge\rho} \ . \tag{2.8}$$

We also define:

$$W^{<\rho} = \{v_\theta \in E \mid \lim_{m\to+\infty} |\rho^{-m} M(\theta,m)v_\theta| = 0\} \ , \tag{2.9}$$

$$W^{>\rho} = \{v_\theta \in E \mid \lim_{m\to-\infty} |\rho^{-m} M(\theta,m)v_\theta| = 0\} \ . \tag{2.10}$$

REMARK 2.2. Using the uniform boundedness principle, it follows that the constants $C^{\le\rho}_{v_\theta}$ and $C^{\ge\rho}_{v_\theta}$ in (2.6) and (2.7) are of the form $C^{\le\rho}_{v_\theta} = C^{\le\rho}_\theta |v_\theta|$ and $C^{\ge\rho}_{v_\theta} = C^{\ge\rho}_\theta |v_\theta|$, respectively.

REMARK 2.3. Notice that

$$W^{\le\rho}_\theta(M_f) = B^+\left(\frac{1}{\rho}M_f\right) \quad , \quad W^{\ge\rho}_\theta(M_f) = B^-\left(\frac{1}{\rho}M_f\right) \ ,$$
$$W^{<\rho}_\theta(M_f) = E^s\left(\frac{1}{\rho}M_f\right) \quad , \quad W^{>\rho}_\theta(M_f) = E^u\left(\frac{1}{\rho}M_f\right) \ .$$

So then, some results for W-spaces follow from results on B-spaces.

We summarize the results in the following theorem. The standing hypothesis $B = E_0$ in [**SS74, SS76a, SS76b**] is trivially replaced by $W^\rho = E_0$.

THEOREM 2.4. *Let $M_f : E \to E$ be a vector bundle automorphism. Given $\rho > 0$, assume $W^\rho = E_0$.*
Then:

(a) $W^{<\rho} = W^{\leq\rho}$ *and* $W^{>\rho} = W^{\geq\rho}$.
(b) *There exists a constant $K \geq 1$ such that the sets $W^{\leq\rho}$ and $W^{\geq\rho}$ are characterized by:*

$$v_\theta \in W^{\leq\rho} \quad \Leftrightarrow \quad \forall m \geq 0 \ |M(\theta, m)v_\theta| \leq K\rho^m |v_\theta| \tag{2.11}$$

$$v_\theta \in W^{\geq\rho} \quad \Leftrightarrow \quad \forall m \leq 0 \ |M(\theta, m)v_\theta| \leq K\rho^m |v_\theta| \ . \tag{2.12}$$

To make explicit the uniform bound, we will write $W^{\leq\rho} = W^{\leq\rho,K}$, $W^{\geq\rho} = W^{\geq\rho,K}$.
(c) $W^{\leq\rho}$, $W^{\geq\rho}$ *are closed subsets of E.*
(d) *The functions $\theta \to \dim W_\theta^{\leq\rho}$ and $\theta \to \dim W_\theta^{\geq\rho}$ are upper semicontinuous. That is, for all $\alpha \in \mathbb{R}$ the sets $\{\theta \in \mathcal{P} \mid \dim W^{\leq\rho} < \alpha\}$ and $\{\theta \in \mathcal{P} \mid \dim W_\theta^{\geq\rho} < \alpha\}$ are open.*
(e) *There exists $C > 0$ and $\lambda < \rho < \mu$ such that the linear spaces $W^{\leq\rho}$ and $W^{\geq\rho}$ are characterized by the following growth rates:*

$$v_\theta \in W^{\leq\rho} \quad \Leftrightarrow \quad \forall m \geq 0 \ , |M(\theta, m)v_\theta| \leq C\lambda^m |v_\theta| \tag{2.13}$$

$$v_\theta \in W^{\geq\rho} \quad \Leftrightarrow \quad \forall m \leq 0 \ , |M(\theta, m)v_\theta| \leq C\mu^m |v_\theta| \ . \tag{2.14}$$

So $W^{\leq\lambda} = W^{\leq\lambda,C} = W^{<\rho} = W^{\leq\rho}$, $W^{\geq\mu} = W^{\geq\mu,C} = W^{>\rho} = W^{\geq\rho}$.
(f) *If for all $\theta \in \mathcal{P}$ $E_\theta = W_\theta^{\leq\rho} \oplus W_\theta^{\geq\rho}$, then $E = W^{\leq\rho} \oplus W^{\geq\rho}$ as a Whitney sum of two subbundles.*
(g) *If f is chain-recurrent, then $E = W^{\leq\rho} \oplus W^{\geq\rho}$ as a Whitney sum of two subbundles.*

REMARK 2.5. Theorem 2.4 holds if we restrict the action of M_f to the bundle $\hat{E} = E_{|\hat{P}} = \Pi^{-1}\hat{P}$, where $\hat{P} \subset \mathcal{P}$ is an f-invariant closed set of $\mathcal{P}$. For instance, we can take $\hat{P} = R(f)$, the recurrent set of f, that is the set of chain-recurrent points of f.

Proof: From Remark 2.3, it is enough to consider the case $\rho = 1$. So, the standing hypothesis will be $B = E_0$.

(a) $E^s = B^+$ and $E^u = B^-$.

We have to prove that if $v_\theta \in B_\theta^+$, then $\lim_{m\to+\infty} |M(\theta, m)v_\theta| = 0$. We denote $v_m = M(\theta, m)v_\theta$, and notice that $\{v_m\}_{m\geq 0}$ is bounded. For any convergent subsequence $\{v_{m_k}\}_{k\geq 0}$, the limit $\bar{v}_{\bar{\theta}}$ belongs to the omega limit set of v_θ, that is compact and invariant. That is $\bar{v}_{\bar{\theta}} \in B = E_0$ and $|\bar{v}_{\bar{\theta}}| = 0$. This proves (a).

(b) There exists a constant $K \geq 1$ such that the sets B^+ and B^- are characterized by:

$$v_\theta \in B^+ \quad \Leftrightarrow \quad \forall m \geq 0 \ |M(\theta, m)v_\theta| \leq K|v_\theta| \ ; \tag{2.15}$$

$$v_\theta \in B^- \quad \Leftrightarrow \quad \forall m \leq 0 \ |M(\theta, m)v_\theta| \leq K|v_\theta| \ . \tag{2.16}$$

To prove (b) we use the following lemma [**CI99**].

LEMMA 2.6. *Under the assumption $B = E_0$, there exists $K \geq 1$ such that for all $v_\theta \in E$, and m, n with $0 \leq m \leq n$:*

$$|M(\theta, m)v_\theta| \leq K(|v_\theta| + |M(\theta, n)v_\theta|),$$
$$|M(\theta, -m)v_\theta| \leq K(|v_\theta| + |M(\theta, -n)v_\theta|).$$

Proof of the lemma: We will prove the first inequality by contradiction.

Suppose it is false, then there exists a sequence of vectors $\{v_k = v_{\theta_k}\}_{k \geq 0} \subset E \setminus E_0$ and two sequences of indices $\{m_k\}_{k\geq 0}$, $\{n_k\}_{k\geq 0}$ with $0 \leq m_k \leq n_k$ such that

$$|M(\theta_k, m_k)v_k| > k(|v_k| + |M(\theta_k, n_k)v_k|) \ . \tag{2.17}$$

We can assume that $|v_k| = 1$, and m_k satisfies

$$|M(\theta_k, m_k)v_k| = \max_{0 \leq m \leq n_k} |M(\theta_k, m)v_k| \ .$$

We assume also that the sequence of unit vectors

$$w_k = \frac{1}{|M(\theta_k, m_k)v_k|} \cdot M(\theta_k, m_k)v_k \in E_{f^{m_k}(\theta_k)}$$

is convergent (taking subsequences, if necessary): $f^{m_k}(\theta_k) \to \theta_0$, $w_k \to w_0$.

Henceforth,

$$|M(\theta_k, m_k)v_k| > k(1 + |M(\theta_k, n_k)v_k|)$$

and $m_k \to +\infty$ because the right hand side tends to infinite. Notice also that

$$\left(1 - k\left|M(f^{m_k}(\theta_k), n_k - m_k)^{-1}\right|^{-1}\right)|M(\theta_k, m_k)v_k| > k \ ,$$

from where we see that $n_k - m_k \to +\infty$.

Then, for all m such that $-m_k \leq m \leq n_k - m_k$

$$\begin{aligned} |M(f^{m_k}(\theta_k), m)w_k| &= \frac{|M(\theta_k, m + m_k)v_k|}{|M(\theta_k, m_k)v_k|} \\ &\leq \frac{1}{|M(\theta_k, m_k)v_k|} \max_{0 \leq \bar{m} \leq n_k} \cdot |M(\theta_k, \bar{m})v_k| \\ &= 1 \ . \end{aligned}$$

Taking limits $k \to \infty$, we see that

$$|M(\theta_0, m)w_0| \leq 1$$

for all $m \in \mathbb{Z}$, so $w_0 \in B$, which is a contradiction with the assumption $B = E_0$. So we are done with the proof of Lemma 2.6. $\square$

The proof of (b) is now very easy. If $v_\theta \in B^+$, then $v_\theta \in E^s$ and we can take $\lim_{n\to\infty}$ in the RHS of

$$|M(\theta, m)v_\theta| \le K(|v_\theta| + |M(\theta, n)v_\theta|)$$

in Lemma 2.6, where $0 \le m \le n$. Then, for all $m \ge 0$, $|M(\theta, m)v_\theta| \le K|v_\theta|$. This proves (b), that is the uniform bounds in the linear subspaces of positively and negatively bounded orbits.

(c) B^+, B^- are closed subsets.
(d) The functions $\theta \to \dim B^+_\theta$ and $\theta \to \dim B^-_\theta$ are upper semicontinuous.

The proofs of (c) and (d) are straightforward [**SS74**].

(e) There exists $C > 0$ and $\lambda < 1 < \mu$ such that the stable and unstable spaces are characterized by the following growth rates:

$$v_\theta \in E^s \quad \Leftrightarrow \quad \forall m \ge 0 \ , |M(\theta, m)v_\theta| \le C\lambda^m |v_\theta| \tag{2.18}$$

$$v_\theta \in E^u \quad \Leftrightarrow \quad \forall m \le 0 \ , |M(\theta, m)v_\theta| \le C\mu^m |v_\theta| \ . \tag{2.19}$$

To prove (e) we use the following lemma (see lemma 6 in [**SS74**]).

LEMMA 2.7. *Under the assumption $B = E_0$, there exists $N > 0$ such that:*

- *for all $v_\theta \in B^+$, if $m \ge N$ then $|M(\theta, m)v_\theta| \le \frac{1}{2}|v_\theta|$;*
- *for all $v_\theta \in B^-$, if $m \ge N$ then $|M(\theta, -m)v_\theta| \le \frac{1}{2}|v_\theta|$.*

Proof of the lemma: If this were not the case, we could construct a sequence of vectors $\{v_{\theta_k}\} \subset B^+$ and a sequence of times $\{m_k\}$ with $m_k \longrightarrow +\infty$ such that

$$|M(\theta_k, m_k)v_{\theta_k}| > \frac{1}{2}|v_{\theta_k}| \ .$$

From the uniform bound in (b) we have that for all $m \ge 0$, $k > 0$,

$$|M(\theta_k, m)v_{\theta_k}| \le K|v_{\theta_k}| \ .$$

Notice that we can choose $|v_{\theta_k}| = \frac{1}{K}$ and taking subsequences we can assume that $v_{\theta_k} \longrightarrow v_\theta \in B^+$.

If we define $w_{\bar\theta_k} = M(\theta_k, m_k)v_{\theta_k}$, notice that $\frac{1}{2K} < |w_{\bar\theta_k}| \le 1$, and taking again subsequences we can assume that $w_{\bar\theta_k} \longrightarrow w_{\bar\theta}$.

Fixed, k, for all $m \ge -m_k$:

$$|M(\bar\theta_k, m)w_{\bar\theta_k}| = |M(\theta_k, m + m_k)v_{\theta_k}| \le K|v_{\theta_k}| = 1 \ .$$

Taking limits we obtain that for all $m \in \mathbb{Z}$ we have $|M(\bar\theta, m)w_{\bar\theta}| \le 1$, and hence $|w_{\bar\theta}| = 0$. This is a contradiction with that fact that $\frac{1}{2K} < |w_{\bar\theta_k}|$. This finishes the proof of the lemma. □

To prove the exponential growth rate characterization of B^+ in (e), take any $v_\theta \in B^+$ and $m \ge 0$. We write $m = jN + m_0$, with $m_0 \in \{0, 1, \dots, N-1\}$

(N given by the lemma). Then:

$$\begin{aligned}|M(\theta,m)v_\theta| &= |M(f^{jN}(\theta),m_0)M(f^{(j-1)N}(\theta),N)\dots M(\theta,N)v_\theta| \\ &\le |M(f^{jN}(\theta),m_0)|\left(\tfrac{1}{2}\right)^j|v_\theta| \\ &= \left(\tfrac{1}{2}\right)^{-m_0/N}|M(f^{jN}(\theta),m_0)|\left(\tfrac{1}{2}\right)^{m/N}|v_\theta| \\ &\le C\lambda^m|v_\theta|\ ,\end{aligned}$$

where

$$C=\max_{\bar\theta\in\mathcal{P}}\ \max_{m_0=0,\dots,N-1}\left(\frac{1}{2}\right)^{-m_0/N}|M(\bar\theta,m_0)|$$

and $\lambda=\sqrt[N]{\dfrac{1}{2}}<1$.

(f) If for all $\theta\in\mathcal{P}$ $E_\theta=E^s_\theta\oplus E^u_\theta$, then $E=E^s\oplus E^u$ as a Whitney sum of two subbundles.

Let us prove now (f), and assume then that $E_\theta=E^s_\theta\oplus E^u_\theta$ for all $\theta\in\mathcal{P}$. Notice that the rank functions $\theta\to\dim E^s_\theta$ and $\theta\to\dim E^u_\theta$ are upper semicontinuous. Taking limits $\theta\to\theta_0$, the ranks $\dim E^s_{\theta_0}$ and $\dim E^u_{\theta_0}$ could be bigger than $\dim E^s_\theta$ and $\dim E^u_\theta$, respectively. But this is not possible, because E^s_θ and E^u_θ span the whole E_θ. Henceforth, both closed linear subspaces E^s_θ and E^u_θ have constant rank, and, therefore, they are (continuous) subbundles.

(g) If f is chain-recurrent, then $E=E^s\oplus E^u$ as a Whitney sum of two subbundles.

The point (g) is a deep result in [**SS76a**] (see also [**Sel75, Sel76**]). This result implies fairly easily that, under the hypothesis of chain recurrence for the underlying dynamics – a mild condition of recurrence – we have that quasi-hyperbolicity implies hyperbolicity.

Here, we will present a proof under the stronger condition that f is regionally-recurrent. This is enough in many cases, but the reader should read [**SS76a**] for a complete proof in the case of chain recurrence. We follow the proof in [**CI99**], where this result was stated for flows.

So, assume that $\mathcal{P}=\Omega(f)$. Let $\theta\in\mathcal{P}$ any point, and consider a sequence of points $\{\theta_k\}_{k\ge0}\subset\mathcal{P}$ and a sequence of indices $\{n_k\}_{k\ge0}\subset\mathbb{N}$, such that

$$\theta_k\to\theta\ ,\ n_k\to+\infty\ ,\ f^{n_k}(\theta_k)\to\theta\ ,$$

when $k\to+\infty$.

For each k, we consider a subspace $E_k\subset E_{\theta_k}$ such that

$$\dim E^s_\theta+\dim E_k=\dim E_\theta=n\ ,\ E^s_\theta\oplus\lim_k E_k=E_\theta\ . \tag{2.20}$$

We will see know that there exists a positive constant $C>0$ such that

$$\|M(f^{n_k}(\theta_k),-m)_{|M(\theta_k,n_k)E_k}\|\le C \text{ for all } 0\le m\le n_k, \tag{2.21}$$

that is to say,

$$|M(\theta_k, n_k - m)v_k| \leq C|M(\theta_k, n_k)v_k| \text{ for all } v_k \in E_k \text{ and } 0 \leq m \leq n_k.$$

If this were not true, there would exist a sequence of unit vectors $\{v_k\}_{k\geq 0}$, with $v_k \in E_k$ and $|v_k| = 1$, and a sequence of indices $\{m_k\}_{k\geq 0}$, with $0 \leq m_k \leq n_k$, such that

$$|M(\theta_k, n_k - m_k)v_k| \geq k|M(\theta_k, n_k)v_k| \ . \tag{2.22}$$

From Lemma 2.6,

$$\begin{aligned} |M(\theta_k, n_k - m_k)v_k| &= |M(f^{n_k}(\theta_k), -m_k)M(\theta_k, n_k)v_k| \\ &\leq K\left(|M(\theta_k, n_k)v_k| + |v_k|\right) \ , \end{aligned} \tag{2.23}$$

and using (2.22) we obtain that

$$|M(\theta_k, n_k)v_k| \leq \frac{K}{k-K} \ .$$

Using again Lemma 2.6, we obtain that for all $0 \leq m \leq n_k$,

$$|M(\theta_k, m)v_k| \leq K(|v_k| + |M(\theta_k, n_k)v_k| \leq K + \frac{K^2}{k-K} \ .$$

Taking limits $k \to +\infty$ (passing to subsequences, so that $v_k \to v_\theta$), we obtain that for all $m \geq 0$

$$|M(\theta, m)v_\theta| \leq K \ .$$

Hence, $|v_\theta| = 1$ and $v_\theta \in B^+ = E^s$, but $v_\theta \in \lim_k E_k$! This is a contradiction with the construction (2.20). So, the claim (2.21) is proved.

Notice that we can assume that the subspaces constructed in (2.20) satisfy

$$\lim_k M(\theta_k, n_k)E_k = E_0 \subset E_\theta \ .$$

Taking limits in (2.21), we see that for all $m \geq 0$

$$\|M(\theta, -m)_{|E_0}\| \leq C \ ,$$

from where $E_0 \subset E^u_\theta$. Then, counting dimensions,

$$\dim E^u_\theta \geq \dim E_0 = \dim E_k = \dim E - \dim E^s_\theta \ ,$$

and since $E^s_\theta \cap E^u_\theta = \{0_\theta\}$, we obtain $E^s_\theta \oplus E^u_\theta = E_\theta$. Since this splitting works for any $\theta \in \mathcal{P}$, (g) is proved by appealing (f).

These arguments finish the proof of Theorem 2.4. □

Remark 2.8. When M_f induces an invariant continuous splitting in stable and unstable subbundles $E = E^s \oplus E^u$ it is said to be hyperbolic. When it satisfies the weaker condition $B = E_0$, it is said to be quasi-hyperbolic or quasi-Anosov.

The previous result states that continuity of the stable and unstable bundles in the definition of hyperbolicity is redundant, and we just need quasi-hyperbolicity and the splitting condition. The previous result also shows that, under mild properties of recurrence on f, hyperbolic vector bundle automorphisms covering f are hyperbolic.

The previous remark motivates the following definitions which try to capture different notions of dichotomy much weaker than the ones considered so far.

DEFINITION 2.9. *Let $0 < \lambda \leq \mu$. We say that M_f is quasi-(λ, μ)-dichotomic if $W^\rho = E_0$ for $\rho \in [\lambda, \mu]$. We say that M_f is (λ, μ)-dichotomic if there exists an invariant splitting $E = W^{\leq\lambda} \oplus W^{\geq\mu}$. Notice that in this case the growth rates can be made uniform and the invariant splitting is continuous (see Theorem 2.4).*

If $\lambda = \mu = \rho$ we will write ρ instead of (ρ, ρ) in the previous definitions.

2.1.1. Lyapunov characteristic numbers. The asymptotic rate of growth of an orbit of the cocycle is given by its Lyapunov multipliers. These are very weak notions of exponential dichotomy. They are associated to each orbit and, even then, they do not require uniformity.

The importance of these notions is that Oseledec's theorem shows that they happen for almost all orbits of an invariant measure. On the other hand, they are strong enough that they can allow to obtain many conclusions. See [**Pol93, BP01**] for recent surveys and tutorials. One of the questions that we will study in this paper (See specially Section 4) is the extent to which the properties of individual orbits determine the spectrum.

DEFINITION 2.10. *For each $\theta \in \mathcal{P}$ and $v_\theta \in E_\theta$, the* forward Lyapunov multiplier *and the* backward Lyapunov multiplier *of v_θ are given by the limits*

$$\lambda_s^+(v_\theta) = \limsup_{m\to+\infty} |M(\theta, m)v_\theta|^{\frac{1}{m}} \ , \ \lambda_i^-(v_\theta) = \liminf_{m\to-\infty} |M(\theta, m)v_\theta|^{\frac{1}{m}} \ , \tag{2.24}$$

respectively. Besides these two characteristic multipliers*, we define also,*

$$\lambda_i^+(v_\theta) = \liminf_{m\to+\infty} |M(\theta, m)v_\theta|^{\frac{1}{m}} \ , \ \lambda_s^-(v_\theta) = \limsup_{m\to-\infty} |M(\theta, m)v_\theta|^{\frac{1}{m}} \ . \tag{2.25}$$

Recall also that the Lyapunov (characteristic) exponents are just the logarithms of the corresponding Lyapunov (characteristic) multipliers.

Lyapunov multipliers on E_θ satisfy the following properties:

a) $\lambda_s^+(v_\theta) \in \mathbb{R} \cup \{0\}, \lambda_i^-(v_\theta) \in \mathbb{R} \cup \{+\infty\}$ for $v_\theta \in E_\theta$;
b) $\lambda_s^+(\alpha v_\theta) = \lambda_s^+(v_\theta), \lambda_i^-(\alpha v_\theta) = \lambda_i^-(v_\theta)$ for $v_\theta \in E_\theta$ and $\alpha \in \mathbb{C} \setminus \{0\}$;
c) $\lambda_s^+(v_\theta + w_\theta) \leq \max\{\lambda_s^+(v_\theta), \lambda_s^+(w_\theta)\}$,
$\lambda_i^-(v_\theta + w_\theta) \geq \min\{\lambda_i^-(v_\theta), \lambda_i^-(w_\theta)\}$ for $v_\theta, w_\theta \in E_\theta$;
d) $\lambda_s^+(0_\theta) = 0, \lambda_i^-(0_\theta) = +\infty$.

These properties imply that *there are finitely many distinct Lyapunov multipliers* over E_θ.

The following concept plays an important role in the Lyapunov theory.

DEFINITION 2.11. *We say that a vector* $v_\theta \in E_\theta$ *is* regular *when its forward and backwards Lyapunov exponents are the same.*

That is,

$$\lambda_s^+(v_\theta) = \lambda_s^-(v_\theta)$$

Related with the existence of Lyapunov exponents, it is important to recall the Oseledec's theorem [**Ose68, Rue79, Pol93, BP01**]. We will not give a formal statement of the theorem, and refer to the references above.

We recall that Oseledec's theorem ensures that

COROLLARY 2.12. *With the notations above,*

If μ *is probability measure invariant under* f*. then, for* μ *almost every point* θ*, there is a finite set of Lyapunov exponents reached on vectors in* E_θ*. Furthermore, for almost every* θ *the set of forwards and backwards Lyapunov exponents is the same. For each Lyapunov exponent, there is at least a regular vector (in the sense of Definition 2.11) having this Lyapunov exponent.*

The set of Lyapunov exponents is invariant under f*. In particular, if* μ *is ergodic, then, the set of Lyapunov constants is constant a.e.* μ*.*

2.1.2. Spaces characterized by rates of growth. When dealing with spectral properties of transfer operators and Lyapunov multipliers it is natural to consider the following subspaces of E_θ (see Theorem A.14).

DEFINITION 2.13. *Given a positive constants* ρ*, we define the* Lyapunov sets

$$L^{<\rho} = \{v_\theta \in E \mid \lambda_s^+(v_\theta) < \rho\} \ , \ L^{>\rho} = \{v_\theta \in E \mid \lambda_i^-(v_\theta) > \rho\} \ . \tag{2.26}$$

Obviously, Lyapunov sets are invariant linear subspaces of E*.*

It is obvious that

$$W^{\le \rho-\varepsilon} \subset L^{<\rho} \subset W^{<\rho} \subset W^{\le\rho} \ , \ W^{\ge \rho+\varepsilon} \subset L^{>\rho} \subset W^{>\rho} \subset W^{\ge\rho} \ .$$

for $\varepsilon > 0$.

PROPOSITION 2.14. *Let* M_f *be a vector bundle automorphism. Let* ρ *be a positive number. Let* $\theta \in \mathcal{P}$ *be a point. Hence, for* $\varepsilon > 0$ *small enough:*

$$W_\theta^{\le \rho-\varepsilon} = L_\theta^{<\rho} \ , \ W_\theta^{\ge\rho+\varepsilon} = L_\theta^{>\rho} \ .$$

Proof: We will prove $W_\theta^{\le\rho-\varepsilon} = L_\theta^{<\rho}$. Since the number of Lyapunov multipliers over E_θ is finite, we define

$$\lambda_{\max} = \max\{\lambda_s^+(v_\theta) \mid v_\theta \in L_\theta^{<\rho}\} < \rho \ .$$

Hence, for $\varepsilon > 0$ small enough we have $\lambda_{\max} < \rho - \varepsilon < \rho$ and, then, for any $v_\theta \in L_\theta^{<\rho}$ we have

$$\lim_{m\to+\infty} \frac{|M(\theta, m)v_\theta|}{(\rho-\varepsilon)^m} = 0 \ ,$$

from where $v_\theta \in W_\theta^{\le\rho-\varepsilon}$. □

2.2. Quasi-dichotomies and point spectrum

In this section we will review some arguments of Mañé that will be used heavily later. In particular, the following Lemma relates the existence of orbits of the cocycle satisfying certain growth rates with the existence of point spectrum. This result is the well known Mañé's lemma ([**Mn78, CL99**]). The original proofs are based on studying the spectrum on spaces of continuous sections. We will rather study the spectrum in spaces of bounded sections. Later, we will show that the spectrum is the same in both spaces.

LEMMA 2.15. *Let M_f be a vector bundle automorphism and $\rho > 0$.*

$$M_f \text{ is quasi-}\rho\text{-dichotomic} \Leftrightarrow \mathrm{Spec}_P(\mathcal{M}_f, \Gamma_B(E)) \cap \mathcal{S}_\rho = \emptyset \ .$$

Proof: Recall that M_f is quasi-ρ-dichotomic means, see Definition 2.9, that

$$W^\rho = W^{\leq \rho} \cap W^{\geq \rho} = E_0.$$

Suppose first that $W^\rho \neq E_0$. So, there exists a non zero vector v_{θ_0} in $W^\rho_{\theta_0}$ supported in some point θ_0. We will refer to v_{θ_0} as a *Mañé vector* of growth rate ρ.

There are two possibilities: either θ_0 is aperiodic or θ_0 is periodic.

If θ_0 is aperiodic, for any $z \in \mathbb{C}$ with $|z| = \rho$ the section $v = v(\theta)$ defined by

$$\hat{v}(\theta) = \begin{cases} z^{-k} M(\theta_0, k) v_{\theta_0}, & \text{if } \theta = f^k(\theta_0),\ k \in \mathbb{Z}, \\ 0, & \text{otherwise}, \end{cases} \tag{2.27}$$

is bounded and satisfies $\mathcal{M}_f v = zv$. Hence: $\mathcal{S}_\rho \subset \mathrm{Spec}_P(\mathcal{M}_f, \Gamma_B(E))$.

Suppose now that θ_0 is periodic, that is there exists $N > 0$ such that $f^N(\theta_0) = \theta_0$. In such a case, $W^{\leq\rho}_{\theta_0} \cap W^{\geq\rho}_{\theta_0}$ is a non-zero subspace of E_{θ_0} that is invariant under $M(\theta_0, N)$. By assumption, there exist constants C_1, C_2 such that for each $v_{\theta_0} \in W^{\leq\rho}_{\theta_0} \cap W^{\geq\rho}_{\theta_0}$ and $k \geq 0$:

$$|M(\theta_0, N)^k v_{\theta_0}| = |M(\theta_0, kN) v_{\theta_0}| \leq C_1 \rho^{kN} |v_{\theta_0}| \ ,$$

$$|M(\theta_0, N)^{-k} v_{\theta_0}| = |M(\theta_0, -kN) v_{\theta_0}| \leq C_2 \rho^{-kN} |v_{\theta_0}| \ .$$

Applying the spectral radius formula, we see that all the eigenvalues of $M(\theta_0, N)$ in $W^{\leq\rho}_{\theta_0} \cap W^{\geq\rho}_{\theta_0}$ have modulus ρ^N. Let $z \in \mathbb{C}$ be such that z^N is an eigenvalue of $M(\theta_0, N)$ and v_{θ_0} be an eigenvector, with $|v_{\theta_0}| = 1$. The section defined by (2.27) is bounded and satisfies $\mathcal{M}_f v = zv$. Notice that in this case we prove that $\mathcal{M}_f$ has eigenvalues in the circle of radius ρ.

In both cases, we obtain the claimed result on the point spectrum.

Suppose now that $\mathrm{Spec}_P(\mathcal{M}_f, \Gamma_B(E)) \cap \mathcal{S}_\rho \neq \emptyset$. Under this assumption we will prove that there exists a Mañé vector of growth rate ρ.

Let $z \in \mathrm{Spec}_P(\mathcal{M}_f, \Gamma_B(E))$, with $|z| = \rho$. Let v be a bounded eigensection of z, with $\|v\|_\infty = 1$. We can suppose that v is supported in an orbit $\{\theta_m = f^m(\theta_0)\}_{m \in \mathbb{Z}}$, with $v(\theta_0) \neq 0$. Let $v_m = v(\theta_m)$ be the vectors supported on such orbit. Notice that $M(\theta_0, m)v_0 = z^m v_m$, for all $m \in \mathbb{Z}$. Hence:

$$\frac{1}{\rho^m}|M(\theta_0, m)v_0| = |v_m| \leq 1 \ ,$$

for all $m \in \mathbb{Z}$, so $v_0 \in W^\rho \setminus E_0$. □

REMARK 2.16. Notice that for a Mañé vector v_0 of growth rate ρ, we have the estimates $\lambda_s^+(v_0) \leq \rho$ and $\lambda_i^-(v_0) \geq \rho$.

REMARK 2.17. If for certain $0 < \lambda \leq \mu$ there exists a non zero vector v_{θ_0} in $W^{\leq\lambda} \cap W^{\geq\mu}$ and supported on an aperiodic point θ_0, then we can take any $z \in \mathcal{A}_{\lambda,\mu}$ in the definition of (2.27). As a result, we prove that $\mathcal{A}_{\lambda,\mu} \subset \mathrm{Spec}_P(\mathcal{M}_f, \Gamma_B(E))$.

Notice also that if such a vector is supported in a periodic point, then the growth rates λ, μ are necessarily equal (applying the spectral radius formula).

2.3. Dichotomies and spectrum

In this section we will see how invariant subbundles satisfying growth rates can be constructed from spectral properties of the transfer operator associated to a vector bundle automorphism.

2.3.1. Spectral characterization of invariant splittings. The following is a fairly general result about transfer operators, where the existence of a gap in the spectrum of the transfer operator (Functional Analysis) is related with the existence of an invariant splitting of the bundle (Dynamics). This result can be found in many places, such as [**Mat68, HPS77, Mn78, LS90, CL99**]). Notably, the essential argument can be found already in [**Mat68**]. Note, however that some of the above papers state the theorem for $\Gamma_{C^0}(E)$ in place of $\Gamma_B(E)$. The version with $\Gamma_B(E)$ is somewhat easier (see [**HPS77**]).

THEOREM 2.18. *Let $M_f : E \to E$ be a vector bundle automorphism. Assume that*

$$\mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E)) \cap \mathcal{A}_{\lambda,\mu} = \emptyset \ ,$$

where $0 < \lambda \leq \mu$. Denote by $P^{<\lambda} = P_B^{<\lambda}, P^{>\mu} = P_B^{>\mu}$ the projections associated to this spectral gap. Then, it is possible to find a continuous invariant splitting

$$E = E^{<\lambda} \oplus E^{>\mu} \tag{2.28}$$

such that the corresponding projections over the bundles, $\Pi^{<\lambda}, \Pi^{>\mu}$, satisfy for any $v \in \Gamma_B(E)$:

$$(P^{<\lambda}v)(\theta) = \Pi_\theta^{<\lambda}v(\theta) \ , \ (P^{>\mu}v)(\theta) = \Pi_\theta^{>\mu}v(\theta) \ . \tag{2.29}$$

The splitting is characterized by the following growth rates: for all $\varepsilon > 0$ small enough

$$E^{<\lambda} = W^{\leq\lambda-\varepsilon} = L^{<\lambda}, \ E^{>\mu} = W^{\geq\mu+\varepsilon} = L^{>\mu} \ . \tag{2.30}$$

(See Definitions 2.1 and 2.13).

Moreover, the rates of growth can be made uniform: there exists a positive constant C_ε *such that* $W^{\leq\lambda-\varepsilon} = W^{\leq\lambda-\varepsilon, C_\varepsilon}, W^{\geq\mu+\varepsilon} = W^{\geq\mu+\varepsilon, C_\varepsilon}$.

The regularity of the invariant subbundles is determined by the following:

- *If* f^{-1} *is Lipschitz, with* $L = \mathrm{Lip}(f^{-1})$*, then the splitting is Hölder, with exponent* $\alpha \leq \frac{\log(\lambda/\mu)}{\log L}$.
- *If all the objects are* C^r*, with* $L = \mathrm{Lip}(f^{-1})$ *and* $r \leq \frac{\log(\lambda/\mu)}{\log L}$*, then the splitting is* C^r.

Conversely, if there is a splitting (2.28) *of* E *in linear subspaces* $E^{<\lambda}$ *and* $E^{>\mu}$ *satisfying* (2.30)*, then both linear subspaces are vector subbundles that satisfy all the other properties, and there is a gap in the spectrum:* $\mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E)) \cap \mathcal{A}_{\lambda,\mu} = \emptyset$.

Remark 2.19. We call attention to the remarkable formula (2.29). The left hand side is constructed using functional analysis whereas the right hand side is purely geometrical.

Remark 2.20. The case $\lambda \leq 1 \leq \mu$ corresponds to obtaining a *stable bundle* $E^s = E^{<\lambda}$, and an *unstable bundle* $E^u = E^{>\mu}$.

Remark 2.21. If E and M are the complexification of a real vector bundle $\tilde{E}$ and a vector bundle automorphism $\tilde{M}_f$ on $\tilde{E}$, respectively, then $\overline{E^{<\lambda}} = E^{<\lambda}$ and $\overline{E^{>\mu}} = E^{>\mu}$, and we obtain an invariant real splitting $\tilde{E} = \tilde{E}^{<\lambda} + \tilde{E}^{>\mu}$ (see Theorem A.13).

Now, we start the proof of Theorem 2.18.

Proof: The proof we present here follows [**HPS77**]. We split the proof in several steps. In the following, we denote $\Gamma_B^{<\lambda} = P^{<\lambda}(\Gamma_B)$ and $\Gamma_B^{>\mu} = P^{>\mu}(\Gamma_B)$.

- *b-linearity of spectral projections.* The first point is to realize that the spectral projections are $b(\mathcal{P}, \mathbb{C})$-linear.

  Lemma 2.22. *For any* $\rho \in b(\mathcal{P}, \mathbb{C})$ *and* $v \in \Gamma_B(E)$*:*

$$P^{<\lambda}(\rho v) = \rho P^{<\lambda} v \ , \ P^{>\mu}(\rho v) = \rho P^{>\mu} v \ . \tag{2.31}$$

  Proof of the lemma: To prove Lemma 2.22, notice that

$$\mathcal{M}_f^m(\rho v)(\theta) = \rho(f^{-m}(\theta)) \mathcal{M}_f^m v(\theta) \ .$$

  Using the Banach algebra properties of spaces of bounded functions, and using Theorem A.14 we obtain that

$$\begin{aligned} v \in \Gamma_B^{<\lambda}(E) \quad &\Rightarrow \quad \limsup_{m\to\infty} \|\mathcal{M}_f^m v\|_\infty^{\frac{1}{m}} < \lambda \\ &\Rightarrow \quad \limsup_{m\to\infty} \|\mathcal{M}_f^m \rho v\|_\infty^{\frac{1}{m}} \leq \limsup_{m\to\infty} \|\rho\|_\infty^{\frac{1}{m}} \|\mathcal{M}_f^m v\|_\infty^{\frac{1}{m}} < \lambda \\ &\Rightarrow \quad \rho v \in \Gamma_B^{<\lambda}(E) \ . \end{aligned}$$

Repeating the argument for $P^{>\mu}$, we obtain that

(2.32) $$v \in \Gamma_B^{<\lambda}(E) \Rightarrow \rho v \in \Gamma_B^{<\lambda}(E)\ ,\ v \in \Gamma_B^{>\mu}(E) \Rightarrow \rho v \in \Gamma_B^{>\mu}(E)\ .$$

Finally, consider any $v \in \Gamma_B(E)$ and its spectral decomposition $v = v^{<\lambda} + v^{>\mu}$. Then, since $\rho v = \rho v^{<\lambda} + \rho v^{>\mu}$ and (2.32) we obtain the claimed result. □

- *Localization of spectral projections.* The second step in the proof is to check that the spectral projections are local operators, in the sense that for a bounded section v, and a base point $\theta_0 \in \mathcal{P}$, $(P^{<\lambda}v)(\theta_0)$ and $P^{>\mu}v(\theta_0)$ depend only on $v(\theta_0)$.

LEMMA 2.23. *Let v be a bounded section. Let $\theta_0 \in \mathcal{P}$ be any base point. If $v(\theta_0) = 0$ then $(P^{<\lambda}v)(\theta_0) = 0$ and $(P^{>\mu}v)(\theta_0) = 0$.*

Proof of the lemma: We have just to consider the bounded function $\varphi : \mathcal{P} \to \mathbb{C}$ defined by

$$\varphi(\theta) = \begin{cases} 1, & \text{if } \theta = \theta_0, \\ 0, & \text{otherwise}\ , \end{cases}$$

and apply Lemma 2.22. □

- *Bundle projections.* We define the bundle automorphisms $\Pi_{\theta_0}^{<\lambda}$ and $\Pi_{\theta_0}^{>\mu}$ on the fiber E_{θ_0}, as

$$\Pi_{\theta_0}^{<\lambda} v_{\theta_0} = (P^{<\lambda}v)(\theta_0)\ ,\ \Pi_{\theta_0}^{>\mu} v_{\theta_0} = (P^{>\mu}v)(\theta_0)\ ,$$

where v is a bounded section such that $v(\theta_0) = v_{\theta_0}$. Notice that the definitions do not depend on the bounded section that we take (Lemma 2.23). The bundle automorphisms are obviously linear.

- *Splitting.* From the properties of the spectral projections (Theorem A.14), we prove that the bundle automorphisms are in fact bundle projections. Hence, we obtain an invariant splitting $E_\theta = E_\theta^{<\lambda} \oplus E_\theta^{>\mu}$ in linear subspaces just defining $E_{\theta_0}^{<\lambda} = \Pi_{\theta_0}^{<\lambda} E_{\theta_0}$ and $E_{\theta_0}^{>\mu} = \Pi_{\theta_0}^{>\mu} E_{\theta_0}$ for each $\theta_0 \in \mathcal{P}$. It is easy to see that

$$E_{\theta_0}^{<\lambda} = \{v(\theta_0) \mid v \in \Gamma_B^{<\lambda}(E)\}\ ,\ E_{\theta_0}^{>\mu} = \{v(\theta_0) \mid v \in \Gamma_B^{>\mu}(E)\}\ .$$

- *Rates of growth.* From the characterization of the rates of growth in the spectral subspaces (Theorem A.15), we obtain that for $\varepsilon > 0$ small enough there exists a positive constant C_ε such that

$$\begin{aligned} \Gamma_B^{<\lambda}(E) &= \{v \in \Gamma_B(E) \mid \forall m \geq 0\ \|\mathcal{M}_f^m v\|_\infty \leq C_\varepsilon(\lambda - \varepsilon)^m \|v\|_\infty\}\ , \\ \Gamma_B^{>\mu}(E) &= \{v \in \Gamma_B(E) \mid \forall m \geq 0\ \|\mathcal{M}_f^{-m} v\|_\infty \leq C_\varepsilon(\mu + \varepsilon)^{-m} \|v\|_\infty\}\ . \end{aligned}$$

We claim that

(2.33) $$E_{\theta_0}^{<\lambda} = W_{\theta_0}^{\leq\lambda-\varepsilon, C_\varepsilon} = L_{\theta_0}^{<\lambda}\ ,\ E_{\theta_0}^{>\mu} = W_{\theta_0}^{\geq\mu+\varepsilon, C_\varepsilon} = L_{\theta_0}^{>\mu}\ .$$

To prove this claim, take any $v_{\theta_0} \in E^{<\lambda}_{\theta_0}$. Notice that the bounded section

$$v(\theta) = \begin{cases} v_{\theta_0}, \text{ if } \theta = \theta_0, \\ 0, \text{ otherwise }, \end{cases} \tag{2.34}$$

belongs to $\Gamma^{<\lambda}_B(E)$ and $v(\theta_0) = v_{\theta_0}$. We are lead to

$$\forall m \geq 0 \ |M(\theta, m) v_\theta| \leq \|\mathcal{M}^m_f v\|_\infty \leq C_\varepsilon (\lambda - \varepsilon)^m \|v\|_\infty = C_\varepsilon (\lambda - \varepsilon)^m |v_\theta| \ ,$$

and the inclusion $E^{<\lambda}_{\theta_0} \subset W^{\leq \lambda-\varepsilon, C_\varepsilon}_{\theta_0}$ is proved. Finally, to prove that $L^{<\lambda}_{\theta_0} \subset E^{<\lambda}_{\theta_0}$, take $v_{\theta_0} \in L^{<\lambda}_{\theta_0}$ and define the bounded section v by (2.34). Since

$$\limsup_{m \to \infty} \|\mathcal{M}^m_f v\|^{\frac{1}{m}}_\infty = \limsup_{m \to \infty} |M(\theta_0, m) v_{\theta_0}|^{\frac{1}{m}} < \lambda$$

then $v \in \Gamma^{<\lambda}_B(E)$, and $v_{\theta_0} \in E^{<\lambda}_{\theta_0}$.

Remark 2.24. Notice that the uniformity of the rates of growth that characterizes the invariant subbundles comes from the boundedness of the spectral projections (cf. Theorem 2.4).

One consequence of the characterization of the spaces by the rates of growth (2.30) is that the mappings that to a point θ in the manifold $\mathcal{P}$ associate the $E^{<\lambda}_\theta$ and the $E^{>\mu}_\theta$ are closed. Since for each $\theta \in \mathcal{P}$ those spaces split the fiber E_θ, then the splitting $E_\theta = E^{<\lambda}_\theta \oplus E^{>\mu}_\theta$ is continuous (see Theorem 2.4).

Another proof that also yields Hölder regularity is obtained using the invariant section theorem (see [**HP70, HPS77**]). Here, we will give the details only for the regularity of $E^{<\lambda}$. Similar arguments give the claim for $E^{>\mu}$. The idea of the proof is to show that the vector bundle automorphism M_f induces a bundle morphism $\hat{M}^{<\lambda}_f$ on the Grassmannian bundle $\mathcal{G}_{n^{<\lambda}}$. Notice that the corresponding transfer operator, $\hat{\mathcal{M}}^{<\lambda}_f$, acts on sections of this Grassmannian bundle, and there exists an invariant section $\hat{E}^{<\lambda}$ which corresponds to $E^{<\lambda}$. In fact, for $\varepsilon > 0$ small enough we can construct a natural metric (see the section on adapted metrics below) for which this map is a contraction by a factor $\frac{\lambda-\varepsilon}{\mu+\varepsilon} < 1$, and there is one and only one invariant section. Since this also acts on Hölder sections of exponent α and it is a contraction of exponent $\frac{\lambda-\varepsilon}{\mu+\varepsilon} L^\alpha$, where $L = \mathrm{Lip}(f^{-1})$. Hence, there is a C^α section which is unique provided that $\alpha \leq \frac{\log(\lambda/\mu)}{\log L}$.

To prove the converse, just note that any splitting satisfying (2.30) is, at least, continuous. Then, the other properties follow immediately once we define the spectral projections $P^{<\lambda}$ and $P^{>\mu}$ by

$$(P^{<\lambda} v)(\theta) = \Pi^{<\lambda}_\theta v(\theta) \ , \ (P^{>\mu} v)(\theta) = \Pi^{>\mu}_\theta v(\theta) \ ,$$

and check their properties using Theorem A.14. □

REMARK 2.25. The bootstrap in the regularity of the splitting follows from the invariant section theorem. It depends only on the growth rates (2.30).

REMARK 2.26. The arguments of the proof of Theorem 2.18 work if we restrict the action of M_f to the bundle $E_{|\mathcal{P}_0}$, where $\mathcal{P}_0 \subset \mathcal{P}$ is a f-invariant set without isolated points. This produces an invariant splitting of E on $\mathcal{P}_0$.

We note that the invariant sections theorem [**HP70, HPS77**] works when the base set is just a metric space. So, the subbundles on $\mathcal{P}_0$ extend to subbundles on $\mathrm{cl}(\mathcal{P}_0)$. See Theorem 3.39.

REMARK 2.27. If the base manifold $\mathcal{P}$ is a torus $\mathbb{T}^d$, and the motion f is a rotation $f(\theta) = t_\omega(\theta) = \theta + \omega$ with $\omega \in \mathbb{R}^d$, then the invariant splitting is as smooth as the transfer operator, because the Lipschitz constant of $f^{-1}(\theta) = \theta - \omega$ is 1. This observation is important for Part 3.

Notice that the existence of annular gaps of the transfer operator has consequences on the dynamics of the corresponding cocycle. This motivates the following definition.

DEFINITION 2.28. *The annular hull of the spectrum of a transfer operator $\mathcal{M}_f$ is decomposed in annuli*

$$\mathcal{A}\mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E)) = \bigcup_{i=1}^{k} \mathcal{A}_i \ .$$

where the $\mathcal{A}_i$ is a pairwise disjoint set of annuli.

$$\mathcal{A}_i = \mathcal{A}_{\lambda_i^-, \lambda_i^+}$$

with $\lambda_i^+ < \lambda_{i+1}^- \leq \lambda^+ i + 1$. The annuli $\mathcal{A}_i$ are called the spectral annuli.

Each annulus $\mathcal{A}_i$ has associated an invariant subbundle E_i, characterized by the rates of growth of its orbits, and $E = \bigoplus_{i=1}^k E_i$. Notice that

$$\mathcal{A}\mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E_i)) = \mathcal{A}_i \ .$$

We call multiplicity *of the spectral annulus $\mathcal{A}_i$ the rank of the corresponding invariant subbundle E_i.*

REMARK 2.29. Since $E = \bigoplus_{i=1}^k E_i$, the maximum number of spectral annuli is n, the rank of the bundle E. That is to say, the maximum number of gaps in the spectrum is $n - 1$.

REMARK 2.30. We point out that the invariant bundles constructed above as the images of spectral projections are not the only possible invariant bundles. For example, if we take the product of two systems, the sum of any pair of invariant subbundles – including the trivial subbundle – for each system is an invariant subbundle for the product system. It is easy to construct examples for which some of the invariant subbundles are not a spectral subset. For example, if we take two identical systems, the product of a spectral space from the first and the zero space in the other is invariant, but it is not a spectral subset. Another example can be obtained by taking $f_* : T\mathbb{T}^4 \to T\mathbb{T}^4$ where $f = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$ where $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. The bundles $\{e_1\} \oplus \{0\}$ and $\{0\} \otimes \{e_1\}$ where e_1 is an eigenvalue of A — are non-trivial subbundles of $\{e_1\} \oplus \{e_1\}$ which is the spectral subbundle.

Note that by taking the product of a system with itself we do not change the spectrum — hence, the number of spectral subbundles — but by adding a system to itself we can produce arbitrarily large number of invariant subbundles. This observation plays a role in the theory of non-resonant invariant manifolds (See e.g. [**dlL97, CFdlL03a, CFdlL03b, HdlLb, HdlL04**]).

REMARK 2.31. We will refer to the invariant subbundles constructed above from spectral projections as *spectral subbundles*.

The spectral theory of transfer operators acting on continuous sections is similar to that on bounded sections. A first dividend of Theorem 2.18 is that the existence of gaps on $\mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E))$ implies their existence on $\mathrm{Spec}(\mathcal{M}_f, \Gamma_{C^0}(E))$.

PROPOSITION 2.32. *Under the conditions of Theorem 2.18:*

$$\mathrm{Spec}(\mathcal{M}_f, \Gamma_{C^0}(E)) \cap \mathcal{A}_{\lambda,\mu} = \emptyset \ .$$

Proof: Once we know that the splitting (2.28) of Theorem 2.18 is continuous we note that $P^{<\lambda}$ and $P^{>\mu}$ send continuous functions into continuous functions. Then $\Gamma_{C^0}(E)$ is invariant under the spectral projections $P^{<\lambda}$ and $P^{>\mu}$. We conclude that the annulus of radii λ, μ does not intersect $\mathrm{Spec}(\mathcal{M}_f, \Gamma_{C^0}(E))$ and that the projections on $\Gamma_{C^0}(E)$ are the restrictions of the projections on $\Gamma_{C^0}(E)$ (see Corollary A.16). □

REMARK 2.33. As a corollary we obtain that

$$\mathcal{A}\mathrm{Spec}(\mathcal{M}_f, \Gamma_{C^0}(E)) \subset \mathcal{A}\mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E)) \ .$$

This result will be completed in Section 3.6.

2.3.2. Adapted metric, cone fields and existence of a spectral gap. We have seen that a spectral gap $\mathcal{A}_{\lambda,\mu}$ produces an invariant splitting $E = E^{<\lambda} \oplus E^{>\mu}$. We will write $M^{<\lambda} = M_{|E^{<\lambda}}$ and $M^{>\mu} = M_{|E^{>\mu}}$, the restriction of the vector bundle automorphism over such a subbundles.

The following device is sometimes called the adapted metric or the Lyapunov metric (see e.g. [**Mat68**]), and it is a metric adapted to the invariant splitting. This is a geometric construct that parallels the construction of adapted norms in functional analysis covered in Section A.3.

PROPOSITION 2.34. *Assume that*

$$\mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E)) \cap \mathcal{A}_{\lambda,\mu} = \emptyset \ ,$$

where $0 < \lambda \le \mu$*. Then, for every* $\varepsilon > 0$ *small enough we can find a metric* $\|\cdot\|_\theta$ *on the fibers* E_θ *such that*

$$\|M^{<\lambda}\|_{C^0} \le \lambda - \varepsilon \ ,$$
$$\|(M^{>\mu})^{-1}\|_{C^0} \le (\mu + \varepsilon)^{-1} \ .$$

If the bundle and M *are analytic, the metric can be chosen analytic on* θ*.*

Proof: Taking $\varepsilon > 0$ small enough, for $v_\theta \in E_\theta$ we define first the norms of its projections on $E_\theta^{<\lambda}$ and $E_\theta^{>\mu}$ by

$$\|v_\theta^{<\lambda}\|^2 = \sum_{k=0}^{\infty} (\lambda - \varepsilon)^{-2k} |M(\theta, k) v_\theta^{<\lambda}|^2 \tag{2.35}$$

and

$$\|v_\theta^{>\mu}\|^2 = \sum_{k=0}^{\infty} (\mu + \varepsilon)^{2k} |M(\theta, -k) v_\theta^{>\mu}|^2 \ . \tag{2.36}$$

Notice that both series are convergent, from the characterization of the invariant splitting by growth rates. Finally, we define

$$\|v_\theta\|^2 = \|v^{<\lambda}\|^2 + \|v^{>\mu}\|^2 \ . \tag{2.37}$$

It is very easy to show that $\|\cdot\|$ continuous with respect to θ and satisfies

$$\|M^{<\lambda}(\theta)\| \le \lambda - \varepsilon \ , \ \|(M^{>\mu})^{-1}(f^{-1}(\theta)\| \le (\mu + \varepsilon)^{-1} \ , \tag{2.38}$$

for every $\theta \in \mathcal{P}$.

If we smooth the metric, we obtain an analytic metric for which (2.38) is satisfied with slightly worse constants. □

Remark 2.35. We note that if the original norm satisfied the parallelogram law, (i.e. are Riemannian metrics coming from an inner product) it is easy to check that the norms (2.35), (2.36), and hence that in (2.37) also do. Hence, if we chose a norm in E_θ which comes from a Riemannian metric, we obtain a norm that also comes from a Riemannian metric.

The hypothesis on the existence of the spectral gap in C^0 can be checked with a finite computation. As it is quite standard in hyperbolicity theory, the tool is the construction of suitable cone fields. .

Proposition 2.36. *Suppose we can find cone fields* $\mathcal{C}_\theta^{<\lambda}$, $\mathcal{C}_\theta^{>\mu}$ *such that*

$$M(\theta)\mathcal{C}_\theta^{<\lambda} \subset \mathcal{C}_{f(\theta)}^{<\lambda} \ , \ M^{-1}(f^{-1}(\theta))\mathcal{C}_\theta^{>\mu} \subset \mathcal{C}_{f^{-1}(\theta)}^{>\mu} \tag{2.39}$$

and a metric such that

$$\begin{aligned} |M(\theta)v| &\ge \mu|v| \ , \ v \in \mathcal{C}_\theta^{>\mu}, \\ |M^{-1}(f^{-1}(\theta))v| &\ge \lambda^{-1}|v| \ , \ v \in \mathcal{C}_\theta^{<\lambda} \ , \end{aligned} \tag{2.40}$$

and that, for every θ, there are spaces $\hat{E}_\theta^{<\lambda} \subset \mathcal{C}_\theta^{<\lambda}$ and $\hat{E}_\theta^{>\mu} \subset \mathcal{C}_\theta^{>\mu}$ such that

$$\hat{E}_\theta^{<\lambda}, \hat{E}_\theta^{>\mu} \text{ span } E \ . \tag{2.41}$$

Then

$$\mathrm{Spec}(\mathcal{M}_f, \Gamma_{C^0}(E)) \cap \mathcal{A}_{\lambda,\mu} = \emptyset \ .$$

Proof: The proof of the Proposition 2.36 is quite straightforward because it implies that there exist cones in B on which $\|\mathcal{M}_f^{>\mu}\| \geq \mu$ and $\|\mathcal{M}_f^{<\lambda^{-1}}\| \geq \lambda^{-1}$. The cone fields $\mathcal{C}^{<\lambda}$, $\mathcal{C}^{>\mu}$ are spanning and the spectral radius formula tells us that the spectrum satisfies the result claimed. □

In numerical applications, *it is not too difficult* to verify the hypothesis of Proposition 2.36. If the conclusion is true, one can compute the stable and unstable splittings by iterating frames. Since the mapping on frames is a contraction, a calculation with finite precision is enough to give the cones. A discussion of the implementation of these calculations is in [**HdlL04**].

2.4. Lyapunov multipliers and spectral annuli

In this section we continue the study of the asymptotic growth rates of the orbits of a vector bundle automorphism undertaken in Section 2.1. We will exploit here the spectral properties of the associated transfer operator.

The results of this section are highly inspired in [**SS78**], which developed a spectral theory for lineal skew-product flows. The spectrum considered in [**SS78**], the so called *Sacker-Sell spectrum*, is defined using dynamical properties rather than functional analysis. The fact that there is relation between Sacker-Sell spectrum and the functional analysis spectrum of the transfer operator was established in [**CS80**] (see also [**Joh80**]), incidentally this gave functional analysis proofs of several results in [**SS78**].

In this paper, we will also adopt the point of view of trying to obtain results using with preference "soft" functional analysis rather than "hard" analysis based on the dynamics. See also [**Swa81**].

The first result we consider is the translation of the uniformization lemma in [**SS78**] (see also [**Fen72**]) to the language of the spectral theory of transfer operators. We will use the notation introduced in Section 2.1.

PROPOSITION 2.37. *Let $M_f : E \to E$ be a vector bundle automorphism. Let ρ be a positive number. The following four statements are equivalent:*

(a) $W^{\leq\rho} = E$ *and* $W^{\geq\rho} = E_0$*;*
(b) $L^{<\rho} = E$*;*
(c) *for all* $\theta \in \mathcal{P}$, $\lim_{m\to+\infty} \rho^{-m}|M(\theta,m)| = 0$*;*
(d) $r_s(\mathcal{M}_f, \Gamma_B(E)) < \rho$.

Analogously, the following four statements are equivalent:

(a') $W^{\geq\rho} = E$ *and* $W^{\leq\rho} = E_0$*;*
(b') $L^{>\rho} = E$*;*
(c') *for all* $\theta \in \mathcal{P}$, $\lim_{m\to-\infty} \rho^{-m}|M(\theta,m)| = 0$*;*

(d') $r_i(\mathcal{M}_f, \Gamma_B(E)) > \rho$.

Proof: We will first prove the equivalence among (a),(b), (c) and (d).

Notice that (d)⇒(a) and (d)⇒(b) are immediate consequences of the characterization of spectral subbundles undertaken in Theorem 2.18.

Assume (a) $E = W^{\leq\rho}$ and $W^{\geq\rho} = E_0$. Obviously, $W^{\leq\rho} \cap W^{\geq\rho} = E_0$, and Theorem 2.4 (e) implies $E = W^{\leq\lambda}$ for any $\lambda < \rho$ close enough, with uniform growth rates. Hence, for all $\theta \in \mathcal{P}$, $v_\theta \in E_\theta$, and for all $m \geq 0$

$$|M(\theta, m)v_\theta| \leq C\lambda^m |v_\theta| \ .$$

That is to say, $|M(\theta, m)| \leq C\lambda^m$. This proves (a)⇒(c).

Assume (b) $E = L^{<\rho}$. Let $\theta \in \mathcal{P}$ be any base point. Since there are finitely many Lyapunov multipliers in E_θ, then Proposition 2.14 concludes that for any $\varepsilon = \varepsilon(\theta)$ small enough $W_\theta^{\leq\rho-\varepsilon} = L_\theta^{<\rho} = E_\theta$. Hence, using the uniform boundedness principle, there exists a constant $C_{\varepsilon,\theta}$ such that

$$|M(\theta, m)| \leq C_{\varepsilon,\theta}(\rho - \varepsilon)^m \ ,$$

for any $m \geq 0$. This proves (b)⇒(c).

It only remains to prove (c)⇒(d). To do so, assume that (c) is true, and we will bound $\rho^{-m}M(\theta, m)$ uniformly in θ. To do so, we use the following argument [**SS78**]. From (c), for each $\theta \in \mathcal{P}$ there exists $m_\theta > 0$ and an open neighborhood U_θ of θ such that

$$\frac{M(\bar\theta, m_\theta)}{\rho^{m_\theta}} < \frac{1}{2}$$

for any $\bar\theta \in U_\theta$. Since $\mathcal{P}$ is compact, we can cover it with a finite number of open neighborhoods $U_1, \ldots, U_k$ such that

$$\frac{M(\bar\theta, m_i)}{\rho^{m_i}} < \frac{1}{2}$$

for any $\bar\theta \in U_i$, $i = 1, \ldots, k$, where $m_1 \leq \cdots \leq m_k$. Given any $\theta \in \mathcal{P}$ and $m \geq 0$, we can construct a finite sequence $m_{i_1}, \ldots, m_{i_l}, m_{i_{l+1}}$ such that

$$s_l = m_{i_1} + \cdots + m_{i_l} \leq m < m_{i_1} + \cdots + m_{i_{l+1}} = s_{l+1} \leq (l+1)m_k$$

and $f^{s_j}(\theta) \in U_{i_{j+1}}$ for $j = 0, \ldots, l$, that is,

$$|M(f^{s_j}(\theta), m_{j+1})| \leq \frac{1}{2}\rho^{m_{j+1}} \ .$$

Therefore,

$$\begin{aligned} |M(\theta, m)| &= |M(f^{s_l}(\theta), m - s_l)M(f^{s_{l-1}}(\theta), m_{i_l}) \ldots M(f^{s_1}(\theta), m_{i_2})M(\theta, m_{i_1})| \\ &\leq |M(f^{s_l}(\theta), m - s_l)| \left(\frac{1}{2}\right)^l \rho^{s_l} \\ &\leq K \left(\frac{1}{2}\right)^{\frac{m}{m_k} - 1} \rho^m \ , \end{aligned}$$

where

$$K = \max_{\theta \in \mathcal{P}} \max_{0 \le m \le m_k} \frac{|M(\theta, m)|}{\rho^m} ,$$

and we use that $l > \frac{m}{m_k} - 1$. Hence,

$$\|\mathcal{M}_f^m\|^{\frac{1}{m}} = \sup_{\theta \in \mathcal{P}} |M(\theta, m)|^{\frac{1}{m}} \le K^{\frac{1}{m}} \left(\frac{1}{2}\right)^{\frac{1}{m_k} - \frac{1}{m}} \rho \xrightarrow{m \to +\infty} \left(\frac{1}{2}\right)^{\frac{1}{m_k}} \rho < \rho .$$

Hence, $r_s(\mathcal{M}_f, \Gamma_B(E)) < \rho$ and (d) is proved. We are done with the proof of Proposition 2.37. □

In the analysis of the spectral annuli in terms of Lyapunov multipliers we have the following result. This proof summarizes the dividends obtained from the the relationship between spectrum (functional analysis) and Lyapunov multipliers (dynamics).

THEOREM 2.38. *Let $M_f : E \to E$ be a vector bundle automorphism such that we can write the annular hull of the spectrum on bounded sections as:*

$$\mathcal{A}\mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E)) = \bigcup_{j=1}^{k} \mathcal{A}_{\lambda_j^-, \lambda_j^+} .$$

Let

$$E = \bigoplus_{j=1}^{k} E^j$$

be the corresponding invariant splitting. Hence:

(a) *For each spectral subbundle E^j, the four characteristic multipliers of any of its non-zero vectors $v \in E^j \setminus E_0$ belong to $[\lambda^-, \lambda^+]$. In particular, if $\lambda^+ = \lambda^- = \lambda$, the four characteristic multipliers are equal to λ and the limits*

$$\lim_{m \to +\infty} |M(\theta, m) v_\theta|^{\frac{1}{m}} = \lim_{m \to -\infty} |M(\theta, m) v_\theta|^{\frac{1}{m}} = \lambda$$

exist.

(b) *For each spectral subbundle E^j there exist $v, w \in E^j$, such that $\lambda_s^+(v) = \lambda_j^+$ and $\lambda_i^-(w) = \lambda_j^-$.*

(c) *Each spectral subbundle E^j is characterized by*

$$E^j = \{ v_\theta \in E_\theta \mid \lambda_s^+(v_\theta) \le \lambda_j^+ \ , \ \lambda_i^-(v_\theta) \ge \lambda_j^- \}$$

(d) *If $v_\theta = \sum_{j=k_1}^{k_2} v_\theta^j$, with $v_\theta^j \in E_\theta^j$ for all j, then:*

$$\lambda_{k_2}^- \le \lambda_i^+(v_\theta) \le \lambda_s^+(v_\theta) \le \lambda_{k_2}^+ \ , \ \lambda_{k_1}^- \le \lambda_i^-(v_\theta) \le \lambda_s^-(v_\theta) \le \lambda_{k_1}^+$$

In particular, for any vector $v_\theta \in E$ its four characteristic multipliers lie in the spectral annuli.

Proof: For the proof of the sentences (a) and (b) we have just to analyze one spectral subbundle, so we suppose that

$$\mathcal{A}\mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E)) = \mathcal{A}_{\lambda^-,\lambda^+} \ .$$

Given $\theta \in \mathcal{P}$ and $v_\theta \in E_\theta \setminus \{0_\theta\}$, we will prove that $\lambda^- \leq \lambda_i^+(v_\theta)$ and $\lambda_s^+(v_\theta) \leq \lambda^+$. The other inequalities are proved using similar arguments.

Since $\mathcal{A}\mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E)) = \mathcal{A}_{\lambda^-,\lambda^+}$, then $E = L^{<\lambda}$ for all $\lambda > \lambda^+$. Hence, $\lambda_s^+(v_\theta) < \lambda$ for all $\lambda > \lambda^+$ and then $\lambda_s^+(v_\theta) \leq \lambda^+$.

Notice also that $E = L^{>\mu}$ for all $\mu < \lambda^-$. Fixed $\mu < \lambda^-$, from the characterization of spectral subbundles, for all $\varepsilon > 0$ small enough (i.e., such that $\mu + \varepsilon < \lambda^-$) there exists a positive constant C_ε such that

$$\forall v_\theta \in E \ \forall m \geq 0 \ |M(\theta, -m)v_\theta| \leq C_\varepsilon (\mu + \varepsilon)^{-m} |v_\theta| \ .$$

Fixed $v_\theta \in E_\theta$ and $m \geq 0$, since $M(\theta, m)v_\theta \in E_{f^m(\theta)}$, then

$$|v_\theta| = |M(f^m(\theta), -m)M(\theta, m)v_\theta| \leq C_\varepsilon (\mu + \varepsilon)^{-m} |M(\theta, m)v_\theta|$$

and, hence,

$$|M(\theta, m)v_\theta|^{\frac{1}{m}} \geq \left(\frac{1}{C_{\mu,\varepsilon}} |v_\theta| \right)^{\frac{1}{m}} (\mu + \varepsilon) \ .$$

Taking $\liminf\limits_{m \to +\infty}$ in this inequality we obtain that $\lambda_i^+(v_\theta) \geq \mu + \varepsilon$ for all $\mu < \lambda^-$ and $\varepsilon > 0$ small enough, so then $\lambda^- \leq \lambda_i^+(v_\theta)$. (a) is proved.

To prove (b), that the boundary radii of the spectral annulus are reached as Lyapunov multipliers of some vectors, we have just to apply Proposition 2.37.

The statement (c) is straightforward.

To prove (d), we will show that, if

$$\theta = v_\theta^{<k_2} + v_\theta^{k_2}$$

with

$$v_\theta^{<k_2} \in \bigoplus_{j=1}^{k_2 - 1} E^j$$

and $v_\theta^{k_2} \in E^{k_2}$, then

$$\lambda_{k_2}^- \leq \lambda_i^+(v_\theta) \leq \lambda_s^+(v_\theta) \leq \lambda_{k_2}^+.$$

First, notice that:

$$\lambda_s^+(v_\theta) \leq \max\{\lambda_s^+(v_\theta^{<k_2}), \lambda_s^+(v_\theta^{k_2})\} = \lambda_s^+(v_\theta^{k_2}) \leq \lambda_{k_2}^+ \ .$$

Second, take any $\varepsilon > 0$ such that $\lambda_{k_2 - 1}^+ < \lambda_{k_2}^- - \varepsilon < \lambda_{k_2}^-$. Since $v_\theta^{<k_2} \in E^{<\lambda_{k_2}^- - \varepsilon}$ then

$$\lim_{m \to +\infty} \frac{|M(\theta, m)v_\theta^{<k_2}|}{(\lambda_{k_2}^- - \varepsilon)^m} = 0 \ .$$

Since $v_\theta^{k_2} \in E^{>\lambda_{k_2}^- - \varepsilon}$ then

$$\lim_{m\to+\infty} \frac{|M(\theta,m)v_\theta^{k_2}|}{(\lambda_{k_2}^- - \varepsilon)^m} = \infty \ .$$

Henceforth, for all $\varepsilon > 0$ small enough

$$\lim_{m\to+\infty} \frac{|M(\theta,m)v_\theta|}{(\lambda_{k_2}^- - \varepsilon)^m} = \infty$$

and then $\lambda_i^+(v_\theta) \geq \lambda_{k_2}^-$. □

2.4.1. Bounds of the spectrum. In this section we will bound the spectrum of a transfer operator $\mathcal{M}_f$ associated to a vector bundle automorphisms M_f, under the assumption that f is uniquely ergodic (i.e. it admits only one invariant measure). We will establish the results for the spectrum on the space of bounded sections, because the corresponding results for the other spaces follows from this paper.

The following is a well known result that gives estimates on the size of the spectrum. See also [**CS81**].

Proposition 2.39. *Let M_f be a vector bundle automorphism over a uniquely ergodic homeomorphism f. Let μ be its corresponding invariant measure. Then:*

$$\mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E)) \subset \mathcal{A}_{\rho_-,\rho_+} \ ,$$

where

(2.42)

$$\exp\left(-\int_{\mathcal{P}} \log|M(\theta)^{-1}| \ d\mu\right) \ = \ \rho_- \ \leq \ \rho_+ = \exp\left(\int_{\mathcal{P}} \log|M(\theta)| \ d\mu\right) \ .$$

Proof: We will prove the inequality $r_s(\mathcal{M}_f, \Gamma_B(E)) \leq \rho_+$, because the other one follows immediately by inverting.

The spectral radius formula is

$$\log r_s = \lim_{m\to\infty} \frac{1}{m} \log \|\mathcal{M}_f^m\|_\infty$$

Notice that

$$\frac{1}{m}\log\|\mathcal{M}_f^m\|_\infty \leq \frac{1}{m}\log\max_{\theta\in\mathcal{P}}|M(\theta,m)| \leq \max_{\theta\in\mathcal{P}} \frac{1}{m}\sum_{i=0}^{m-1}\log|M(f^i(\theta))| \ .$$

Since f is uniquely ergodic, it is well known in ergodic theory (see [**Wal82**]) that the time average of the continuous function $\log|M(\theta)|$

$$\frac{1}{m}\sum_{i=0}^{m-1}\log|M(f^i(\theta))|$$

converges uniformly in θ when $m \to +\infty$ to the space average

$$\int_{\mathcal{P}} \log|M(\theta)| \ d\mu \ .$$

The inequality $r_s \leq \rho_+$ is follows immediately. □

REMARK 2.40. Jensen's inequality together with Proposition 2.39 implies that:

$$\left(\int_{\mathcal{P}} |M(\theta)^{-1}|\ d\mu\right)^{-1} \ \leq\ \rho_- \ \leq\ \rho_+ \ \leq\ \left(\int_{\mathcal{P}} |M(\theta)|\ d\mu\right)\ .$$

The above result is enough to locate the spectrum in some situations (cf. [**CS81**]).

COROLLARY 2.41. *Assume that the mapping f is uniquely ergodic.*

If the rank of the bundle E is $n = 1$, then the spectrum is the circle of radius $\rho = \rho_- = \rho_+$.

If $M(\theta)$ is an isometry for all $\theta \in \mathcal{P}$, that is $|M(\theta)| = 1$ for all $\theta \in \mathcal{P}$, then the spectrum is the unit circle.

COROLLARY 2.42. *In the conditions of Proposition 2.39 we have for every n*

$$\rho_+ = \exp\left(\int_{\mathcal{P}} \frac{1}{n} \log|M(\theta, n)|\ d\mu\right)\ . \tag{2.43}$$

Taking infimum in n in (2.43) *and using Oseledec's theorem, we obtain*

$$\rho_+ \leq \exp \lambda^+(M, \mu) \tag{2.44}$$

where λ^+ denotes the maximal Lyapunov exponent with respect to the invariant measure μ.

The proof of Corollary 2.42 consists in observing that the spectral radius of $\mathcal{M}_f$ is the same as the spectral radius of $\mathcal{M}_f^n$. On the other hand, $\mathcal{M}_f^n$ is induced by the cocycle $M(\theta, n)$. If we apply (2.42) to $M(\theta, n)$, we obtain (2.43). □

REMARK 2.43. If the spectrum is decomposed in n spectral annuli, then they are in fact spectral circles. This is a direct consequence of the previous corollary and the fact that there are invariant subbundles associated to the spectral annuli, that are 1-dimensional. Notice that the proof of this results uses essentially that the invariant measure is uniquely ergodic.

It is straightforward that if we multiply a transfer operator by a constant number $\alpha \in \mathbb{C}^*$, then the spectrum is multiplied by such constant number. When the motion on the base is uniquely ergodic, we can generalize this result.

PROPOSITION 2.44. *Let $M_f : E \to E$ be a vector bundle automorphism. Let $\sigma : \mathcal{P} \to \mathbb{C}$ be a non-vanishing complex function. Assume one of the following hypotheses:*

- *The modulus of σ is constant, with $\hat{\sigma} = |\sigma(\theta)|$ for all $\theta \in \mathcal{P}$;*
- *f is uniquely ergodic and $\hat{\sigma} = \exp\left(\int \log|\sigma(\theta)|\ d\mu\right)$, where μ is the invariant measure. ($\hat{\sigma}$ is the geometric average).*

Then:

$$\mathcal{A}\mathrm{Spec}(\sigma\mathcal{M}_f, \Gamma_B(E)) \Leftrightarrow \hat{\sigma}\mathcal{A}\mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E)) \ .$$

REMARK 2.45. Recall that under the assumption of the existence of an unique invariant measure the homeomorphism f is APD, and so the spectrum involved is rotationally invariant.

Proof: Since $\widehat{(\frac{1}{\sigma})} = \frac{1}{\hat{\sigma}}$, we have just to prove for $\lambda > 0$, if $\mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E)) \cap \mathcal{S}_\lambda = \emptyset$ then $\mathrm{Spec}((\sigma\mathcal{M})_f, \Gamma_B(E)) \cap \mathcal{S}_{\hat{\sigma}\lambda} = \emptyset$.

From the assumption $\mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E)) \cap \mathcal{S}_\lambda = \emptyset$, we know that there is an invariant splitting $\Gamma_B(E) = \Gamma_B^{<\lambda}(E) \oplus \Gamma_B^{>\lambda}(E)$ of the space of bounded sections. Let $\mathcal{M}_f^{<\lambda}$ and $\mathcal{M}_f^{>\lambda}$ be the corresponding restrictions of the transfer operator on these closed Banach subspaces. Since this spectral subspaces are Γ_B-linear subspaces, they are also invariant under the action of $\sigma\mathcal{M}_f$. Notice that the splitting $E = E^{<\lambda} \oplus E^{>\mu}$ is invariant for both M and σM.

Then, under the assumptions on σ (constant modulus) or f (unique ergodicity),

$$\begin{aligned} r_s((\sigma\mathcal{M})_f, \Gamma_B^{<\lambda}(E)) &= \lim_{m\to\infty} \|\sigma(f^{-1}(\theta))\ldots\sigma(f^{-m}(\theta))\|_\infty^{\frac{1}{m}} \, \|\left(\mathcal{M}_f^{<\lambda}\right)^m\|_\infty^{\frac{1}{m}} \\ &= \hat{\sigma}\; r_s(\mathcal{M}_f, \Gamma_\infty^{<\lambda}) \\ &< \hat{\sigma}\lambda \ . \end{aligned}$$

The same arguments are used to prove

$$r_i((\sigma\mathcal{M})_f, \Gamma_B^{>\lambda}) \quad > \quad \hat{\sigma}\lambda \ .$$

Therefore, $\mathrm{Spec}((\sigma\mathcal{M})_f, \Gamma_B(E)) \cap \mathcal{S}_{\hat{\sigma}\lambda} = \emptyset$. The rest of the proof is straightforward. □

The following result gives a characterization of the spectral radius ρ in terms of the maximal Lyapunov exponents of invariant measures. For the proof we refer to [**CL99**, Thorem 8.15, p. 269]. See also [**JPS87, LS90**].

THEOREM 2.46. *With the notations above we have*

$$\rho(\mathcal{M}_f) = \sup \lambda^+(M, \mu) \tag{2.45}$$

where the supremum is taken over all the ergodic measures of f.

Of course, from Theorem 2.46 one can obtain similar formulas for the spectral radius of the inverse. If the spectrum is contained in different annuli, we can obtain a formula for the edges of the annuli by applying (2.45) to the cocycle restricted to the invariant bundle.

2.5. Dependence of the spectrum on parameters

In this section we study the persistence of the spectral gap and the associated invariant subbundles under perturbations of vector bundle automorphisms. This dependence was undertaken, for instance, in [**HPS77, SS78**].

As we will see, stronger results can be obtained for transfer operators over rotations, see Part 3, that lead to the perturbation theory of normally hyperbolic invariant tori for quasi-periodic systems, see Chapter 15.

Given two C^r bundles E and F defined over the same base manifold $\mathcal{P}$, we denote by $\mathrm{Mor}_{C^r}(E, F)$ the Banach space of C^r vector bundle maps from E to F, with the C^r-topology. As usual, we use trivialization charts to introduce the topology. If we fix the dynamics on the base manifold with a C^r diffeomorphism $f : \mathcal{P} \to \mathcal{P}$, we denote by $\mathrm{Mor}_{C^r,f}(E, F)$ the space of C^r vector bundle maps over f. Both $\mathrm{Mor}_{C^r}(E, F)$ and $\mathrm{Mor}_{C^r,f}(E, F)_f$ are closed subspaces of $C^r(E, F)$. The definitions can be extended to other categories, such as analytic, Sobolev, etc.

REMARK 2.47. As was pointed out in [**HPS77**], the transfer operators $\mathcal{M}_f$, $\bar{\mathcal{M}}_{\bar{f}}$ associated to two close vector bundle automorphisms M_f, $\bar{M}_{\bar{f}}$ are not necessarily close as operators on $\Gamma_B(E)$. So, we cannot apply general spectral theory to prove that the corresponding spectra are close.

The following is an illustrating example. Consider a trivial bundle $E = \mathcal{P} \times \mathbb{C}^n$. Let $M : \mathcal{P} \to \mathrm{GL}_n(\mathbb{C})$ be the matrix valued continuous map generating two vector bundle automorphisms M_f, M_g, where $f, g : \mathcal{P} \to \mathcal{P}$ are two different homeomorphisms. The distance of the corresponding transfer operators, as acting on $\Gamma_B(E)$, can be estimated from below as follows. Let $\theta_0 \in \mathcal{P}$ such that $f(\theta_0) \neq g(\theta_0)$. Take any $v_0 \in E_{\theta_0}$ such that $|v_0| = 1$ and construct the bounded section

$$v_0(\theta) = \begin{cases} v_0 \text{ if } \theta = \theta_0 \ , \\ 0 \text{ if } \theta \neq \theta_0 \ , \end{cases}$$

whose norm is 1. Then:

$$\begin{aligned} \|\mathcal{M}_f - \mathcal{M}_g\|_{\Gamma_B} &\geq \|\mathcal{M}_f v - \mathcal{M}_g v\|_\infty \\ &= \sup_{\theta \in \mathcal{P}} |M(f^{-1}(\theta)) v(f^{-1}(\theta)) - M(g^{-1}(\theta)) v(g^{-1}(\theta))| \\ &= |M(\theta_0) v_0| \\ &\geq \left| M(\theta_0)^{-1} \right|^{-1} . \end{aligned}$$

The best lower bound, which is general on f,g is:

$$\|\mathcal{M}_f - \mathcal{M}_g\|_{\Gamma_B} \geq \inf_{\theta \in \mathcal{P}} \left| M(\theta)^{-1} \right|^{-1} .$$

With similar arguments, we obtain the same lower bound for $\|\mathcal{M}_f - \mathcal{M}_g\|_{C^0}$.

An exception to the previous comments and constructions is the case in which the base homeomorphisms coincide ($f = \bar{f}$). If M_f and $\bar{M}_f$ are two vector bundle automorphisms over the same base homeomorphism f, then

$$\|\mathcal{M}_f - \bar{\mathcal{M}}_f\|_B \leq \|M - \bar{M}\|_\infty .$$

Since we cannot apply functional analysis (Theorem A.20) to study the perturbation of the spectrum in a general situation, a more dynamical construction has to be performed (cf. [**HPS77**]).

THEOREM 2.48. *Let M_f be a C^0 vector bundle automorphism. Assume that*

$$\operatorname{Spec}(\mathcal{M}_f, \Gamma_B(E)) \cap \mathcal{A}_{\lambda,\mu} = \emptyset \ ,$$

where $0 < \lambda \le \mu$. Let $E = E^{<\lambda} \oplus E^{>\mu}$ be the corresponding continuous invariant splitting.

Then, we can find a C^0 neighborhood $\mathcal{U}$ of M_f in $\operatorname{Mor}_{C^0}(E)$ such that:

$$\bar{M}_{\bar{f}} \in \mathcal{U} \Rightarrow \operatorname{Spec}(\bar{M}_{\bar{f}}, \Gamma_B(E)) \cap \mathcal{A}_{\lambda,\mu} = \emptyset \ .$$

Moreover, the mapping that to $\bar{M}_{\bar{f}}$ associates the continuous splitting $E = \bar{E}^{<\lambda} \oplus \bar{E}^{>\mu}$ is continuous when we give the space of bundle splittings the C^0 topology and $\mathcal{U}$ the C^0 topology.

The map

$$M_f \longrightarrow \mathcal{A}(\operatorname{Spec}(\mathcal{M}_f, \Gamma_B(E)))$$

is continuous, when consider from C^0 vector bundle automorphisms (with the C^0 topology) to compact sets (with Hausdorff topology).

Proof: The following construction of the splitting $E = \bar{E}^{<\lambda} \oplus \bar{E}^{>\mu}$, invariant under $\bar{M}_{\bar{f}}$, appears, *mutatis mutandis*, in [**HPS77**]. We will do here the construction of $\bar{E}^{<\lambda}$, but a similar construction can be also carried for $\bar{E}^{>\mu}$.

First, recall that for $\varepsilon > 0$ small enough, we can construct a norm adapted to the splitting $E = E^{<\lambda} \oplus E^{>\mu}$, invariant under M_f, such that:

$$\|A(\theta)\| \le \lambda - \varepsilon \ , \ \|D(f^{-1}(\theta))^{-1}\| \le (\mu + \varepsilon)^{-1} \ , \tag{2.46}$$

where $A(\theta) = M(\theta)_{|E_\theta^{<\lambda}}$ and $D(\theta) = M(\theta)_{|E_\theta^{>\mu}}$.

We represent $\bar{M}_{\bar{f}}$ relatively to that splitting by

$$\bar{M}_{\bar{f}}(\theta) = \begin{pmatrix} \bar{A}(\theta) & \bar{B}(\theta) \\ \bar{C}(\theta) & \bar{D}(\theta) \end{pmatrix} \ .$$

That is: $\bar{A}(\theta) = \Pi^{<\lambda}_{\bar{f}(\theta)} \bar{M}(\theta) \Pi^{<\lambda}_\theta$, $\bar{B}(\theta) = \Pi^{<\lambda}_{\bar{f}(\theta)} \bar{M}(\theta) \Pi^{>\mu}_\theta$, etc. With this notation, notice that $\bar{B}(\theta)$ and $\bar{C}(\theta)$ are "small" and $\bar{A}(\theta)$ and $\bar{D}(\theta)$ are "close" to $A(\theta)$ and $D(\theta)$, respectively (using the C^0 topology).

We will write the invariant bundle $\bar{E}^{<\lambda}$ for $\bar{M}_{\bar{f}}$ as the graph of a linear map $W_\theta : E^{<\lambda}_\theta \to E^{>\mu}_\theta$. A point $(v_\theta, W_\theta v_\theta)$, with $v_\theta \in E^{<\lambda}_\theta$, gets mapped by $\bar{M}(\theta)$ onto

$$(\bar{A}(\theta) v_\theta + \bar{B}(\theta) W_\theta v_\theta, \bar{C}(\theta) v_\theta + \bar{D}(\theta) W_\theta v_\theta) \ . \tag{2.47}$$

We see that the fact that the bundle $\bar{E}^{<\lambda}$ is invariant under $\bar{M}_{\bar{f}}$ just means that the second component of (2.47) is $W_{\bar{f}(\theta)}$ applied to the first component. That is

$$W_{\bar{f}(\theta)}(\bar{A}(\theta) + \bar{B}(\theta) W_\theta) = \bar{C}(\theta) + \bar{D}(\theta) W_\theta \ ,$$

or, equivalently,

$$\mathcal{T}_{\bar{M}_{\bar{f}}}(W) = W \ , \tag{2.48}$$

where
(2.49)
$$\begin{array}{rccc} \mathcal{T}: & \mathrm{Mor}_{C^0}(E,E)\times \mathrm{Mor}_{C^0,\mathrm{id}}(E^{<\lambda},E^{>\mu}) & \longrightarrow & \mathrm{Mor}_{C^0,\mathrm{id}}(E^{<\lambda},E^{>\mu}) \\ & (\bar{M}_{\bar{f}},W) & \longrightarrow & \mathcal{T}_{\bar{M}_{\bar{f}}}(W) \end{array}$$

is defined by

$$\Big(\mathcal{T}_{\bar{M}_{\bar{f}}}(W)\Big)_\theta = \bar{D}(\theta)^{-1}(W_{\bar{f}(\theta)}(\bar{A}(\theta)+\bar{B}(\theta)W_\theta)-\bar{C}(\theta)) \ .$$

We note that when $\bar{M}_{\bar{f}} = M_f$ the operator reduces to

$$\big(T_{M_f}(W)\big)_\theta = D(\theta)^{-1}W_{f(\theta)}A(\theta) \ .$$

Using the adapted norm (2.46), this is a contraction in $\mathrm{Mor}_{C^0,\mathrm{id}}(E^{<\lambda},E^{>\mu})$, whose fixed point is $W(\theta)=0$. If $\bar{M}_{\bar{f}}$ is close enough to M_f, then $\bar{T}_{\bar{M}_{\bar{f}}}$ is a contraction in $\mathrm{Mor}_{C^0,\mathrm{id}}(E^{<\lambda},E^{>\mu})$ and we construct the invariant bundle $\bar{E}^{<\lambda}$ by using the fixed point theorem.

From here, we obtain that $\mathrm{Spec}(\mathcal{M}_{\bar{f}},\Gamma_B(E))\cap\mathcal{A}_{\lambda,\mu}=\emptyset$ if $\bar{M}_{\bar{f}}$ is close enough to M_f. □

COROLLARY 2.49. *Let $M^{\varepsilon}_{f_\varepsilon}: E\to E$ be a C^r family of C^r vector bundle automorphisms on E, with corresponding base C^r diffeomorphisms $f_\varepsilon:\mathcal{P}\to\mathcal{P}$, where $\varepsilon\in B_0 = B(0,\varepsilon_0)\subset\mathbb{R}^p$ and $\varepsilon_0>0$. This means that $M: E\times B_0\to E$ defined by $M(\theta,\varepsilon)v_\theta = M^\varepsilon(\theta)v_\theta$ is C^r jointly in θ and ε.*

Suppose that the Lipschitz constants of the C^r diffeomorphisms f_ε^{-1} are uniformly bounded: $\mathrm{Lip}(f_\varepsilon^{-1})\le L$. Suppose that

1) *For $\varepsilon=0$, $\mathrm{Spec}(\mathcal{M}^0_{f_0},\Gamma_B(E))\cap\mathcal{A}_{\lambda,\mu}=\emptyset$, where $0<\lambda<\mu$;*
2) *The global Lipschitz constant satisfies $\frac{\lambda}{\mu}L^r<1$.*

Then, there exists $0<\varepsilon_1<\varepsilon_0$ such that for all $\varepsilon\in B_1=B(0,\varepsilon_1)$:

a) *$\mathrm{Spec}(\mathcal{M}^\varepsilon_{f_\varepsilon},\Gamma_B(E))\cap\mathcal{A}_{\lambda,\mu}=\emptyset$;*
b) *The corresponding invariant splitting $E=E^{<\lambda}_\varepsilon\oplus E^{>\mu}_\varepsilon$ is C^r;*
c) *The splitting is C^r jointly in θ and ε.*

Proof: a) is a direct consequence of the previous theorem. b) follows from the invariant section theorem (see Theorem 2.18), using hypothesis 2).

To prove c), we define the extended bundle $\hat{E}=E\times\bar{B}_1$ over $\hat{\mathcal{P}}=P\times\bar{B}_1$, and the extended vector bundle automorphism $\hat{M}:\hat{E}\to\hat{E}$ over $\hat{f}:\hat{P}\to\hat{P}$, by

$$\hat{f}(\theta,\varepsilon)=(f_\varepsilon(\theta),\varepsilon), \hat{M}(\theta,\varepsilon)(v_\theta,\varepsilon)=(M^\varepsilon(\theta)v_\theta,\varepsilon) \ .$$

Notice that

$$\mathrm{Spec}(\hat{\mathcal{M}}_{\hat{f}},\Gamma_B(\hat{E}))\cap\mathcal{A}_{\lambda,\mu}=\emptyset \ ,$$

and the Lipschitz constant of $\hat{f}^{-1}$ is $\hat{L}=\max(1,\sup_{|\varepsilon|\le|\varepsilon_1|}L_\varepsilon)\le\max(1,L)$, that satisfies $\frac{\lambda}{\mu}\hat{L}<1$. We have just to apply Theorem 2.18 to prove (c). □

REMARK 2.50. We emphasize that the constructions of this section can be implemented numerically (see [**HdlL04**]).

REMARK 2.51. We emphasize that the spectrum is only claimed to be continuous in open sets around a gap. The open set depends a lot on the properties of the spectral projections.

When we consider the closure of sets where the gaps exit, it could well happen that the spectrum at the endpoint. This may happen because the norm of the spectral projections blow up.

In [**HdlL05a**] it is conjectured that this phenomenon happens in the examples that were considered in [**HdlLa**]. Namely, in the cocycles associated to some normally hyperbolic manifolds it happens that the spectrum remains continuous in an open interval of parameters. Nevertheless, the norm of the spectral projections blows up as we approach the boundary. As a consequence, the spectrum at the boundary is larger than the limit (the gap is filled).

Part 2

Mather theory of transfer operators

In this part we go over the Mather theory of transfer operators, that is the application of Spectral Theory to study dynamical properties of cocycles.

In Section 3 we repeat some of the arguments of [**Mat68**] in the slightly more general framework of vector bundle automorphisms. Notice that, in principle, the spectrum of a bounded linear operator depends heavily on the Banach space on which it is acting. We have found the study in the lower level of sections, just bounded sections, very fruitful. This point of view simplifies many arguments. As we will see, the spectrum of the transfer acting on other spaces, such as continuous sections or L^p sections ($p > 1$), coincides with that on bounded sections (see also [**CS80**]).

An important ingredient is a device known as Mather localization (see also [**CL99**]), that lies in finding approximate bounded eigensections of a simple type, that means supported on orbits. This is the content of Lemma 3.1. We have refined the arguments in [**Mat68**] so they can be applied to prove the equality of approximate point spectrum in bounded and continuous sections (Theorem 3.7), and, as we will see in Section 4, to obtain a characterization of the spectrum based on the behavior of the transfer on orbits at least in some cases. The invariance under rotation of the spectrum in the case that the homeomorphism on the base manifold has a dense set of aperiodic orbits is also a corollary (Theorem 3.11). Another ingredient is the characterization of spectral projections carried out in Theorem 3.14. The generality of the framework presented here let us work in a great variety of spaces of sections, and will be also very useful in the further Part 3 and Part 4. To prove this theorem, we use heavily some arguments in [**Mn78**] (see also [**CL99**]), that we summarize in Lemma 2.15. As a corollary of Theorem 3.14, we obtain the equality of the spectrum in bounded and continuous sections (Theorem 3.25). This theorem also contains a similar result for L^p sections. Notice that these equalities among the spectra reduce the study the action of the transfer operator on bounded sections, that are the simpler ones. For instance, repeating some arguments in [**Swa81**], that are supported in a deep result in [**SS76a**], we prove in Theorem 3.19 that, if the dynamics on the base manifold is chain-recurrent and does not have periodic orbits, then the full spectrum is point spectrum (when acting on bounded sections). Section 3.7 is devoted to study the spectrum of the transfer operator acting on spaces of C^r sections. As we will see, making smaller the space (taking C^r sections instead of bounded or continuous sections) the spectrum of the operator grows. The results will be improved, including also Sobolev regularities, in Part 3, in the special case of transfer operators over rotations.

In Section 2.5 we study the persistence of the spectral gaps and the associated invariant subbundles under perturbations of vector bundle automorphisms (see also [**HPS77**]).

In Section 3.9 we will study the spectral implication of lifting of the cocycle to a covering space. That is to say, what is the relationship between the spectra of a vector bundle automorphism and its lifting when covering the base manifold. We obtain, for instance, that if the dynamics on the base manifold is chain-recurrent and has a dense set of aperiodic orbits, then both spectra coincide.

Finally, the last section is devoted to the spectral theory of triangular transfer operators, that arise for instance when considering push forward operators acting on jets in Part 4, or the construction of a great variety of invariant manifolds [**dlL97, CFdlL03a, CFdlL03b, CFdlL05, HdlLb, HdlL04, HdlL05a**]).

CHAPTER 3

Mather theory for the spectrum of vector bundle maps

In this section we repeat the arguments in [**Mat68**] in the slightly more general framework of vector bundle automorphisms. Even if the notation changes, most of the ideas come from [**Mat68**]. Nevertheless, stating them in the greater generality of vector bundles will allow us to perform induction arguments, either by passing to invariant subbundles, that are not necessarily trivial (see also Part 3, or by passing to tangent bundles (see Part 4). Moreover, some of the constructions of approximate eigensections, etc. will be performed in such a way that they are of use for later developments. We will also find it convenient to consider at the same time the spaces of continuous sections and the spaces of bounded sections. Later, this will make it easy to shorten significantly some of the arguments.

We will first show that the Weyl spectrum does not depend on whether we consider $\Gamma_B(E)$ or $\Gamma_{C^0}(E)$. Then, we will establish invariance under rotation and finally, the characterization of spectral projections. As a corollary, we will obtain that the spectra on $\Gamma_B(E)$ and $\Gamma_{C^0}(E)$ agree. In the case that f preserves a topological measure μ (i.e. a measure which is positive on non-empty open sets), we will show the same results for L^p sections ($p > 1$).

3.1. Mather localization

Theorem 3.7 below states the equality of the approximate point spectra

$$\mathrm{Spec}_W(\mathcal{M}_f, \Gamma_B(E)) = \mathrm{Spec}_W(\mathcal{M}_f, \Gamma_{C^0}(E)) \ .$$

Since the inclusion $\mathrm{Spec}_W(\mathcal{M}_f, \Gamma_{C^0}(E)) \subset \mathrm{Spec}_W(\mathcal{M}_f, \Gamma_B(E))$ is obvious, it suffices to establish the opposite inclusion. So, given an approximate eigensection $v \in \Gamma_B(E)$ we will produce another one $\tilde{v} \in \Gamma_{C^0}(E)$.

The first task will be to show that we can assume that the bounded approximate eigensection is of a particularly simple kind. Once we have accomplished this, we will fatten up this approximate eigensection to produce a continuous approximate eigensection, that is localized around an orbit.

This device is also known as Mather localization, and it is based on the ideas of [**Mat68**] (see also [**CL99**]), where approximate continuous eigensections of push forward operators of APD diffeomorphisms were considered. A similar argument will occur in Part 3 and Part 4, where other spaces of sections are considered. Since the process of fattening up a function is rather cumbersome, it is to our advantage to get these eigensections to be as simple as possible. Much of this argument will be used again later. Roughly, we will show that it is always possible to obtain approximate eigensections that are supported either on a finite segment of an orbit or on a periodic orbit. Moreover, these eigensections are obtained by letting the map act and multiply by a suitable number.

LEMMA 3.1. *Let $z \in \mathrm{Spec}_W(\mathcal{M}_f, \Gamma_B(E))$. We can find a sequence of approximate eigensections v^n supported on finite segments of orbits $\{f^i(\theta_n)\}_{i\in\mathbb{Z}}$ with $|v^n(\theta_n)| \geq 3/4$, $\|v^n\|_\infty \leq 1$. Moreover, we can assume that θ_n tends to $\theta_* \in \mathcal{P}$ and the approximate eigensections satisfy one of the following sets of properties, depending on the periodicity properties of the points around θ_*:*
(a) *If for all open set $U \subset \mathcal{P}$ such that $\theta_* \in U$, the function $p_{|U}$ is not bounded, then:*

- (i) *$v = v^n$ is supported in the finite segment of orbit $\{f^i(\theta_n)\}_{i=-N}^N$, with $N = [1/\varepsilon_n]$;*
- (ii) *there exist constants $\gamma_{n,-N}, \cdots, \gamma_{n,N-1} \in [\frac{1}{2}, \frac{3}{2}]$, such that*
$$v(f^{i+1}(\theta_n)) = \frac{\gamma_{n,i}}{z} M(f^i(\theta_n))v(f^i(\theta_n)) ,$$
for $i = -N, \cdots, N-1$;
- (iii) $\|\mathcal{M}_f v - zv\|_\infty \leq 2\varepsilon_n |z| \xrightarrow{n\to\infty} 0$.

(b) *If there exists an open set $U \subset \mathcal{P}$ such that $\theta_* \in U$ and $p_{|U}$ is bounded, and in particular $p(\theta_*) = N_*$, then:*

- (i') *$v = v^n$ is supported in the periodic point $\theta_n \in U$, whose minimal period N does not depend on n and it is multiple of N_*;*
- (ii') *there exists a complex constant z_n such that*
$$v(f^{i+1}(\theta_n)) = \frac{1}{z_n} M(f^i(\theta_n))v(f^i(\theta_n)) ,$$
for $i = 0, \cdots N-1$; (In particular, $v(\theta_n)$ is an eigenvector of $M(\theta_n, N)$ whose eigenvalue is z_n^N).
- (iii') $\|\mathcal{M}_f v - zv\|_\infty \leq |z_n - z| \xrightarrow{n\to\infty} 0$.

Moreover, if f is APD, it is the first alternative **(a)** *that holds.*

Proof: By the assumption that $z \in \mathrm{Spec}_W(\mathcal{M}_f, \Gamma_B(E))$ we can find a sequence of approximate eigensections w^n. For each of these eigensections, we can find a point θ_n such that

$$|w^n(\theta_n)| \geq 3/4. \tag{3.1}$$

By compactness of $\mathcal{P}$ we can find a point θ_* which is an accumulation of $\{\theta_n\}_n$. By passing to a subsequence, we can assume that it is a limit.

A first localization of the approximate eigensection w^n is just considering the bounded section restricted to the orbit of θ_n, so w^n is a bounded sequence of vectors $\{w^n(f^i(\theta_n))\}_{i\in\mathbb{Z}}$.

Using Thychonov theorem, we note that we can pass to a further subsequence in such a way that $w^n(f^i(\theta_n)) \xrightarrow{n\to\infty} w_i$, for all $i \in \mathbb{Z}$. Note that, because of our assumptions $|w_i| \leq 1$, $|w_0| \geq 3/4$, and $\Pi(w_i) = f^i(\theta_*)$. Since

$$|M(f^i(\theta_n))w^n(f^i(\theta_n)) - zw^n(f^{i+1}(\theta_n))| \xrightarrow{n\to\infty} 0 \ ,$$

we have

$$M(f^i(\theta_*))w_i = zw_{i+1} \ , \tag{3.2}$$

for all $i \in \mathbb{Z}$.

If $\{f^i(\theta_*)\}_{i\in\mathbb{Z}}$ is an aperiodic section, we can consider w_i to be a section supported on $\{f^i(\theta_*)\}_{i\in\mathbb{Z}}$. Note that in this case, w_i is an eigenvector and that z is in the point spectrum.

On the other hand, if $f^N(\theta_*) = \theta_*$, we have to argue further. We have to distinguish depending on whether

$$w_{i+N} = w_i \quad \forall i \in \mathbb{Z} \tag{3.3}$$

whether this is not the case.

If (3.3) holds, again, we can consider $\{w_i\}_{i\in\mathbb{Z}}$ as a section on the orbit and it is an eigenvalue for an eigenvalue z.

If (3.3) fails, we cannot consider w_i as a section supported on the finite set $\{f^i(\theta_*)\}_{i\in\mathbb{Z}}$. On the other hand, we have that the sequence w_{Nj} satisfies

$$z^{-N}M(f^{N-1}(\theta_*))\cdots M(\theta_*)\, w_{Nj} = w_{N(j+1)}$$

Since the sequence w_{Nj} is bounded uniformly in j, we conclude that the only possibly is that w_N is an eigenvalue of modulus 1 of the matrix $z^{-N}M(f^{N-1}(\theta_*))\cdots M(\theta_*)$.

Remark 3.2. As we will see in Section 4, this implies that there is an eigenvalue on the periodic orbit of the same eigenvalue.

Hence, we have shown that if $z \in \mathrm{Spec}_W(\mathcal{M}_f, \Gamma_B(E))$ we have that for some $\eta \in \mathbb{R}$ we have $ze^{i\eta} \in \mathrm{Spec}_W(\mathcal{M}_f, \Gamma_B(E))$

The case $\eta \neq 0$ happens only when the eigenvalue is supported in a periodic orbit.

In particular, if f is NPO, the Weyl spectrum is point spectrum (on bounded sections). One important case of systems satisfying NPO is irrational rotations in the circle, which are the main case considered in Part 3 motivated by [**HdlLb, HdlL04, HdlL05a**].

There are two alternatives for θ_*: either there are points of periods arbitrarily high, possibly infinite, arbitrarily close to θ_*; or there is a neighborhood of θ_* in which all the points are periodic, and the periods are bounded. We will analyze both alternatives separately.

(a) The first alternative is that for all open set $U \subset \mathcal{P}$ such that $\theta_* \in U$, the function $p_{|U}$ is not bounded.

Notice that if f is APD, this is the only possible alternative.

We distinguish two cases:

(a.1) If θ_* is aperiodic, we define $N = [1/\varepsilon]$ and set

$$\begin{aligned} &v(f^i(\theta_*)) = v_i = (1 - |i|/(N+1))_+ w_i \text{ for } |i| \le N \ , \\ &v(\theta) = 0 \text{ otherwise} \ , \end{aligned}$$

where we denote $(t)_+ = \max(t, 0)$. This section is supported in the finite segment $f^{-N}(\theta_*), \cdots, f^N(\theta_*)$. In this case $\theta_n = \theta_*$.

Using (3.2) we have

$$\begin{aligned} |M(f^i(\theta_*))v_i - zv_{i+1})| &= \left| \left(1 - \frac{|i|}{N+1}\right)_+ - \left(1 - \frac{|i+1|}{N+1}\right)_+ \right| |z||w_{i+1}| \\ &\le |z|/(N+1) < |z|\varepsilon \end{aligned}$$

and $M(\theta)v(\theta) - zv(f(\theta)) = 0$ for all points θ not on the orbit of θ_*. Notice also that v satisfies

$$v(f^{i+1}(\theta_*)) = \gamma_i \frac{1}{z} M(f^i(\theta_*))v(f^i(\theta_*)) \ , \tag{3.4}$$

for $i = -N, \cdots, N-1$, where

$$\gamma_i = \frac{(1 - |i+1|/(N+1))_+}{(1 - |i|/(N+1))_+} \ .$$

REMARK 3.3. Notice that in this case we have found a bounded eigensection for z supported on the orbit of θ_*. That is, there exists a bounded sequence $\{w_k\}_{k\in\mathbb{Z}}$ such that $\Pi(w_k) = f^k(\theta_*)$ and

$$M(f^k(\theta_*))w_k = zw_{k+1} \ .$$

Notice also that for a given $\alpha \in \mathbb{R}$, the sequence $\{\bar{w}\}_{k\in\mathbb{Z}}$ defined by $\bar{w}_k = e^{-k\alpha\mathbf{i}} w_k$ satisfies

$$M(f^k(\theta_*))\bar{w}_k = e^{\alpha\mathbf{i}} z\bar{w}_{k+1} \ .$$

That is to say, from a eigenvalue whose bounded eigensection is supported in an aperiodic orbit we obtain a whole circle of eigenvalues.

(a.2) If θ_* is periodic, we can find a point θ_n whose period is $p(\theta_n) > \frac{2}{\varepsilon}$ (possibly infinite), and so close to θ_* that the finite segment of orbit $\{f^i(\theta_n)\}_{i=-N}^N$ with $N = [1/\varepsilon]$, does not leave the trivializations around the corresponding segment of orbit $\{f^i(\theta_*)\}_{i=-N}^N$, and the distance between both segments of orbits is smaller than a small enough $\delta > 0$.

Working on the trivializations – so that we can freely add vectors at different points –, we define the section

$$\begin{aligned} &v(f^i(\theta_n)) = v_i = (1 - |i|/(N+1))_+ \frac{1}{z^i} M(\theta_n, i)w_0 \text{ for } |i| \le N \ , \\ &v(\theta) = 0 \text{ otherwise} \ . \end{aligned} \tag{3.5}$$

Using (3.2), it is easy to show that v is an approximate eigensection:

$$\begin{aligned}|M(f^i(\theta_n))v_i - zv_{i+1})| &\leq \varepsilon \tfrac{1}{|z|^i} |M(\theta_n, i+1)w_0| \\ &\leq \varepsilon \tfrac{1}{|z|^i} (|(M(\theta_n, i{+}1) - M(\theta_*, i{+}1))w_0| \\ &\qquad + |M(\theta_*, i{+}1)w_0|) \\ &\leq \varepsilon|z| \; (\eta_{i+1}(\delta) + 1) \leq 2\varepsilon|z|,\end{aligned}$$

where η_i is the modulus of continuity of $\frac{1}{z^i}M(\cdot, i)$, and we choose $\delta > 0$ small enough. Obviously, for all the points θ not on the segment of orbit of θ_*, $M(\theta)v(\theta) - zv(f(\theta)) = 0$.

It is obvious from the construction that v satisfies (3.4).

Remark 3.4. Instead of using local trivializations, a more geometric construction is using a connector T on the bundle E (see Definition 1.23 in Part 1).

We define the section $v(f^i(\theta_n)) = (1 - |i|/(N+1))_+ \frac{1}{z^i} M(\theta_n, i) \mathrm{T}_{\theta_*, \theta_n} w_0$ if $|i| \leq N$ and $v(\theta) = 0$ otherwise.

The main reason why we have used the local trivializations is that the typography is significantly easier since we can use additive notation.

Remark 3.5. Notice that if $p(\theta_*) = N_*$, then the sequence

$$w_{N_* k} = \frac{1}{z^{kN_*}} M(\theta_*, N_* k) w_0 = \left(\frac{1}{z^{N_*}} M(\theta_*, N_*) \right)^k w_0$$

is bounded ($|w_i| \leq 1$ for all $i \in \mathbb{Z}$). So then, the matrix $M(\theta_*, N_*)$ has eigenvalues of modulus $|z|^{N_*}$, and w_0 is in the space spanned for the corresponding eigenvectors.

Notice also that if $\tilde{z} = e^{\alpha \mathbf{i}} z$ with $\alpha \in \mathbb{R}$, then the definition (3.5) with $\tilde{z}$ instead of z also produces an approximate eigensection, in this case for the approximate eigenvalue $\tilde{z} = e^{\alpha \mathbf{i}} z$.

In summary, if $z \in \mathbb{C}$ is such that z^{N_*} is an eigenvalue of $M(\theta_*, N_*)$, then z is an eigenvalue of $\mathcal{M}_f$, and it produces a whole circle of radius $|z|$ of approximate eigenvalues.

(b) The second alternative is that there exist an open set $U \subset \mathcal{P}$ such that $\theta_* \in U$ and $p_{|U}$ is bounded.

Hence, we can consider the points θ_n satisfying (3.1) inside this neighborhood U, so the corresponding approximate eigensection can be restricted to their orbits, that are periodic (and the periods are bounded). We can assume that all the minimal periods are equal to N, $p(\theta_n) = N$, and this implies that N is a multiple of $p(\theta_*) = N_*$: $N = N_* \ell$ for some natural ℓ.

Thychonov theorem implies

$$\begin{aligned} &M(f^i(\theta_*))w_i = zw_{i+1} \quad 0 \leq i < N-1 \;, \\ &M(f^{N-1}(\theta_*))w_{N-1} = zw_0 \;, \end{aligned} \tag{3.6}$$

but, however, there is no reason why $w_{i+N_*} = w_i$. Hence, we cannot consider $\{w_i\}_{i=0}^{N-1}$ as a function over the periodic orbit $\{f^i(\theta_*)\}_{i=0}^{N-1}$. Using (3.6), we obtain that z^N and w_0 are an eigenvalue and a corresponding eigenvector of $M(\theta_*, N) = (M(\theta_*, N_*))^\ell$. For large enough n, $M(\theta_*, N)$ is close to $M(\theta_n, N)$. Hence, we can find a z_n close to z so that z_n^N is an eigenvalue of $M(\theta_n, N)$. Let $\tilde{v}_0$ be an eigenvector corresponding to such eigenvalue. Now, we construct the function $\tilde{v}(f^i(\theta_n)) = \frac{1}{z_n^i} M(\theta_n, i)\tilde{v}_0$ and zero otherwise. This function satisfies $\mathcal{M}_f \tilde{v} - z_n \tilde{v} = 0$ and is supported in a periodic orbit. Finally, consider $v = \tilde{v}/\|\tilde{v}\|_\infty$. This satisfies the second set of properties.

REMARK 3.6. Since $z^N = z^{N_* l}$ is an eigenvalue of $M(\theta_*, N) = (M(\theta_*, N_*))^l$, then $z = e^{2\pi \frac{j}{N}\mathbf{i}}\tilde{z}$,where $j \in \{0, 1, \cdots, l-1\}$ and $\tilde{z}^{N_*}$ is an eigenvalue of $M(\theta_*, N_*)$. Obviously, $\tilde{z}$ is an eigenvalue of $\mathcal{M}_f$, whose bounded eigensection is supported in a periodic orbit.

Conversely, if $\tilde{z}^{N_*}$ is an eigenvalue of $M(\theta_*, N_*)$ and $j = 0, 1, \dots, N-1$, we will see that $z = e^{2\pi \frac{j}{N}\mathbf{i}}\tilde{z}$ is an approximate eigenvalue of $\mathcal{M}_f$. To do so, notice that if $\tilde{w}_0$ is an eigenvector of $M(\theta_*, N_*)$ for $\tilde{z}^{N_*}$, then the finite sequence $\{w_i\}_{i=0}^{N-1}$ defined by

$$w_i = \frac{1}{z^i} M(\theta_*, i)\tilde{w}_0$$

satisfies (3.6). Although $w_{i+N} = w_i$, there is no reason why $w_{i+N_*} = w_i$, but with the same arguments above we prove that z is an approximate eigenvalue of $\mathcal{M}_f$.

In summary, for a periodic point θ_*, of minimal period N_*, if it can be approached by periodic orbits of minimal period N, then for each complex value z such that z^{N_*} is an eigenvalue of $M(\theta_*, N_*)$ we produce a regular N-polygon of approximate eigenvalues of $\mathcal{M}_f$ with one vertex in z.

□

3.2. Equality of Weyl spectra for bounded, continuous and L^p sections

The main result of this subsection is

THEOREM 3.7. *Let $M_f : E \to E$ be a vector bundle automorphism. Then:*

$$\mathrm{Spec}_W(\mathcal{M}_f, \Gamma_B(E)) = \mathrm{Spec}_W(\mathcal{M}_f, \Gamma_{C^0}(E)) \tag{3.7}$$

In the case that f preserves a measure μ we have that

$$\mathrm{Spec}_W(\mathcal{M}_f, \Gamma_B(E)) \supset \mathrm{Spec}_W(\mathcal{M}_f, \Gamma_{L^p(\mu)}(E)) \tag{3.8}$$

for $p > 1$.

If, furthermore, μ is a topological measure

$$\mathrm{Spec}_W(\mathcal{M}_f, \Gamma_B(E)) = \mathrm{Spec}_W(\mathcal{M}_f, \Gamma_{L^p(\mu)}(E)) \tag{3.9}$$

for $p > 1$.

REMARK 3.8. The inclusion $\mathrm{Spec}_W(\mathcal{M}_f, \Gamma_{C^0}(E)) \subset \mathrm{Spec}_W(\mathcal{M}_f, \Gamma_{L^p(\mu)}(E))$ for $p = 2$ was proved in [**CS80**] for push-forward operator of a diffeomorphism over a smooth compact Riemannian manifold, preserving the Lebesgue measure induced by the metric, and under the assumption that f is APD. This, without the assumption of APD, is a particular case of Theorem 3.7

[**CS80**] quote in a remark that their result holds for vector bundle maps + APD + measure absolutely continuous w.r.t Lebesgue, even for the case p= 1. They do not present a proof.

Proof: Since $\mathrm{Spec}_W(\mathcal{M}_f, \Gamma_{C^0}(E)) \subset \mathrm{Spec}_W(\mathcal{M}_f, \Gamma_B(E))$, it suffices to establish the opposite inclusion to prove the first part of Theorem 3.7.

Let $z \in \mathrm{Spec}_W(\mathcal{M}_f, \Gamma_B(E))$. Given an approximate eigensection $v \in \Gamma_B(E)$ we will produce another one $\tilde{v} \in \Gamma_{C^0}(E)$. Notice that we can assume that the bounded approximate eigensection is of a particularly simple kind, that is satisfying one of the set of properties (a) and (b) of Lemma 3.1. Then, we will fatten up this localized approximate eigensection to produce a continuous approximate eigensection. This will prove inclusion $\mathrm{Spec}_W(\mathcal{M}_f, \Gamma_B(E)) \subset \mathrm{Spec}_W(\mathcal{M}_f, \Gamma_{C^0}(E))$.

Once we obtain a continuous approximate eigensection, we will obtain a L^p approximate eigensection with $p > 1$ under the assumption that the measure μ is topological. This will prove inclusion $\mathrm{Spec}_W(\mathcal{M}_f, \Gamma_B(E)) \subset \mathrm{Spec}_W(\mathcal{M}_f, \Gamma_{L^p(\mu)}(E))$.

Proof of $\mathrm{Spec}_W(\mathcal{M}_f, \Gamma_B(E)) \subset \mathrm{Spec}_W(\mathcal{M}_f, \Gamma_{C^0}(E)) \subset \mathrm{Spec}_W(\mathcal{M}_f, \Gamma_{L^p(\mu)}(E))$

The proof will start in the same way as the proof of Lemma 3.1. We will start by considering a sequence of approximate eigensections v^n such that $|v^n(\theta_n)| \geq 3/4$ and assume also that θ_n converges to θ_*.

We will analyze the alternatives (a) and (b) of Lemma 3.1 separately.
(a) In this alternative, the period function was unbounded in any neighborhood of θ_*.

Proceeding as in Lemma 3.1, we obtain a bounded approximate eigensection v supported in a finite segment of orbit $\{f^i(\theta_n)\}_{i=-N}^{N}$, with $f^i(\theta_n) \neq f^j(\theta_n)$ for $i \neq j$ with $|i| \leq N, |j| \leq N$, with $N = [1/\varepsilon]$. The error as approximate eigensection is smaller than $2\varepsilon|z|$. To produce a continuous approximate eigensection, we just replace each of the vectors on the orbit by a function that has support in a small neighborhood chosen in such a way that it is still an approximate eigensection.

Notice that we can pick coordinates around each of the points in the finite segment $\{f^i(\theta_n)\}_{i=-N}^{N}$ in such a way that the bundle is trivialized.

Pick $\rho : \mathbb{R}^+ \to \mathbb{R}$ continuous such that $\rho(0) = 1$, $\rho(t) = 0$ if $t \geq \delta$ with $\delta > 0$ small enough that if $d(\theta, \theta_n) \leq \delta$ then $f^i(\theta)$ is in the neighborhood of $f^i(\theta_n)$ where the trivialization is valid, for all $|i| \leq N$. We can also assume that these trivializing neighborhoods do not overlap. (We will later assume further smallness properties for δ.) Set

$$\rho^0(\theta) = \rho(d(\theta, \theta_n)) \text{ and } \rho^i(\theta) = \rho^0(f^{-i}(\theta)) \ . \tag{3.10}$$

Then, for each $|i| \leq N$, we set in the trivialization on the bundle around $f^i(\theta_n)$

$$\tilde{v}(\theta) = \rho^i(\theta) v(f^i(\theta_n)) \ , \tag{3.11}$$

and the section is zero away of these trivializations. This section is continuous and $\tilde{v}(\theta_n) \geq 3/4$.

Then, working in coordinates, we have for θ with $d(f^{-i}(\theta), \theta_n) < \delta$:

$$\begin{aligned} |M(\theta)\tilde{v}(\theta) - z\tilde{v}(f(\theta))| &= \rho^i(\theta)|M(\theta)v(f^i(\theta_n)) - zv(f^{i+1}(\theta_n))| \\ &\leq |M(\theta) - M(f^i(\theta_n))||v(f^i(\theta_n))| \\ &\quad + |M(f^i(\theta_n))v(f^i(\theta_n) - zv(f^{i+1}(\theta_n))| \\ &\leq \eta_i(\delta) + 2\varepsilon|z| \ , \end{aligned} \tag{3.12}$$

where η_i is the modulus of continuity of $M \circ f^i$, and we use the estimates in Lemma 3.1. We note that both terms are small. In fact, we can make the first term arbitrarily small by making δ sufficiently small. Clearly, outside of these neighborhoods the difference is zero.

This completes the construction of the continuous approximate eigensection in the alternative (a).

Remark 3.9. We note that the above construction is hard to generalize for $p = 1$. On one hand, the error of the approximate eigensection is the sum of the error at all the steps, which should be small. On the other hand, the sum of the error at all the step should be of order 1 because this is what allows to get the error to be small.

Of course, this is not the only possible constructions of eigensections. Clearly, we have $\|M_f v - \lambda v\|_{L^1} \leq \|M_f v - \lambda v\|_{C^0}$. It is perhaps possible to choose sets for smoothing tailored for the function, etc.

We will show now that, under the assumption that μ is a topological measure, a multiple of such continuous approximate eigensection is also an $L^p(\mu)$ approximate eigensection. This is not completely trivial because the continuous approximate eigensections may have very small L^p norm, but we will perform a careful construction.

We will compute now the L^p norm of the continuous approximate eigensection defined in (3.11) and the L^p norm of the error as approximate eigensection. Using the invariance of the measure, we have

(3.13)

$$\begin{aligned} \|\tilde{v}\|_{L^p}^p = \sum_{i=-N}^{N} |v(f^i(\theta_n))|^p \int (\rho^i(\theta))^p \, d\mu = \int (\rho^0)^p d\mu \sum_{i=-N}^{N} |v(f^i(\theta_n))|^p \\ \geq (3/4)^p \int (\rho^0(\theta))^p d\mu > 0 \ , \end{aligned}$$

where the last bound is obtained just because the sum contains a term of size 3/4. Notice also that this bound is positive, because the measure is topological and the function ρ^0 is strictly positive in a open neighborhood. Therefore: $\|\tilde{v}\|_{L^p} \geq (3/4)\|\rho^0\|_{L^p} > 0$.

Bounding each term in (3.12) as in (3.13), we obtain:

$$\begin{aligned} \|\mathcal{M}_f\tilde{v} - z\tilde{v}\|^p_{L^p} &\leq 2^{p-1}\int (\rho^0)^p d\mu \sum_{i=-N}^{N} |v(f^i(\theta_n))|^p \sup_{\theta\in\operatorname{supp}\rho^0} |M(f^i(\theta)) - M(f^i(\theta_n))|^p \\ &+ 2^{p-1}\int (\rho^0)^p d\mu \sum_{i=-N-1}^{N} |M(f^i(\theta_n))v(f^i(\theta_n)) - zv(f^{i+1}(\theta_n))|^p \end{aligned} \tag{3.14}$$

The first term in (3.14) can be made arbitrary small by making the support of ρ^0 sufficiently small since $\sup_{\theta\in\operatorname{supp}\rho^0} |M(f^i(\theta)) - M(f^i(\theta_n))|$ can be made arbitrary small. The second term can be bounded from above by $2^{p-1}\int(\rho^0)^p d\mu\ \varepsilon^p(2N+2)$. Recalling that we took $N = [1/\varepsilon]$, we obtain that, for ε sufficiently small, this term can be made arbitrarily small with respect to $\|\tilde{v}\|^p_{L^p}$, provided that $p > 1$.

So, by multiplying the continuous approximate eigensection $\tilde{v}$ by $\|\rho^0\|^{-1}_{L^p}$ we obtain an L^p approximate eigensection. These arguments complete the analysis of the first alternative.

Remark 3.10. We can also construct the L^p approximate eigensection from the localized bounded approximate eigensection by defining a suitable step eigensection (that is, constant in small trivializing neighborhoods around the points of the finite segment of orbit).

(b) We make now the construction of the continuous and L^p approximate eigensection under the alternative (b) of Lemma 3.1.

A key observation is that there exists an open neighborhood $V \subset U$ of θ_n in which all the periods of the points $\theta \in V$ are multiple of N (the minimal period of θ_n): $p(\theta)/p(\theta_n) \in \mathbb{N}$. If this were not the case, we could construct a sequence of periodic points $\bar{\theta}_k$ tending to θ_n and whose periods are all equal to $\bar{N}$ (notice that there is a finite number of possible periods around θ_n), and such that the period $\bar{N}$ is not a multiple of N. But this is not possible, because

$$\bar{\theta}_k = f^{\bar{N}}(\bar{\theta}_k) \xrightarrow{k\to\infty} \theta_n = f^{\bar{N}}(\theta_n)\ ,$$

so $\bar{N}$ is a period of θ_n, whose minimal period is N.

Following the arguments of the alternative (a), we construct a continuous section by

$$\tilde{v}(\theta) = \rho^i(\theta)v(f^i(\theta_n))\ , \tag{3.15}$$

for θ near enough to $f^i(\theta_n)$, with $0 \le i < N$. Recall that $\rho(t) = 0$ for $t \ge \delta$, and we choose δ small enough to be able of defining this section through trivializations and such that $d(\theta, \theta_n) < \delta$ implies $\theta \in V$. This choice assures that we construct a continuous section (and we do not double the definition of the vector supported on one point).

To check that the continuous section $\tilde{v}$ defined by (3.15) is an approximate eigensection, we proceed as in the alternative (a). The key point is that for a given θ such that $d(f^{-i}(\theta), \theta_n)) < \delta$, the corresponding orbit is periodic of period multiple of N.

To produce an L^p approximate eigensection in the case that μ is a topological measure we proceed just as in the alternative (a). Notice that in this case, the second term in (3.14) can be bounded from above by $2^{p-1}\int(\rho^0)^p d\mu \varepsilon^p N$, but in this case N does not depend on ε. Hence, we produce approximate eigensection even in the case $p = 1$.

This finishes the proof of the first part of Theorem 3.7.

Proof of $\operatorname{Spec}_W(M_f, \Gamma_{L^p(\mu)}(E)) \subset \operatorname{Spec}_W(M_f, \Gamma_B(E))$.

It suffices to show that given an approximate eigensection in L^p we can produce an approximate eigensection in $\Gamma_B(E)$. Notice that for this part of the argument, we do not need the assumption that the invariant measure μ is topological.

Let v be an L^p approximate eigensection with $\|v\|_{L^p} = 1$, $\|\mathcal{M}_f v - zv\|_{L^p} = \varepsilon$. Since C^0 is dense in L^p we can assume without loss of generality that v is C^0. Notice, however, that v is not in principle an approximate eigensection in C^0.

Denote $e(\theta) = M(\theta)v(\theta) - zv(f(\theta))$ and for $N \in \mathbb{N}$, that will specify later, denote

$$r(\theta) = \begin{cases} \dfrac{1}{|v(\theta)|^p} \displaystyle\sum_{i=-2N}^{2N} |e(f^i(\theta))|^p & \text{if } v(\theta) \neq 0 \\ 0 & \text{otherwise} \end{cases} \tag{3.16}$$

$$s(\theta) = \begin{cases} \dfrac{1}{|v(\theta)|^p} \displaystyle\sum_{i=-2N}^{2N} |v(f^i(\theta)|^p & \text{if } v(\theta) \neq 0 \\ 0 & \text{otherwise} \end{cases} \tag{3.17}$$

We consider the measure $\nu = |v(\cdot)|^p \mu$, which is by definition absolutely continuous with respect to μ. Note that $\nu(\mathcal{P}) = 1$ and $\int r d\nu \le (4N+1)\varepsilon^p$, $\int s d\nu = (4N+1)$. We can take, in particular, $(4N+1) = \varepsilon^{-\alpha}$ for some $\alpha > 0$, that will be specified later.

If we apply Chebyshev inequality to both positive and ν-integrable functions r and s, we obtain

$$\nu(r(\theta) \ge a) \le \frac{1}{a}\int r(\theta) d\nu \le \frac{1}{a}\varepsilon^{p-\alpha},$$

$$\nu(s(\theta) \geq b) \leq \frac{1}{b} \int s(\theta) d\nu = \frac{1}{b} \varepsilon^{-\alpha},$$

for positive values a,b. By taking $a = \varepsilon^{p-\alpha-\delta}$ and $b = \varepsilon^{-\alpha-\delta}$ for some $\delta > 0$, we get that r is greater than $\varepsilon^{p-\alpha-\delta}$ in a set Ω_r of ν-measure less than ε^{δ} and s is greater than $\varepsilon^{-\alpha-\delta}$ in a set Ω_s of ν-measure less than ε^{δ}.

At this point, we will assume that α, δ are so small that $p - \alpha - \delta > 0$, i.e. $1 - \alpha/p - \delta/p > 0$. Since $p > 1$, we can also assume that they are so small that $\alpha(1 - 1/p) - \delta/p > 0$. This choice fixes also N in (3.16) and (3.17).

Since $\nu(\mathcal{P}) = 1$, $\nu(\Omega_r) \leq \varepsilon^{\delta}$ and $\nu(\Omega_s) \leq \varepsilon^{\delta}$, we obtain $\nu(\mathcal{P} - \Omega_r - \Omega_s) > 0$ for ε sufficiently small. Hence, there exist $\theta_0 \in \mathcal{P} - \Omega_s - \Omega_r$ for which $v(\theta_0) \neq 0$. Denote $\theta_i = f^i(\theta_0)$.

Bounding each of the terms – they are all positive – in r by their sum, we have

$$|e(\theta_i)|^p \leq |M((\theta_i)v(\theta_i) - zv(\theta_{i+1})|^p \leq |v(\theta_0)|^p \varepsilon^{p-\alpha-\delta} \quad i = -2N, \ldots, 2N \ . \tag{3.18}$$

Similarly for s,

$$|v(\theta_i)|^p \leq |v(\theta_0)|^p \varepsilon^{-\alpha-\delta} \quad i = -2N, \ldots, 2N \ . \tag{3.19}$$

If we denote $\tilde{v} = v / \max_{i=-N,\ldots,N} |v(\theta_i)|$ we have:

$$\begin{aligned} &|M(\theta_i)\tilde{v}(\theta_i) - z\tilde{v}(\theta_{i+1})| \leq \varepsilon^{1-\alpha/p-\delta/p} \text{ for all } -2N \leq i \leq 2N \\ &|\tilde{v}(\theta_i)| \leq 1 \text{ for all } -N \leq i \leq N \\ &|\tilde{v}(\theta_j)| = 1 \text{ for some } -N \leq j \leq N \\ &|\tilde{v}(\theta_i)| \leq \varepsilon^{-\alpha/p-\delta/p} \text{ for all } -2N \leq i \leq 2N \end{aligned} \tag{3.20}$$

There are two possibilities. Either $\{\theta_i\}_{i=-2N}^{2N}$ contains a periodic orbit of period less than N or not.

In the case that we have a periodic orbit of period less or equal than N, we have produced an approximate eigensection in $\Gamma_B(E)$, so we are done.

In the later case, we proceed in a manner similar to the proof of the first part of Theorem 3.7, that used localization arguments. We consider $\hat{v}(\theta_i) = (1 - 2|i-j|/N)_+ \tilde{v}(\theta_i)$ for $i = -2N, \ldots, 2N$, and $\hat{v}(\theta) = 0$ otherwise, where j is as in (3.20).

Notice that

$$\begin{aligned} M(\theta_i)\hat{v}(\theta_i) - z\hat{v}(\theta_{i+1}) = &\left(1 - \frac{2|i-j|}{N}\right)_+ [M(\theta_i)\tilde{v}(\theta_i) - z\tilde{v}(\theta_{i+1})] \\ &+ \left(\left(1 - \frac{2|i-j|}{N}\right)_+ - \left(1 - \frac{2|i+1-j|}{N}\right)_+\right) z\tilde{v}(\theta_{i+1}) \ . \end{aligned} \tag{3.21}$$

Using (3.20), we can bound from above the size of the first term in the R.H.S. by $\varepsilon^{1-\alpha/p-\delta/p}$ and, since $4N + 1 = \varepsilon^{-\alpha}$, the size of the second term can be bounded by $10|z|\varepsilon^{\alpha-\alpha/p-\delta/p}$. According to the way that we chose the α, δ, both bounds have positive powers of ε.

With these arguments we complete the proof of Theorem 3.7. □

3.3. Invariance of the spectrum under rotations

The following theorem is a restatement of a result in [**Mat68**] for the generality of bundle automorphisms.

THEOREM 3.11. *Let $M_f : E \to E$ be a vector bundle automorphism over an APD homeomorphism f. Then, the Weyl spectrum on bounded sections is rotationally invariant, that is*

$$z \in \mathrm{Spec}_W(\mathcal{M}_f, \Gamma_B(E)) \Rightarrow e^{\alpha \mathbf{i}} z \in \mathrm{Spec}_W(\mathcal{M}_f, \Gamma_B(E))$$

for all $\alpha \in \mathbb{R}$.

Of course, the same happens for all the other spaces of sections Γ for which the Weyl spectrum is the same as that in $\Gamma_B(E)$.

Proof: Let $z \in \mathrm{Spec}_W(\mathcal{M}_f, \Gamma_B(E))$, and let α be any real number.

By Mather's localization, we can find an approximate eigensection v supported in a finite segment of an aperiodic orbit $\{f^j(\theta)\}_{j=-N}^{N}$.

$$|M(f^j(\theta))v(f^j(\theta)) - zv(f^{j+1}(\theta))| \leq \varepsilon$$

for $j = -N-1, \dots, N$, otherwise the difference is zero.

To construct an approximate eigensection for $e^{\alpha \mathbf{i}}$, we define

$$\tilde{v}(f^j(\theta)) = e^{-\alpha j \mathbf{i}} v(f^j(\theta))$$

for $j = -N, \cdots, N$ and zero otherwise. With this election:

$$|M(f^j(\theta))\tilde{v}(f^j(\theta)) - e^{-\alpha \mathbf{i}} z \tilde{v}(f^{j+1}(\theta))|$$

$$= |M(f^j(\theta))e^{-\alpha j \mathbf{i}} v(f^j(\theta)) - e^{-\alpha \mathbf{i}} z e^{\alpha(j+1)\mathbf{i}} v(f^{j+1}(\theta))| \leq \varepsilon \ ,$$

and this proves the theorem. □

REMARK 3.12. Recall that if f is not APD then we can find an open set such that $f^N(\theta) = \theta$, for a certain $N \geq 1$. In that case, it is possible to show that the conclusion of Theorem 3.11 could be false (see Lemma 3.1 and Section 4).

As an immediate consequence of Theorems 3.11 and Corollary A.27, we conclude the same property for the full spectrum.

THEOREM 3.13. *Let $M_f : E \to E$ a vector bundle automorphism over and APD homeomorphism f. Then, the full spectrum on bounded sections is rotationally invariant, that is*

$$z \in \mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E)) \Rightarrow e^{\alpha \mathbf{i}} z \in \mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E))$$

for all $\alpha \in \mathbb{R}$. The same happens for all the other spaces of sections Γ that have the same Weyl spectrum as in $\Gamma_B(E)$.

In particular, according to Theorem 3.7 we can take $\Gamma = \Gamma_{C^0}(E)$. If f preserves a topological measure μ, we can also take $\Gamma = \Gamma_{L^p(\mu)}(E)$ for $p > 1$. In Part 3 we will show that there are some other spaces Γ for which $\mathcal{M}_f$ has the same Weyl spectrum as in $\Gamma_B(E)$, in the case of rotations $f(\theta) = \theta + \omega$ in the torus $\mathcal{P} = \mathbb{T}^d$.

3.4. Spectral subbundles

Theorem 2.18 is crucial in Mather theory since it relates abstract constructs at the functional analysis level of $\mathcal{M}_f$ to the geometric setting of the invariant subbundles of M_f. In this subsection, we generalize Theorem 2.18, and this generalization will make possible to work with other spaces of sections. To do so, we exploit the results reviewed in Section 2.1.

THEOREM 3.14. *Let $M_f : E \to E$ be a vector bundle automorphism. Let $\Gamma \subset \Gamma_B(E)$ be a Banach space of sections, with a norm $\|\cdot\|_\Gamma$, for which we assume:*

(a) *$\mathcal{M}_f$ defines a bounded linear operator in Γ (in particular, Γ is invariant under the transfer operator).*
(b) *Evaluation at one point is a well defined operation. That is*
  (b.1) *There exists a positive constant K such that $|v(\theta_0)| \leq K\|v\|_\Gamma$ for all $v \in \Gamma$ and $\theta_0 \in \mathcal{P}$;*
  (b.2) *For all $v_{\theta_0} \in E_{\theta_0}$ there exists $v \in \Gamma$ such that $v(\theta_0) = v_{\theta_0}$.*
(c) $\mathrm{Spec}_P(\mathcal{M}_f, \Gamma_B(E)) \subset \mathrm{Spec}(\mathcal{M}_f, \Gamma)$.

Assume furthermore that

$$\mathrm{Spec}(\mathcal{M}_f, \Gamma) \cap \mathcal{A}_{\lambda,\mu} = \emptyset \ ,$$

where $0 < \lambda \leq \mu$. Denote by $P^{<\lambda}, P^{>\mu}$ the projections associated to the spectral decomposition $\Gamma = \Gamma^{<\lambda} \oplus \Gamma^{>\mu}$.

Then, it is possible to find a continuous invariant splitting

$$E = E^{<\lambda} \oplus E^{>\mu} \tag{3.22}$$

such that the corresponding projections over the bundles, $\Pi^{<\lambda}, \Pi^{>\mu}$, satisfy for any $v \in \Gamma$:

$$(P^{<\lambda}v)(\theta) = \Pi_\theta^{<\lambda} v(\theta) \ , \ (P^{>\mu}v)(\theta) = \Pi_\theta^{>\mu} v(\theta) \ . \tag{3.23}$$

The splitting is characterized by the following growth rates: for all $\varepsilon > 0$ small enough

$$E^{<\lambda} = W^{\leq \lambda-\varepsilon} = L^{<\lambda}, \ E^{>\mu} = W^{\geq \mu+\varepsilon} = L^{>\mu} \ . \tag{3.24}$$

(See Definitions 2.1 and 2.13).

Moreover, the rates of growth can be made uniform: there exists a positive constant C_ε such that $W^{\leq\lambda-\varepsilon} = W^{\leq\lambda-\varepsilon, C_\varepsilon}, W^{\geq\mu+\varepsilon} = W^{\geq\mu+\varepsilon, C_\varepsilon}$.

The rest of consequences in Theorem 2.18 also hold.

REMARK 3.15. We do not assume that the topology in Γ is that induced by $\Gamma_B(E)$, although condition (b) implies that the inclusion $\Gamma \subset \Gamma_B(E)$ is continuous: for all $v \in \Gamma$, $\|v\|_\infty \leq K\|v\|_\Gamma$. Since the norm in Γ can be different to the sup-norm, there is no reason why $\mathrm{Spec}_W(\mathcal{M}_f, \Gamma) \subset \mathrm{Spec}_W(\mathcal{M}_f, \Gamma_B(E))$. Notice, however that $\mathrm{Spec}_P(\mathcal{M}_f, \Gamma) \subset \mathrm{Spec}_P(\mathcal{M}_f, \Gamma_B(E))$ does hold, because topology does not enter in the definition of the point spectrum.

REMARK 3.16. At the moment, the only spaces Γ for which we have proved $\mathrm{Spec}_P(\mathcal{M}_f, \Gamma_B(E)) \subset \mathrm{Spec}(\mathcal{M}_f, \Gamma(E))$ are $\Gamma_{C^0}(E)$ and, of course, $\Gamma_B(E)$. In Section 3.7 we will consider the spectrum on the space of C^r sections.

In Part 3 we will see this inclusion is in fact an equality for spaces of higher regularities (C^r, Sobolev), in the case of vector bundle maps over rotations.

In Part 4, these arguments imply the fact that such circular gaps do not exist for certain constrained spaces. Therefore, for the constrained spaces, Theorem 3.14 is only an intermediate step in the proof - by contradiction – that its hypotheses are not verified for these spaces.

Nevertheless for $\Gamma_{C^0}(E)$ the hypothesis are indeed verified often and indeed there are gaps. Later on, we will use Theorem 3.14 to show that the gaps in $\Gamma_{C^0}(E)$ and $\Gamma_B(E)$ are the same.

REMARK 3.17. In principle the decomposition of the bundle E into subbundles $E^{<\lambda}$, $E^{>\mu}$ could depend on the space we are considering the spectral projections. Nevertheless, (3.24) shows that this decomposition is unique (it is expressed independently of the space considered). This is the reason why we did not include this possible dependence in the notation. As a conclusion: $\mathcal{A}\mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E)) \subset \mathcal{A}\mathrm{Spec}(\mathcal{M}_f, \Gamma)$.

Now, we start the proof of Theorem 3.14.

Proof: We split the proof in several steps.

- *Definition of the spectral subbundles.* We start the proof by defining the spectral subbundles. For each $\theta_0 \in \mathcal{P}$, let
$$E^{<\lambda}_{\theta_0} = \{v(\theta_0) \mid v \in \Gamma^{<\lambda}\}\ ,\ E^{>\mu}_{\theta_0} = \{v(\theta_0) \mid v \in \Gamma^{>\mu}\}\ .$$
Obviously, both of them are linear subspaces of E_{θ_0}. Notice that the linear spaces $E^{<\lambda}$ and $E^{>\mu}$ are invariant under M_f.
- *Rates of growth.* The spectral subspaces are characterized by
$$\begin{aligned} \Gamma^{<\lambda} &= \{v \in \Gamma \mid \forall m \geq 0\ \|\mathcal{M}_f^m v\|_\Gamma \leq \hat{C}_\varepsilon(\lambda-\varepsilon)^m \|v\|_\Gamma\}\ , \\ \Gamma^{>\mu} &= \{v \in \Gamma \mid \forall m \geq 0\ \|\mathcal{M}_f^{-m} v\|_\Gamma \leq \hat{C}_\varepsilon(\mu+\varepsilon)^{-m} \|v\|_\Gamma\}\ , \end{aligned} \tag{3.25}$$
where $\varepsilon > 0$ is small enough. Therefore, for all $v \in \Gamma^{<\lambda}$ and for all $m \geq 0$
$$\begin{aligned} |M(\theta_0, m)v(\theta_0)| = |(\mathcal{M}_f^m v)(f^m(\theta_0))| &\leq K\|\mathcal{M}_f^m v\|_\Gamma \\ &\leq K\hat{C}_\varepsilon(\lambda-\varepsilon)^m\|v\|_\Gamma\ , \end{aligned}$$

where we use property (b.1). Hence,

$$v_{\theta_0} \in E^{<\lambda}_{\theta_0} \Rightarrow \forall m \geq 0 \ |M(\theta_0, m)v_{\theta_0}| \leq C^{<\lambda}_{\varepsilon, v_{\theta_0}} (\lambda - \varepsilon)^m \ , \tag{3.26}$$

where

$$C^{<\lambda}_{\varepsilon, v_{\theta_0}} = K\hat{C}_\varepsilon \cdot \inf\{\|v\|_\Gamma \mid v \in \Gamma^{<\lambda} \ , \ v(\theta_0) = v_{\theta_0}\} \ .$$

This proves that $E^{<\lambda} \subset W^{\leq \lambda - \varepsilon}$ and a similar argument proves also $E^{>\mu} \subset W^{\geq \mu + \varepsilon}$.

Notice also that we have the towers of inclusions

$$E^{<\lambda} \subset W^{\leq \lambda - \varepsilon} \subset L^{<\lambda} \subset W^{\leq \rho} \tag{3.27}$$
$$E^{>\mu} \subset W^{\geq \mu + \varepsilon} \subset L^{>\mu} \subset W^{\geq \rho} \tag{3.28}$$

for any $\rho \in [\lambda, \mu]$.

Notice that property (c) implies

$$\mathrm{Spec}_P(\mathcal{M}_f, \Gamma_B(E)) \cap \mathcal{S}_\rho \subset \mathrm{Spec}(\mathcal{M}_f, \Gamma) \cap \mathcal{A}_{\lambda,\mu} = \emptyset \ ,$$

so there are not Mañé vectors of growth ρ. Hence, Lemma 2.15 implies that $W^\rho = W^{\leq \rho} \cap W^{\geq \rho} = E_0$, incidentally proving that $E^{<\rho} \cap E^{>\rho} = E_0$. So then, the growth rates can be made uniform using Sacker-Sell theory (see Theorem 2.4 part (b)).

- *Splitting.* For $v_{\theta_0} \in E_{\theta_0}$, using (b.2) we construct $v \in \Gamma$ such that $v(\theta_0) = v_{\theta_0}$. Obviously

$$v_{\theta_0} = (P^{<\lambda} v)(\theta_0) + (P^{>\mu} v)(\theta_0)$$

from where we obtain that the invariant linear subspaces $E^{<\lambda}$ and $E^{>\mu}$ span the bundle: $E = E^{<\lambda} + E^{>\mu}$.

This proves that the sum is a Whitney sum, and that the inclusions in the towers (3.27) and (3.28) are in fact equalities.

- *Localization of spectral projections.* Once we know that E splits as $E = E^{<\lambda} \oplus E^{>\mu}$, it is easy to see that

$$\forall v^{<\lambda}_{\theta_0} \in E^{<\lambda} \ \forall v \in \Gamma \ , v(\theta_0) = v^{<\lambda}_{\theta_0} \Rightarrow (P^{<\lambda} v)(\theta_0) = v^{<\lambda}_{\theta_0} \ ,$$

because $v^{<\lambda}_{\theta_0} = v(\theta_0) = (P^{<\lambda} v)(\theta_0) + (P^{>\mu} v)(\theta_0)$ and $(P^{<\lambda} v)(\theta_0) \in E^{<\lambda}_{\theta_0}$, $(P^{>\mu} v)(\theta_0) \in E^{>\mu}_{\theta_0}$. We use the same argument for $E^{>\mu}$. Then, for any $v \in \Gamma$ and $\theta_0 \in \mathcal{P}$, the values of $(P^{<\lambda} v)(\theta_0)$ and $(P^{>\mu} v)(\theta_0)$ depend only on $v(\theta_0)$.

- *Bundle projections.* We define, then, the bundle projections

$$\Pi^{<\lambda}_{\theta_0} v_{\theta_0} = (P^{<\lambda} v)(\theta_0) \ , \ \Pi^{>\mu}_{\theta_0} v_{\theta_0} = (P^{>\mu} v)(\theta_0) \ ,$$

where v is any section in Γ such that $v(\theta_0) = v_{\theta_0}$. So, the functional-geometrical identity (3.23) holds.

The rest of the proof follows the same lines as in Theorem 2.18. □

REMARK 3.18. If the space Γ in Theorem 3.14 is Γ_{C^0}, then the spectral projections are in Γ_{C^0}. Then, the functional-geometrical identity (3.23) implies directly that the spectral subbundles are C^0 (cf. Theorem 2.18).

More generally, the properties of the space of sections Γwe consider translate into properties of the spectral subbundles.

3.5. Approximation of the spectrum

In this section we will see that under mild conditions of recurrence on the dynamics on the base manifold, the full spectrum is approximate point spectrum. A similar theorem was proved in [**Swa81**] for vector bundle flows, and the operators in that paper acted on continuous sections. We obtain this result for the spectrum on bounded sections, and the corresponding results for continuous and for L^p sections follow from the results of this part, which show that the spectrum does not depend on the spaces considered. Similar results will be considered in Section 4.2.

THEOREM 3.19. *Let $M_f : E \to E$ be a vector bundle automorphism.*

(a) *If f is chain-recurrent, then:*

$$\mathcal{A}\mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E)) = \mathcal{A}\mathrm{Spec}_P(\mathcal{M}_f, \Gamma_B(E)) \ .$$

(b) *If f is chain-recurrent and APD, then:*

$$\mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E)) = \mathrm{Spec}_W(\mathcal{M}_f, \Gamma_B(E)) \ .$$

(c) *If $z \in \mathrm{Spec}_W(\mathcal{M}_f, \Gamma_B(E))$ either we have an eigensection supported on an orbit (in this case $z \in \mathrm{Spec}_P(\mathcal{M}_f, \Gamma_B(E))$ or there is $\eta \in \mathbb{R}$ such that there is an eigenvalue $ze^{i\eta}$ whose eigenvalue is supported on a periodic orbit.*

Note that a corollary of part c) is that

COROLLARY 3.20. *In the conditions of Theorem 3.19 We have*

c.1)

$$\mathrm{Spec}_W(\mathcal{M}_f, \Gamma_B(E)) \subset \mathcal{A}\,\mathrm{Spec}_P(\mathcal{M}_f, \Gamma_B(E)) \ .$$

c.2) *If, furthermore, f is NPO, then:*

$$\mathrm{Spec}_W(\mathcal{M}_f, \Gamma_B(E)) = \mathrm{Spec}_P(\mathcal{M}_f, \Gamma_B(E)) \ .$$

Proof: The statement (c) follows from the arguments before Remark 3.2 in the proof of Lemma 3.1. We recall that the gist of the argument is that, given a sequence of approximate eigensections w^n such that $|w_n(\theta_n)| \geq 3/4$, we can, extracting subsequences obtain a $\{w_i\}$ such that $M(f^i(\theta_*))w_i = zw_{i+1}$. If $\{f^i(\theta_*)\}_{i\in\mathbb{Z}}$ is aperiodic, then w_i can be considered as eigensection. In case that $\{f^i(\theta_*)\}_{i\in\mathbb{Z}}$ is periodic, we argued that there is an eigensection of an eigenvalue of the same modulus supported in the periodic orbit.

Statement (b) is an immediate consequence of statement (a), because APD implies that the spectrum involved is rotationally invariant.

It only remains to prove (a). A similar argument can be found in [**Swa81**]. Let ρ be a positive number. If $\mathrm{Spec}_P(\mathcal{M}_f, \Gamma_B(E) \cap S_\rho = \emptyset$, then Lemma 2.15 implies that M_f is quasi-ρ-dichotomic. Since f is chain-recurrent, Theorem 2.4 (g) implies that M_f is ρ-dichotomic and hence, Theorem 2.18 implies that $\mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E) \cap S_\rho = \emptyset$. □

REMARK 3.21. As an immediate consequence of this result we see that if f is chain-recurrent and does not have periodic orbits, the full spectrum is point spectrum. An example is given by vector bundle automorphisms over irrational rotations, see Part 3.

REMARK 3.22. If f is uniquely ergodic, then it is APD and the spectrum is rotationally invariant. Moreover, if the unique invariant measure is topological then f is minimal, and the full spectrum is point spectrum.

In Section 2.4 we characterized the spectral subbundles in terms of Lyapunov multipliers (see Theorem 2.38). Under stronger properties on the motion f on the base manifold $\mathcal{P}$, we can obtain stronger results using the previous Theorem 3.19.

PROPOSITION 3.23. *Let $M_f : E \to E$ be a vector bundle automorphism over an NPO homeomorphism. Let E^j be one spectral subbundle associated to one spectral annulus $\mathcal{A}_{\lambda_j^-,\lambda_j^+}$. Then, there exist $v_j, w_j \in E^j$, such that $\lambda_s^+(v_j) = \lambda_i^+(v_j) = \lambda_j^-$ and $\lambda_i^-(w_j) = \lambda_s^-(w_j) = \lambda_j^+$.*

Proof: Notice that the spectrum is rotationally invariant because f does not have periodic orbits. By analyzing each spectral subbundle, it suffices to consider the case $\mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E)) = \mathcal{A}_{\lambda^-,\lambda^+}$.

Notice that the boundary of the spectrum is included in the Weyl spectrum (Proposition A.26), and Weyl spectrum coincides with point spectrum (Theorem 3.19) because f is NPO.

So, if $z \in \mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E))$ with $|z| = \lambda^-$, then $z \in \mathrm{Spec}_P(\mathcal{M}_f, \Gamma(E))$, and Lemma 2.15 implies the existence of a non zero vector $v_0 \in E$ for which $\lambda_s^+(v_0) \le \lambda^-$. From Theorem 2.38 we know that $\lambda^- \le \lambda_i^+(v_0) \le \lambda_s^+(v_0)$, so then $\lambda_s^+(v_0) = \lambda_i^+(v_0) = \lambda^-$. A similar argument proves that there exists a non zero vector w_0 such that $\lambda_i^-(w_0) = \lambda_s^-(w_0) = \lambda^+$. □

Similar arguments prove the following proposition.

PROPOSITION 3.24. *Let $M_f : E \to E$ be a vector bundle automorphism over an NPO and chain-recurrent homeomorphism f. Then, for all $z \in \mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E))$ there exists a non zero vector $v_0 \in E$ such that*

$$\lambda_s^+(v_0) \le |z| \le \lambda_i^-(v_0) \ .$$

3.6. Equality of spectra for bounded, continuous and L^p sections

In this section we will study the relation between the spectrum of the operator in the low regularity spaces. The main result is the following:

THEOREM 3.25. *Let $M_f : E \to E$ be a vector bundle automorphism. Then:*

$$\mathcal{A}\mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E)) = \mathcal{A}\mathrm{Spec}(\mathcal{M}_f, \Gamma_{C^0}(E)) \ . \tag{3.29}$$

In the case that f preserves a measure μ we have that

$$\mathcal{A}\mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E)) \supset \mathcal{A}\mathrm{Spec}(\mathcal{M}_f, \Gamma_{L^p(\mu)}(E)) \tag{3.30}$$

for $p \geq 1$.

If, furthermore, μ is a topological measure

$$\mathcal{A}\mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E)) = \mathcal{A}\mathrm{Spec}(\mathcal{M}_f, \Gamma_{L^p(\mu)}(E)) \tag{3.31}$$

for $p > 1$.

If μ is non-atomic and topological, then (3.31) *holds even for $p = 1$.*

Suppose that f is an APD homeomorphism. Then:

$$\mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E)) = \mathrm{Spec}(\mathcal{M}_f, \Gamma_{C^0}(E)) \ . \tag{3.32}$$

In the case that f preserves a measure μ we have that

$$\mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E)) \supset \mathrm{Spec}(\mathcal{M}_f, \Gamma_{L^p(\mu)}(E)) \tag{3.33}$$

for $p > 1$.

If, furthermore, μ is a topological measure

$$\mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E)) = \mathrm{Spec}(\mathcal{M}_f, \Gamma_{L^p(\mu)}(E)) \tag{3.34}$$

for $p > 1$.

REMARK 3.26. Similar results were obtained in [**CS80**]. For instance, The equality $\mathcal{A}\mathrm{Spec}(\mathcal{M}_f, \Gamma_{C^0}(E)) = \mathcal{A}\mathrm{Spec}(\mathcal{M}_f, \Gamma_{L^2(\mu)}(E))$ is proved for push-forward operators of measure preserving diffeomorphisms on a smooth compact Riemannian manifold. This follows from Theorem 3.25 since the Riemannian volume is a topological measure and the equality (3.29).

REMARK 3.27. One situation where the assumption in Theorem 3.25 that the measure is non-atomic and topological is satisfied is when the measure is equivalent to Lebesgue.

Notice also that the hypothesis are satisfied if the measure μ is the push forward by a homeomorphism of the another measure that satisfies them. (e.g. if we consider an Anosov system f preserving an smooth measure μ_f and g is a small perturbation, then the homeomorphism given by structural stability will produce a measure μ_g invariant under g and satisfying the hypothesis of our theorem. Even in the case that g preserves a smooth measure, this transported measure may be different.) In dynamical systems, SRB measures often (but not always) are non-atomic and topological.

Proof: We have just to prove the assertions on the annular hull of the spectrum, because the additional property of APD implies that spectrum involved is rotationally invariant (for $p > 1$).

Hence, the only thing that we have to show is that a gap in the spectrum in one of the spaces are also a gap in the other spaces we are considering.

The fact that gaps in $\operatorname{Spec}(\mathcal{M}_f, \Gamma_B(E))$ are also gaps in $\operatorname{Spec}(\mathcal{M}_f, \Gamma_{C^0}(E))$ is just the content of Proposition 2.32:

$$\mathcal{A}\operatorname{Spec}(\mathcal{M}_f, \Gamma_{C^0}(E)) \subset \mathcal{A}\operatorname{Spec}(\mathcal{M}_f, \Gamma_B(E)) \ .$$

If there is such an annular gap in $\operatorname{Spec}(\mathcal{M}_f, \Gamma_{C^0}(E))$, by Theorem 3.14, we have a continuous decomposition $E = E^{<\lambda} \oplus E^{>\mu}$ of the bundle characterized by growth rates. Given a section $v \in \Gamma_B(E)$ (resp. $v \in \Gamma_{L^p(\mu)}(E)$) then if we define $v^<$ and $v^>$ by

$$v^<(\theta) = \Pi_\theta^{<\lambda}(v(\theta)) \ , \ v^>(\theta) = \Pi_\theta^{>\mu}(v(\theta)) \ , \tag{3.35}$$

it is immediate to verify that it is a direct decomposition of $\Gamma_B(E)$ (resp. $\Gamma_{L^p(\mu)}(E)$). Moreover, the pointwise bounds in (3.24) of Theorem 3.14 imply $\|\mathcal{M}_f^m v^<\| \le C_\varepsilon(\lambda - \varepsilon)^m \|v^<\|$ for $m \ge 0$, with the norm understood in the $\Gamma_B(E)$ sense (resp. in the $\Gamma_{L^p(\mu)}(E)$ sense), and an analogous result for $v^>$ and $m \le 0$. Therefore, there is a gap in the spectrum in $\Gamma_B(E)$ (resp. in $\Gamma_{L^p(\mu)}(E)$).

Hence, we have proved

$$\mathcal{A}\operatorname{Spec}(\mathcal{M}_f, \Gamma_B(E)) \subset \mathcal{A}\operatorname{Spec}(\mathcal{M}_f, \Gamma_{C^0}(E)) \ ,$$

that proves (3.32), and

$$\mathcal{A}\operatorname{Spec}(\mathcal{M}_f, \Gamma_{L^p(\mu)}(E)) \subset \mathcal{A}\operatorname{Spec}(\mathcal{M}_f, \Gamma_{C^0}(E))$$

for $p \ge 1$ and without any assumption on the measure μ, proving (3.33).

If μ preserves a topological measure, then it is chain-recurrent, and from Theorems 3.7 and 3.19 we obtain that

$$\begin{aligned}\mathcal{A}\operatorname{Spec}(\mathcal{M}_f, \Gamma_B(E)) &= \mathcal{A}\operatorname{Spec}_W(\mathcal{M}_f, \Gamma_B(E)) \\ &= \mathcal{A}\operatorname{Spec}_W(\mathcal{M}_f, \Gamma_{L^p(\mu)}(E)) \\ &\subset \mathcal{A}\operatorname{Spec}(\mathcal{M}_f, \Gamma_{L^p(\mu)}(E)) \ .\end{aligned}$$

This proves (3.34). Notice that this argument works for only for $p > 1$.

We can prove (3.34) for $p \ge 1$ under the assumption that μ is non-atomic and topological.

If there is an annular gap in $\operatorname{Spec}(\mathcal{M}_f, \Gamma_{L^p(\mu)}(E))$, we note that if $v \in \Gamma_{L^p}^{<\lambda}(E)$, then, for any φ complex function in $L^\infty(\mu)$, using

$$\mathcal{M}_f^m(\varphi v)(\theta) = \varphi(f^{-m}(\theta))\mathcal{M}_f^m v(\theta) \tag{3.36}$$

we conclude that $\varphi v^<$ is also in $\Gamma_{L^p}^{<\lambda}(E)$. The arguments are similar to those in the first step of the proof of Theorem 2.18.

We recall that by a celebrated theorem, any non-atomic probability Borel measure in a complete and separable metric space is measure equivalent to the Lebesgue measure in the unit interval [0,1] (see for instance [**Hal50**] p. 173, or [**Par77**] p. 118). Since all our constructions in the proof will only use measurable functions (the transformations mapping the measure space into the unit interval are often discontinuous) we will just consider that the space we are working into ins the unit interval.

In particular, by Lebesgue's differentiation theorem, for each $\theta \in \mathcal{P}$ we can find a family of positive measure Borel sets $\{B_r(\theta)\}_{r>0}$, with $B_{r_1}(\theta) \subset B_{r_2}(\theta)$ if $r_1 < r_2$, with the property that $\cap_{r>0} B_r(\theta) = \{\theta\}$, and that for any $\Psi \in L^1(\mu)$ we have for μ-almost all θ_0

$$\lim_{r\to 0} \frac{1}{\mu(B_r(\theta_0))} \int_{B_r(\theta_0)} \Psi(\theta)\, d\mu = \Psi(\theta_0)\ . \tag{3.37}$$

The points θ_0 for which (3.37) is satisfied are the Lebesgue points of the function Ψ.

If we take now $\varphi = I_{B_r(\theta_0)}/\mu(B_r(\theta_0))^{1/p}$ where $I_{B_r(\theta_0)}$ is the characteristic function, we have for a fixed $m \geq 0$,

$$\|M_f^m(\varphi v^<)\|^p_{L^p(\mu)} \leq C^p_\varepsilon(\lambda-\varepsilon)^{pm}\|\varphi v^<\|^p_{L^p}\ , \tag{3.38}$$

which is equivalent to

$$\frac{1}{\mu(B_r(\theta_0))}\int_{B_r(\theta_0)} |M(\theta,m)v(\theta)|^p d\mu \leq C^p_\varepsilon(\lambda-\varepsilon)^{pm}\frac{1}{\mu(B_r(\theta_0))}\int_{B_r(\theta_0)}|v(\theta)|^p d\mu\ . \tag{3.39}$$

By Lebesgue's differentiation theorem, for almost all points the limit as r tends to zero exists, so, we get almost everywhere

$$|M(\theta_0,m)v(\theta_0)| \leq C_\varepsilon(\lambda-\varepsilon)^m|v(\theta_0)|\ , \tag{3.40}$$

and we get that there is a set of full measure for which this is satisfied for all m. This tells us that, given that there is a splitting in L^p we can produce a pointwise splitting almost everywhere. The fact that the bounds are uniform in the point, show, as before that the splitting is continuous, hence it extends to the whole set $\mathcal{P}$.

Conversely, we note that if an L^p-section v satisfies (3.40) almost everywhere, it is in $\Gamma^{<\lambda}_{L^p(\mu)}(E)$. Analogously, for the other subspace $\Gamma^{>\mu}_{L^p(\mu)}(E)$.

From here on, the proof is almost the same as that of Theorem 3.14. We argue that the linear space

$$\begin{aligned} E^{<\lambda}_{\theta_0} = \{v_\theta \in E_\theta \mid \exists v \in \Gamma^{<\lambda}_{L^p}(E), v(\theta_0) = v_{\theta_0},\\ \text{for all } m \geq 0\ \theta_0 \text{ is Lebesgue point for } |M(\theta,m)v(\theta)|^p\}\end{aligned} \tag{3.41}$$

is a linear subspace, all the points in it satisfy (3.40). We do a similar construction for $E^{>\mu}$.

The same argument we used Theorem 3.14 shows that, $E^{<\lambda}_\theta$, $E^{>\mu}_\theta$ are a direct decomposition, and that they are characterized by (3.40). This decomposition is defined in the support of the measure μ, which by our assumption that μ is topological is the whole manifold.

Once we have finished the measure theoretical constructions, we can look at the problem in the original space. The bounds (3.40) show that the decomposition is continuous. □

3.7. Spectrum on C^r sections

In this section we will study the spectral properties of the transfer operator when acting on sections with C^r regularities. Given a C^r vector bundle automorphism $M_f : E \to E$, it is natural to consider the action on C^r sections: if $v \in \Gamma_{C^r}(E)$ then $\mathcal{M}_f v \in \Gamma_{C^r}(E)$. Moreover, the Banach algebra properties of C^r functions make $\mathcal{M}_f$ continuous in $\Gamma_{C^r}(E)$. We will consider the spectrum of the transfer operator in $\Gamma_{C^r}(E)$. The results will be improved in Part 3, including Sobolev regularities, in the case of transfer operators over rotations.

The first result is a consequence of the fattening up arguments of Theorem 3.7 to the generalities of C^r sections.

THEOREM 3.28. *Let $M_f : E \to E$ be a C^r vector bundle automorphism. Then:*

$$\mathrm{Spec}_W(\mathcal{M}_f, \Gamma_B(E)) \subset \mathrm{Spec}_W(\mathcal{M}_f, \Gamma_{C^r}(E)) \ . \tag{3.42}$$

Proof: We will use the fattening arguments of Theorem 3.7 to prove the inclusion. That is to say, given $z \in \mathrm{Spec}_W(\mathcal{M}_f, \Gamma_B(E))$, we will consider an approximate eigensection supported in a finite segment of orbit satisfying either (a) or (b) of Lemma 3.1, that will be fattened up to construct a C^r eigensection.

We will analyze here the alternative (a) of Lemma 3.1, because the alternative (b) follows similar lines.

In the alternative (a) we obtained a bounded approximate eigensection v supported in a finite segment of orbit $\{f^i(\theta_0)\}_{i=-N}^N$, with $f^i(\theta_0) \neq f^j(\theta_0)$ for $i \neq j$ with $|i| \leq N, |j| \leq N$, with $N = [1/\varepsilon]$. We will write $\theta_i = f^i(\theta_0)$ and $v_i = v(\theta_i)$. The error as approximate eigensection is smaller than $2\varepsilon|z|$. Then we pick coordinates around each of the points in the finite segment $\{\theta_i\}_{i=-N}^N$ in such a way that the bundle is trivialized.

To fatten the localized eigensection, we will use the device of C^r- bump functions.

LEMMA 3.29. *There exists a C^∞-function $\rho : \mathbb{R}^d \to \mathbb{R}$ satisfying the following conditions:*

a) *ρ is supported in $\bar{B}(0,1)$, the closed unit ball with respect to the Euclidean norm in $\mathbb{R}^d$;*
b) *$\|\rho\|_\infty < \|\mathrm{D}\rho\|_\infty < \cdots < \|\mathrm{D}^r\rho\|_\infty$;*
c) *$\|\mathrm{D}^r\rho\|_\infty = |\mathrm{D}^r\rho(0)| = 1$.*

Proof of Lemma 3.29

Consider the C^∞ function $\varphi : \mathbb{R} \to \mathbb{R}$ defined by

$$\varphi(t) = \frac{1}{1 + \exp\left(\frac{1}{(1-t)^2} - \frac{1}{t^2}\right)} \ , \tag{3.43}$$

in the interval $[0,1]$, $\varphi(t) = 0$ for $t > 1$ and $\varphi(t) = 1$ for $t < 0$. This function satisfies the following set of properties:

- $\forall k \geq 1$, $\varphi^{k)}(0) = 0$;
- $\forall k \geq 1$, $\varphi^{k)}(1) = 0$;
- $1 = \|\varphi\|_\infty < \|\varphi^{1)}\|_\infty < \|\varphi^{2)}\|_\infty < \dots$.

We consider now the C^∞ function $\beta : \mathbb{R}^d \to \mathbb{R}$ as

$$\beta(x) = \varphi(|x|) \ ,$$

which satisfies a) and b). Notice that, for $0 < \delta \leq 1$, the function

$$\beta_\delta(x) = \beta(x/\delta)$$

also satisfies a) and b), because $\|\mathrm{D}^k\beta_\delta\|_\infty = \frac{1}{\delta^k}\|\mathrm{D}^k\beta\|_\infty$ and it is supported in $\bar{B}(0,\delta)$.

Let $x_0 \in B(0,1)$ be any point such that $|\mathrm{D}^r\beta(x_0)| = \|\mathrm{D}^r\beta\|_\infty$. Taking $\delta = (1+|x_0|)^{-1}$, the function

$$\rho(x) = \frac{\delta^r}{\|\mathrm{D}^r\beta\|_\infty}\beta(x_0 - x/\delta) \tag{3.44}$$

satisfies the three conditions a), b) and c). □

From the function ρ defined above, we consider the functions $\rho_\sigma(x) = \rho(x/\sigma)$. Taking σ small enough, for each $|i| \leq N$ we set in the trivialization on the bundle around $f^i(\theta_0)$

$$\tilde{v}(\theta) = \sigma^r\rho_\sigma(f^{-i}(\theta))v_i \ , \tag{3.45}$$

and the section is zero away of these trivializations. This section is C^r and $|\mathrm{D}^r\tilde{v}(\theta_0)| \geq 3/4$.

Then, working in coordinates, we have for $\theta \in f^i(B(\theta_0,\sigma))$,

$$\begin{aligned} M(f^{-1}(\theta))\tilde{v}(f^{-1}(\theta)) - z\tilde{v}(\theta) &= \sigma^r\rho_\sigma(f^{-i}(\theta))\left(M(f^{-1}(\theta))v_{i-1} - zv_i\right) \\ &= \sigma^r\rho_\sigma(f^{-i}(\theta))\left(M(\theta_{i-1})v_{i-1} - zv_i\right) + \\ &\quad \sigma^r\rho_\sigma(f^{-i}(\theta))\left(M(f^{-1}(\theta)) - M(\theta_{i-1})\right)v_{i-1} \ . \end{aligned}$$

For $s = 0,\dots,r$, we have

$$\begin{aligned} &\left|\mathrm{D}^s\left(M(f^{-1}(\theta))\tilde{v}(f^{-1}(\theta)) - z\tilde{v}(\theta)\right)\right| \\ &\leq \sigma^r\left|\mathrm{D}^s\left(\rho_\sigma(f^{-i}(\theta))\right)\right| \ |M(\theta_{i-1})v_{i-1} - zv_i| \ + \\ &\quad \sigma^r\sum_{l=0}^{s-1}\binom{s}{l}\left|\mathrm{D}^l\left(\rho_\sigma(f^{-i}(\theta))\right)\right| \ \left|\mathrm{D}^{s-l}\left(M(f^{-1}(\theta))v_{i-1}\right)\right| \ + \\ &\quad \sigma^r\left|\mathrm{D}^s\left(\rho_\sigma(f^{-i}(\theta))\right)\right| \ \left|M(f^{-1}(\theta)) - M(f^{i-1}(\theta_0))\right| \\ &\leq C_{s,i}\sigma^{r-s}2\varepsilon|z| + \sum_{l=0}^{s-1}\binom{s}{l}C_{l,i}\sigma^{r-l}\|\mathcal{M}_f\|_{C^r} + C_{s,i}\sigma^{r-s}\eta_i(\sigma) \ . \end{aligned}$$

where $C_{s,i}$ are constants depending on the bounds of the derivatives of ρ and f^{-i}, and η_i is the modulus of continuity of $M \circ f^{i-1}$, for $i = -N,\dots,N$. Notice that the first addend is small, and the elements of the second addend can be made arbitrarily small by taking σ small enough. This is also the case of the third addend. Clearly, outside of these neighborhoods the difference is zero.

Hence, we have constructed an approximate C^r eigensection. □

REMARK 3.30. Notice that we cannot use the inclusion argument (1.19) to prove the opposite inclusion, because although $\Gamma_{C^r}(E) \subset \Gamma_B(E)$, the topologies in both spaces do not coincide. In fact, $\mathrm{Spec}_W(\mathcal{M}_f, \Gamma_{C^r}(E)) \subset \mathrm{Spec}_W(\mathcal{M}_f, \Gamma_B(E))$ is not true in general, but in some particular cases it is, as for f being a rotation on a torus, see Part 3.

THEOREM 3.31. *Let $M_f : E \to E$ be a C^r vector bundle automorphism. Assume that*

$$\mathrm{Spec}(\mathcal{M}_f, \Gamma_{C^r}(E)) \cap \mathcal{A}_{\lambda,\mu} = \emptyset \ .$$

Denote by $P^{<\lambda} = P^{<\lambda}_{C^r}, P^{>\mu} = P^{>\mu}_{C^r}$ the projections associated to this spectral gap. Then, it is possible to find a invariant C^r-splitting

$$E = E^{<\lambda} \oplus E^{>\mu} \tag{3.46}$$

such that the corresponding projections over the bundles, $\Pi^{<\lambda}, \Pi^{>\mu}$, satisfy for any $v \in \Gamma_{C^r}$:

$$(P^{<\lambda} v)(\theta) = \Pi^{<\lambda}_\theta v(\theta) \ , \ (P^{>\mu} v)(\theta) = \Pi^{>\mu}_\theta v(\theta) \ . \tag{3.47}$$

Moreover, for all $\varepsilon > 0$ small enough there exists a constant $C_\varepsilon > 0$ such that

$$E^{<\lambda} = W^{\leq \lambda - \varepsilon, C_\varepsilon} = L^{<\lambda} \ , \ E^{>\mu} = W^{\geq \mu + \varepsilon, C_\varepsilon} = L^{>\mu} \ . \tag{3.48}$$

(See Definitions 2.1 and 2.13).

Proof: We have to check that the three properties (a),(b) and (c) of Theorem 3.14 are satisfied by the space of C^r sections $\Gamma = \Gamma_{C^r}(E)$, endowed with the C^r norm:

(a) Since M_f is C^r, if $v \in \Gamma_{C^r}(E)$ then $\mathcal{M}_f v \in \Gamma_{C^r}(E)$. Moreover, the Banach algebra properties of C^r functions make $\mathcal{M}_f$ continuous in $\Gamma_{C^r}(E)$.
(b) For any $v \in \Gamma_{C^r}(E)$ and $\theta_0 \in \mathcal{P}$, obviously $|v_{\theta_0}| \leq \|v\|_{C^r}$. Moreover, given any v_{θ_0} we can construct a C^r section v with $v(\theta_0) = v_{\theta_0}$, using C^r bump functions.
(c) $\mathrm{Spec}_P(\mathcal{M}_f, \Gamma_B(E)) = \mathrm{Spec}_W(\mathcal{M}_f, \Gamma_B(E)) \subset \mathrm{Spec}_W(\mathcal{M}_f, \Gamma_{C^r}(E)) \subset \mathrm{Spec}(\mathcal{M}_f, \Gamma_{C^r}(E))$ (see Theorem 3.19 and Theorem 3.28).

Theorem 3.14 produces an invariant splitting that is C^0 and it is characterized by rates of growth. From the functional-dynamical equalities (3.47) we will see that the splitting is, in fact, C^r. Since $(P^{<\lambda} v)(\theta) = \Pi^{<\lambda}_\theta(v(\theta))$, if $\Pi^{<\lambda}_\theta$ were not C^r we could find a C^r section v for which $P^{<\lambda} v$ would not be C^r, in contradiction with the fact that $P^{<\lambda}$ is a projection on Γ_{C^r}. The same argument works for $\Pi^{>\mu}_\theta$. Hence, E decompose in the Whitney sum $E = E^{<\lambda} \oplus E^{>\mu}$, and this C^r-splitting is invariant. □

REMARK 3.32. In the literature, it is customary to find the theory for the regularity of the splittings corresponding to the spectral gaps in C^0, B sections. It is possible to show that these bundles have some smoothness.

The argument we have carried out assumes that there is a spectral gap in C^r and obtains C^r regularity. Of course, the existence of a spectral gap in C^r will require a separate argument. Later, we will develop tools to study this question.

As an immediate consequence of the previous theorem, we have the following inclusion.

THEOREM 3.33. *Let $M_f : E \to E$ be a C^r vector bundle automorphism. Then:*

$$\mathcal{A}\mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E)) \subset \mathcal{A}\mathrm{Spec}(\mathcal{M}_f, \Gamma_{C^r}(E)) \ . \tag{3.49}$$

REMARK 3.34. If f is chain recurrent, we can prove the inclusion in Theorem 3.33 by using the inclusion in Theorem 3.28, because

$$\begin{aligned}\mathcal{A}\mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E)) &= \mathcal{A}\mathrm{Spec}_W(\mathcal{M}_f, \Gamma_B(E))\\ &\subset \mathcal{A}\mathrm{Spec}_W(\mathcal{M}_f, \Gamma_{C^r}(E))\\ &\subset \mathcal{A}\mathrm{Spec}(\mathcal{M}_f, \Gamma_{C^r}(E))\end{aligned}$$

(see Theorem 3.19).

REMARK 3.35. We can use Theorem 3.31 and Theorem 3.33 as intermediate steps to prove that a gap in the spectrum on bounded sections does not exist in the spectrum on C^r sections. Notice that a gap in the spectrum on bounded sections produces an invariant splitting characterized by rates of growth. If we are able of proving that the splitting is not C^r, then such a gap can not exist in the spectrum on bounded sections. Notice also that the Invariant Section Theorem gives lower bounds of the regularity of the bundles, but still we would need separate arguments to prove that the spectrum on sections with such regularity has the gap. These separate arguments work for transfer operators over rotations, see Part 3.

REMARK 3.36. Notice that if f is APD, the spectrum of the operator acting on bounded sections is rotationally invariant. We do not know if this is also true when acting on smooth sections, except for some examples such as transfer operators over rotations, see Part 3. The key to these proofs is that there is a function in this space such that $\varphi \circ f = e^{\mathbf{i}\eta}\varphi$ for a dense set of η. In the case of rotations, it is enough to take the exponentials.

REMARK 3.37. The inclusion in (3.49) can be strict. One easy example can be obtained by taking $f \in SL(2, \mathbb{Z})$. Such f defines an automorphisms of the torus $\mathbb{T}^2$. We will assume that this automorphism has eigenvalues $|\lambda_+| > 1 > |\lambda_-|$. We will assume that these eigenvalues are irrational, and hence, there are no integer eigenvalues.

If we take $E = \mathbb{R} \times \mathbb{T}^2$ and define

$$M_f\Gamma(\theta) = \Gamma(f\theta)$$

we see that the spectral radius of M_f in C^0 is just 1. Indeed, the spectrum is just the unit circle.

On the other hand, if ∂_u, ∂_s denote respectively the derivative along the unstable and stable eigenvector of f we have that

$$\partial_u^r(M_f^n)\Gamma(\theta) = \partial_u^r\Gamma(f^n\theta)(\lambda_+)^{rn}$$
$$\partial_s^r(M_f^n)\Gamma(\theta) = \partial_s^r\Gamma(f^n\theta)(\lambda_-)^{rn}$$

Hence, we see that the spectral radius in C^r is at least $|(\lambda^+)^r|$ and that the spectral radius of the inverse is at least $|(\lambda_-)^{-r}|$.

Actually, by using trigonometric polynomials, we can show that the Weyl spectrum is the annulus of inner radius λ_- and outer radius λ_+.

If we take

$$\Psi_{N,M} = \sum_{j=-M}^{N} \exp(2\pi\mathbf{i}\langle (f^t)^j(0,1), \theta\rangle)\, \lambda^{-j} \frac{1}{|j-(N+M)/2|}$$

we see that, if $|\lambda_-| \le |\lambda| \le |\lambda_+|$ it is possible to choose N, M so that $\Psi_{N,M}$ becomes an approximate eigenfunction.

3.8. Restriction of the base set

In some previous sections we have taken advantage of a device that consists in restricting the behavior of the transfer operator to sections supported in orbits. This idea will be extended in Section 4.

Here, we will generalize some arguments to analyze the spectrum on sections supported in invariant sets (and orbits are a particular case). The following lemma is obvious.

Lemma 3.38. *Let $M : E \to E$ be a vector bundle automorphism over a homeomorphism $f : \mathcal{P} \to \mathcal{P}$. Let $\mathcal{P}_0 \subset \mathcal{P}$ be a f-invariant set: $f(\mathcal{P}_0) = \mathcal{P}_0$. Consider the vector bundle $E_{|\mathcal{P}_0} = \Pi^{-1}\mathcal{P}_0$, and the restriction of M to $E_{|\mathcal{P}_0}$. Then:*

$$\mathrm{Spec}_W(\mathcal{M}_f, \Gamma_B(E_{|\mathcal{P}_0})) \subset \mathrm{Spec}_W(\mathcal{M}_f, \Gamma_B(E)) \ ,$$

and

$$\mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E_{|\mathcal{P}_0})) \subset \mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E)) \ .$$

Proof: Follows from the fact that the splitting $\Gamma_B(E) = \Gamma_B(E_{|\mathcal{P}_0}) \oplus \Gamma_B(E_{|\mathcal{P}\setminus\mathcal{P}_0})$ in closed Banach subspaces is invariant under $\mathcal{M}_f$. □

The following result is a restatement of Theorem 2.18 to transfer operators restricted to sections supported in invariant sets (see Remark 2.26).

Theorem 3.39. *Let $M : E \to E$ be a vector bundle automorphism over a homeomorphism $f : \mathcal{P} \to \mathcal{P}$. Let $\mathcal{P}_0 \subset \mathcal{P}$ be a f-invariant set without isolated points. Then:*

$$\mathrm{Spec}_W(\mathcal{M}_f, \Gamma_B(E_{|\mathcal{P}_0})) = \mathrm{Spec}_W(\mathcal{M}_f, \Gamma_B(E_{|\mathrm{cl}(\mathcal{P}_0)})) \ ,$$

and

$$\mathcal{A}\mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E_{|\mathcal{P}_0})) = \mathcal{A}\mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E_{|\mathrm{cl}(\mathcal{P}_0)})) \ .$$

REMARK 3.40. In particular, if $\mathcal{P}_0$ is dense in $\mathcal{P}$, then $E_{|\mathrm{cl}(P_0)} = E$. So, if f has a dense invariant set $\mathcal{P}_0$, we can characterize all the spectrum from the behavior on $\mathcal{P}_0$.

Proof: To prove the equality of Weyl spectra, it suffices to prove that if z is an approximate eigenvalue of M_f in $\Gamma_B(E_{|\mathrm{cl}(\mathcal{P}_0)})$, we can construct an approximate eigensection in $\Gamma_B(E_{|\mathcal{P}_0})$. Since $\mathrm{cl}(\mathcal{P}_0)$ is compact, we can construct an approximate eigensection supported in a finite segment of orbit S_* of a certain $\theta_* \in \mathrm{cl}(\mathcal{P}_0)$, by Lemma 3.1. We can find a finite segment S_0 of an orbit of a point $\theta_0 \in \mathcal{P}_0$ arbitrarily close to a corresponding finite segment S_* of the orbit of θ_*. Working in a trivialization supported on a sufficiently small neighborhood of these segments, we can transfer the approximate eigensection supported in S_* to the segment S_0.

To prove the second equality, it suffices to prove that the gaps in the spectrum of M_f in $\Gamma_B(E_{|\mathcal{P}_0}))$ are also gaps when acting on $\Gamma_B(E_{|\mathrm{cl}(\mathcal{P}_0)}))$. The existence of a gap $\mathcal{A}_{\lambda,\mu}$ in the spectrum of M_f in $\Gamma_B(E_{|\mathcal{P}_0}))$ implies the existence of a continuous decomposition of the bundle E in $\mathcal{P}_0$, characterized by uniform rates of growth (that come from the boundedness of the spectral projections): $E_{|\mathcal{P}_0} = E^{<\lambda}_{|\mathcal{P}_0} \oplus E^{>\mu}_{|\mathcal{P}_0}$. See Remark 2.26.

Since the invariant section theorem works[**HP70, HPS77**] even if the base set is just a metric space, we conclude that the invariant splitting of E on the points of $\mathcal{P}_0$ can be extended to an invariant splitting of E on all the points of the closure $\mathrm{cl}(\mathcal{P}_0)$: $E_{|\mathrm{cl}(\mathcal{P}_0)} = E^{<\lambda}_{|\mathrm{cl}(\mathcal{P}_0)} \oplus E^{>\mu}_{|\mathrm{cl}(\mathcal{P}_0)}$ with $E^{<\lambda}_{|\mathrm{cl}(\mathcal{P}_0)} = \mathrm{cl}\left(E^{<\lambda}_{|\mathcal{P}_0}\right)$ and $E^{>\mu}_{|\mathrm{cl}(\mathcal{P}_0)} = \mathrm{cl}\left(E^{>\mu}_{|\mathcal{P}_0}\right)$.

Moreover, these bundles are also characterized by the same rates of growth. In effect, if we fix $m \geq 0$ and $\theta_* \in \mathrm{cl}(\mathcal{P}_0)$, each $v_\theta \in E^{<\lambda}_{\theta_*}$ can be approximated by a sequence $\hat{v}^n \in E^{<\lambda}_{\theta_n}$ with $\theta_n \in \mathcal{P}_0$. Since

$$|M(\theta_n, m)v^n| \leq C_\varepsilon(\lambda - \varepsilon)^m |v^n|$$

then doing $n \to \infty$ we conclude

$$|M(\theta_*, m)v_\theta| \leq C_\varepsilon(\lambda - \varepsilon)^m |v_{\theta_*}| \ .$$

An identical argument works for m negative and the other subbundle.

In particular, if $\mathcal{P}_0$ is dense in $\mathcal{P}$ and $\mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E_{|\mathcal{P}_0}) \cap \mathcal{A}_{\lambda,\mu} = \emptyset$, then there is a continuous splitting on the whole E characterized by rates of growth. This implies that $\mathcal{A}\mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E)) = \mathcal{A}\mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E_{|\mathcal{P}_0})$. □

3.9. Covering the base set

In this section we will study the spectrum of a transfer operator associated to a vector bundle automorphism that covers a vector bundle automorphism.

Let $(E, \pi, \mathcal{P})$ be a vector bundle, and $\tilde{\mathcal{P}}$ be a covering space of $\mathcal{P}$, with projection p and with a finite number k of leaves. The elements of $\mathcal{P}$ are denoted by θ, and those of $\tilde{\mathcal{P}}$ are denoted by $\tilde{\theta}$. Recall that p is a local homeomorphism and $k = \#\{\tilde{\theta} \mid p(\tilde{\theta}) = \theta\}$, for all $\theta \in \mathcal{P}$.

The fibered product

$$\tilde{E} = p^* E = E \times_{\mathcal{P}} \tilde{\mathcal{P}} = \{(v_\theta, \tilde{\theta}) \in E \times \tilde{\mathcal{P}} \mid \theta = \pi(v_\theta) = p(\tilde{\theta})\}$$

is a vector bundle over $\tilde{\mathcal{P}}$, with the obvious projection $\tilde{\pi}$, and also covers E with the obvious projection $\tilde{p}$, and the number of leaves is also k. Notice that $\tilde{p}$ is a vector bundle map over p (it sends the fibers of $\tilde{E}$ to fibers of E, and this is done with isomorphisms): $\pi\circ\tilde{p} = p\circ\tilde{\pi}$.

If E is admits a Finsler norm, we can construct a Finsler metric on $\tilde{E}$, by just defining

$$|(v_\theta, \tilde{\theta})| = |v_\theta| \ .$$

Given a section $v(\theta)$ of E, it has a lift $\tilde{v}(\tilde{\theta})$ defined by

$$\tilde{v}(\tilde{\theta}) = (v(p(\tilde{\theta})), \tilde{\theta}) \in \tilde{E} \ .$$

Let $M_f : E \to \mathcal{P}$ be a vector bundle automorphism over the homeomorphism $f : \mathcal{P} \to \mathcal{P}$. Suppose f is covered by a homeomorphism $\tilde{f} : \tilde{P} \to \tilde{P}$, that is to say, $p\circ\tilde{f} = f\circ p$. From the lift $\tilde{f}$ of f we can construct a lift $\tilde{M}_{\tilde{f}}$ of M_f:

$$\tilde{M}(v_\theta, \tilde{\theta}) = (M(\theta)v_\theta, \tilde{f}(\tilde{\theta})) \ .$$

So, $\tilde{M}_{\tilde{f}}$ is a vector bundle automorphism of $\tilde{E}$, over $\tilde{f}$.

The question for us is the relationship between the spectrum of both $\mathcal{M}_f$ and $\tilde{\mathcal{M}}_{\tilde{f}}$.

THEOREM 3.41. *With the above notation.*

(a)

$$\mathcal{A}\mathrm{Spec}(\tilde{\mathcal{M}}_{\tilde{f}}, \Gamma_B(\tilde{E})) \subset \mathcal{A}\mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E)) \ .$$

(b)

$$\mathrm{Spec}_W(\tilde{\mathcal{M}}_{\tilde{f}}, \Gamma_B(\tilde{E})) \supset \mathrm{Spec}_W(\mathcal{M}_f, \Gamma_B(E)) \ .$$

(c) *If f is chain-recurrent:*

$$\mathcal{A}\mathrm{Spec}(\tilde{\mathcal{M}}_{\tilde{f}}, \Gamma_B(\tilde{E})) = \mathcal{A}\mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E)) = \mathcal{A}\mathrm{Spec}_P(\mathcal{M}_f, \Gamma_B(E)) \ .$$

(d) *If f is chain-recurrent and APD:*

$$\mathrm{Spec}(\tilde{\mathcal{M}}_{\tilde{f}}, \Gamma_B(\tilde{E})) = \mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E)) = \mathrm{Spec}_W(\mathcal{M}_f, \Gamma_B(E)) \ .$$

Proof: The proof of (a) follows from the fact that if $F \subset E$ is an invariant subbundle of M_f, then $\tilde{F} = p^*F \subset \tilde{E}$ is an invariant subbundle of $\tilde{M}_{\tilde{f}}$. In effect,

$$\begin{aligned}
\tilde{v}_{\tilde{\theta}} = (v_\theta, \tilde{\theta}) \in \tilde{F}_{\tilde{\theta}} \ &\Rightarrow \ v_\theta \in F_\theta, \ p(\tilde{\theta}) = \theta \\
&\Rightarrow \ M(\theta)v_\theta \in F_{f(\theta)}, \ p(\tilde{f}(\tilde{\theta})) = f(\theta) \\
&\Rightarrow \ \tilde{M}(\tilde{\theta})\tilde{v}_{\tilde{\theta}} = \tilde{M}(\tilde{\theta})(v_\theta, \tilde{\theta}) = (M(\theta)v_\theta, \tilde{f}(\tilde{\theta})) \in \tilde{F}_{\tilde{\theta}} \ .
\end{aligned}$$

Moreover, if F is characterized by rates of growth, then $\tilde{F}$ is also characterized by the same rates of growth.

Henceforth, if there is a gap $\mathcal{A}_{\lambda,\mu}$ in the spectrum of $\mathcal{M}_f$, it produces an M_f- invariant splitting $E = E^{<\lambda} \oplus E^{>\mu}$ that its lifted to an $\tilde{M}_{\tilde{f}}$-invariant splitting $\tilde{E} = \tilde{E}^{<\lambda} \oplus \tilde{E}^{>\mu}$ characterized by the same rates of growth, so the gap $\mathcal{A}_{\lambda,\mu}$ is also present in the spectrum of $\tilde{\mathcal{M}}_{\tilde{f}}$.

To prove b, notice that if $z \in \mathrm{Spec}_W(\mathcal{M}_f, \Gamma_B(E))$, then we can find an approximate eigensection $v(\theta)$ for every ε: $|M(\theta)v(\theta) - zv(f(\theta))| < \varepsilon$ for all $\theta \in \mathcal{P}$. Lifting v to $\tilde{E}$, we have for all $\tilde{\theta} \in \tilde{\mathcal{P}}$:

$$\begin{aligned} |\tilde{M}(\tilde{\theta})\tilde{v}(\tilde{\theta}) - z\tilde{v}(\tilde{f}(\tilde{\theta}))| &= |\tilde{M}(\tilde{\theta})(v(p(\tilde{\theta})), \tilde{\theta}) - z(v(p(\tilde{f}(\tilde{\theta}))), \tilde{f}(\tilde{\theta}))| \\ &= |M(p(\tilde{\theta}))v(p(\tilde{\theta})) - zv(f(p(\tilde{\theta})))| < \varepsilon \ . \end{aligned}$$

So, $\tilde{v}(\tilde{\theta})$ is an approximate eigensection of $\tilde{\mathcal{M}}_{\tilde{f}}$.

(c) and (d) follow immediately from (a) and (b) and Theorem 3.19 Notice that if f is chain-recurrent, then the lift $\tilde{f}$ is also chain-recurrent. □

3.10. Triangular transfer operators

In this section we study the spectral implications of the existence of an invariant subbundle of a vector bundle automorphism. We emphasize that the invariant subbundle is not necessarily and spectral subbundle. This case of study is very useful for the construction of a great variety of invariant manifolds (see [**dlL97, CFdlL03a, CFdlL03b, HdlLb, HdlL04**]). We will use also these constructions to study spaces of sections of jets in Part 4, which are closely related to some global questions in dynamical systems.

DEFINITION 3.42. *A vector bundle automorphism M_f on E is* upper triangular *with respect to a splitting $E = \bigoplus_{i=1}^{l} E^i$ iff there exist vector bundle maps $M_f^{i,j} : E^j \to E^i$, $i = 1, \ldots, l$, $j = i, \ldots, l$, such that for all $v = v^1 + \cdots + v^l$ with $v^i = E^i_\theta$,*

$$M(\theta)v = \sum_{i=1}^{l} \sum_{j=i}^{l} M^{i,j}(\theta)v^j \ .$$

Pictorially, we will write

$$M(\theta) = \begin{pmatrix} M^{1,1}(\theta) & M^{1,2}(\theta) & \ldots & M^{1,l}(\theta) \\ 0 & M^{2,2}(\theta) & \ldots & M^{2,l}(\theta) \\ \vdots & 0 & \ddots & \vdots \\ 0 & \ldots & & M^{l,l}(\theta) \end{pmatrix} .$$

We will also write $M^i = M^{i,i}$. Notice also that each M_f^i induces a transfer operator $\mathcal{M}_f^i$ in $\Gamma_B(E^i)$.

We recall the following results from Functional Analysis.

DEFINITION 3.43. *Given a Banach space, X which can be decomposed into closed subspaces $X = \bigoplus_{i=1}^{l} X_i$ with the norm in X being equivalent to the supremum of the norms of the projections, we will say that the operator $T : X \to X$ is upper triangular (with respect to the above decomposition) if for $x_i \in X_i$, we can write*

$$T\left(\sum_{i=1}^{l} x_i\right) = \sum_{i=1}^{l}\sum_{j=i}^{l} T_{i,j}x_j$$

with $T_{i,j} : X_j \to X_i$ linear operators.

Note that, in particular, $T_i = T_{i,i} : X_i \to X_i$, hence it makes sense to speak about the spectrum of T_i. A natural question is how is the spectrum (and the Weyl spectrum) of T related to the spectrum of the T_i. Although the answer is trivial in finite dimension, this is not the case in infinite dimensions. Nevertheless, some relations are true [**CFdlL03a**].

LEMMA 3.44. *Assume that T is upper triangular as in Definition 3.43. Then:*

$$\mathrm{Spec}(T, X) \subset \bigcup_{i=1}^{l} \mathrm{Spec}(T_i, X_i)\ ,$$

$$\mathrm{Spec}_W(T, X) \subset \bigcup_{i=1}^{l} \mathrm{Spec}_W(T_i, X_i)$$

and

$$\bigcup_{i=1}^{l}\left(\mathrm{Spec}_W(T_i, X_i)\bigcap_{j=1}^{i-1}\mathrm{Res}(T_j, X_j)\right) \subset \mathrm{Spec}_W(T, X)$$

Proof: For the sake of simplicity, we will prove the theorem for $l = 2$. Then, one uses induction arguments. Let $X = X^1 \oplus X^2$ be a splitting of a Banach space X into closed Banach subspaces X^1 and X^2, and $A : X \to X$ is a bounded linear operator given by

$$A = \begin{pmatrix} A^1 & B \\ 0 & A^2 \end{pmatrix}\ , \tag{3.50}$$

with respect to that splitting (so X^1 is invariant). We have to prove:

$$\mathrm{Spec}(A, X) \subset \mathrm{Spec}(A^1, X^1) \cup \mathrm{Spec}(A^2, X^2)\ , \tag{3.51}$$

$$\mathrm{Spec}_W(A, X) \subset \mathrm{Spec}_W(A^1, X^1) \cup \mathrm{Spec}_W(A^2, X^2) \tag{3.52}$$

and

$$\mathrm{Spec}_W(A^1, X^1) \cup \big(\mathrm{Spec}_W(A^2, X^2) \cap \mathrm{Res}(A^1, X^1)\big) \subset \mathrm{Spec}_W(A, X)\ . \tag{3.53}$$

To prove (3.51), take $z \in \mathrm{Res}(A^1, X^1) \cap \mathrm{Res}(A^2, X^2)$, and we will prove that $z \in \mathrm{Res}(A, X)$. Let $y = y^1 + y^2 \in X$ be any vector, written in X^1 and X^2 components. We have to solve the equation

$$\begin{aligned} A^1x^1 + Bx^2 - zx^1 = y^1 \ , \\ A^2x^2 - zx^2 = y^2 \ , \end{aligned}$$

where $x = x^1 + x^2$. Since $z \in \mathrm{Res}(A^2, X^2)$, we can solve the second equation for x^2. Then, x^1 have to solve the equation $A^1x^1 - zx^1 = y^1 - Bx^2$, that has an unique solution because $z \in \mathrm{Res}(A^1, X^1)$. This proves (3.51).

The inclusion (3.52) is proved in the following way. Let $z \in \mathrm{Spec}_W(A, X)$, so there exist sequences $\{\varepsilon_n\}_n \subset \mathbb{R}$ and $\{x_n = x_n^1 + x_n^2\}_n \subset X$, with $\varepsilon_n \to 0$ and $\|x_n\| = \max\{x_n^1, x_n^2\} \geq 1$, such that

$$\begin{aligned} \|A^1x_n^1 + Bx_n^2 - zx_n^1\| \leq \varepsilon_n \ , \\ \|A^2x_n^2 - zx_n^2\| \leq \varepsilon_n \ . \end{aligned}$$

If $x_n^2 \to 0$, then $\|A^1x_n^1 - zx_n^1\| \leq \varepsilon_n + \|B\|\|x_n^2\|$, and we conclude that $z \in \mathrm{Spec}_W(A^1, X^1)$. Otherwise we conclude that $z \in \mathrm{Spec}_W(A^2, X^2)$. Hence, (3.52) is proved.

For the proof of the last inclusion (3.53), first notice that $\mathrm{Spec}_W(A^1, X^1) \subset \mathrm{Spec}_W(A, X)$ is obvious, Notice also that from an approximate eigenvector x^2 of $z \in \mathrm{Spec}_W(A^2, X^2)$ we can produce an approximate eigenvector $x = x^1 + x^2$ of z for A if, for instance, we can find x^1 such that $A^1x^1 + Bx^2 - zx^1 = 0$. This is the case if $z \in \mathrm{Spec}_W(A^2, X^2) \cap \mathrm{Res}(A^1, X^1)$. □

The inclusions in Lemma 3.44 above can be strict. Also, all the other inclusions may be false. Of course, a particular case in which the inclusions are equalities is when T is diagonal, that is to say, the splitting $X = \bigoplus_{i=1}^l X_i$ is invariant under T. Another special case that will be useful for us and in which the inclusions are equalities is the following.

LEMMA 3.45. *Assume that T is upper triangular as in Definition 3.43. Assume also that for every i,* $\mathrm{Spec}(T_i, X_i) = \mathrm{Spec}_W(T_i, X_i)$. *Then:*

$$\mathrm{Spec}(T, X) = \mathrm{Spec}_W(T, X) = \bigcup_{i=1}^{l} \mathrm{Spec}(T_i, X_i) \ .$$

Proof: Since

$$\operatorname{Spec}(T,X) \subset \bigcup_{i=1}^{l} \operatorname{Spec}(T_i,X_i) = \bigcup_{i=1}^{l} \left(\operatorname{Spec}(T_i,X_i) \bigcap_{j=1}^{i-1} \operatorname{Res}(T_j,X_j) \right)$$

$$= \bigcup_{i=1}^{l} \left(\operatorname{Spec}_W(T_i,X_i) \bigcap_{j=1}^{i-1} \operatorname{Res}(T_j,X_j) \right) \subset \operatorname{Spec}_W(T,X)$$

$$\subset \operatorname{Spec}(T,X) \ ,$$

all the inclusions are equalities. □

As an immediate corollary of the previous general results, the spectrum of triangular transfer operators is clarified. For instance, we have

PROPOSITION 3.46. *Let M_f be an upper triangular vector bundle automorphism as in Definition 3.42. Suppose that f is chain-recurrent and APD. Then:*

$$\operatorname{Spec}(\mathcal{M}_f, \Gamma_B(E)) = \bigcup_{i=1}^{l} \operatorname{Spec}(\mathcal{M}_f^i, \Gamma_B(E^i)) \ .$$

Proof: It is an immediate consequence of Lemma 3.45, because if f is chain-recurrent and APD then the full spectrum of a transfer operator is Weyl spectrum. □

The following result is important in the study of the whiskered tori in Hamiltonian systems. For some applications see [**dlLGJV05, HdlLb**].

PROPOSITION 3.47. *Let $M_f : E \to E$ be a vector bundle automorphism over a uniquely ergodic homeomorphism f. Let μ be its corresponding invariant measure. Suppose there exists a splitting*

$$E = \bigoplus_{i=1}^{n} E^i \tag{3.54}$$

in subbundles of rank 1.

With respect to the decomposition (3.54)*, the representation of M_f is:*

$$M(\theta) = \begin{pmatrix} a_1(\theta) & b_{12}(\theta) & \dots & b_{1n}(\theta) \\ 0 & a_2(\theta) & \dots & b_{2n}(\theta) \\ \vdots & 0 & \ddots & \vdots \\ 0 & \dots & & a_n(\theta) \end{pmatrix} . \tag{3.55}$$

Then:

$$\operatorname{Spec}(\mathcal{M}_f, \Gamma_B(E)) = \bigcup_{i=1}^{n} \mathcal{S}_{\rho_i}$$

where

$$\rho_i = \exp \int_{\mathcal{P}} \log |a_i(\theta)| \; d\mu \ .$$

Proof: Since f is uniquely ergodic and the rank of E^i is 1, then Corollary 2.41 implies that

$$\mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E^i)) = \mathcal{S}_{\rho_i} \ .$$

Notice that the full spectrum is in fact Weyl spectrum, because it is a circle. So, we get the result applying Proposition 3.45. □

CHAPTER 4

Characterization of the spectrum by behavior on orbits

In this section, we will show how the spectrum and Weyl spectrum in $\Gamma_B(E)$ can be characterized by the behavior of individual orbits. Since we have already shown that the spectrum in $\Gamma_B(E)$ is the same as in other spaces, the same results will apply to other spaces, in particular to continuous or to L^p sections.

This question is very natural from different points of view. From the point of view of dynamical systems, there are two notions of hyperbolicity, the one developed in [**HP70, HPS77**] based on contraction properties on bundles (which can be shown to be spectral properties using adapted metrics) and the theory of [**Fen72, Fen74, Fen77**] which emphasizes the rates of growth of periodic orbits. The present results show that, the two characterizations of regularity are equivalent. From the point of view of PDE's, the paper [**Vis96**] contains lower bounds of the spectrum in terms of Lyapunov numbers of certain orbits. In this section we will show that, in some cases (L^p spaces, the purely kinematic operators, some assumptions on recurrence) the converse is also true.

4.1. The spectrum on bounded sequences supported on orbits

Let $M_f : E \to E$ be a vector bundle automorphism.

DEFINITION 4.1. *Given an aperiodic point $\theta \in \mathcal{P}$ we define*

$$b_{f,\theta}(E) = \{v \in \prod_{i\in\mathbb{Z}} E_{f^i(\theta)} \mid \|v\|_\infty = \sup_{i\in\mathbb{Z}} |v_i| < \infty\} \tag{4.1}$$

where v_i denote the components in the direct product.

Given a periodic point $\theta \in \mathcal{P}$ of minimal period N, we define

$$b_{f,\theta}(E) = \{v \in \prod_{i=0}^{N-1} E_{f^i(\theta)} \mid \|v\|_\infty = \sup_{0\le i<N} |v_i| < \infty\} \ . \tag{4.2}$$

DEFINITION 4.2. *For each $\theta \in \mathcal{P}$, we define a pointwise transfer operator acting on the spaces above, by*

$$(m_{f,\theta}v)_i = M(f^{i-1}(\theta))v_{i-1} \ ,$$

where in the N-periodic case, the indices i are understood (mod N).

The intuition behind these definitions is that $m_{f,\theta}$ is the restriction of $\mathcal{M}_f$ on orbits. Once we have fixed θ, we will denote $M_i = M(f^i(\theta))$ for $i \in \mathbb{Z}$. The pointwise transfer operators $m_{f,\theta}$ are also known as *weighted shift operators* (see [**CL99**]). (The weights are the M_i's, and we shift the indices).

Even if for one periodic orbit it does not make a difference what norm we use in $b_{f,\theta}$, it will be quite important to specify the norm when we make statements that are uniform on a set of periodic orbits.

We will use the notation

(4.3)

$$\Sigma_{P,\theta} = \mathrm{Spec}_P(m_{f,\theta}, b_{f,\theta}(E)) \ , \quad \text{and} \ \bigcup_{\theta\in\mathcal{P}} \Sigma_{P,\theta} \subset \mathrm{Spec}_P(\mathcal{M}_f, \Gamma_B(E))$$

$$\Sigma_{W,\theta} = \mathrm{Spec}_W(m_{f,\theta}, b_{f,\theta}(E)) \ , \quad \text{and} \ \bigcup_{\theta\in\mathcal{P}} \Sigma_{W,\theta} \subset \mathrm{Spec}_W(\mathcal{M}_f, \Gamma_B(E))$$

$$\Sigma_{\theta} = \mathrm{Spec}(m_{f,\theta}, b_{f,\theta}(E)) \ , \quad \text{and} \ \bigcup_{\theta\in\mathcal{P}} \Sigma_{\theta} \subset \mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E)) \ .$$

We note that when θ is periodic, $b_{f,\theta}(E)$ is finite dimensional and $\Sigma_{P,\theta} = \Sigma_{W,\theta} = \Sigma_\theta$. We will show now that, when θ is aperiodic $\Sigma_\theta = \mathcal{A}\Sigma_\theta$. This, of course, is impossible when θ is periodic.

Proposition 4.3. *With the notations above*

i) *If θ is aperiodic, then $\Sigma_{P,\theta}$, $\Sigma_{W,\theta}$ and Σ_θ are invariant under rotations.*

ii) *If θ is periodic of minimal period N,*

$$\Sigma_\theta = \{z \in \mathbb{C} \mid z^N \text{ is eigenvalue of } M(\theta, N)\} \tag{4.4}$$

so that, in particular, Σ_θ is invariant under rotations by $\exp(2\pi\mathbf{i}/N)$.

Proof: To prove i) we note that if θ is aperiodic and $v \in b_{f,\theta}(E)$ satisfies

$$\|m_{f,\theta} v - zv\| \le \varepsilon \ , \qquad \|v\| = 1 \ , \tag{4.5}$$

then, $\tilde{v}$ defined by

$$\tilde{v}_k = e^{-k\alpha\mathbf{i}} v_k \tag{4.6}$$

satisfies

$$\|m_{f,\theta}\tilde{v} - ze^{\alpha\mathbf{i}}\tilde{v}\| \le \varepsilon \ , \qquad \|\tilde{v}\| = 1 \ . \tag{4.7}$$

Hence, given any sequence of approximate eigensections for z, we can construct another such sequence for $e^{\alpha\mathbf{i}}z$. The proof also works for the point spectrum, taking $\varepsilon = 0$. Notice that since the Weyl spectrum is rotationally invariant, so is the full spectrum. This proves i).

The same construction works for the periodic orbits of minimal period N if we restrict α to satisfy $e^{\alpha N\mathbf{i}} = 1$ so that the function $e^{\alpha N\mathbf{i}j}$ can be considered as a function on the orbit.

To prove ii) we note that if z^N is an eigenvalue of $M(\theta, N)$ with eigenvector v_0 then $z \neq 0$ (we assume that M_f is automorphism) and setting

$$\tilde{v} = \left(v_0, \frac{1}{z}M(\theta,1)v_0, \dots, \frac{1}{z^{N-1}}M(\theta, N-1)v_0\right) \tag{4.8}$$

we have $\tilde{v} \neq 0$ and

$$\begin{aligned} m_{f,\theta}\tilde{v} &= \left(\frac{1}{z^{N-1}}M(\theta,N)v_0, M(\theta,1)v_0, \dots, \frac{1}{z^{N-2}}M(\theta, N-1)v_0\right) \\ &= z\tilde{v} \ . \end{aligned} \tag{4.9}$$

Hence, the R.H.S. of ii) is contained in the L.H.S.

Conversely, if $m_{f,\theta}\tilde{v} = z\tilde{v}$ in $b_{f,\theta}(E)$ for a periodic point θ with $p(\theta) = N$, we have $m^N_{f,\theta}\tilde{v} = z^N\tilde{v}$, which in coordinates means

$$M(f^i(\theta), N)v_i = z^N v_i \ , \tag{4.10}$$

for all $i = 0, \dots, N-1$. Notice that one of the v_i is non-zero, and then we define $w_0 = M(f^i(\theta), N-i)v_i$, that is obviously non-zero. Then:

$$\begin{aligned} M(\theta,N)w_0 = M(\theta,N)M(f^i(\theta),N-i)v_i &= M(f^{i+N}(\theta),N-i)M(f^i(\theta),N)v_i \\ &= z^N M(f^i(\theta),N-i)v_i = z^N w_0 \ , \end{aligned}$$

and z^N is an eigenvalue of $M(\theta, N)$.

Remark 4.4. If M_f is not invertible and we find $z = 0$, we conclude from (4.10) that some of the $M(f^i(\theta))$ is not invertible, hence, 0 belongs to the spectrum of $M(\theta, N)$.

Hence, the L.H.S. of ii) is contained in the R.H.S. and we conclude the proof of Proposition 4.3. □

4.2. Characterization of the Weyl spectrum

In this section we will characterize the Weyl spectrum on the space of bounded sections, and we will finish the analysis started in Section 3.1. Motivated by the findings in Lemma 3.1, we will distinguish the following subsets of $\mathcal{P}$:

- the set of aperiodic points:
$$A(f) = \{\theta \in \mathcal{P} \mid p(\theta) = \infty\} \ ;$$
- the set of periodic points:
$$P(f) = \{\theta \in \mathcal{P} \mid p(\theta) < \infty\} \ ;$$
- the set of strongly periodic points:
$$B(f) = \{\theta \in \mathcal{P} \mid \exists U \subset \mathcal{P}, \text{ open }, \theta \in U, p_{|U} \text{ bounded}\} \ .$$

Moreover, for a given point $\theta \in \mathcal{P}$, we define its set of periods as

$$P(\theta) = \{N \mid \exists \{\theta_n\}_{n>0}, \theta_n \to \theta, p(\theta_n) = N\} \ .$$

Notice that if $\theta \in B(f)$ then $P(\theta)$ is a finite set and all its elements are (finite) multiple of $p(\theta)$.

The following theorem summarizes Lemma 3.1 and Proposition 4.3. As we will see, the Weyl spectrum can be characterized from the point spectrum.

THEOREM 4.5. *Let $M_f : E \to E$ be a vector bundle automorphism. Then:*

$$\begin{aligned}\mathrm{Spec}_W(\mathcal{M}_f, \Gamma_B(E)) &= \bigcup_{\theta \in A(f)} \Sigma_{P,\theta} \cup \bigcup_{\theta \in P(f) \setminus B(f)} \mathcal{A}\Sigma_{P,\theta} \cup \bigcup_{\substack{\theta \in B(f) \\ N \in P(\theta)}} \mathcal{A}_N \Sigma_{P,\theta} \\ &\subset \bigcup_{\theta \in A(f)} \Sigma_{P,\theta} \cup \bigcup_{\theta \in P(f)} \mathcal{A}\Sigma_{P,\theta} \ .\end{aligned}$$

Proof: We follow the proof of Lemma 3.1, and, in particular, the remarks 3.3, 3.5 and 3.6, stating them using the notation introduced above. See also the proof of Theorem 3.19.

Given $z \in \mathrm{Spec}_W(\mathcal{M}_f, \Gamma_B(E))$, recall that from a sequence of approximate eigensections supported on orbits we can find a point $\theta_* \in \mathcal{P}$ and a sequence of vectors $\{w_i\}_{i \in \mathbb{Z}}$ such that

$$\Pi(w_i) = f^i(\theta_*) \ , M(f^i(\theta_*))w_i = z w_{i+1} \text{ for all } i \in \mathbb{Z} \ . \tag{4.11}$$

We distinguish three cases:

(a.1) $\theta_* \in A(f)$, that is θ_* is aperiodic. Henceforth, $z \in \Sigma_{P,\theta_*}$. Obviously, $\Sigma_{P,\theta_*} \subset \mathrm{Spec}_W(\mathcal{M}_f, \Gamma_B(E))$.

(a.2) $\theta_* \in P(f) \setminus B(f)$, that is θ_* is periodic but can be approximated by orbits of period arbitrarily high. Notice that if $p(\theta_*) = N_*$, then the sequence, indexed by k, given by

$$w_{N_* k} = \frac{1}{z^{kN_*}} M(\theta_*, N_* k) w_0 = \left(\frac{1}{z^{N_*}} M(\theta_*, N_*) \right)^k w_0$$

is bounded ($|w_i| \le 1$ for all $i \in \mathbb{Z}$). Then, the matrix $M(\theta_*, N_*)$ has eigenvalues of modulus $|z|^{N_*}$ and w_0 is in the space spanned for the corresponding eigenvectors. That is to say, $z \in \mathcal{A}\Sigma_{P,\theta_*}$.

Conversely, let $z \in \Sigma_{P,\theta_*}$ and $w_0 \neq 0$ such that $M(\theta_*, N_*)w_0 = z^{N_*} w_0$. Define $\tilde{z} = e^{\alpha \mathbf{i}} z$, where $\alpha \in \mathbb{R}$. Take θ_0 close enough θ_* whose period is high enough, say $p(\theta_0) > 2N$. Then, the bounded section

$$\begin{aligned} &v(f^i(\theta_0)) = v_i = (1 - |i|/(N+1))_+ \frac{1}{\tilde{z}^i} M(\theta_0, i) w_0 \text{ for } |i| \le N \ , \\ &v(\theta) = 0 \text{ otherwise} \ , \end{aligned} \tag{4.12}$$

defines an approximate bounded eigensection of $\tilde{z}$. In summary, if $z \in \Sigma_{P,\theta_*}$ with $\theta_* \in P(f) \setminus B(f)$, then it produces a whole circle of radius $|z|$ of approximate eigenvalues, that is $\mathcal{A}\Sigma_{P,\theta_*} \subset \mathrm{Spec}_W(\mathcal{M}_f, \Gamma_B(E))$.

(b) $\theta_* \in B(f)$, (i.e. there exist an open set containing θ_* in which the periods of the orbits are bounded).

If $p(\theta_*) = N_*$, recall we can obtain approximate eigensections supported in periodic orbits of period N, with $N = N_*\ell$. (4.11) reads now

$$M(f^i(\theta_*))w_i = zw_{i+1} \quad 0 \leq i < N-1 \ , \qquad M(f^{N-1}(\theta_*))w_{N-1} = zw_0 \ . \tag{4.13}$$

As a result, we obtain that $z^N = z^{N_*\ell}$ is an eigenvalue of $M(\theta_*, N) = M(\theta_*, N_*)^\ell$. This means that $z = e^{2\pi \frac{j}{N}\mathbf{i}}\tilde{z}$, where $\tilde{z}^{N_*}$ is an eigenvalue of $M(\theta_*, N_*)$ and $j \in \{0, 1, \cdots, l-1\}$. That is to say, z is one of the vertices of the regular N-polygon generated by $\tilde{z} \in \Sigma_{P,\theta_*}$.

Conversely, if $\tilde{z} \in \Sigma_{P,\theta_*}$, that is to say $\tilde{z}^{N_*}$ is an eigenvalue of $M(\theta_*, N)$, and N is one of the periods of θ_*, we see that $z = e^{2\pi \frac{j}{N}\mathbf{i}}\tilde{z}$ is an approximate eigenvalue of $\mathcal{M}_f$. Notice that θ_* can be approached by periodic points θ_n, of period N. Moreover, if $\tilde{w}_0$ is an eigenvector of $M(\theta_*, N_*)$ for $\tilde{z}$, then the finite sequence $\{w_i\}_{i=0}^{N-1}$ defined by

$$w_i = \frac{1}{z^i} M(\theta_*, i)\tilde{w}_0$$

satisfies (4.13). Again, there is no reason why $w_{i+N_*} = w_i$, but we can transport these vectors to the N-periodic orbit θ_n and obtain an approximate eigenvector of $\mathcal{M}_f$ for z. That is $\mathcal{A}_N\Sigma_{P,\theta_*} \subset \mathrm{Spec}_W(\mathcal{M}_f, \Gamma_B(E))$.

The analysis of the three cases completes the proof of Theorem 4.5. □

Remark 4.6. This result not only characterizes the Weyl spectrum of a transfer operator by the behavior of orbits, but also summarizes some of the results that we had obtained before.

Notice that the rotational part of the spectrum is contained in the aperiodic orbits and the periodic orbits that can be approached by orbits with period arbitrarily high. In particular, if f is APD, then the Weyl spectrum is invariant under rotations, because $B(f) = \emptyset$ (see Theorem 3.13).

Notice also that if f does not have periodic orbits then the Weyl spectrum can be computed from the point spectrum on the (aperiodic) orbits (see remark 3.2).

As a corollary of Theorem 4.5, we obtain the following result, that is stronger than lemma 2.15 (see [**CL99**], where a general result on Weyl spectrum is used).

COROLLARY 4.7. *Let M_f be a vector bundle automorphism. Given a positive constant ρ:*

$$M_f \text{ is quasi-}\rho\text{-dichotomic} \Leftrightarrow \operatorname{Spec}_W(\mathcal{M}_f, \Gamma_B(E)) \cap \mathcal{S}_\rho = \emptyset \ .$$

Proof: The proof of the "if" part is Lemma 2.15. For the proof of the "only if" part, suppose that $\rho = 1$, and we will find a non-trivial bounded orbit for M_f (the general result is obtained by scaling).

Let $z \in \operatorname{Spec}_W(\mathcal{M}_f, \Gamma_B(E))$ with $|z| = 1$. From Theorem 4.5, $z \in \Sigma_{P,\theta}$ with $\theta \in A(f)$ or $z \in \mathcal{A}\Sigma_{P,\theta}$ with $\theta \in P(f)$.

In the first case, since $\Sigma_{P,\theta}$ is invariant under rotations we can assume that $z = 1$, and then there exist $\{v_i\}_{i\in\mathbb{Z}} \in b_{f,\theta}(E)$ such that

$$M(f^i(\theta))v_i = v_{i+1} \ .$$

That is $v_i = M(\theta, i)v_0$ is an orbit, and it is bounded.

If θ is periodic, with $p(\theta) = N$, then $M(\theta, N)$ has eigenvalues on the unit circle. The corresponding eigenvectors produce bounded orbits for M_f.
□

4.3. The spectrum on l^p sequences supported on orbits

We consider in this section the pointwise transfer operators acting on l^p spaces of sequences (see [**CL99**] for related results).

Let $M_f : E \to E$ be a vector bundle automorphism.

DEFINITION 4.8. *Given an aperiodic point $\theta \in \mathcal{P}$ we define*

$$l^p_{f,\theta}(E) = \{v \in \prod_{i\in\mathbb{Z}} E_{f^i(\theta)} \mid \|v\|^p_{l^p} = \sum_{i\in\mathbb{Z}} |v_i|^p < \infty\} \tag{4.14}$$

where v_i denote the components in the direct product.

Given a periodic point $\theta \in \mathcal{P}$ of minimal period N, we define

$$l^p_{f,\theta}(E) = \{v \in \prod_{i=0}^{N-1} E_{f^i(\theta)} \mid \|v\|^p_{l_p} = \sum_{0\le i<N} |v_i|^p < \infty\} \ . \tag{4.15}$$

When θ is periodic, $b_{f,\theta}(E)$ and $l^p_{f,\theta}(E)$ are the same space and the norms are equivalent. Nevertheless, the constants that give the equivalence in the norms are not uniform in the period. Hence, making statements that are valid for all periods is not automatic.

We will study here the Weyl spectrum on spaces of l^p sequences, and we will see that it is the same as that on bounded sequences. As a consequence, if the vector bundle automorphism $M_f : E \to E$ is defined over an homeomorphism f that is APD and has an invariant measure μ that is topological, then the spectrum is rotationally invariant and for $p > 1$

$$\begin{aligned}\mathrm{Spec}(\mathcal{M}_f, \Gamma_{L^p}(E)) &= \mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E)) \\ &= \mathrm{Spec}_W(\mathcal{M}_f, \Gamma_B(E)) \\ &= \bigcup_{\theta \in A(f)} \Sigma_{P,\theta} \cup \bigcup_{\theta \in P(f)\setminus B(f)} \mathcal{A}\Sigma_{P,\theta} \cup \bigcup_{\substack{\theta \in B(f) \\ N \in P(\theta)}} \mathcal{A}_N \Sigma_{P,\theta} \ .\end{aligned}$$

So, the fact that $\Sigma_{W,\theta} \subset \mathrm{Spec}_W(m_{f,\theta}, l^p_{f,\theta}(E))$ has further applicability in the study of the spectrum of the transfer operator acting on L^p sections. Another interesting example is the case in which the topological measure is ergodic, because this implies that f has a dense orbit θ_0. So, in this case:

$$\begin{aligned}\mathrm{Spec}(M_f, \Gamma_{L^p}(E)) &= \mathrm{Spec}(M_f, \Gamma_B(E) \\ &= \Sigma_{W,\theta_0} \\ &= \mathrm{Spec}_W(m_{f,\theta}, l^p_{f,\theta_0}(E)) \ .\end{aligned}$$

This is the result, which we now state formally.

THEOREM 4.9. *Let $M_f : E \to E$ be a vector bundle automorphism. Assume that f has an aperiodic orbit $\theta \in \mathcal{P}$. Then, for $p > 1$,*

$$\mathrm{Spec}_W(m_{f,\theta}, l^p_{f,\theta}(E)) = \mathrm{Spec}_W(m_{f,\theta}, b_{f,\theta}(E)) \ .$$

In particular, $\mathrm{Spec}(m_{f,\theta}, l^p_{f,\theta}(E))$ *is invariant under rotations.*

REMARK 4.10. The fact that $\mathrm{Spec}_W(m_{f,\theta}, l^p_{f,\theta}(E))$ is rotationally invariant can be proved directly just noting that the map $(v_k)_k \to (\bar{v}_k)_k$ defined by $\bar{v}_k = e^{-\alpha k \mathbf{i}} v_k$ is an isometry in $l^p_{f,\theta}(E)$ for all $\alpha \in \mathbb{R}$ that for an approximate eigensequence v of z produces an approximate eigensequence $\bar{v}$ of $e^{\alpha \mathbf{i}} z$.

Proof: Firstly, from a bounded approximate eigensequence v of z we will produce an l^p approximate eigensequence $\bar{v}$. Recall that using localization arguments, we can assume that the bounded eigensequence is supported in a finite segment of the orbit:

- $v_i = 0$ for $|i| > N = \left[\frac{1}{\varepsilon}\right]$;
- $\|v\|_b = 1$, $|v_0| \geq \frac{3}{4}$;
- $\|m_{f,\theta} v - z v\|_b \leq 2|z|\varepsilon$.

(Possibly, to obtain the sequence centered in θ we would have to shift the indices i, but this do not change the spectrum). Notice that

$$\|v\|^p_{l^p} = \sum_{i=-N}^{i=N} |v_i|^p \geq \left(\frac{3}{4}\right)^p$$

so $\|v\|_{l^p} \geq \frac{3}{4}$, and

$$\begin{aligned}\|m_{f,\theta}v - zv\|_{l^p}^p &= \sum_{i=-N}^{N+1} |M_{i-1}v_{i-1} - zv_i|^p \\ &\leq \sum_{i=-N}^{N+1} (2|z|\varepsilon)^p = (2|z|\varepsilon)^p(2N+2) \\ &\leq 2^{p+2}|z|^p\varepsilon^{p-1} .\end{aligned}$$

So then, v is a l^p approximate eigensequence, provided that $p > 1$. This argument proves the inclusion

$$\mathrm{Spec}_W(m_{f,\theta}, l^p_{f,\theta}(E)) \supset \mathrm{Spec}_W(m_{f,\theta}, b_{f,\theta}(E)) .$$

For the proof of the opposite inclusion, we will use similar arguments to those in the proof of Theorem 3.7. Let v be an l^p approximate eigensequence with $\|v\|_{l^p} = 1$, $\|\mathcal{M}_f v - zv\|_{l^p} = \varepsilon$. Notice that v is bounded, but it is not in principle a bounded approximate eigensequence.

For $k \in \mathbb{Z}$, denote $e_k = M_{k-1}v_{k-1} - zv_k$ and

$$r_k = \begin{cases} \dfrac{1}{|v_k|^p} \displaystyle\sum_{i=-2N}^{2N} |e_{k+i}|^p & \text{if } v_k \neq 0 , \\ 0 & \text{otherwise} , \end{cases} \tag{4.16}$$

$$s_k = \begin{cases} \dfrac{1}{|v_k|^p} \displaystyle\sum_{i=-2N}^{2N} |v_{k+i}|^p & \text{if } v(\theta) \neq 0 , \\ 0 & \text{otherwise} , \end{cases} \tag{4.17}$$

where $N \in \mathbb{N}$ will specify later. This setting defines three sequences

$$e = (e_k)_{k\in\mathbb{Z}} \ , \ r = (r_k)_{k\in\mathbb{Z}} \ , \ s = (s_k)_{k\in\mathbb{Z}} \ .$$

We consider the finite measure in $\mathbb{Z}$ given by $\nu(\{i\}) = \nu_i = |v_i|^p$, for $i \in \mathbb{Z}$. Note that $\nu(\mathbb{Z}) = \sum_{i\in\mathbb{Z}} \nu_i = 1$,

$$\int r d\nu = \sum_{k\in\mathbb{Z}} r_k\nu_k \leq \sum_{k\in\mathbb{Z}} \sum_{i=-2N}^{2N} |e_{k+i}|^p \leq (4N+1)\varepsilon^p$$

and

$$\int s d\nu = \sum_{k\in\mathbb{Z}} s_k\nu_k = \sum_{k\in\mathbb{Z}} \sum_{i=-2N}^{2N} 1 = (4N+1) .$$

We can take, in particular, $(4N+1) = \varepsilon^{-\alpha}$ for some $\alpha > 0$, that will be specified later.

For positive values a,b,

$$\sum_{r_k\geq a} |v_k|^p \leq \frac{1}{a}\sum_k r_k|v_k|^p \leq \frac{1}{a}\varepsilon^{p-\alpha},$$

and

$$\sum_{s_k \geq b} |v_k|^p \leq \frac{1}{b} \sum_k s_k |v_k|^p = \frac{1}{b} \varepsilon^{-\alpha},$$

By taking $a = \varepsilon^{p-\alpha-\delta}$ and $b = \varepsilon^{-\alpha-\delta}$ for some $\delta > 0$, we get that r is greater than $\varepsilon^{p-\alpha-\delta}$ in a set of indices $\Omega_r \subset \mathbb{Z}$ of ν-measure less than ε^{δ} and s is greater than $\varepsilon^{-\alpha-\delta}$ in a set of indices $\Omega_s \subset \mathbb{Z}$ of ν-measure less than ε^{δ}.

We assume that α, δ are so small that $p-\alpha-\delta > 0$, i.e. $1-\alpha/p-\delta/p > 0$. Since $p > 1$, we can also assume that they are so small that $\alpha(1-1/p)-\delta/p > 0$. This choice fixes also N in (4.16) and (4.17).

Since $\nu(\mathcal{P}) = 1$, $\nu(\Omega_r) \leq \varepsilon^{\delta}$ and $\nu(\Omega_s) \leq \varepsilon^{\delta}$, we obtain $\nu(\mathcal{P}-\Omega_r-\Omega_s) > 0$ for ε sufficiently small. Hence, there exist $k \in \mathcal{P} - \Omega_s - \Omega_r$ for which $v_k \neq 0$. Shifting indices, we can assume that $k = 0$.

We have

$$|e_i|^p \leq |M_{i-1} v_{i-1} - z v_i|^p \leq |v_0|^p \varepsilon^{p-\alpha-\delta} \quad i = -2N, \ldots, 2N \tag{4.18}$$

and

$$|v_i|^p \leq |v_0|^p \varepsilon^{-\alpha-\delta} \quad i = -2N, \ldots, 2N \ . \tag{4.19}$$

If we denote $\tilde{v}_i = v_i / \max_{k=-N,\ldots,N} |v_k|$ we have:

$$\begin{aligned} &|M_{i-1}\tilde{v}_{i-1} - z\tilde{v}_{i+1}| \leq \varepsilon^{1-\alpha/p-\delta/p} \text{ for all } -2N \leq i \leq 2N \\ &|\tilde{v}_i| \leq 1 \text{ for all } -N \leq i \leq N \\ &|\tilde{v}_j| = 1 \text{ for some } -N \leq j \leq N \\ &|\tilde{v}_i| \leq \varepsilon^{-\alpha/p-\delta/p} \text{ for all } -2N \leq i \leq 2N \end{aligned} \tag{4.20}$$

We proceed again using localization arguments. We consider

$$\hat{v}_i = (1 - 2|i-j|/N)_+ \tilde{v}_i \text{ for } i = -2N, \ldots, 2N \ ,$$

and $\hat{v}_i = 0$ otherwise, where j is as in (4.20). Notice that

$$\begin{aligned} M_i\hat{v}_i - z\hat{v}_{i+1} &= \left(1 - \frac{2|i-j|}{N}\right)_+ (M_i\tilde{v}_i - z\tilde{v}_{i+1}) \\ &\quad + \left(\left(1 - \frac{2|i-j|}{N}\right)_+ - \left(1 - \frac{2|i+1-j|}{N}\right)_+\right) z\tilde{v}_{i+1} \ . \end{aligned} \tag{4.21}$$

Using (4.20), we can bound from above the size of the first term in the R.H.S. by $\varepsilon^{1-\alpha/p-\delta/p}$ and, since $4N+1 = \varepsilon^{-\alpha}$, the size of the second term can be bounded by $10|z|\varepsilon^{\alpha-\alpha/p-\delta/p}$. According to the way that we chose the α, δ, both bounds have positive powers of ε. Henceforth, $\hat{v}$ is a bounded approximate eigensequence of z.

With these arguments we complete the proof of Theorem 4.9. □

4.4. Characterization of the spectrum by behavior on a dense orbit

In this section we will characterize the spectrum of a transfer operator $\mathcal{M}_f$ in case that f is topologically transitive. The spectrum is determined by the behavior of the transfer on a dense orbit.

THEOREM 4.11. *Let $M_f : E \to E$ be a vector bundle automorphism. Suppose that the orbit of $\theta_0 \in \mathcal{P}$ is dense in $\mathcal{P}$. Then:*

$$\begin{array}{c} \mathrm{Spec}_W(\mathcal{M}_f, \Gamma_B(E)) = \Sigma_{W,\theta_0} \\ || \\ \mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E)) = \Sigma_{\theta_0} \end{array} \tag{4.22}$$

REMARK 4.12. Since we have shown that the spectrum on bounded sections is the same as the spectrum in the space of continuous sections, the equalities (4.22) hold in the space of continuous sections.

REMARK 4.13. The hypothesis of topological transitiveness holds if f has an invariant measure μ that is topological and ergodic. In such a case, the equalities hold in the space of L^p sections, with $p > 1$.

Proof: Note that the presence of a dense orbit forces that $\mathrm{Spec}_W(\mathcal{M}_f, \Gamma_B(E))$ is rotationally invariant and so is $\mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E))$.

Note also that as the orbit by θ_0 is not periodic, then Σ_{W,θ_0} and Σ_{θ_0} are also rotationally invariant.

The equalities of Weyl spectra and of full spectrum are immediate consequences of Theorem 3.39, since an orbit is a particular case of invariant set.

Finally, the fact that the full spectrum is Weyl spectrum follows directly from the fact that topological transitivity implies APD and chain-recurrence (see Theorem 3.19). □

4.5. Characterization of the spectrum by behavior on periodic orbits

Our next result is a variation of Theorem 3.19, Theorem 4.5, which characterize the spectrum in terms of behavior of individual orbits.

The conclusions are somewhat better since we show that it suffices to use only periodic orbits. On the other hand, the hypothesis require a very strong recurrence property (namely specification introduced in Definition 1.30) and, more importantly, that we are dealing with one dimensional bundles.

THEOREM 4.14. *Let $M_f : E \to E$ be a vector bundle automorphism covering a homeomorphism f which satisfies the specification property, and assume that there is an continuous splitting of E in bundles of dimension 1 for which M_f is upper triangular.*

We define the periodic spectrum *as*

$$\Sigma_{\mathrm{per}} = \overline{\bigcup_{\theta_0 \in P(f)} \Sigma_{\theta_0}} \ .$$

Then:

$$\mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E)) = \mathrm{Spec}_W(\mathcal{M}_f, \Gamma_B(E)) = \Sigma_{\mathrm{per}} \ .$$

Proof: Let $E = \bigoplus_{i=1}^n E^i$ the decomposition in bundles of dimension 1 for which M_f is upper triangular. Since f is chain-recurrent and APD, Theorem 3.19 asserts that

$$\mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E)) = \mathrm{Spec}_W(\mathcal{M}_f, \Gamma_B(E))$$

and

$$\mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E^i)) = \mathrm{Spec}_W(\mathcal{M}_f, \Gamma_B(E^i))$$

for all $i = 1, \dots, n$. Moreover, Proposition 3.46 asserts that

$$\mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E)) = \bigcup_{i=1}^n \mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E^i)).$$

If Σ^i_{per} denotes the periodic spectrum on each subbundle, then

$$\Sigma_{\mathrm{per}} = \bigcup_{i=1}^n \Sigma^i_{\mathrm{per}},$$

because for finite-dimensional triangular matrices the eigenvalues are the diagonal elements. In summary, what we have to prove is that

$$\mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E^i)) = \Sigma^i_{\mathrm{per}}$$

for all $i = 1, \dots, n$.

Henceforth, we will assume from now on that the bundle E is 1 dimensional, M is bounded away from zero. Notice that the spectrum is a full annulus, i.e. without gaps, $\mathrm{Spec}(M_f, \Gamma_B(E) = \mathcal{A}_{\lambda_-,\lambda_+}$. We will show first that the boundaries of the spectrum can be approached by the spectrum on periodic orbits. Then, we will show that Σ_{per} is annularly convex, that is, if $z_1, z_2 \in \Sigma_{\mathrm{per}}$, then $\mathcal{A}_{|z_1|,|z_2|} \subset \Sigma_{\mathrm{per}}$.

First of all, notice that Theorem 2.46 asserts that the boundary radii of the spectrum can be approached by "regular" Lyapunov multipliers λ, for which there exist a point θ_0 such that

$$\lim_{N \to \infty} \frac{1}{N} \log |M(\theta_0, N)| = \log \lambda \ . \tag{4.23}$$

We will assume that θ_0 is aperiodic, otherwise (4.23) says that λ is the spectral radius of the transfer on the periodic orbit by θ_0, whose spectrum is a regular polygon centered in the origin. Choose now a segment of the aperiodic orbit of length N which will take sufficiently large as indicated later. Choose also ε, which later we will make small, and denote K_ε the natural number given by the specification property in Definition 1.30. Hence, for all $T \geq T_{\varepsilon,N} := K_\varepsilon + N$ there exists a T-periodic orbit by a certain $\varphi_{\varepsilon,N,T}$ such that

$$\mathrm{d}(f^i(\theta_0), f^i(\varphi_{\varepsilon,N,T})) < \varepsilon$$

for $i = 0, 1, \dots, N$. Then:

(4.24)

$$\begin{aligned}\left|\frac{1}{T}\log|M(\varphi,T)| - \log\lambda\right| \leq \quad & \left|\frac{1}{T}\left(\log|M(\varphi,T)| - \log|M(\varphi,N)|\right)\right| \\ & + \left|\frac{1}{T}\left(\log|M(\varphi,N)| - \log|M(\theta_0,N)|\right)\right| \\ & + \left|\left(\frac{1}{N} - \frac{1}{T}\right)\log|M(\theta_0,N)|\right| \\ & + \left|\frac{1}{N}\log|M(\theta_0,N)| - \log\lambda\right|.\end{aligned}$$

The first and third terms of the RHS above can be bounded by $\frac{T-N}{T}\log_+\|M\|_\infty$ that can be made small by taking $T = K_\varepsilon + N$ and N large. The second term can be bounded by $\frac{N}{T}\eta(\varepsilon) \leq \eta(\varepsilon)$, where η is the modulus of continuity of the function $\log|M(\theta)|$, so it can be made small by taking ε small. Finally, the fourth term can be made small by taking N large. In summary, we can make all the terms in the RHS above as small as desired if we take ε small enough and N large enough, and $T = K_\varepsilon + N$. We conclude that the spectrum of the T-periodic orbits φ_T contains eigenvalues whose modulus can be made as close as desired to λ, that can be made as close as we want to the radii of the spectrum, λ_-, λ_+.

Remark 4.15. The bound of the second term in (4.24) by $\eta(\varepsilon)$ is the only point in which we use the commutativity of the products in $M(\theta, N)$ coming from the fact that the bundle is 1D.

Now, we turn to the proof of the annular convexity.

Let z_1, z_2 be eigenvalues corresponding to periodic orbits whose starting points are θ_1, θ_2 respectively. Denote by T_1, T_2 the minimal periods of these orbits. We will assume for the sake of convenience that $|z_1| \leq |z_2|$. We fix $z \in \mathcal{A}_{z_1,z_2}$.

We will first discuss the case that the two orbits are different.

Now, we pick an orbit segments corresponding to iterating the first order N_1 times and the second orbit N_2 times. Later we will choose N_1, N_2 to be sufficiently large but their ratio approaches a fixed value.

We fix $\varepsilon > 0$ and apply the specification property to obtain a periodic orbit of period $T = N_1 + N_2 + 2K_\varepsilon$ shadowing the periodic orbits, where K_ε is the jump required by the specification property.

Again, we compute the multiplier going around the orbit. Proceeding as before, we note that, up to terms that go to zero when $\varepsilon \to 0$ and

$N_1, N_2 \to \infty$, the Lyapunov exponent around the orbit is

$$\frac{N_1 \cdot T_1}{N_1 \cdot T_1 + N_2 \cdot T_2} \log|z_1| + \frac{N_2 \cdot T_2}{N_1 \cdot T_1 + N_2 \cdot T_2} \log|z_2| \tag{4.25}$$

If we take N_1 and N_2 large enough but their ratio approaches a limit, we can obtain that the value in (4.25) approaches any desired value in the interval $[\log|z_1|, \log|z_2|]$.

Hence, for any fixed ε we can obtain a sequence of periodic orbits whose Lyapunov exponent converges to any prescribed value in $[\log|z_1|, \log|z_2|]$ up to an error $\eta(\varepsilon)$. Again, passing to a sequence of ε taking to zero, we can get a sequence of periodic orbits whose Lyapunov exponent approaches any desired value.

Since the minimal period of the orbits in the sequence constructed grows to infinity (notice that the period has to be larger than N_1T_1), the spectrum of the periodic orbits accumulates in the whole annulus of radii $|z_1| \le |z_2|$.

In the case that the two periodic orbits are the same (and hence, $|z_1| = |z_2|$), the only thing that we need to do is to show that Σ_{per} includes the whole circle of this radius. This can be easily achieved by taking any point outside of the periodic orbit and shadowing an orbit that consists on repeating N times the periodic orbit and then visiting the other point. Proceeding as before, we see that the average Lyapunov exponent is not affected much, but the period grows to infinity which ensures that the spectrum accumulates in the whole circle. □

REMARK 4.16. Notice that along the proof, we have also shown that, in the hypothesis of Theorem 4.14, Σ_{per} is invariant under rotations.

This, also follows because specification implies that the system is APD (Proposition 1.35) and, we have shown that this implies that the spectrum is invariant under rotations (Theorem 3.11).

REMARK 4.17. Recall that the size of the gaps in the spectrum gives the regularity of the spectral subbundles, as a consequence of the invariant section theorem (see Theorem 2.18). Theorem 4.14 transfer the growth behavior on periodic orbits (their Lyapunov multipliers) to regularity of the invariant subbundles.

A theorem very similar to Theorem 4.14 without the assumption that the bundles are 1-dimensional is claimed in [**Ham94**] for transitive Anosov systems (that indeed satisfy specification). This goes under the name *periodic bunching* . From the point of view of dynamical systems, this is interesting since in [**Ano69, HK90**] it is shown that the periodic orbits also give obstructions to regularity.

Unfortunately, the argument presented in [**Ham94**] (also used in [**Has94**]) is not complete. The $k(\varepsilon, \delta)$ constructed in p. 304 by appealing to the subadditive ergodic theorem for the invariant measure μ could depend on μ and, indeed, simple examples show that this dependence could take place. This invalidates the argument presented there.

We do not know whether indeed the result is true for Anosov systems. Nevertheless, the following examples, indicate that the eventual proof of this result (if indeed there is one) cannot just rely on soft arguments.

The following example is a caricature of the situation in which periodic orbits of a dynamical system approximate an aperiodic orbit. The spectrum in the aperiodic orbit is significantly larger. It could well be true that the result claimed in [**Ham94**] is true for dynamical systems on manifolds. Nevertheless, the Example 4.18 shows that the proof will have to use some delicate structure of the dynamical system (e.g. that it is defined on a manifold) and cannot be based just on properties true for measurable spaces

EXAMPLE 4.18. Consider the space $M = \mathbb{N} \bigcup_n \mathbb{N}/2^n$ Denote its elements by (n, i), $n \in \mathbb{N} \cup \{\infty\}$, $i \in \mathbb{N}/2^n$ (with the obvious meaning $\mathbb{N}/2^\infty = \mathbb{N}$) Define on it the topology in which $(n, i) \to (\infty, i)$ as $n \to \infty$. Define on it the dynamical system $f(n, i) = (n, i+1)$. Consider a trivial bundle $E = M \times \mathbb{R}^2$ and the bundle map $\mathcal{T}_{(n,i)} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ when $0 \le i \le 2^n - 2$ $M(n, 2^n - 1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ Then, the Lyapunov multipliers of the periodic orbits are $(2{\cdot}3)^{1/2}$. Nevertheless, the aperiodic orbit has Lyapunov multipliers $2, 3$ and indeed the spectrum is $\{2 \le |z| \le 3\}$.

Part 3

Transfer operators over rotations

In this part, we will study the spectrum of transfer operators when the motion of the base is a translation on the torus.

In this special case, it is possible to obtain significantly sharper results than in the general case discussed above.

Transfer operators over rotations appear naturally in many applications. For example, many systems in celestial mechanics are systems that are subject to external perturbations which are quasi-periodic. The linearization of such systems leads naturally to transfer operators over rotations. Indeed, some of the results discussed here are useful in the study of persistence of normally hyperbolic invariant manifolds for such systems.

CHAPTER 5

Corollaries from the general theory

In this chapter we obtain several results on transfer operators defined from vector bundle maps over rotations on a torus. These results are just corollaries of the general theory.

5.1. Some spectral equalities and inclusions

The following is a corollary of the results in Part 2.

COROLLARY 5.1. *Let $M_\omega : E \to E$ be a C^r vector bundle automorphism over a rotation ω. Assume that ω is irrational. Then:*

(5.1)

$$\begin{array}{ccccc}
\mathrm{Spec}(\mathcal{M}_\omega, \Gamma_B(E)) & = & \mathrm{Spec}_W(\mathcal{M}_\omega, \Gamma_B(E)) & = & \mathrm{Spec}_P(\mathcal{M}_\omega, \Gamma_B(E)) \\
\| & & \| & & \\
\mathrm{Spec}(\mathcal{M}_\omega, \Gamma_{L^p}(E)) & = & \mathrm{Spec}_W(\mathcal{M}_\omega, \Gamma_{L^p}(E)) & & \\
\| & & \| & & \\
\mathrm{Spec}(\mathcal{M}_\omega, \Gamma_{C^0}(E)) & = & \mathrm{Spec}_W(\mathcal{M}_\omega, \Gamma_{C^0}(E)) & & \\
\cap & & \cap & & \\
\mathcal{A}\mathrm{Spec}(\mathcal{M}_\omega, \Gamma_{C^r}(E)) & \supset & \mathrm{Spec}_W(\mathcal{M}_\omega, \Gamma_{C^r}(E)) & &
\end{array}$$

where $p > 1$.

Moreover, the spectrum acting on bounded sections is rotationally invariant, and the same happens to the other spaces of sections that have the same Weyl spectrum as in $\Gamma_B(E)$.

Proof: The fact that the Weyl spectrum (or approximate point spectrum) on bounded sections is rotationally invariant follows from localization arguments for C^0 sections given in [**Mat68**]. In Part 2, they are generalized to other regularities. The key ingredient is that irrational rotations does not have periodic orbits (all the orbits through the irrational flow are aperiodic). This implies that the full spectrum on bounded sections is also rotationally invariant, because the boundary of the spectrum is approximate point spectrum. We will present an extended result (that works in other spaces) in Section 5.2 (Theorem 3.13), that uses specifically that the motion on the base torus is an irrational rotation.

The full spectrum on bounded sections is approximate point spectrum because all the orbits through rotations are non wandering (and so, rotations are chain recurrent). This result is proved in [**Swa81, HdlL03a**], using the findings in [**SS76a, Sel75, CI99**] (quasi-hyperbolicity implies hyperbolicity if the dynamics is chain recurrent).

The approximate point spectrum on bounded sections is point spectrum comes from the fact that irrational rotations does not have periodic orbits. It is a direct application of Thychonov theorem. We will review these results in Section 6.

The inclusion of $\mathrm{Spec}_W(\mathcal{M}_\omega, \Gamma_B(E))$ in $\mathrm{Spec}_W(\mathcal{M}_\omega, \Gamma_{C^0}(E))$ is proved by fattening the localized bounded approximate eigensections to produce continuous approximate eigensections. A similar argument is used to prove the inclusion in $\mathrm{Spec}_W(\mathcal{M}_\omega, \Gamma_{L^p}(E))$, and it is crucial that $p > 1$ and the Lebesgue measure is topological.

The inclusion $\mathrm{Spec}_W(\mathcal{M}_\omega, \Gamma_{C^0}(E)) \subset \mathrm{Spec}_W(\mathcal{M}_\omega, \Gamma_B(E))$ is straightforward, because $\Gamma_{C^0}(E)$ is a closed Banach subspace of $\Gamma_B(E)$. The proof of the inclusion $\mathrm{Spec}_W(\mathcal{M}_\omega, \Gamma_{L^p}(E)) \subset \mathrm{Spec}_W(\mathcal{M}_\omega, \Gamma_B(E))$, that is producing a bounded approximate eigensection from a L^p approximate eigensection, is more involved and uses Chebyshev inequality.

The equalities among the Weyl spectra imply that the corresponding spectra is rotationally invariant.

A gap in the spectrum on bounded sections produces an invariant splitting that is, at least, continuous. This splitting in the bundle produces a splitting in the space of continuous sections, that is also characterized by rates of growth, so the gap does exist also when acting on continuous sections. This proves $\mathrm{Spec}(\mathcal{M}_\omega, \Gamma_{C^0}(E)) \subset \mathrm{Spec}(\mathcal{M}_\omega, \Gamma_B(E))$.

Notice also that a gap in the spectrum on continuous functions produces a continuous invariant splitting satisfying rates of growth (this is a general result, but it is also a direct application of Theorem 3.14). The pointwise rates of growth imply similar rates of growth for bounded and L^p sections (this argument works for $p \geq 1$). This proves the inclusion of $\mathrm{Spec}(\mathcal{M}_\omega, \Gamma_B(E))$ and $\mathrm{Spec}(\mathcal{M}_\omega, \Gamma_{L^p}(E)$ in $\mathrm{Spec}(\mathcal{M}_\omega, \Gamma_{C^0}(E))$.

With the previous arguments we have finished the prove of the equalities in Corollary 5.1. Let us now to go through the prove of the inclusion when considering higher regularities.

The inclusion of $\mathrm{Spec}_W(\mathcal{M}_\omega, \Gamma_B(E))$ in $\mathrm{Spec}_W(\mathcal{M}_\omega, \Gamma_{C^r}(E))$ is carried out using again fattening arguments. We emphasize that these arguments use bump functions, that are not aware when working in the analytic category.

Finally, for proving $\mathrm{Spec}(\mathcal{M}_\omega, \Gamma_B(E)) \subset \mathcal{A}\mathrm{Spec}(\mathcal{M}_\omega, \Gamma_{C^r}(E))$ we use that a gap in the spectrum in $\Gamma_{C^r}(E)$ produces an invariant splitting characterized by rates of growth (by appealing to Theorem 3.14), and so the gap is also present in the spectrum in $\Gamma_B(E)$. □

Remark 5.2. From the results in Part 2, it does not follow that the spectrum of on C^r sections is invariant under complex rotations for transfer operators over a general map and, as far as we know, this is still an open problem.

We will prove later in Theorem 3.13 that that the spectrum is indeed invariant under rotations using an argument which uses heavily that the map on the base is an ergodic rotation of the torus.

REMARK 5.3. For $p = 1$, we have the equality

$$\mathrm{Spec}(\mathcal{M}_\omega, \Gamma_B(E)) = \mathcal{A}\mathrm{Spec}(\mathcal{M}_\omega, \Gamma_{L^1}(E)) ,$$

because the Lebesgue measure is non-atomic and topological (see Theorem 3.25 in Part 2). At this point, we do not know if the spectrum on L^1 sections is rotationally invariant for a general base map. When the motion on the base is an irrational rotation, the invariance under rotations follows from Theorem 3.13.

REMARK 5.4. Notice also that for ω irrational, we have

$$\mathrm{Spec}(\mathcal{M}_\omega, \Gamma_B(E)) = \bigcup_{\theta \in \mathbb{T}^d} \mathrm{Spec}_P(m_{\omega,\theta}, b_{\omega,\theta}(E)) .$$

(This is trivial if ω is rational).

Moreover, if ω is ergodic, then for any $\theta \in \mathbb{T}^d$:

$$\mathrm{Spec}(\mathcal{M}_\omega, \Gamma_B(E)) = \mathrm{Spec}(m_{\omega,\theta}, b_{\omega,\theta}(E)) = \mathrm{Spec}_W(m_{\omega,\theta}, b_{\omega,\theta}(E)) .$$

The goal of this paper is to complete the diagram (5.1) as much as possible, under the extra assumption that the base motion is a rotation.

In many cases, we will prove the inclusions are in fact equalities, including also the action on Sobolev spaces. We will also pay attention to the analytic case. For the sake of completeness and for making this part more legible, we will repeat some of the arguments.

5.2. Invariance of the spectrum under rotations

The invariance of the spectrum under rotations was proved for C^0 sections in [**Mat68**], under the assumption of existence of a dense set of aperiodic orbits, that is obviously satisfied for irrational rotations. In Part 2 there is a proof for bounded sections (obtaining as a corollary the corresponding result for continuous and L^p sections, $p > 1$). The proofs presented in those papers do not generalize easily for spaces of sections that have more regularity, so we use a different method which, however depends heavily on the fact that the motion on the base is an irrational rotation.

THEOREM 5.5. *Let $M_\omega : E \to E$ be a vector bundle automorphism over an irrational rotation. Let $\Gamma \subset \Gamma(E)$ be a Banach space of sections (with a norm $\|\cdot\|_\Gamma$), such that:*

a) *Γ is invariant under $\mathcal{M}_\omega$, and it defines a continuous linear operator in Γ;*
b) *Γ is invariant under the multiplication operators by the functions $e_k : \mathbb{T}^d \to \mathbb{C}$ defined by*

$$e_k(\theta) = e^{2\pi \mathbf{i} k\cdot\theta} ,$$

where $k \in \mathbb{Z}^d$, defining also continuous linear operators in Γ*.*
Then, $\mathrm{Spec}(\mathcal{M}_\omega, \Gamma)$ *is invariant under rotations, i.e.:*

$$z \in \mathrm{Spec}(\mathcal{M}_\omega, \Gamma) \Rightarrow e^{\alpha \mathbf{i}} z \in \mathrm{Spec}(\mathcal{M}_\omega, \Gamma)$$

for all $\alpha \in \mathbb{R}$*.*

Proof: Since the boundary of the spectrum is Weyl spectrum, it is enough to prove that $\mathrm{Spec}_W(\mathcal{M}_\omega, \Gamma)$ is invariant under rotations. We observe that, for any section $v : \mathbb{T}^d \to E$ in Γ, we have:

$$\mathcal{M}_\omega(e_k v)(\theta) = e^{2\pi \mathbf{i} k\cdot\omega} e_k(\theta) \mathcal{M}_\omega v(\theta) \ .$$

Therefore, if v_ε is an ε-eigensection for λ, $e_k v_\varepsilon$ is an ε-eigensection for $\lambda e^{-2\pi \mathbf{i} k\cdot\omega}$. Since $\{e^{-2\pi \mathbf{i} k\cdot\omega}\}_{k\in\mathbb{Z}^d}$ is dense in the unit circle and $\mathrm{Spec}_W(\mathcal{M}_\omega, \Gamma)$ is closed, we obtain the result claimed. $\square$

REMARK 5.6. This result proves the invariance under rotations of the spectrum on L^1 sections, C^r sections and analytic sections, facts that do not follow from the general results in [**Mat68, HdlL03a**].

CHAPTER 6

Spectral theory for transfer operators over rotations in spaces of bounded sections

In this section we clarify the spectral theory for transfer operators over rotations, when acting on bounded sections. As we will prove, all the spectrum is point spectrum (when acting on bounded sections).

In the analysis of the transfer operators over a rotation, the numerical properties of the frequency vector $\omega \in \mathbb{R}^d$ play an important role. We distinguish the following cases:

- ω is *rational*: $\omega \in \mathbb{Q}^d$, and then the orbit $\{m\omega \pmod 1 \mid m \in \mathbb{Z}\} \subset \mathbb{T}^d$ is periodic;
- ω is *irrational* : ω is not rational, and then the orbit $\{m\omega \pmod 1 \mid m \in \mathbb{Z}\} \subset \mathbb{T}^d$ is not periodic, and the set $\{k \cdot \omega \pmod 1 \mid k \in \mathbb{Z}^p\}$ is dense in $\mathbb{T}^1$.
- ω is *ergodic* : $k \cdot \omega \notin \mathbb{Z}$ for all $k \in \mathbb{Z}^d \setminus \{0\}$, and then the orbit $\{m\omega \pmod 1 \mid m \in \mathbb{Z}\} \subset \mathbb{T}^d$ is dense in $\mathbb{T}^d$;
- ω is *Diophantine*: there exist constants $C > 0, \tau \geq 1$ such that for all $k \in \mathbb{Z}^d$ and $n \in \mathbb{N} \setminus \{0\}$

$$|k \cdot \omega - n| \geq C|k|_1^{-\tau} \ .$$

A rotation that is not ergodic is called *resonant*.

6.1. Rational rotations

From the dynamical point of view, a cocycle over a rational rotation $\omega = \frac{p}{q}$, $p \in \mathbb{Z}, q \in \mathbb{N} \setminus \{0\}$, corresponds to iterating the matrix $M_q(\theta) = M(\theta, q)$. The dynamical analysis is straightforward: for each $\theta \in \mathbb{T}^d$, we have to compute the eigenvalues of $M_q(\theta)$. Two are the main reasons to undertake the spectral analysis of the corresponding transfer operator: from the numerical point of view, this analysis is useful to compute the spectrum of a transfer operator over an irrational rotation, by approximating the irrational frequency by rational frequencies (see [**HdlL04**]); this analysis also leads to prove existence of normally hyperbolic invariant manifolds of skew products over rational rotations which are foliated by periodic orbits (see [**HdlLb**]).

In the analysis of the spectrum of transfer operators over rational rotations we consider the eigenvalues of composition of matrices. For a given matrix A, we denote by $\mathrm{Eig}(A)$ the set of its eigenvalues. The next result is elementary but useful for our analysis.

PROPOSITION 6.1. *Let $M_0, M_1, \ldots, M_{q-1}$ be matrices in $M_n(\mathbb{C})$. We denote by $\hat{M}_q$ the nq-matrix defined by*

$$\hat{M}_q = \begin{pmatrix} 0 & & \cdots & & M_{q-1} \\ M_0 & 0 & \cdots & & 0 \\ & M_1 & \cdots & & 0 \\ & & \ddots & & \vdots \\ & & & M_{q-2} & 0 \end{pmatrix} . \tag{6.1}$$

Then:

$$z \in \mathrm{Eig}(\hat{M}_q) \Leftrightarrow z^q \in \mathrm{Eig}(M_{q-1} \ldots M_0) \ .$$

In short: $\mathrm{Eig}(\hat{M}_q) = \sqrt[q]{\mathrm{Eig}(M_{q-1} \ldots M_0)}$, *where we are supposed to take all the possible determinations of the q root.*

Proof: Given an eigenvector $\hat{v} = (v_0, \ldots, v_{q-1})$ of $z \neq 0$ for $\hat{M}_q$, it is immediate that v_0 is non zero and it is an eigenvector for $M_{q-1} \ldots M_0$, whose eigenvalue is z^q. If $z = 0$, then some of the $M_0, \ldots, M_{q-1}$ have zero determinant, and then 0 is also eigenvalue for $M_{q-1} \ldots M_0$.

On the contrary, if $z^q \neq 0$ is an eigenvalue of $M_{q-1} \ldots M_0$, whose associated eigenvector is v_0, then we can construct one eigenvector $\hat{v}$ for $\hat{M}_q$, whose eigenvalue is z, by

$$v_1 = \frac{1}{z} M_0 v_0 \ , \ v_2 = \frac{1}{z} M_1 v_1 \ , \ \ldots \ , v_{q-1} = \frac{1}{z} M_{q-2} v_{q-2} \ .$$

If $z = 0$, then one of the matrices M_i has zero determinant, and so $\hat{M}_q$. □

REMARK 6.2. Notice that if $z \in \mathrm{Eig}(\hat{M}_q)$ then $\exp(2\pi \mathbf{i} \frac{p}{q}) z \in \mathrm{Eig}(\hat{M}_q)$. Hence, the eigenvalues of $\hat{M}_q$ are distributed in at most n circles (in fact, n regular q-polygons).

REMARK 6.3. The previous result is the basis of the parallel shooting method that is widely used in solving badly behaved equations (for instance, in the computation of periodic orbits in celestial mechanics, or the computation of high period solutions in discrete systems).

We start now to analyze the spectrum of the transfer operator over a rational rotation $\omega = \frac{p}{q}$. The first result gives the spectrum when acting on bounded functions.

THEOREM 6.4. *Let $M_\omega : E \to E$ be a vector bundle automorphism over a rational rotation $\omega = \frac{p}{q}$. Then:*

$$\begin{aligned} \mathrm{Spec}(\mathcal{M}_\omega, \Gamma_B(E)) &= \mathrm{Spec}_W(\mathcal{M}_\omega, \Gamma_B(E)) = \mathrm{Spec}_P(\mathcal{M}_\omega, \Gamma_B(E)) \\ &= \bigcup_{\theta \in \mathbb{T}^d} \mathrm{Eig}(\hat{M}_q(\theta)) \ , \end{aligned}$$

where

$$\hat{M}_q(\theta) : \prod_{i=0}^{q-1} E_{\theta+i\frac{p}{q}} \to \prod_{i=0}^{q-1} E_{\theta+i\frac{p}{q}}$$

is the linear map represented by the matrix $\hat{M}_q(\theta)$ *associated to* $M_0 = M(\theta), M_1 = M(\theta+\omega), \ldots, M_{q-1} = M(\theta+(q-1)\omega)$ *as in* (6.1).

Proof: For a given $\theta_0 \in \mathbb{T}^d$, let $z \in \mathrm{Eig}(\hat{M}_q(\theta_0))$ be an eigenvalue and $\hat{v} = (v_0, v_1, \ldots, v_{q-1})$ be a corresponding eigenvector. Denoting $\theta_i = \theta_0 + i\frac{p}{q}$ and $M_i = M(\theta_0 + i\frac{p}{q})$, we have

$$M_0 v_0 = z v_1 \ , \ \ldots \ , \ M_{q-2} v_{q-2} = z v_{q-1} \ , \ldots \ , M_{q-1} v_{q-1} = z v_0 \ ,$$

and the bounded section

$$v(\theta) = \begin{cases} v_i \text{ , if } \theta = \theta_0 + i\frac{p}{q}, \ i = 0, \ldots, q-1 \\ 0, \text{ otherwise.} \end{cases}$$

defines an eigensection for $\mathcal{M}_\omega$ of z. So $z \in \mathrm{Spec}_P(\mathcal{M}_\omega, \Gamma_B(E))$.

To prove the other inclusion, take $z \in \bigcap_{\theta\in\mathbb{T}^d}(\mathbb{C} \setminus \mathrm{Eig}(\hat{M}_q(\theta)))$. Notice that for each $\theta \in \mathbb{T}^d$ the matrix $\hat{M}_q(\theta) - z\mathrm{Id}$ is invertible, and the map

$$\theta \longrightarrow (\hat{M}_q(\theta) - z\mathrm{I})^{-1} \tag{6.2}$$

is continuous. We have to see that $z \in \mathrm{Res}(\mathcal{M}_\omega, \Gamma_B(E))$, that is to say, the equation

$$\mathcal{M}_\omega v - z v = \eta \tag{6.3}$$

is solvable for each $\eta \in \Gamma_B(E)$ (and as a result we can control the sup-norm of v by the sup-norm of η). From η, we construct the vector function

$$\hat{\eta}(\theta) = (\eta(\theta), \eta(\theta + \frac{p}{q}), \ldots, \eta(\theta + (q-1)\frac{p}{q}))^\top \ .$$

We define the vector function

$$\hat{v}(\theta) = (\hat{M}_q(\theta) - z\mathrm{I})^{-1}\hat{\eta}(\theta) = (v_0(\theta), v_1(\theta)), \ldots, v_{q-1}(\theta))^\top \ .$$

Since $\hat{M}_q(\theta + \frac{p}{q})\hat{R}_q\hat{v}(\theta) - z\hat{R}_q\hat{v}(\theta) = \hat{\eta}(\theta + \frac{p}{q})$, where $\hat{R}_q$ is the shift matrix

$$\hat{R}_q = \begin{pmatrix} 0 & I & \ldots & 0 \\ & 0 & \ddots & 0 \\ & & & 0 \\ & & & I \\ I & & & 0 \end{pmatrix} ,$$

we have $v_i(\theta + \frac{p}{q}) = v_{i+1}(\theta)$ (assuming $v_q(\theta) = v_0(\theta)$. Hence, $v(\theta) = v_0(\theta)$ solves (6.3). Notice that $\|v\|_\infty \le \|(\hat{M}_q(\theta) - z\mathrm{I})^{-1}\|_\infty \ \|\eta\|_\infty$. □

Remark 6.5. Notice that the $\mathrm{Spec}(\mathcal{M}_\omega, \Gamma_B(E))$, with $\omega = \frac{p}{q}$, is invariant under multiplication by $\exp(2\pi\mathbf{i}\frac{p}{q})$.

6.2. Irrational rotations

Since irrational rotations do not have periodic orbits, we see that approximate eigenvalues in the space of bounded sections are, in fact, eigenvalues. Even more, since irrational rotations are chain recurrent, the full spectrum on bounded sections is Weyl spectrum. So that, for irrational rotations, all the spectrum on bounded sections consists of eigenvalues.

THEOREM 6.6. *Let $M_\omega : E \to E$ be a vector bundle automorphism over an irrational rotation ω. Then:*

$$\mathrm{Spec}(\mathcal{M}_\omega, \Gamma_B(E)) = \mathrm{Spec}_W(\mathcal{M}_\omega, \Gamma_B(E)) = \mathrm{Spec}_P(\mathcal{M}_\omega, \Gamma_B(E)) \ .$$

Proof: For the first equality, notice that irrational rotations are chain recurrent and all the orbits are aperiodic, and so the full spectrum is Weyl spectrum. This result is proved in Part 2 (see [**Swa81**], when acting on continuous sections), using the findings in [**SS76a, Sel75, CI99**].

For the second equality, we can apply directly a result in Part 2. We will repeat here the argument, for the sake of completeness. The essential property is that the dynamics on the base $\mathbb{T}^d$ does not have periodic orbits.

Given $z \in \mathrm{Spec}_W(\mathcal{M}_\omega, \Gamma_B(E))$, for any $\varepsilon_n > 0$ we can construct an ε_n-approximate eigensection v_n supported in an orbit $\{\theta_{n,k} = \theta_n + k\omega\}_{k\in\mathbb{Z}}$, such that, say $|v_n(\theta_n)| \geq \frac{3}{4}$, $|v_n(\theta_{n,k})| \leq 1$ for all $k \in \mathbb{Z}$ and

$$|M(\theta_{n,k})v_{n,k} - zv_{n,k+1}| \leq \varepsilon_n \ , \tag{6.4}$$

where we denote $v_{n,k} = v_n(\theta_{n,k})$.

If D_E denotes the unit disk bundle of E, that is $D_E = \{v \in E \mid |v| \leq 1\}$, we identify v_n as an element $\{v_{n,k}\}_{k\in\mathbb{Z}}$ of $D_E^{\mathbb{Z}}$. Using Thychonov theorem, $D_E^{\mathbb{Z}}$ is compact when it is endowed with the product topology. Moreover, $D_E^{\mathbb{Z}}$ is metrizable.

Taking now a sequence $\varepsilon_n \to 0$, and the corresponding orbit-supported ε_n-eigensections v_n we suppose that, without loss of generality, that:

- θ_n converges to θ_*, using compactness of $\mathbb{T}^d$;
- v_n converges componentwise to v_*, using compactness of $D_E^{\mathbb{Z}}$.

Hence, for all $k \in \mathbb{Z}$, $v_{n,k} \to v_{*,k}$ and, in particular, $\theta_{n,k} = \theta_n + k\omega \to \theta_{*,k} = \theta_* + k\omega$. That is, v_* is also supported in an orbit. We have just proved that the set

$$\{\{v_k\}_{k\in\mathbb{Z}} \in D_E^{\mathbb{Z}} \mid \forall k \in \mathbb{Z}\ \pi(f(v_k)) = \pi(v_{k+1})\} \subset D_E^{\mathbb{Z}}$$

is closed in $D_E^{\mathbb{Z}}$, so compact.

Finally, taking limits again in (6.4) we see that for all $k \in \mathbb{Z}$

$$|M(\theta_{*,k})v_{*,k} - zv_{*,k+1}| = 0 \ .$$

Since θ_* is not periodic, $\{v_{*,k}\}_{k\in\mathbb{Z}}$ produces an eigensection v_* for z, supported in the orbit of θ_*: $v(\theta_{*,k}) = v_{*,k}$ and zero otherwise. Notice that $v(\theta_*) \geq \frac{3}{4}$. Thus, $z \in \mathrm{Spec}_P(\mathcal{M}_\omega, \Gamma_B(E))$. □

REMARK 6.7. When one considers transfer operators action on spaces of sections of bundles that satisfy certain geometric constrains, the presence of non Weyl spectrum is possible. This annoying fact was discovered in [**dlL93**]. See Part 4 for more results.

The following proposition is very useful for the analysis of the Weyl spectrum. It says that we can localize the eigensections in finite segments of orbits (see [**Mat68, CL99, HdlL03a**]). A similar argument will be used when analyzing the Weyl spectrum in other spaces of sections.

LEMMA 6.8. *Let $M_\omega : E \to E$ be a vector bundle automorphism over an irrational rotation. Let $z \in \mathrm{Spec}(\mathcal{M}_\omega, \Gamma_B(E))$ be a spectral value. Hence, for all $\varepsilon > 0$ there exist a bounded ε-eigensection of z supported in a finite number of points.*

Proof: We know from Theorem 3.19 that z is an eigenvalue of $\mathcal{M}_\omega$. Let $v \in \Gamma_B(E)$ be a bounded eigensection of z, supported in the orbit $\{\theta_k = \theta_* + k\omega\}_{k\in\mathbb{Z}}$ of $\theta_* \in \mathbb{T}^d$, such that $|v(\theta_*)| = 1$, and

$$M_\omega(\theta_k)v(\theta_k) - zv(\theta_{k+1}) = 0 \ , |v(\theta_k| \geq 1$$

for all $k \in \mathbb{Z}$. We will write $v_k = v(\theta_k)$, $M_k = M_\omega(\theta_k)$.

We are going to construct from v a finitely supported approximate eigensection w. Given $\varepsilon > 0$, we define $N = [\frac{1}{\varepsilon}]$ and the section w is defined by

$$w(\theta) = \begin{cases} w_k = \left(1 - \frac{|k|}{N+1}\right)_+ v_k \ , \text{ if } \theta = \theta_k \ , \ k \in \mathbb{Z}, \\ 0, \text{ otherwise.} \end{cases}$$

where we denote $t_+ = \max\{0, t\}$ for any real number t. This bounded section, which is supported in $K = 2N + 1$ points, is an approximate eigensection of z, because

$$\begin{aligned} |M_{*,k}w_{*,k} - zw_{*,k+1}| &= \left| \left(1 - \frac{|k|}{N+1}\right)_+ M_k v_k - \left(1 - \frac{|k+1|}{N+1}\right)_+ z v_{k+1} \right| \\ &\leq \left| \left(1 - \frac{|k|}{N+1}\right)_+ - \left(1 - \frac{|k+1|}{N+1}\right)_+ \right| |z||v_{k+1}| \\ &< |z|\varepsilon \ . \end{aligned}$$

This finishes the proof of the localization lemma. □

6.3. Ergodic rotations

The following result gives bounds for the spectrum of a continuous transfer operator over an ergodic rotation. It follows directly from Corollary 2.41 because ergodic rotations are uniquely ergodic.

PROPOSITION 6.9. *Let $M_\omega : E \to E$ be a vector bundle automorphism over an ergodic rotation. Then:*

$$\mathrm{Spec}(\mathcal{M}_\omega, \Gamma_B(E)) \subset \{\rho_- \leq |z| \leq \rho_+\} \ ,$$

where

$$\exp\left(-\int_{\mathbb{T}^d}\log|M(\theta)^{-1}|\ d\theta\right) \ = \ \rho_- \ \leq \rho_+ = \exp\left(\int_{\mathbb{T}^d}\log|M(\theta)|\ d\theta\right)\ .$$

The above result is enough to locate the spectrum in some situations (see [**CS81**]).

COROLLARY 6.10. *If the rank of the bundle E is $n = 1$, then the spectrum is the circle of radius $\rho = \rho_- = \rho_+$.*

If $M(\theta)$ is an isometry for all $\theta \in \mathbb{T}^d$, that is $|M(\theta)| = 1$ for all $\theta \in \mathbb{T}^d$, then the spectrum is the unit circle.

REMARK 6.11. If the spectrum is decomposed in n spectral annuli, then they are in fact spectral circles. This is a direct consequence of the previous corollary and the fact that there are invariant subbundles associated to the spectral annuli, that are 1-dimensional. Notice that this is not necessarily true for rotations that are not ergodic.

The following result is again an immediate consequence of the findings in Part 2.

PROPOSITION 6.12. *Let $M_\omega : E \to E$ be a vector bundle automorphism over an ergodic rotation. Let $\alpha : \mathcal{P} \to \mathbb{C}$ be a non-vanishing continuous function and let*

$$\hat{\alpha} = \exp\int_{\mathcal{P}}\log|\alpha(\theta)|\ d\theta$$

be its geometric average. Then:

$$\mathrm{Spec}((\alpha\mathcal{M})_\omega, \Gamma_B(E)) = \hat{\alpha}\ \mathrm{Spec}(\mathcal{M}_\omega, \Gamma_B(E))\ .$$

6.4. Computation of the spectrum

In this section we will study a method to compute the spectrum of a transfer operator over an irrational rotation by approaching the irrational frequencies by rational ones. We refer the readers to Part 2 for general results, and to [**HdlL04**] for other numerical applications.

As a result of the main theorems of this paper, it is enough to analyze the spectrum of the transfer operator when acting on bounded sections. Moreover, since spectral values are eigenvalues, we only need to compute point spectrum.

The key idea is the following. In Proposition 6.8, we located approximate eigensections on finite number of points. An approximate eigensection supported in K points of an orbit $\{\theta_k = \theta_0 + k\omega\}_{k=0}^{K-1}$, for a certain $\theta_0 \in \mathbb{T}^d$ is just determined by the vectors supported on those points $\{v_k = v_{\theta_k}\}_{k=0}^{K-1}$ (and $v_k = 0$ if $k < 0$ or $k > K$). Notice that, if the bundle is trivial $E = \mathbb{T}^d \times \mathbb{C}^n$, the condition

$$|M(\theta_k)v_k - \lambda v_{k+1}| < \varepsilon$$

implies that the vector $(v_0, v_1, \ldots, v_{K-1})$ is an approximate eigenvector of the matrix

$$\hat{M}_k(\theta_0) = \begin{pmatrix} 0 & & \ldots & & M_{K-1} \\ M_0 & 0 & \ldots & & 0 \\ & M_1 & \ldots & & 0 \\ & & \ddots & & \vdots \\ & & & M_{K-2} & 0 \end{pmatrix}, \tag{6.5}$$

where $M_k = M(\theta_k)$. This is the type of matrices appearing in the analysis of the spectrum of transfer operators over rational rotations.

We have just to take into account that M_{k-1} sends the fibber $E_{\theta_{k-1}}$ to E_{θ_k}, and that E_{θ_K} can be far from E_{θ_0}. So, we have to choose $K = q$ such that $|p - q\omega|$ is small enough so that we have θ_q and θ_0 close enough to transport vectors in E_{θ_q} to E_{θ_0}.

In the general set up we can use the connector introduced in [**HPPS70**] (see Definition 1.23 in Part 1), that is a 'local' parallel transport connecting close fibbers. That is, a *connector* T in a vector bundle E is a continuous (we will often assume that they are more differentiable) family of isomorphisms $\mathrm{T}_{\theta,\theta'} : E_\theta \to E_{\theta'}$ defined in neighborhood of the diagonal Δ in $\mathbb{T}^d \times \mathbb{T}^d$ such that $\mathrm{T}_{\theta,\theta} = \mathrm{Id}_{E_\theta}$. If, for instance, we endow $\mathbb{T}^d$ with the distance induced by the Euclidean distance in $\mathbb{R}^d$, we fix $\delta > 0$ small enough such that the connector $\mathrm{T} = \mathrm{T}_{\theta,\theta'}$ is defined when the points $\theta, \theta' \in \mathbb{T}^d$ satisfy $d(\theta, \theta') \le \delta$.

We can also consider local coordinates, or in order to simplify notation, we can imbed the vector bundle into a trivial one.

Proposition 6.13. *Let M_ω be a vector bundle automorphism over an irrational rotation. Then:*

> *If $z \in \mathbb{C}$ is an approximate ε-eigenvalue of $\mathcal{M}_\omega$, we can construct a ε'-eigenvector for $\hat{M}_q(\theta_0)$ (see (6.5)), with where $\theta_0 \in \mathbb{T}^d$ and $d(q\omega, p) \le \varepsilon$ is small enough.*

Proof: Proposition 6.8 shows that we can construct a bounded approximate ε''-section for z, v, with $\varepsilon' = |z|\varepsilon$ and that is supported in a finite number of points $\{\theta_k = \theta_0 + k\omega\}_{k=0}^{K-1}$, for a suitable $\theta_0 \in \mathbb{T}^d$, where $K = 2\left[\frac{1}{\varepsilon}\right] + 1$.

Taking q large enough, such that $|p - \omega q| < \delta$ for a suitable $p \in \mathbb{Z}^d$ and $q > K$, we define the vector $w = (v_0, \ldots, v_{q-1})$. Notice that $v_k = 0$ if $K \le k < q-1$, and $\|w\|_\infty = 1$. For $k = 0, \ldots, q-2$, we have $|M_k v_k - z v_{k+1}| \le |z|\varepsilon$. For $k = q-1$, $|M_{q-1} v_{q-1} - z v_0| = |z v_0| \le |z|\varepsilon$. So, w is a ε'-eigenvector of $\hat{M}_q(\theta_0)$. □

Remark 6.14. Using connectors, one can deal with the hat matrix $\hat{M}'_q(\theta_0)$ defined from $M'_k = \mathrm{T}^{-1}_{\theta'_{k+1},\theta_{k+1}} M_k \mathrm{T}_{\theta'_k,\theta_k}$, where $\theta'_k = \theta_0 + k\frac{p}{q}$. Notice that $\theta'_q = \theta_0$.

The following proposition deals with the opposite construction: We want to obtain an approximate eigenvalue of an infinite dimensional matrix from an approximate eigenvalue of a finite dimensional matrix.

PROPOSITION 6.15. *Let M_ω be a vector bundle automorphism over an irrational rotation.*

> *If $z \in \mathbb{C}$ is an approximate ε-eigenvalue of $\hat{M}_q(\theta_0)$ (see (6.5)), where $\theta_0 \in \mathbb{T}^d$ and $|q\omega - p|$ is small enough, we can construct a bounded ε'-eigensection of $\mathcal{M}_\omega$ for z.*

Proof: Let $|q\omega - p| < \varepsilon_1$ and $\frac{1}{q} < \varepsilon_2$. Let $v = (v_0, \dots, v_{q-1})$ be an ε-eigenvector of $\hat{M}_q(\theta_0)$, and let $I \in \{0, \dots, q-1\}$ be an index such that

$$|v_I| = \max_{k=0,\dots,q-1} |v_k| = 1 \ .$$

We consider the bounded section

$$w(\theta) = \begin{cases} w_k = (1 - |k - I|\varepsilon_2)_+ v_{\bar{k}} & \text{if } \theta = \theta_k, k \in \mathbb{Z} \ , \\ 0, \ \text{otherwise,} & \end{cases}$$

where $\bar{k}$ means $k \pmod q$. Notice that w is supported in $2\left[\dfrac{1}{\varepsilon_2}\right] + 1$ points. Since k runs on $|k - I| < \frac{1}{\varepsilon_2} < q$, then $-q < k < 2q - 1$. Hence,

$$d(\theta'_k, \theta_k) \le |k| \left| \frac{p}{q} - \omega \right| < 2\varepsilon_1,$$

and w is well defined if $2\varepsilon_1 < \delta$ (i.e., the connectors are well defined, or the trivialization neighborhoods). Moreover,

$$d(\theta_{\bar{k}}, \theta_k) = d(k\omega, \bar{k}\omega) = \frac{|k - \bar{k}|}{q} d(q\omega, \mathbb{Z}^d) \le \varepsilon_1 \ .$$

To see that w is an approximate eigensection of z, we make the estimates

$$\begin{aligned} |M_k w_k - z w_{k+1}| &= |(1 - |k-I|\varepsilon_2)_+ M_k v_{\bar{k}} - (1 - |k+1-I|\varepsilon_2)_+ z v_{\overline{k+1}}| \\ &\le |M_k - M_{\bar{k}}| \ |v_{\bar{k}}| + |M_{\bar{k}} v_{\bar{k}} - z v_{\overline{k+1}}| + |z|\varepsilon_2 \\ &\le \eta(\varepsilon_1) + \varepsilon + |z|\varepsilon_2 \ , \end{aligned}$$

where η is the modulus of continuity of M.

Hence, we can construct an approximate ε'-eigensection, with $\varepsilon' = \eta(\varepsilon_1) + \varepsilon + |z|\varepsilon_2$. □

REMARK 6.16. We emphasize that recurrence properties of rotations are essential in the arguments and they can be generalized to more general dynamics.

In summary, the previous two propositions lead to the following theorem.

THEOREM 6.17. *Let M_ω be a vector bundle automorphism over an irrational rotation. Let $\{\frac{p_i}{q_i}\}_i$ be a sequence of rational approximants of ω, such that $|q_i\omega - p_i| = d(q_i\omega, \mathbb{Z}^d) \to 0$ when $i \to \infty$. Then:*

$$z \in \mathrm{Spec}(\mathcal{M}_\omega, \Gamma_B(E)) \quad \Leftrightarrow \quad \forall i \; \exists \theta^i \mid z \text{ is } \varepsilon_i\text{-eigenvalue of } \hat{M}_{q_i}(\theta^i) \; , \\ \text{and } \varepsilon_i \to 0 \; .$$

Moreover, if the rotation ω is ergodic, we can fix the initial angle: $\theta^i = \theta_0$, for all i.

Proof: Suppose that z is an ε_i-eigenvalue of $\hat{M}_{q_i}(\theta^i)$ and that $\varepsilon_i \to 0$ when $i \to \infty$. Given $\varepsilon > 0$, we take i such that $\varepsilon_i \leq \varepsilon$, and $d(q_i\omega, p_i)$ is small enough. Using Proposition 6.15, we can construct a bounded ε_i'-eigensection for $\mathcal{M}_\omega$, with ε_i' and notice that $\varepsilon_i' \to 0$ when $\varepsilon_i \to 0$. Then $z \in \mathrm{Spec}_W(\mathcal{M}_\omega, \Gamma_B(E))$.

Let z be a spectral value of $\mathcal{M}_\omega$, so it is an eigenvalue. Given i, let $\varepsilon = \frac{4}{q_i}$, and using Proposition 6.8 we construct a bounded ε'-eigensection supported in $2N + 1 \leq 2\frac{1}{\varepsilon} + 1 < q_i$ points, where $\varepsilon' = |z|\varepsilon$. In Proposition 6.13 we produce an ε'-eigenvalue of $\hat{M}_{q_i}(\theta_i)$. So, $\varepsilon_i = \frac{4}{q_i}|z|$. □

CHAPTER 7

Spectral theory for transfer operators over rotations in spaces of C^r sections

We start now the task of analyzing the spectrum of the transfer operators over rotations when acting on different spaces of sections.

The main results will be to show that for rotating transfer operators, the spectrum is largely independent of the space considered. (In the analytic case, our results are not totally complete.)

We emphasize that the equality of the spectra for spaces of different regularities is definitely false when the base is not a rotation.

In this section we analyze the spectrum in spaces of smooth functions. From now on, $E \xrightarrow{\Pi} \mathbb{T}^d$ is a C^r vector bundle over the torus, with $r \in \mathbb{N}$, and $\mathcal{M}_\omega$ is a transfer operator generated by a C^r vector bundle map $M : E \to E$ over a rotation $\omega \in \mathbb{R}^d$.

The results for $r = 0$ follow immediately from those in Part 2. Nevertheless, these are a particular case of the following results in the case that the motion on the base manifold is a rotation.

Remark 7.1. We call attention to the fact that the only properties of C^r Banach spaces of functions that are used in the arguments in this section are just:

- The spaces admit a structure of Banach algebra.
- It is possible to construct bump functions in these spaces.
- Evaluation at one point is a well defined operation.
- Composition on the left with a rotation does not increase the norm.

Hence, all the results that we will establish in this section have analogues for the spaces that satisfy these all the properties above. Of course, some particular results may use only a subset of these properties.

In Section 8 we will present the case of Sobolev spaces.

7.1. A technical device: fattening sections

One of the main devices of this section is the fattening of bounded sections to obtain smooth sections. This will be used in the analysis of spectral gaps and of the Weyl spectrum.

Moreover, we can control the C^r norms with sup-norms of the initial bounded section. This is the subject of the following proposition.

PROPOSITION 7.2. *Let* $E \xrightarrow{\Pi} \mathcal{P}$ *be a* C^r *bundle. Then, there exists a constant* K_r *such that for any* $v_{\theta_0} \in E_{\theta_0}$ *there exists* $v \in \Gamma_{C^r}(E)$ *such that:*

$$v(\theta_0) = v_{\theta_0} \ , \ \|v\|_{C^r} \leq K_r |v_{\theta_0}| \ .$$

Proof: We split the proof in several steps. First, we construct bump functions.

- *Construction of bump functions.*

LEMMA 7.3. *There exists a* C^∞ *function* $\varphi : \mathbb{R} \to \mathbb{R}$ *satisfying the following conditions:*

a) $\forall t \leq 0 \ \varphi(t) = 1$ *(and therefore* $\forall k \geq 1 \ , \varphi^{k)}(0) = 0$*);*
b) $\forall t \geq 1 \ \varphi(t) = 0$ *(and therefore* $\forall k \geq 1 \ , \varphi^{k)}(1) = 0$*);*
c) $1 = \|\varphi\|_\infty < \|\varphi^{1)}\|_\infty < \|\varphi^{2)}\|_\infty < \dots$.

Proof: We have just to construct a function satisfying a), b) and $1 = \|\varphi\|_\infty$, because c) follows from the repeated application of the Mean Value Theorem.

A well known example of such a function is given by

$$\varphi(t) = \frac{1}{1 + \exp\left(\frac{1}{(1-t)^2} - \frac{1}{t^2}\right)} \ , \tag{7.1}$$

in the interval $[0, 1]$, and $\varphi(t) = 0$ otherwise. □

The following result is a small modification of the previous result with some quantitative statements valid for finite regularities.

LEMMA 7.4. *For each* $r \geq 0$*, there exists a* C^∞*-function* $\rho : \mathbb{R}^d \to \mathbb{R}$ *satisfying the following conditions:*

a) ρ *is supported in* $\bar{B}(0, 1)$*, the closed unit ball with respect to the Euclidean norm in* $\mathbb{R}^d$*;*
b) $\|\rho\|_\infty < \|\mathrm{D}\rho\|_\infty < \dots < \|\mathrm{D}^r\rho\|_\infty$*;*
c) $\|\mathrm{D}^r\rho\|_\infty = |\mathrm{D}^r\rho(0)| = 1$.

Proof: From the function φ of the previous Lemma 7.3, we define the function $\beta : \mathbb{R}^d \to \mathbb{R}$ as

$$\beta(x) = \varphi(|x|) \ ,$$

which satisfies a) and b). Notice that, for $0 < \sigma \leq 1$, the function

$$\beta_\sigma(x) = \beta(x/\sigma)$$

also satisfies a) and b), because $\|\mathrm{D}^k\beta_\sigma\|_\infty = \frac{1}{\sigma^k}\|\mathrm{D}^k\beta\|_\infty$ and it is supported in $\bar{B}(0, \sigma)$.

Let $x_0 \in B(0, 1)$ be any point such that $|\mathrm{D}^r\beta(x_0)| = \|\mathrm{D}^r\beta\|_\infty$. Taking $\sigma = (1 + |x_0|)^{-1}$, the function

$$\rho(x) = \frac{\sigma^r}{\|\mathrm{D}^r\beta\|_\infty}\beta(x_0 - x/\sigma) \tag{7.2}$$

satisfies the three conditions a), b) and c). □

- *Construction of a fat section.* We are going to construct the section v of the proposition. To do so, first we fix a finite atlas $\{U_i\}_{i=1,\dots,p}$, whose Lebesgue radius is R. We set $\sigma = \min\{1, R\}$ and define

$$v(\theta) = \begin{cases} \beta_\sigma(\theta) v_{\theta_0} \text{ , if } \theta \in B(\theta_0, \sigma) \subset U_{\nu(\theta_0)} \text{ ,} \\ 0, \text{ otherwise ,} \end{cases} \tag{7.3}$$

where we are writing the formulae in the chart $U_{\nu(\theta_0)}$. Obviously $v(\theta_0) = v_{\theta_0}$. Since

$$\|D^s v_{|U}\|_\infty = \frac{1}{\sigma^s} \|D^s \beta\|_\infty |v_{\theta_0}|_\infty \text{ .}$$

The existence of K_r follows immediately from the equivalence between the Finsler norm and the "local" norms.

□

REMARK 7.5. A similar argument can be used to fatten a bounded section supported on a finite number of points, i.e., a finite set of vectors attached to different base points.

7.2. Equality of Weyl spectra for C^r and bounded sections

We analyze here the Weyl spectrum in $\Gamma_{C^r}(E)$.

THEOREM 7.6. *Let $M_\omega : E \to E$ be a C^r vector bundle automorphism over a rotation. Then:*

$$\mathrm{Spec}_W(\mathcal{M}_\omega, \Gamma_B(E)) = \mathrm{Spec}_W(\mathcal{M}_\omega, \Gamma_{C^r}(E)) \text{ .} \tag{7.4}$$

Proof: To prove that $\mathrm{Spec}_W(\mathcal{M}_\omega, \Gamma_B(E)) \subset \mathrm{Spec}_W(\mathcal{M}_\omega, \Gamma_{C^r}(E))$, we will "fatten" a bounded eigensection to produce a C^r eigensection. Recall that we can choose the approximate bounded eigensections of a particularly simple form, that are supported in finite segments of orbits. Hence, let $z \in \mathrm{Spec}_W(\mathcal{M}_\omega, \Gamma_B(E))$, and for a given $\varepsilon > 0$ let v be a bounded ε-eigensection supported in the finite segment of orbit $\{\theta_k = \theta_0 + k\omega\}_{k=0,\dots,K}$. This argument works in both rational case and irrational cases, but in the rational case we have in fact a bounded eigensection (that is $\varepsilon = 0$) supported in a periodic orbit.

To fatten v, we use the bump function ρ constructed in Lemma 3.29. That is to say, ρ is a C^r function supported in the unit ball of $\mathbb{R}^d$, and such that:

$$\|\rho\|_\infty < \|\mathrm{D}\rho\|_\infty < \cdots < \|\mathrm{D}^r \rho\|_\infty = |\mathrm{D}^r \rho(0)| = 1 \text{ .}$$

We denote $\rho_\sigma(\theta) = \rho(\theta/\sigma)$, that is supported in the ball of radius σ. Taking a small enough $\sigma > 0$, so that the balls $B(\theta_k, \sigma)$ are included in the trivializing charts and do not overlap, and that satisfies other smallness conditions that we will specify later, we define the C^r section

$$w(\theta) = \begin{cases} \sigma^r \rho_\sigma(\theta - \theta_k) v_k, \text{ if } |\theta - \theta_k| \le \sigma, \\ 0, \text{ otherwise.} \end{cases}$$

Obviously, $\|w\|_\infty < \|\mathrm{D}w\|_\infty < \cdots < \|\mathrm{D}^r w\|_\infty = 1$. We have to estimate now the derivatives of $M(\theta-\omega)w(\theta) - zw(\theta)$, where $|\theta-\theta_k| \le \sigma$. To do so, for any $0 \le s \le r$:

$$
\begin{aligned}
&|\mathrm{D}^s_\theta(M(\theta-\omega)w(\theta) - zw(\theta))| \\
&\quad = \sigma^r |\mathrm{D}^s_\theta(\rho_\sigma(\theta-\theta_k)(M(\theta-\omega)v_{k-1} - zv_k))| \\
&\quad \le \sum_{l=0}^{s-1} \binom{s}{l} \sigma^{r-l}\|M\|_{C^r} + \ \sigma^{r-s}|M(\theta-\omega)v_{k-1} - zv_k| \\
&\quad \le ((1+\sigma)^r - 1)\,\|M\|_{C^r} + \varepsilon + \eta(\sigma) \ ,
\end{aligned}
$$

where η is the modulus of continuity of M. If we take σ small enough, we obtain that $\|\mathcal{M}_\omega w - zw\|_{C^r} \le 2\varepsilon$. This establishes $\mathrm{Spec}_W(\mathcal{M}_\omega, \Gamma_B(E)) \subset \mathrm{Spec}_W(\mathcal{M}_\omega, \Gamma_{C^r}(E))$.

Remark 7.7. The previous arguments can be extended for general transfer operators Part 2. So, the inclusion is true for other dynamics on the base manifold.

Since $\Gamma_{C^0}(E) \subset \Gamma_B(E)$ (as a Banach subspace) and $\Gamma_{C^0}(E)$ is invariant under $\mathcal{M}_\omega$, it is straightforward that $\mathrm{Spec}_W(\mathcal{M}_\omega, \Gamma_{C^0}(E)) \subset \mathrm{Spec}_W(\mathcal{M}_\omega, \Gamma_B(E))$. To prove $\mathrm{Spec}_W(\mathcal{M}_\omega, \Gamma_{C^r}(E)) \subset \mathrm{Spec}_W(\mathcal{M}_\omega, \Gamma_B(E))$ for $r \ge 1$ we can not use the previous inclusion argument, because the topologies in both spaces are different. Since for $r = 0$ this result is proved, we will use an induction argument. Suppose then that $r \ge 1$ and that we have proven the inclusion for $r-1$.

Notice that if $\{v_n\}_n$ is a sequence of C^r approximate eigensections of z, i.e.,

$$
\|\mathcal{M}_\omega v_n - zv_n\|_{C^r} \to 0 \ , \ \|v_n\|_{C^r} = 1 \ ,
$$

we can assume that $\|v_n\|_{C^{r-1}} \to 0$. Otherwise, since

$$
\|\mathcal{M}_\omega v_n - zv_n\|_{C^{r-1}} \le \|\mathcal{M}_\omega v_n - zv_n\|_{C^r}
$$

we would have that v_n is an approximate eigensection on C^{r-1} and we could, by the induction hypothesis construct an approximate bounded eigensection.

Hence, we will assume that we have a C^r section v with $\|v\|_{C^r} = 1$, such that $\|v\|_{C^{r-1}} \le \varepsilon$ and $\|\mathcal{M}_\omega v - zv\|_{C^r} \le \varepsilon$. In those circumstances, there is a point θ_0 such that $|\mathrm{D}^r v(\theta_0)| = 1$.

We write again $\theta_k = \theta_0 + k\omega$. Since v is a C^r approximate eigensection we have

$$
\begin{aligned}
\varepsilon &\geq \max_{\theta\in\mathbb{T}^d} |\mathrm{D}^r\left(M(\theta-\omega)v(\theta-\omega) - zv(\theta)\right)| \\
&\stackrel{*}{=} \max_{\theta\in\mathbb{T}^d} |\mathrm{D}^r(Mv)(\theta-\omega) - z\mathrm{D}^r v(\theta)| \\
&\geq |\mathrm{D}^r(Mv)(\theta_{k-1}) - z\mathrm{D}^r v(\theta_k)| \\
&\geq |M(\theta_{k-1})\mathrm{D}^r v(\theta_{k-1}) - z\mathrm{D}^r v(\theta_k)| - \left|\sum_{s=1}^{r}\binom{r}{s}\mathrm{D}^s M(\theta_{k-1})\mathrm{D}^{r-s}v(\theta_{k-1})\right| \\
&\geq |M(\theta_{k-1})\mathrm{D}^r v(\theta_{k-1}) - z\mathrm{D}^r v(\theta_k)| - \varepsilon(2^r-1)\|M\|_{C^r}\ .
\end{aligned}
$$

Therefore,

$$
|M(\theta_{k-1})\mathrm{D}^r v(\theta_{k-1}) - z\mathrm{D}^r v(\theta_k)| \quad\leq\quad (1+(2^r-1)\|M\|_{C^r})\varepsilon\ .
$$

REMARK 7.8. Equality (*) is crucial in our argumentation, and uses that the motion on the base manifold, a torus, is a rotation. It fails in general.

Since $|\mathrm{D}^r v(\theta_0)| = 1$, we can find r unit vectors $e_1, \dots e_r \in E_{\theta_0}$ such that $|\mathrm{D}^r v(\theta_0)(e_1,\dots,e_r)| = 1$ (using the trivialization charts). We set

$$
w(\theta) = \begin{cases} \mathrm{D}^r v(\theta_k)(e_1,\dots,e_r) \text{ if } k\in\mathbb{Z} \\ 0,\ \text{otherwise.}\end{cases}
$$

Hence, $\|w\|_\infty = 1$ and

$$
\begin{aligned}
\|\mathcal{M}_\omega w - zw\|_\infty &\leq \max_{k\in\mathbb{Z}} |(M(\theta_{k-1})\mathrm{D}^r v(\theta_{k-1}) - z\mathrm{D}^r v(\theta_k))(e_1,\dots,e_r)| \\
&\leq (1+(2^r-1)\|M\|_{C^r})\varepsilon\ .
\end{aligned}
$$

So, we have found a bounded approximate eigensection and we are done with the proof of Theorem 7.6. ☐

7.3. Spectral gaps in spaces of C^r sections

We will prove a refinement of Theorem 2.18 for rotations. We will obtain a characterization of spectral projections for spaces of C^r sections in terms of bundles.

THEOREM 7.9. *Let $M_\omega : E \to E$ be a C^r vector bundle automorphism over a rotation ω. Assume that*

$$
\mathrm{Spec}(\mathcal{M}_\omega, \Gamma_{C^r}(E)) \cap \mathcal{A}_{\lambda,\mu} = \emptyset\ .
$$

Denote by $P^{<\lambda} = P^{<\lambda}_{C^r}, P^{>\mu} = P^{>\mu}_{C^r}$ the projections associated to this spectral gap. Then, it is possible to find a invariant C^r-splitting

$$
E = E^{<\lambda} \oplus E^{>\mu} \tag{7.5}
$$

such that the corresponding projections over the bundles, $\Pi^{<\lambda}, \Pi^{>\mu}$, satisfy for any $v \in \Gamma_{C^r}(E)$:

$$(P^{<\lambda}v)(\theta) = \Pi_\theta^{<\lambda} v(\theta) \ , \ (P^{>\mu}v)(\theta) = \Pi_\theta^{>\mu} v(\theta) \ . \tag{7.6}$$

Moreover, for all $\varepsilon > 0$ small enough there exists a constant $C_\varepsilon > 0$ such that

$$E^{<\lambda} = W^{\leq \lambda - \varepsilon, C_\varepsilon} = L^{<\lambda} \ , \ E^{>\mu} = W^{\geq \mu + \varepsilon, C_\varepsilon} = L^{>\mu} \ . \tag{7.7}$$

Proof: A direct proof of this result follows from Theorem 3.14, and this is in fact the content of Theorem 3.31, that is general. However, we will do here a proof *a la* [**HPS77**] (whose proof works for the case $\Gamma_B(E)$). We will emphasize the arguments that use the rotations, that can not be extended in general. This is one of the motivations of Theorem 3.31. We split the proof in several steps (see the proof of Theorem 2.18).

- *C^r-linearity of spectral projections.* The first point is to realize that the spectral projections are $C^r(\mathbb{T}^d, \mathbb{C})$-linear.

Lemma 7.10. *For any $\rho \in C^r(\mathbb{T}^d, \mathbb{C})$ and $v \in \Gamma_{C^r}(E)$:*

$$P^{<\lambda}(\rho v) = \rho P^{<\lambda} v \ , \ P^{>\mu}(\rho v) = \rho P^{>\mu} v \ . \tag{7.8}$$

Proof of the lemma: To prove Lemma 7.10, notice first that $\mathcal{M}_\omega(\rho v)(\theta) = \rho(\theta - \omega)\mathcal{M}_\omega v(\theta)$, and then

$$\mathcal{M}_\omega^m(\rho v)(\theta) = \rho(\theta - m\omega)\mathcal{M}_\omega^m v(\theta) \ .$$

The key point is that the C^r norms of ρ and their iterates remain constant: $\|\rho \circ t_\omega^{-m}\|_{C^r} = \|\rho\|_{C^r}$ (this is not true for general transfer operators!). Hence, using the Banach algebra properties of spaces of C^r functions, and using general spectral theory (see for instance the survey in Part 2) we obtain that

$$\begin{aligned} v \in \Gamma_{C^r}^{<\lambda}(E) \quad &\Leftrightarrow \quad \limsup_{m\to\infty} \|\mathcal{M}_\omega^m v\|_{C^r}^{\frac{1}{m}} < \lambda \\ &\Rightarrow \quad \limsup_{m\to\infty} \|\mathcal{M}_\omega^m \rho v\|_{C^r}^{\frac{1}{m}} \leq \limsup_{m\to\infty} 2^{\frac{r}{m}} \|\rho\|_{C^r}^{\frac{1}{m}} \|\mathcal{M}_\omega^m v\|_{C^r}^{\frac{1}{m}} < \lambda \\ &\Rightarrow \quad \rho v \in \Gamma_{C^r}^{<\lambda}(E) \ . \end{aligned}$$

Repeating the argument for $P^{>\mu}$, we obtain that

$$v \in \Gamma_{C^r}^{<\lambda}(E) \Rightarrow \rho v \in \Gamma_{C^r}^{\lambda}(E) \ , \ v \in \Gamma_{C^r}^{>\mu}(E) \Rightarrow \rho v \in \Gamma_{C^r}^{>\mu}(E) \ . \tag{7.9}$$

Finally, consider any $v \in \Gamma_{C^r}(E)$ and its spectral decomposition $v = v^{<\lambda} + v^{>\mu}$. Then, since $\rho v = \rho v^{<\lambda} + \rho v^{>\mu}$ and (7.9) we obtain the claimed result. ☐

- *Localization of spectral projections.* The second step in the proof is to check that the spectral projections are local operators, in the sense that for a C^r section v, and a base point $\theta_0 \in \mathbb{T}^d$, $(P^{<\lambda}v)(\theta_0)$

and $P^{>\mu}v(\theta_0)$ depend only on $v(\theta_0)$. This is proved by using standard techniques of differential geometry, involving bump functions and local charts.

LEMMA 7.11. *Let v be a C^r section. Let $U \subset \mathbb{T}^d$ be an open set. If $v_{|U} \equiv 0$ then $(P^{<\lambda}v)_{|U} \equiv 0$ and $(P^{>\mu}v)_{|U} \equiv 0$.*

Proof of the lemma: Given $\theta_0 \in U$, let $\varphi : \mathbb{T}^d \to \mathbb{R}$ be a C^∞ function such that $\varphi(\theta_0) = 0$ and for all $\theta \notin U$ $\varphi(\theta) = 1$. Notice that $v = \varphi v$. Then:

$$(P^{<\lambda}v)(\theta_0) = (P^{<\lambda}\varphi v)(\theta_0) = \varphi(\theta_0)(P^{<\lambda}v)(\theta_0) = 0 \ ,$$

and we repeat the argument for $(P^{>\mu}v)(\theta_0) = 0$. □

LEMMA 7.12. *Let v be a C^r section. Let $\theta_0 \in \mathbb{T}^d$ be any base point. If $v(\theta_0) = 0$ then $(P^{<\lambda}v)(\theta_0) = 0$ and $(P^{>\mu}v)(\theta_0) = 0$.*

Proof of the lemma: Take a trivialization $U \times \mathbb{C}^d$ of the bundle, with $\theta_0 \in U$, and let $v_{|U} = \sum_{i=1}^n v_i(\theta)e_i$ be the corresponding representation of v ($\{e_i\}_{i=1,\dots,d}$ is the standard base of $\mathbb{C}^d$, that we consider as a local frame of the bundle E, and the coefficients $v_i(\theta)$ are C^r functions). Notice that $v_i(\theta_0) = 0$ for $i = 1, \dots n$. Hence,

$$(P^{<\lambda}v)(\theta_0) = (P^{<\lambda}v_{|U})(\theta_0) = \sum_{i=1}^{n} v_i(\theta_0)(P^{<\lambda}e_i)(\theta_0) = 0 \ .$$

□

- *Bundle projections.* We define the bundle maps $\Pi_{\theta_0}^{<\lambda}$ and $\Pi_{\theta_0}^{>\mu}$ on the fiber E_{θ_0}, as

$$\Pi_{\theta_0}^{<\lambda} v_{\theta_0} = (P^{<\lambda}v)(\theta_0) \ , \ \Pi_{\theta_0}^{>\mu} v_{\theta_0} = (P^{>\mu}v)(\theta_0) \ ,$$

  where v is a C^r section such that $v(\theta_0) = v_{\theta_0}$. Notice that the definitions do not depend on the C^r section that we take (Lemma 7.12), and these kind of sections in fact do exist (using bump functions and trivialization charts). The bundle maps are obviously linear.
- *Splitting.* We define the linear subspaces $E_{\theta_0}^{<\lambda} = \Pi_{\theta_0}^{<\lambda} E_{\theta_0}$ and $E_{\theta_0}^{>\mu} = \Pi_{\theta_0}^{>\mu} E_{\theta_0}$. It is easy to see that

$$E_{\theta_0}^{<\lambda} = \{v(\theta_0) \mid v \in \Gamma_{C^r}^{<\lambda}(E)\} \ , \ E_{\theta_0}^{>\mu} = \{v(\theta_0) \mid v \in \Gamma_{C^r}^{>\mu}(E)\} \ .$$

  From the functional analytical properties of the spectral projections, we prove that the bundle maps are in fact bundle projections, and that $E_{\theta_0} = E_{\theta_0}^{<\lambda} \oplus E_{\theta_0}^{>\mu}$. This defines an invariant splitting in linear subspaces.

  Much more, the bundle maps $\Pi_\theta^{<\lambda}$ and $\Pi_\theta^{>\mu}$ are C^r in θ. Since $(P^{<\lambda}v)(\theta) = \Pi_\theta^{<\lambda}(v(\theta))$, if $\Pi_\theta^{<\lambda}$ were not C^r we could find a C^r section v for which $P^{<\lambda}v$ would not be C^r, in contradiction with the fact that $P^{<\lambda}$ is a projection on $\Gamma_{C^r}(E)$. The same argument works for $\Pi_\theta^{>\mu}$. Hence, E decompose in the Whitney sum $E = E^{<\lambda} \oplus E^{>\mu}$, and this C^r-splitting is invariant.

- *Rates of growth.* From the characterization of the rates of growth in the spectral subspaces, we obtain that for $\varepsilon > 0$ small enough there exists a positive constant $\hat{C}_\varepsilon$ such that

$$\begin{aligned} \Gamma_{C^r}^{<\lambda}(E) &= \{v \in \Gamma_{C^r}(E) \mid \forall m \geq 0 \ \|M_\omega^m v\|_{C^r} \leq \hat{C}_\varepsilon (\lambda - \varepsilon)^m \|v\|_{C^r}\} \ , \\ \Gamma_{C^r}^{>\mu}(E) &= \{v \in \Gamma_{C^r}(E) \mid \forall m \geq 0 \ \|M_\omega^{-m} v\|_{C^r} \leq \hat{C}_\varepsilon (\mu + \varepsilon)^{-m} \|v\|_{C^r}\} \ . \end{aligned} \tag{7.10}$$

We claim that

$$E_{\theta_0}^{<\lambda} = W_{\theta_0}^{\leq \lambda - \varepsilon, C_\varepsilon} = L_{\theta_0}^{<\lambda} \ , \ E_{\theta_0}^{>\mu} = W_{\theta_0}^{\geq \mu + \varepsilon, C_\varepsilon} = L_{\theta_0}^{>\mu} \ , \tag{7.11}$$

for a suitable constant C_ε that we will specify later.

To prove this claim, take any $v_{\theta_0} \in E_{\theta_0}^{<\lambda}$. From Proposition 7.2, there exists a C^r section w such that $w(\theta_0) = v_{\theta_0}$ and $\|w\|_{C^r} \leq K_r |v_{\theta_0}|$, where K_r is a universal constant that depends only on the bundle and r. Notice that

$$v_{\theta_0} = \Pi_{\theta_0}^{<\lambda} v_{\theta_0} = \Pi_{\theta_0}^{<\lambda}(w(\theta_0)) = (P^{<\lambda} w)(\theta_0) \ ,$$

and then the C^r section $v = P^{<\lambda} w \in \Gamma_{C^r}^{<\lambda}(E)$ satisfies $v(\theta_0) = v_{\theta_0}$. Moreover,

$$\|v\|_{C^r} \leq \|P^{<\lambda}\| \cdot \|w\|_{C^r} \leq K_r \|P^{<\lambda}\| |v_{\theta_0}| \ .$$

Using a similar argument, we prove that for all $v_{\theta_0} \in E_{\theta_0}^{>\mu}$ there exists a C^r section $v \in \Gamma_{C^r}^{>\mu}(E)$ such that $v(\theta_0) = v_{\theta_0}$ and

$$\|v\|_{C^r} \leq K_r \|P^{>\mu}\| |v_{\theta_0}| \ .$$

Hence, by defining $C_\varepsilon = \hat{C}_\varepsilon K_r \max\{\|P^{<\lambda}\|, \|P^{>\mu}\|\}$ we are lead to the inclusions $E_{\theta_0}^{<\lambda} \subset W_{\theta_0}^{\leq \lambda - \varepsilon, C_\varepsilon}$ and $E_{\theta_0}^{>\mu} \subset W_{\theta_0}^{\geq \mu + \varepsilon, C_\varepsilon}$.

The inclusions $W_{\theta_0}^{\leq \lambda - \varepsilon, C_\varepsilon} \subset L_{\theta_0}^{<\lambda}$ and $W_{\theta_0}^{\geq \mu + \varepsilon, C_\varepsilon} \subset L_{\theta_0}^{>\mu}$ are obvious.

Finally, notice that

$$L_\theta^{<\lambda} \cap L_\theta^{>\mu} = \{0\} \ , \tag{7.12}$$

because otherwise

$$\mathrm{Spec}_W(\mathcal{M}_\omega, \Gamma_{C^r}(E)) \cap \mathcal{A}_{\lambda,\mu} = \mathrm{Spec}_W(\mathcal{M}_\omega, \Gamma_B(E)) \cap \mathcal{A}_{\lambda,\mu} \neq \emptyset \ ,$$

in contradiction with the main hypothesis of the theorem (we apply Theorem 7.6 and Mañé lemma in [**Mn78, HdlL03a**]). This proves that $E^{<\lambda} = L^{<\lambda}$ and $E^{>\mu} = L^{>\mu}$.

□

REMARK 7.13. We emphasize that the argument we have carried we used projections in C^r and their smoothness is completely independent of the C^0, B theory which included a bootstrap on the regularity of the splittings appealing to the invariant section theorem Part 2. In the present case, the spectral invariant bundles are created out of the spectral projections.

REMARK 7.14. We also call attention to the fact that the present argument uses in an essential way that the motion on the base is a rotation. The case of a general motion on the base is discussed in Part 2 and it is based on Theorem 3.14. We will use Theorem 3.14 to prove the corresponding result in Sobolev spaces.

REMARK 7.15. One of the conclusions of the general theory is that, in systems over a typical map, there can be no gaps in the C^r spectrum for r large, see Section 3.7.

REMARK 7.16. Notice also that some of the arguments of the proof of Theorem 3.31 are not easily generalizable to study other spaces of sections. Specifically, those arguments involving bumps functions do not apply in the analytic case. The theory for this case will be developed later.

7.4. Equality of spectra for C^r and bounded sections

In this section we will prove the following.

THEOREM 7.17. *Let $M_\omega : E \to E$ be a C^r vector bundle automorphism over a rotation ω. Then:*

$$\operatorname{Spec}(\mathcal{M}_\omega, \Gamma_B(E)) = \operatorname{Spec}(\mathcal{M}_\omega, \Gamma_{C^r}(E)) \ . \tag{7.13}$$

Proof: To prove that

$$\operatorname{Spec}(\mathcal{M}_\omega, \Gamma_B(E)) \subset \operatorname{Spec}(\mathcal{M}_\omega, \Gamma_{C^r}(E)) \ , \tag{7.14}$$

we use that the spectrum in spaces of bounded sections coincides with Weyl spectrum in spaces of bounded sections, and this spectrum coincides with Weyl spectrum in spaces of C^r sections. In summary:

$$\begin{aligned}\operatorname{Spec}(\mathcal{M}_\omega, \Gamma_B(E)) = \operatorname{Spec}_W(\mathcal{M}_\omega, \Gamma_B(E)) &= \operatorname{Spec}_W(\mathcal{M}_\omega, \Gamma_{C^r}(E)) \\ &\subset \operatorname{Spec}(\mathcal{M}_\omega, \Gamma_{C^r}(E)) \ .\end{aligned}$$

This argument works in both rational and irrational rotations.

REMARK 7.18. If ω is irrational, a different argument can be used. It involves the fact that in this case the spectra is rotationally invariant. Hence, to prove (7.14) it suffices to show that if $\operatorname{Spec}(\mathcal{M}_\omega, \Gamma_{C^r}(E)) \cap \mathcal{A}_{\lambda,\mu} = \emptyset$ then $\operatorname{Spec}(\mathcal{M}_\omega, \Gamma_B(E)) \cap \mathcal{A}_{\lambda,\mu} = \emptyset$. Suppose then that $\operatorname{Spec}(\mathcal{M}_\omega, \Gamma_{C^r}(E)) \cap \mathcal{A}_{\lambda,\mu} = \emptyset$. By Theorem 2.18 we can find an invariant C^r-splitting $E = E^{<\lambda} \oplus E^{>\mu}$ satisfying rates of growth $E^{<\lambda} = L^{<\lambda}$ and $E^{>\mu} = L^{>\mu}$, and Theorem 9.1 implies that $\operatorname{Spec}(\mathcal{M}_\omega, \Gamma_B(E)) \cap \mathcal{A}_{\lambda,\mu} = \emptyset$.

We are going to prove now that

$$\operatorname{Spec}(\mathcal{M}_\omega, \Gamma_{C^r}(E)) \subset \operatorname{Spec}(\mathcal{M}_\omega, \Gamma_B(E)) \ . \tag{7.15}$$

This is equivalent to show that if $z \in \mathrm{Res}(\mathcal{M}_\omega, \Gamma_B(E))$, then given any $\eta \in \Gamma_{C^r}(E)$ we can solve the equation

$$M(\theta-\omega)\phi(\theta-\omega) - z\phi(\theta) = \eta(\theta) \tag{7.16}$$

in $\Gamma_{C^r}(E)$ and, moreover, $\|\phi\|_{C^r} \le C\|\eta\|_{C^r}$, where C is a positive constant depending only on M and ω.

If ω is rational, we use the following argument to prove that z is a resolvent value $\Gamma_{C^r}(E)$. We repeat the argument in Theorem 6.4, and realize that the map (6.2) is in fact C^r, because of the differentiable dependence on parameters for resolvent of matrices. Then we obtain uniform bounds on the derivatives, and the estimates on C^r norms follow from the Banach algebra properties of the spaces of smooth functions.

Suppose now that ω is irrational. Since the spectrum is rotationally invariant, from $z \in \mathrm{Res}(\mathcal{M}_\omega, \Gamma_B)$ we produce a gap $\mathrm{Spec}(\mathcal{M}_\omega, \Gamma_B) \cap \mathcal{S}_\rho = \emptyset$, where $\rho = |z|$. Hence, there is a C^r invariant splitting $E = E^{<\rho} \oplus E^{>\rho}$ (see Theorem 9.1, and recall that the smoothness of the subbundles comes from the invariant section theorem).

We study the equation (7.16) by taking projections over those invariant subbundles. So, we look for $\phi^{<\rho} = \Pi^{<\rho}\phi \in C^r$ and $\phi^{>\rho} = \Pi^{>\rho}\phi \in C^r$ satisfying the equations

$$M^{<\rho}(\theta-\omega)\phi^{<\rho}(\theta-\omega) - z\phi^{<\rho}(\theta) = \eta^{<\rho}(\theta)\ , \tag{7.17}$$

$$M^{>\rho}(\theta-\omega)\phi^{>\rho}(\theta-\omega) - z\phi^{>\rho}(\theta) = \eta^{>\rho}(\theta)\ , \tag{7.18}$$

where $\eta^{<\rho} = \Pi^{<\rho}\eta \in C^r$ and $\eta^{>\rho} = \Pi^{>\rho}\eta \in C^r$ are given and is given, and $M^{<\rho}(\theta), M^{>\rho}(\theta)$ denote the action of M on the corresponding invariant subbundles.

Note that because the projections are C^r and the angles are bounded from below, there exists a constant C such that

$$\|\eta^{<\rho}\|_{C^r} \le C\|\eta\|_{C^r}\ ,\ \|\eta^{>\rho}\|_{C^r} \le C\|\eta\|_{C^r}$$

We solve (7.17) and (7.18) by setting

$$\phi^{<\rho}(\theta) = -\sum_{m=0}^{\infty} z^{-(m+1)} M^{<\rho}(\theta - m\omega, m)\eta^{<\rho}(\theta - m\omega)\ , \tag{7.19}$$

$$\phi^{>\rho}(\theta) = \sum_{m=1}^{\infty} z^{m-1} M^{>\rho}(\theta + m\omega, -m)\eta^{>\rho}(\theta + m\omega)\ . \tag{7.20}$$

We have to prove that these series converge in C^r. We analyze (7.19), because we can use similar arguments to analyze (7.20).

Since for $\varepsilon > 0$ small enough we can bound

$$\|M^{<\rho}(\theta - m\omega, m)\eta^{<\rho}(\theta - m\omega)\|_\infty \le C_\varepsilon(\rho - \varepsilon)^m \|\eta^{<\rho}\|_\infty\ ,$$

and the series (7.19) converges in C^0.

To estimate the derivatives, just take into account that, applying Leibniz rule, we have that

$$\mathrm{D}^j(M^{<\rho}(\theta-\omega)\dots M^{<\rho}(\theta-m\omega))$$

contains a sum of $C_{j,m} \leq m^j$ terms, each of which is the product of j factors of the form

$$F_l^{<\rho} = M^{<\rho}(\theta - a_l\omega)\dots M^{<\rho}(\theta - b_l\omega) \ , \tag{7.21}$$

where $a_l \leq b_l$ and $\sum_{l=1}^{j}(b_l - a_l + 1) \geq m - j$ and not more that j factors which are derivatives of M of order not longer than j.

For $\varepsilon > 0$ small enough we have $\|F_l^i\| \leq C_\varepsilon(\rho-\varepsilon)^{b_l - a_l + 1}$, and then $\|\mathrm{D}^j M^{<\rho}(\theta - m\omega, \theta)\|_\infty$ can be bounded by

$$m^j C_{\varepsilon,j,\rho}(\rho-\varepsilon)^{m-j}\|M\|_{C^j}^j \ .$$

This leads to bounds

$$\|\mathrm{D}^j(M^{<\rho}(\theta-m\omega,m)\eta^{<\rho}(\theta-m\omega))\|_\infty \leq m^j C_{\varepsilon,j,M,\omega}(\rho-\varepsilon)^{m-j}\|\eta\|_{C^j}.$$

Hence, we conclude that (7.19) and (7.20) indeed give C^r solutions. □

As a corollary of the results of this section we obtain the following.

Corollary 7.19. *Let $\mathcal{M}_\omega$ be a C^r transfer operator over a rotation. Then:*

$$\mathrm{Spec}(\mathcal{M}_\omega, \Gamma_{C^r}) = \mathrm{Spec}_W(\mathcal{M}_\omega, \Gamma_{C^r}) \ . \tag{7.22}$$

7.5. Perturbation of the spectrum

In this section we study the perturbation of the spectrum in C^r spaces.

Proposition 7.20. *Let $\mathcal{M}_\omega$ be a C^r transfer operator over a rotation ω. Assume that for some real numbers $0 < \lambda < \mu$*

$$\mathrm{Spec}(\mathcal{M}_\omega, \Gamma_B(E)) \cap \mathcal{A}_{\lambda,\mu} = \emptyset \ .$$

Let $E = E^{<\lambda} \oplus E^{>\mu}$ the corresponding C^r invariant splitting.

Then, if $\bar{M}_\omega$ is C^r close enough to M_ω, then

$$\bar{M}_\omega \in \mathcal{U} \Rightarrow \mathrm{Spec}(\bar{M}_{\bar{f}}, \Gamma_B(E)) \cap \mathcal{A}_{\lambda,\mu} = \emptyset \ .$$

Moreover, the mapping that to $\bar{M}_\omega$ associates the C^r splitting $E = \bar{E}^{<\lambda} \oplus \bar{E}^{>\mu}$ is C^r when we give the space of bundles splitting the C^r topology and $\mathcal{U}$ the C^r topology.

Remark 7.21. Notice that we can use standard theory of perturbation of the spectrum because if $\bar{M}_\omega$ is C^r close to M_ω, then the corresponding transfer operators are close as linear bounded operators. The crucial fact is that the motion on the base manifold is the same for both $\bar{M}_\omega$ and M_ω. For other general results on perturbation of the spectrum, see [**HPS77**] and Part 2.

REMARK 7.22. We emphasize also that the fact of considering resonant (non ergodic) frequency vectors allows us to produce perturbation results by adding the parameters to the frequency vector.

Proof: From the results of this paper

$$\mathrm{Spec}(\mathcal{M}_\omega, \Gamma_{C^r}(E)) \cap \mathcal{A}_{\lambda,\mu} = \emptyset \ ,$$

and the invariant splitting $E = E^{<\lambda} \oplus E^{>\mu}$ is constructed from the projections $P^{<\lambda}_{C^r}$ and $P^{>\mu}_{C^r}$ on $\Gamma_{C^r}(E)$.

If M_ω and $\bar{M}_\omega$ are C^r-close, then the corresponding transfer operators are also close when considered as acting on C^r-sections. Applying standard perturbation theory of spectrum [**Kat76**], if $\bar{M}_\omega$ is C^r-close enough to M_ω, then

$$\mathrm{Spec}(\bar{\mathcal{M}}_\omega, \Gamma_{C^r}(E)) \cap \mathcal{A}_{\lambda,\mu} = \emptyset \ ,$$

and the corresponding projections $\bar{P}^{<\lambda}_{C^r}$ and $\bar{P}^{>\mu}_{C^r}$ are close to $P^{<\lambda}_{C^r}$ and $P^{>\mu}_{C^r}$, respectively. Hence, the corresponding splitting $E = \bar{E}^{<\lambda} \oplus \bar{E}^{>\mu}$ is C^r-close to $E = \bar{E}^{<\lambda} \oplus E^{>\mu}$. □

CHAPTER 8

Spectral theory for transfer operators over rotations in spaces of Sobolev sections

There are many spaces which are useful in analysis. Clearly, giving detailed proofs of results on the spectrum for transfer operators is out of question. Nevertheless, as pointed in Remark 7.1, the results proved here depend only on a few properties of the spaces.

In this section, we will go briefly over the Sobolev spaces, which play an important role in analysis. All the properties of Sobolev spaces that we will use can be found in [**Tay97**]. A standard comprehensive reference for Sobolev spaces is [**Ada75**].

8.1. Sobolev spaces

We will use the following Sobolev spaces. In the following, $\mathcal{P}$ is a compact Riemannian manifold, and let μ be the associated Lebesgue measure.

DEFINITION 8.1. *For $r \in \mathbb{N}$ and $p \in [1, \infty]$, we define the Sobolev space*

$$W^{r,p} = \{\varphi : \mathcal{P} \to \mathbb{C} \mid \partial^\alpha \varphi \in L^p \ \forall 0 \le |\alpha| \le r\} \ , \tag{8.1}$$

where the derivatives are taken in the sense of distributions. The norm

$$\|\varphi\|_{W^{r,p}} = \sup_{0 \le |\alpha| \le r} \|\partial^\alpha \varphi\|_{L^p} \tag{8.2}$$

makes $W^{r,p}$ a Banach space.

In Part 2, we have the result that the Sobolev spaces satisfy the Banach algebra and evaluation at points properties provided that

$$\frac{1}{p} - \frac{r}{d} < 0$$

(The property of evaluation at points could have been done in slight more generality appealing to the notion of essential values, but this is not worth the trouble for us).

As for the other properties of Sobolev spaces mentioned in Remark 7.1, we just note that the possibility of fattening sections (Proposition 7.2) is true for Sobolev spaces since it is possible to construct functions with arbitrarily small support.

We also note that for $\mathcal{P} = \mathbb{T}^d$, the property

$$\|\varphi \circ t_\omega\|_{W^{r,p}(\mathbb{T}^d)} = \|\varphi\|_{W^{r,p}(\mathbb{T}^d)}$$

is obvious from the definition.

8.2. Spectral theorems in Sobolev sections

We will quickly go over the proofs in Section 7 and formulate corresponding results for Sobolev spaces. The definitions of Sobolev bundles, Sobolev vector bundle maps, Sobolev sections, etc. are given as usual.

The following result corresponds to Theorem 7.6.

THEOREM 8.2. *Let $M_\omega : E \to E$ be a $W^{s,q}$ vector bundle automorphism over a rotation ω, where*

$$1/q - s/d < 0 \ , \ q \in]1, \infty[\ . \tag{8.3}$$

Then, for all r, p, with

$$r \le s \ , \ p \in]1, \infty[\ , \ \frac{1}{q} - \frac{s}{d} \le \frac{1}{p} - \frac{r}{d} < 0 \ , \tag{8.4}$$

we have

$$\mathrm{Spec}_W(\mathcal{M}_\omega, \Gamma_B) = \mathrm{Spec}_W(\mathcal{M}_\omega, \Gamma_{W^{s,q}}) \ . \tag{8.5}$$

Proof: The proof does not require any modification from the proof presented for Theorem 7.6, we just note that the cut-off functions work just as well in Sobolev spaces. Note that we do not need that the sections form a Banach algebra, only that the transfer operator defines a bounded multiplication. □

The following is an analogue of Theorem 7.9.

THEOREM 8.3. *Let $M_\omega : E \to E$ be a $W^{s,q}$ vector bundle automorphism over a rotation ω, where s, q satisfy* (8.3)*. Assume that*

$$\mathrm{Spec}(\mathcal{M}_\omega, \Gamma_{W^{r,p}}) \cap \mathcal{A}_{\lambda,\mu} = \emptyset \ ,$$

for r, p satisfying (8.4)*. Denote by $P^{<\lambda} = P^{<\lambda}_{W^{r,p}}$, $P^{>\mu} = P^{>\mu}_{W^{r,p}}$ the projections associated to this spectral gap.*

Then, it is possible to find a invariant $W^{r,p}$-splitting

$$E = E^{<\lambda} \oplus E^{>\mu} \tag{8.6}$$

such that the corresponding projections over the bundles, $\Pi^{<\lambda}, \Pi^{>\mu}$, satisfy for any $v \in \Gamma_{W^{r,p}}$:

$$(P^{<\lambda} v)(\theta) = \Pi^{<\lambda}_\theta v(\theta) \ , \ (P^{>\mu} v)(\theta) = \Pi^{>\mu}_\theta v(\theta) \ . \tag{8.7}$$

Moreover, for all $\varepsilon > 0$ small enough there exists a constant $C_\varepsilon > 0$ such that

$$E^{<\lambda} = W^{\le \lambda - \varepsilon, C_\varepsilon} = L^{<\lambda} \ , \ E^{>\mu} = W^{\ge \mu + \varepsilon, C_\varepsilon} = L^{>\mu} \ . \tag{8.8}$$

Proof: We will check the three hypothesis (a),(b) and (c) of Theorem 3.14 to be satisfied by the space of $W^{r,p}$ sections $\Gamma = \Gamma_{W^{r,p}}(E)$:

(a) Since M_ω is $W^{s,q}$, if $v \in \Gamma_{W^{r,p}}(E)$ then $\mathcal{M}_\omega v \in \Gamma_{W^{r,p}}(E)$. Moreover, the properties of the multiplication operator and the composition operator (in this case with t_ω), make $\mathcal{M}_\omega$ continuous in $\Gamma_{W^{r,p}}(E)$ (see Lemma 1.10).

(b) For any $v \in \Gamma_{W^{r,p}}(E)$ and $\theta_0 \in \mathbb{T}^d$, obviously $|v_{\theta_0}| \leq \|v\|_{W^{r,p}}$. Moreover, given any v_{θ_0} we can construct a C^r section v with $v(\theta_0) = v_{\theta_0}$, using bump functions (see Lemma 7.2), and that section is obviously $W^{r,p}$.

This is possibly the most delicate step in the proof, but it is overcome in Sobolev spaces by taking into account that, under the hypothesis (8.4), the sections are continuous and can be evaluated uniquely.

(c) The inclusion $\mathrm{Spec}_P(\mathcal{M}_\omega, \Gamma_B(E)) \subset \mathrm{Spec}(\mathcal{M}_\omega, \Gamma_{W^{r,p}}(E))$ follows from Theorem 8.2.

Remark 8.4. Theorem 8.3 works in the generality of transfer operators, not just for rotations.

The last result we consider for Sobolev spaces is about equality of spectra.

Theorem 8.5. *Let $M_\omega : E \to E$ be a $W^{s,q}$ vector bundle automorphism over a rotation ω, where s, q satisfy* (8.3). *Let r, p satisfying* (8.4). *Then:*

$$\mathrm{Spec}(\mathcal{M}_\omega, \Gamma_B) = \mathrm{Spec}(\mathcal{M}_\omega, \Gamma_{W^{r,p}}) \ . \tag{8.9}$$

Proof: The crucial fact is that we can appeal to the invariant section theorem for Sobolev spaces [**dlL01b**] to obtain that the spectral subbundles coming from a spectral gap on bounded sections are Sobolev. The rest of the proof follows the lines of the proof of Theorem 7.17. □

CHAPTER 9

Spectral theory for transfer operators over rotations in spaces of analytic sections

In this section we study transfer operators over rotations acting on analytic sections.

9.1. Spaces of analytic sections

A complex band of size $\zeta > 0$ of the d-dimensional torus $\mathbb{T}^d$ is

$$\mathbb{T}^d_\zeta = (\mathbb{T} \times [-\zeta, \zeta])^d = \{\theta + \mathbf{i}\varphi \mid \theta \in \mathbb{T}^d \ , \ \varphi \in [-\zeta, \zeta])^d\} \ .$$

The space of analytic functions in this complex torus is

$$A_\zeta = \{f : \mathbb{T}^d_\zeta \to \mathbb{C} \text{ continuous, and analytic in the interior of } \mathbb{T}^d_\zeta\} \ .$$

We also consider the spaces of bounded functions and continuous functions

$$B_\zeta = \{f : \mathbb{T}^d_\zeta \to \mathbb{C} \text{ bounded}\} \ , \ C^0_\zeta = \{f : \mathbb{T}^d_\zeta \to \mathbb{C} \text{ continuous}\} \ .$$

These three spaces are endowed with the sup-norm

$$\|\varphi\|_{\infty,\zeta} = \sup_{z \in \mathbb{T}^d_\zeta} |\varphi(z)| \ .$$

If v is analytic, then the open mapping theorem implies

$$\|\varphi\|_{\infty,\zeta} = \sup_{z \in \partial\mathbb{T}^d_\zeta} |\varphi(z)| \ .$$

Notice also that $A_\zeta \subset C^0_\zeta \subset B_\zeta$, and the inclusions are closed.

9.2. Spectral theorems in spaces of analytic sections

The first result gives the regularity of the spectral subbundles in the analytic case.

THEOREM 9.1. *Let $M_\omega : E \to E$ be an analytic vector bundle automorphism over a rotation ω. Assume that*

$$\mathrm{Spec}(\mathcal{M}_\omega, \Gamma_B(E)) \cap \mathcal{A}_{\lambda,\mu} = \emptyset \ .$$

Then:

(a) *The continuous splitting $E = E^{<\lambda} \oplus E^{>\mu}$ produced in Theorem 2.18 is analytic.*
(b) *For $\zeta > 0$ small enough*

$$\mathrm{Spec}(\mathcal{M}_\omega, \Gamma_{A_\zeta}(E)) \cap \mathcal{A}_{\lambda,\mu} = \emptyset \ .$$

Proof: The existence of a gap in the spectrum on bounded sections implies the existence of a continuous invariant splitting $E = E^{<\lambda} \oplus E^{>\lambda}$, characterized by rates of growth.

The representation of the vector bundle map M_ω with respect to the invariant splitting is

$$M_\omega(\theta) = \begin{pmatrix} A(\theta) & 0 \\ 0 & D(\theta) \end{pmatrix} ,$$

where we write $A = M^{<\lambda}$ and $D = M^{>\mu}$.

For $\varepsilon > 0$ small enough, we construct an adapted metric such that

$$\|A\|_\infty \leq \lambda - \varepsilon \ , \|D^{-1}\|_\infty \leq (\mu + \varepsilon)^{-1} \ ,$$

where the supremum is taken on the (real) torus $\mathbb{T}^d$.

By smoothing the subbundles, we can obtain analytic subbundles $\bar{E}^{<\lambda}$ and $\bar{E}^{>\mu}$ arbitrarily close to $E^{<\lambda}$ and $E^{>\mu}$, respectively. Notice that these are not necessarily invariant under M. The representation of M with respect to the analytic splitting $E = \bar{E}^{<\lambda} \oplus \bar{E}^{>\mu}$ is

$$M(\theta) = \begin{pmatrix} \bar{A}(\theta) & \bar{B}(\theta) \\ \bar{C}(\theta) & \bar{D}(\theta) \end{pmatrix} .$$

Smoothing the adapted metric, and choosing a small enough $\zeta > 0$, we get

$$\|\bar{A}\|_{\infty,\zeta} \leq \lambda - \frac{\varepsilon}{2} \ , \ \|\bar{D}^{-1}\|_{\infty,\zeta} \leq (\mu + \frac{\varepsilon}{2})^{-1}$$

and

$$\|\bar{B}\|_{\infty,\zeta} \ , \ \|\bar{C}\|_{\infty,\zeta}$$

are arbitrarily small.

In the space of bundle maps $W_\theta : \bar{E}^{<\lambda}_\theta \to E^{>\mu}_\theta$, analytic in $\mathbb{T}^d_\zeta$, we consider the fixed point equation

$$W_\theta = \bar{D}(\theta)^{-1}(W_{\theta+\omega}(\bar{A}(\theta) + \bar{B}(\theta)W_\theta) - \bar{C}(\theta)$$

that has an unique solution by the fixed point theorem. The graph of W_θ defines an analytic invariant bundle $E^{<\lambda,\zeta}$. A similar construction defines $E^{>\mu,\zeta}$. These analytic subbundles defined in $\mathbb{T}^d_\zeta$, $E^{<\lambda,\zeta}$ and $E^{>\mu,\zeta}$ are characterized by rates of growth, so then they coincide with $E^{<\lambda}$ and $E^{>\mu}$, respectively, when evaluated over the real torus.

From this, we conclude the proof of the Theorem. □

We have seen that gaps in the spectrum on bounded sections produces gaps in the spectrum on analytic sections. We will see now that these gaps can be smaller, and their size is decreased in a length of order ζ, the size of the analyticity band. To do so, we have to analyze the boundary of the spectrum on analytic sections.

An observation is that a vector bundle map M_ω over a real rotation $\omega \in \mathbb{R}^d$ in the complex torus $\mathbb{T}^d_\zeta$ (given by $\theta + \mathbf{i}\varphi \to (\theta+\omega) + \mathbf{i}\varphi$), induces a family of vector bundle maps $M_{\omega,\varphi}$ on the real torus $\mathbb{T}^d$ by $M_{\omega,\varphi}(\theta) = M_\omega(\theta + \mathbf{i}\varphi)$.

With these preliminaries, it is now very easy to use inclusion arguments:

$$\partial\mathrm{Spec}(\mathcal{M}_\omega, \Gamma_{A_\zeta}(E)) \subset \mathrm{Spec}_W(\mathcal{M}_\omega, \Gamma_{A_\zeta}(E)) \tag{9.1}$$

$$\subset \mathrm{Spec}_W(\mathcal{M}_\omega, \Gamma_{b_\zeta}(E)) \tag{9.2}$$

$$= \mathrm{Spec}_P(\mathcal{M}_\omega, \Gamma_{b_\zeta}(E)) \tag{9.3}$$

$$= \bigcup_{\varphi\in[\zeta,\zeta]^d} \mathrm{Spec}_P(\mathcal{M}_{\omega,\varphi}, \Gamma_B(E)) \tag{9.4}$$

$$= \bigcup_{\varphi\in[\zeta,\zeta]^d} \mathrm{Spec}(\mathcal{M}_{\omega,\varphi}, \Gamma_B(E)) \tag{9.5}$$

$$\subset \mathrm{Spec}(\mathcal{M}_{\omega,0}, \Gamma_B(E)) + O(\zeta)\ . \tag{9.6}$$

The notation $A \subset B + O(\zeta)$ means that there exist a constant $C > 0$ such that for any $a \in A$ there exists $b \in B$ with $d(a, b) \le C\zeta$.

The inclusion (9.1) is a general result in spectral theory (see Proposition A.26). The inclusion (9.2) follows because A_ζ is a closed subspace of B_ζ. The equality (9.3) follows from Thychonov theorem (cf. Theorem 3.19). The equality (9.4) needs a moment of reflection, but it is easy and shows the usefulness of going down to bounded sections. The equality (9.5) corresponds to Theorem 3.19, that uses that rotations are chain recurrent. Finally, the last inclusion (9.6) follows from standard perturbation theory of spectrum, since the distance between $M_{\omega,0}$ and $M_{\omega,\varphi}$ is $O(\zeta)$. We have then the following Theorem.

THEOREM 9.2. *Let $\mathcal{M}_\omega$ be an analytic transfer operator over a rotation. Then, for $\zeta > 0$ small enough:*

$$\partial\mathrm{Spec}(\mathcal{M}_\omega, \Gamma_{A_\zeta}(E)) \subset \mathrm{Spec}(\mathcal{M}_\omega, \Gamma_B(E)) + O(\zeta)\ . \tag{9.7}$$

REMARK 9.3. Notice that for an analytic approximate eigensection v_ε of z in the band of size ζ, we have $\|v_\varepsilon\|_{A_\zeta} = 1$ and $\|\mathcal{M}_\omega v_\varepsilon - z v_\varepsilon\|_{A_\zeta} < \varepsilon$. From the open mapping theorem, the maximum is taken in the boundary, so

$$\max_{\theta\in\mathbb{T}^d\ ,|\varphi|_\infty\le\zeta} |v_\varepsilon(\theta + \mathbf{i}\varphi)| = \max_{\theta\in\mathbb{T}^d\ ,|\varphi|_\infty=\zeta} |v_\varepsilon(\theta + \mathbf{i}\varphi)|\ ,$$

and then we obtain a continuous approximate eigensection of z for some of the $M_{\omega,\varphi}$ with $|\varphi|_\infty = \zeta$ (notice that the boundary $\{\theta + \mathbf{i}\varphi \mid |\varphi|_\infty = \zeta\}$ is compact). As a result,

$$\mathrm{Spec}_W(\mathcal{M}_\omega, \Gamma_{A_\zeta}(E)) \subset \bigcup_{|\varphi|_\infty=\zeta} \mathrm{Spec}_W(\mathcal{M}_{\omega,\varphi}, \Gamma_B(E))\ .$$

This estimate is sharper than the produced with (9.2),(9.3),(9.4),(9.5).

REMARK 9.4. In this section we have proved that gaps in the spectrum on bounded sections produce gaps in the spectrum on analytic sections, and that these gaps can be a little bit smaller. Compared with the results we have obtained for the spectrum on other spaces, such as the equality of the spectrum with that on bounded sections, or the fact that the complete spectrum is Weyl spectrum, the results in the section are not totally complete.

We have not proved that gaps of the spectrum on analytic sections produce analytic invariant bundles! The reason is that some of the arguments we have used to prove the corresponding theorems for continuous, C^r and Sobolev sections involve the construction of bump functions, that are not available for analytic functions. We have not been able to find an alternative argument. This is something that needs a clarification.

On the other hand, we have shown that the existence of splittings, implies the existence of a spectral gap.

CHAPTER 10

Reducibility and almost reducibility

In the classification of vector bundle automorphisms a natural question is what are the conjugation classes, and what are the simplest conjugation classes. That is to say, given two vector bundle automorphisms $M_f, N_f : E \to E$ over the same homeomorphism f, we say that they are conjugate if there exists a vector bundle automorphism $P : E \to E$ over the identity such that $P(f(\theta))^{-1}M(\theta)P(\theta) = N(\theta)$. We also say that M_f is reducible to N_f. It is obvious that the spectra of the corresponding transfer operators coincide.

The goal of this section is to study the spectral implications of being M_f reducible to a simple N_f. As we will see, these implications are the same if M_f is almost reducible to N_f (see Definition 10.2 and Theorem 10.4).

10.1. Some abstract results

In this section we consider the previous circle of ideas in an abstract level.

DEFINITION 10.1. *Let $A, B : X \to X$ two bounded linear operators in a Banach space X. We say that A, B are conjugate iff there exists a continuous isomorphism $P : X \to X$ such that*

$$P^{-1}AP = B \ . \tag{10.1}$$

We will say also that A is reducible to B.

It is obvious from the definition that $\mathrm{Spec}(A, X) = \mathrm{Spec}(B, X)$. A similar notion, that will have the same spectral implications, is the following.

DEFINITION 10.2. *Let $A, B : X \to X$ two bounded linear operators in a Banach space X. We will say that A, B are almost conjugate iff for all $\varepsilon > 0$ there exists a continuous isomorphism $P_\varepsilon : X \to X$ such that*

$$\|P_\varepsilon^{-1}AP_\varepsilon - B\| \le \varepsilon \ . \tag{10.2}$$

Almost conjugation defines an equivalence relation in the Banach space of bounded linear operators in X.

REMARK 10.3. A simple but illuminating example is the following. Consider the 2×2 matrices

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \ , \ B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ .$$

For all $\varepsilon > 0$, the matrices

$$P_\varepsilon = \begin{pmatrix} \varepsilon^{-1} & 0 \\ 0 & 1 \end{pmatrix}$$

define an almost conjugation between A, B, because

$$\begin{pmatrix} \varepsilon^{-1} & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \varepsilon^{-1} & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & \varepsilon \\ 0 & 0 \end{pmatrix} .$$

In general, the representative elements of the almost conjugation equivalence relation in the set of finite dimensional matrices are diagonal matrices. In particular, a $n \times n$ matrix with complex entries is almost conjugate to a diagonal matrix $\mathrm{diag}(\lambda_1 \mathrm{Id}_{n_1}, \lambda_2 \mathrm{Id}_{n_2}, \ldots, \lambda_k \mathrm{Id}_{n_k})$, where the λ_i's are the eigenvalues, whose (algebraic) multiplicities are the n_i's.

This is the main result of this section.

Theorem 10.4. *Let $A, B : X \to X$ two bounded linear operators in a Banach space X. Then:*

$$A, B \text{ are almost conjugate} \Rightarrow \mathrm{Spec}(A, X) = \mathrm{Spec}(B, X) .$$

Proof: Let $\lambda \in \mathrm{Res}(B, X)$. We will show that $\lambda \in \mathrm{Res}(A, X)$, that is to say for any given $y \in X$ there exists an unique $x \in X$ solving the equation $(A - \lambda\mathrm{Id})x = y$.

For $\varepsilon > 0$ small enough (we will fix its size later), let P_ε be an almost conjugation between A, B, that is to say $R_\varepsilon = P_\varepsilon^{-1} A P_\varepsilon - B$ satisfies $\|R_\varepsilon\| < \varepsilon$. Then, from

$$\begin{aligned} y &= (A - \lambda\mathrm{Id})x \\ &= P_\varepsilon (P_\varepsilon^{-1} A P_\varepsilon - \lambda\mathrm{Id}) P_\varepsilon^{-1} x \\ &= P_\varepsilon (B - \lambda\mathrm{Id} + R_\varepsilon) P_\varepsilon^{-1} x \\ &= P_\varepsilon (B - \lambda\mathrm{Id})(\mathrm{Id} + (B - \lambda\mathrm{Id})^{-1} R_\varepsilon) P_\varepsilon^{-1} x , \end{aligned}$$

where we use that $B - \lambda\mathrm{Id}$ is invertible, we obtain

$$x = P_\varepsilon (\mathrm{Id} + (B - \lambda\mathrm{Id})^{-1} R_\varepsilon)^{-1} (B - \lambda\mathrm{Id})^{-1} P_\varepsilon^{-1} y ,$$

where we choose ε small enough such that $\|(B - \lambda\mathrm{Id})^{-1} R_\varepsilon\| < 1$. □

Remark 10.5. An equivalent notion of almost reducibility is considering also approximate inverses. That is to say, A, B are almost conjugate if for all $\varepsilon > 0$ there exist continuous isomorphisms $P_\varepsilon, Q_\varepsilon : X \to X$ such that

$$\|Q_\varepsilon A P_\varepsilon - B\| \le \varepsilon , \|Q_\varepsilon P_\varepsilon - \mathrm{Id}\| \le \varepsilon .$$

Remark 10.6. The equality of the Weyl spectrum of two almost conjugate operators A, B does not follow from Definition 10.2. We would control the condition number of the approximate conjugacies $\mathrm{cond}(P_\varepsilon) = \|P_\varepsilon\| \|P_\varepsilon^{-1}\|$.

REMARK 10.7. The equality of the Weyl spectrum of two almost conjugate transfer operators over rotations, (or in general over APD and chain recurrent homeomorphisms) is a consequence of the non existence of spectrum that is not Weyl spectrum.

10.2. Reducibility to constant coefficients

If the bundle E is trivial, $E = \mathcal{P} \times \mathbb{C}^n$, a vector bundle automorphism M_f is identified with a couple (f, M), where $f : \mathcal{P} \to \mathcal{P}$ is a homeomorphism and $M : \mathcal{P} \to \mathrm{GL}_n(\mathbb{C})$ is a continuous matrix valued map, giving rise to the linear skew product

$$\begin{aligned} \bar{x} &= M(\theta)x \ , \\ \bar{\theta} &= f(\theta) \ . \end{aligned}$$

The simplest conjugation classes are those whose representatives are constant type linear skew products, that is to say, $M(\theta) = M_0$ for a constant matrix $M_0 \in \mathrm{GL}_n(\mathbb{C})$.

DEFINITION 10.8. *Let M_f a linear skew product. We say that it is C^0* reducible *when we can write*

$$P(f(\theta))^{-1} M(\theta) P(\theta) = \Lambda$$

where Λ is a constant matrix and $P : \mathcal{P} \to \mathrm{GL}_n(\mathbb{C})$ is C^0.

REMARK 10.9. The geometric meaning of P is that it is an isomorphism on each fiber E_θ in such a way that it reduces the linear skew product to constant.

REMARK 10.10. Analogous definitions of reducibility can be set for higher regularities (C^r, Sobolev, analytic, etc.), but all of them imply C^0 reducibility. Moreover, C^0 reducibility has implications on the spectrum on continuous sections, but as a result of this paper the same implications also work in the other spaces (for cocycles over rotations).

One motivation for the study of reducibility in the context of transfer operators over rotations is the theory of small divisors, since it allows to solve easily equations for sections $\Delta : \mathbb{T}^d \to \mathbb{C}^n$, like

$$M(\theta - \omega)\Delta(\theta - \omega) - \Delta(\theta) = R(\theta), \tag{10.3}$$

given a section $R : \mathbb{T}^d \to \mathbb{C}^n$. These equations appear in KAM theory as well as in the theory of computation of invariant manifolds [**HdlLb, HdlL04**]. If M_ω is reducible, we can rewrite (10.3) as

$$P(\theta)\Lambda P(\theta - \omega)^{-1}\Delta(\theta - \omega) - \Delta(\theta) = R(\theta) \ ,$$

which is equivalent to

$$\Lambda\hat{\Delta}(\theta - \omega) - \hat{\Delta}(\theta) = \hat{R}(\theta) \ ,$$

where $\hat{R}(\theta) = P(\theta)^{-1}R(\theta)$ and the unknown is $\hat{\Delta}(\theta) = P(\theta)^{-1}\Delta(\theta)$. We see that $\hat{\Delta}$ satisfies an equation involving translations and multiplication by constant coefficients. Hence, it can be analyzed using Fourier coefficients

$$e^{2\pi \mathbf{i} k\omega}\Lambda\hat{\delta}_k - \hat{\delta}_k = \hat{r}_k \ .$$

With this motivation it has been shown that reducibility happens frequently when $M(\theta)$ is a small perturbation of a constant matrix and ω is Diophantine [**JS92, Eli98**].

REMARK 10.11. Reducibility is a very strong property that allows us to solve equations (10.3) even in the case that 1 is in the spectrum of $\mathcal{M}_\omega$. If Λ has eigenvalues in the unit circle, but they satisfy Diophantine conditions, then (10.3) (or similar) can be solved, but the solution Δ is not in the same space as the known term R. At it happens in small divisor problems, we loose some degree of smoothness. This appears all the time in KAM theory. See [**dlL01b**].

REMARK 10.12. In the theory of reducibility of real cocycles over rotations is usual considering covering tori $(2\mathbb{T}^d) = \mathbb{R}^d/(2\mathbb{Z})^d$ of the base torus $\mathbb{T}^d$, if one does not want to complexify the system. (This is similar to the well known Floquet theory). In Part 2 was proved that lifting a cocycle to a covering bundle does not change the spectrum.

It is obvious from the definition that reducibility is closely related with the existence of point spectrum in C^0.

PROPOSITION 10.13. *Let M_ω be a C^r cocycle over an ergodic rotation ω. Suppose that it is C^0 reducible to the constant cocycle Λ_ω. Then:*

$$\text{Spec}(\mathcal{M}_\omega, \Gamma_{C^0}(E)) = \{ze^{\alpha \mathbf{i}} \mid z \in \text{Eig}(\Lambda) \ , \alpha \in \mathbb{R}\} \ . \tag{10.4}$$

Moreover, $\text{Spec}_P(\mathcal{M}_\omega, \Gamma_{C^0}(E))$ *is dense in* $\text{Spec}(\mathcal{M}_\omega, \Gamma_{C^0}(E))$.

Proof: We have to compute the spectrum on C^0 of the transfer operator Ł_ω associated to the constant vector bundle automorphism Λ_ω. If λ is an eigenvalue of Λ, whose eigenvector is v_0, then, for each $k \in \mathbb{Z}^d$, the section

$$v_k(\theta) = e^{2\pi \mathbf{i} k\cdot\theta} v_0$$

is an eigensection whose eigenvalue is $z = e^{-2\pi \mathbf{i} k\cdot\omega}\lambda$. Then, we produce a set of eigenvalues filling densely a circle of radius $|\lambda|$. This proves one inclusion of the theorem.

To prove the other inclusion, we will see that if $z \in \mathbb{C}$ satisfies $|z| \neq |\lambda|$ for all $\lambda \in \text{Eig}(\Lambda)$, then z is in the resolvent set $\text{Res}(\text{Ł}_\omega, \Gamma_{C^0})$. We assume that Λ is a Jordan normal form. Given a continuous section $\eta : \mathbb{T}^d \to \mathbb{C}^n$, we have to compute a continuous section $v : \mathbb{T}^d \to \mathbb{C}^n$ such that $\Lambda v(\theta - \omega) - zv(\theta) =$

$\eta(\theta)$. For each Jordan block we have to solve an equation like

$$\begin{pmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & & 1 \\ 0 & \dots & & & \lambda \end{pmatrix} \begin{pmatrix} v^1(\theta-\omega) \\ v^2(\theta-\omega) \\ \vdots \\ \\ v^m(\theta-\omega) \end{pmatrix} - z \begin{pmatrix} v^1(\theta) \\ v^2(\theta) \\ \vdots \\ \\ v^m(\theta) \end{pmatrix} = \begin{pmatrix} \eta^1(\theta) \\ \eta^2(\theta) \\ \vdots \\ \\ \eta^m(\theta) \end{pmatrix}.$$

Since $|z| \neq |\lambda|$, this equation has a continuous solution. □

REMARK 10.14. Notice that C^0 reducibility implies that $\mathrm{Spec}(\mathcal{M}_\omega, \Gamma_{C^0})$ consists of a finite number of circles (the *spectral circles*):

$$\mathrm{Spec}(\mathcal{M}_\omega, \Gamma_{C^0}(E)) = \bigcup_{i=1}^{k} \mathcal{S}_{\rho_i} \ ,$$

where the ρ_i's are the moduli of the eigenvalues of Λ. Of course, since we have equality of spectra for all C^r (and for bounded sections), we obtain the same equality (10.4) for $\mathrm{Spec}(\mathcal{M}_\omega, \Gamma_{C^r})$ (that, of course, does not mean C^r reducibility).

We obtain then a C^r invariant splitting $E = E^1 \oplus \cdots \oplus E^k$. The multiplicity of each spectral circle $\mathcal{S}_{\rho_i}$ (the rank of the corresponding bundle) is the sum of the (algebraic) multiplicities of the eigenvalues of Λ with modulus ρ_i. Notice also that these C^r spectral subbundles contain C^0 invariant subbundles corresponding to the eigenvalues.

REMARK 10.15. In the terminology of Sacker and Sell [**SS75**], a reducible cocycle has pure point spectrum, that is to say, the spectrum is a set of circles. If the number of circles is the rank of the bundle, then it is said that the cocycle satisfies the full spectrum property.

REMARK 10.16. We also point out that there are topological obstructions to reducibility which depend only on the topology of the embedding $M : \mathbb{T}^d \to \mathrm{GL}_n(\mathbb{C})$. We claim that if M_ω is reducible then M retracts to the identity. Indeed, if Λ_t is an homotopy of invertible matrices from $\Lambda_0 = \Lambda$ and $\Lambda_1 = \mathrm{Id}$, then $M_t(\theta) = P(\theta + (1-t)\omega)\Lambda_t P(\theta)^{-1}$ defines a homotopy between $M_0(\theta) = M(\theta)$ and $M_1(\theta) = \mathrm{Id}$. Notice also that this topological obstruction to reducibility holds in an open set.

It is not difficult to produce cocycles $M_\omega : \mathbb{T} \to \mathrm{GL}_n(\mathbb{C})$ which are not retractable to the identity since they have positive index around the zero matrix. These cycles can not be reducible. A family of examples is

$$M_{\omega,\lambda}(\theta) = \begin{pmatrix} \cos(2\pi\theta) & -\sin(2\pi\theta) \\ \sin(2\pi\theta) & \cos(2\pi\theta) \end{pmatrix} \cdot \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \tag{10.5}$$

where $\lambda \geq 1$, and $\omega \in \mathbb{R}$. Herman [**Her83**] minorated the maximal Lyapunov multiplier by $\frac{\lambda}{2} + \frac{2}{\lambda}$, incidentally proving that they have not a hyperbolic structure. In the language of spectral theory, the spectrum in the case

$\lambda > 1$ is a "fat" annulus containing the unit circle, while in the case $\lambda = 1$ it is the unit circle.

Young [**You97**] proved that if the rotation number ω satisfies the Brjuno condition then, $M_{\omega,\lambda}$ is non uniformly hyperbolic and has Lyapunov multipliers close to λ, provided that λ is large enough.

REMARK 10.17. Herman [**Her83**] produced examples of homotopically trivial vector bundle automorphisms over rotations on $\mathbb{R}^2$ (and preserving the standard symplectic structure) with "fat" spectrum.

Other obstructions to reducibility that happen in open sets of maps are considered in [**Eli02**].

On the other hand, results on the abundance of reducibility on cocycles taking values on compact groups can be found in [**Kri99b, Kri99a**]. Relations of reducibility with renormalization group have been studied in [**Ryc92, KLD05**].

REMARK 10.18. Numerical evidence of non uniform hyperbolicity (i.e. "fat" spectrum) appears also in [**BS98, HdlL05a, HdlLa**].

In [**BS98**] there is an exhaustive numerical investigation of the linear system given Hill's equation with quasi-periodic forcing.

In [**HdlL05a, HdlLa**], the study is based on continuation of invariant tori in quasi-periodically forced systems (either conservative such as the quasi-periodic standard map or dissipative such as the quasi-periodic Hénon map). The cocycles studied in [**HdlL05a, HdlLa**] are not explicit, but rather appear as the linearized dynamics around an invariant torus. In [**HdlL05a, HdlLa**] it is argued that the spectral properties of these cocycles are closely related to the breakdown of the exponential dichotomies (or the normal hyperbolicity) of the invariant torus.

In [**HdlLa**] it is observed that the appearance of fat spectrum is accompanied by very remarkable universal properties. We believe that this is something that deserves a more detailed mathematical study.

Another important concept related to this circle of ideas is almost or quasi reducibility. This concept was introduced by Bylov [**Byl62**] for linear differential equations.

DEFINITION 10.19. *Let M_f a linear skew product over $f : \mathcal{P} \to \mathcal{P}$, in the trivial bundle $E = \mathcal{P} \times \mathbb{C}^n$. We say that it is C^0* almost reducible *if it is almost reducible to a a constant matrix Λ. That is to say, for all $\varepsilon > 0$ there exists an isomorphism (over the identity) P_ε such that*

$$\|P_\varepsilon(f(\theta))^{-1}M(\theta)P_\varepsilon(\theta) - \Lambda\|_{C^0} \leq \varepsilon \ .$$

REMARK 10.20. The following example clarifies the definition. Consider a 1-dimensional cocycle $R_{\omega,\alpha}(\theta) = e^{2\pi \mathbf{i}\alpha}$ in $\mathbb{T}^d \times \mathbb{C}$ over an ergodic rotation $\omega \in \mathbb{R}^d$, and $\alpha \in \mathbb{R}$. We will see that it is almost reducible to $R_{\omega,0}$. To do so, consider the bundle transformations $P_k(\theta) = e^{2\pi \mathbf{i}k\cdot\theta}$, where $k \in \mathbb{Z}^d$. Then,

$$P_k(\theta+\omega)^{-1}R_{\omega,\alpha}(\theta)P_k(\theta) = e^{-(2\pi \mathbf{i}k\cdot(\theta+\omega))}e^{2\pi \mathbf{i}\alpha}e^{2\pi \mathbf{i}k\cdot\theta} = e^{2\pi \mathbf{i}(\alpha - k\cdot\omega)} \ .$$

Since ω is ergodic, we can make $k \cdot \omega$ as close as we want to α (in $\mathbb{T}$), and so $P_k(\theta+\omega)^{-1}R_{\omega,\alpha}(\theta)P_k(\theta)$ and $R_{\omega,0}$ can be arbitrarily close.

Theorem 10.4 in Section 10.1 asserts that almost reducibility has the same spectral implications of reducibility, that is to say that M_f has pure point spectrum (the spectrum of $\mathcal{M}_f$ is a set of circles whose radii are the moduli of the eigenvalues of Λ).

REMARK 10.21. If M_ω is almost reducible to a constant cocycle Λ_ω, then

$$\mathrm{Spec}(\mathcal{M}_\omega, \Gamma_{C^0}(E)) = \bigcup_{i=1}^{k} \mathcal{S}_{\rho_i} \ ,$$

where the ρ_i's are the moduli of the eigenvalues of Λ. Let n_i the rank of the spectral bundle E_i associated to $\mathcal{S}_{\rho_i}$. As a result of the Remarks 10.3 and 10.20 we can almost reduce the system to the diagonal cocycle Δ_ω, where $\Delta = \mathrm{diag}(\rho_1 \mathrm{Id}_{n_1}, \dots, \rho_k \mathrm{Id}_{n_k})$.

In [**Cop77, Wan90**] there are spectral conditions for almost reducibility for the case of non autonomous linear differential equations: these correspond to the pure point spectrum property. The results belong to the continuous category (and work for time-continuous systems). For results on almost reducibility in the analytic category, see [**Eli01**] and the references therein.

REMARK 10.22. An almost reducible cocycle over a rotation can not be non homotopically trivial. We repeat the arguments of Remark 10.16. If $P_\varepsilon(\theta+\omega)^{-1}M(\theta)P_\varepsilon(\theta) = \Lambda + R_\varepsilon(\theta)$, we write

$$M(\theta) = P_\varepsilon(\theta+\omega)\Lambda(\mathrm{Id} + \Lambda^{-1}R_\varepsilon(\theta))P_\varepsilon(\theta)^{-1} \ ,$$

and we connect it with the identity via

$$M_t(\theta) = P_\varepsilon(\theta+(1-t)\omega)\Lambda_t(\mathrm{Id} + (1-t)\Lambda^{-1}R_\varepsilon(\theta))P_\varepsilon(\theta)^{-1} \ ,$$

provided that $\|R_\varepsilon\| \le \varepsilon$ is small enough.

So, the cocycle (10.5) for $\lambda = 1$ is not almost reducible, although the spectrum is concentrated in the unit circle. This is an example of cocycle with pure point spectrum that is not almost reducible. This makes a difference between the theory of almost reducibility of non autonomous linear differential equations [**Cop77, Wan90**] and the discrete version.

REMARK 10.23. For numerical applications of both concepts of reducibility and almost reducibility to the computation of invariant tori, see [**HdlL04**].

10.3. Reducibility to triangular systems

Another important class of cocycles are those that have triangular form, that is $M : E \to E$ over $f : \mathcal{P} \to \mathcal{P}$ of the form

$$M(\theta) = \begin{pmatrix} a_1(\theta) & b_{12}(\theta) & \dots & b_{1n}(\theta) \\ 0 & a_2(\theta) & \dots & b_{2n}(\theta) \\ \vdots & 0 & \ddots & \vdots \\ 0 & \dots & & a_n(\theta) \end{pmatrix} . \tag{10.6}$$

If f is uniquely ergodic then

$$\mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E)) = \bigcup_{i=1}^{n} \mathcal{S}_{\rho_i}$$

where

$$\rho_i = \exp \int_{\mathcal{P}} \log |a_i(\theta)| \, d\mu \ .$$

See Proposition 3.47.

Definition 10.24. *We say that M_f is triangular if it is reducible to a triangular form like in* (10.6). *We say that M_f is almost triangular if it is almost reducible to a triangular form like in* (10.6).

As a consequence of Theorem 10.4, the (almost) triangular transfer operators over a uniquely ergodic homeomorphism have pure point spectrum.

The notion of (almost) triangular vector bundle automorphisms extend that of (almost) reducibility to constant coefficients. Notice that in this case the bundle E does not need to be trivial. Moreover, the natural condition on the dynamics on the base manifold is that of unique ergodicity (in the case of rotations, these are ergodic), while in the theory of reducibility to constant coefficients the rotations have to satisfy Diophantine conditions.

Part 4

Interactions with Geometry

In this part we study the interactions of the spectrum with the geometry.

In a first chapter we consider multilinear transfer operators, acting on tensor fields, that appear, when constructing invariant manifolds.

In a second part we study the consequences on the spectrum of the preservation of a geometrical structure (such as symplectic or volume, that appear in Hamiltonian mechanics and fluid mechanics, respectively) or a symmetry (if the cocycle is reversible).

In a third part we study the spectral consquences when one considers transfer operators acting on a space of functions that satisfy some differential constraint (e.g. vector fields that are zero divergence, symplectic or forms which are closed or exact). We show that in all those cases, we obtain a non-gaps phenomenon.

CHAPTER 11

Multilinear transfer operators

In this section we analyze the spectrum of transfer operators acting on tensor bundles. We consider purely contravariant tensors and purely covariant tensors. In particular, if E is a vector bundle over $\mathcal{P}$, and M_f is a vector bundle automorphism in E, we consider the induced vector bundle automorphisms on the bundles $T^\ell_0(E) \simeq L^\ell(E^*;\mathbb{C})$ and $T^0_\ell(E) \simeq L^\ell(E;\mathbb{C})$, that we will call push-forward vector bundle automorphisms and pull-back bundle automorphisms, respectively (see definitions below). A particular important case is the dual vector bundle automorphism. We also analyze the action on symmetric tensors and antisymmetric tensors.

We also study certain operators arising from the study of high order expansions of invariant manifolds attached to a compact invariant manifold, such as the stable manifolds and unstable manifolds. These studies play an important role in the computation of whiskers of invariant tori undertaken in [**HdlLb, HdlL04**]. In this case, the transfer operators act on multilinear sections. That is, the underlying vector bundle is $L^\ell(F;E)$, where E and F are vector bundles over $\mathcal{P}$.

The goal of this section is to investigate the spectrum of those transfer operators and their relation with the spectrum of the transfer operators acting on E-sections (and F-sections).

Of course, since tensor bundles and multilinear bundles have finite rank, all the results of this paper apply, in particular, the invariance of the spectrum under rotations in the APD case and the equality of the spectrum for different spaces.

11.1. Tensor bundles and multilinear bundles

Given a vector bundle E of rank n over a manifold $\mathcal{P}$, we will consider, for $l \in \mathbb{N}$:

- The ℓ-contravariant tensor bundle $T^\ell(E) \simeq L^\ell(E^*;\mathbb{C})$ and the ℓ-covariant tensor bundle $T_\ell(E) \simeq L^\ell(E;\mathbb{C})$, that have rank n^ℓ;
- The ℓ-contravariant symmetric tensor bundle $S^\ell(E) \simeq L^\ell_s(E^*;\mathbb{C})$ and the ℓ-covariant symmetric tensor bundle $S_\ell(E) \simeq L^\ell_s(E;\mathbb{C})$, that have rank $\binom{n+\ell-1}{\ell}$;

- The ℓ-contravariant alternate tensor bundle $\bigwedge^\ell(E) \simeq L_a^\ell(E^*;\mathbb{C})$ and the ℓ-covariant alternate tensor bundle $\bigwedge_\ell(E) \simeq L_a^\ell(E;\mathbb{C})$, that have rank $\binom{n}{\ell}$.

If E is Finslered, and we will suppose it is, we construct Finsler metrics on all of these vector bundle automorphisms, using the standard constructions of tensor norms fiberwise.

As usual, we will denote vectors with subindices and 1-forms with superindices.

The permutation group of order ℓ, $\mathfrak{S}_\ell$, acts on $T^\ell(E)$ fiberwise. That is, for $\sigma \in \mathfrak{S}_\ell$ and $w_\theta \in T^\ell(E_\theta)$, we define

$$\sigma w_\theta(v^1,\dots,v^\ell) = w_\theta(v^{\sigma(1)},\dots,v^{\sigma(\ell)}) \ ,$$

for all $v^1,\dots,v^\ell \in E_\theta^*$. The symmetrization map Sym and alternation map Alt are also defined fiberwise, by

$$\mathrm{Sym}(w_\theta) = \frac{1}{\ell!}\sum_{\sigma\in\mathfrak{S}_\ell}\sigma w_\theta \ , \ \mathrm{Alt}(w_\theta) = \frac{1}{\ell!}\sum_{\sigma\in\mathfrak{S}_\ell}\mathrm{sign}(\sigma)\cdot\sigma w_\theta \ ,$$

where $\mathrm{sign}(\sigma)$ denotes the signature of the permutation σ. Both are projections in $T^\ell(E)$. Obviously $S^\ell(E) = \mathrm{Sym}(T^\ell(E))$ and $\bigwedge^\ell(E) = \mathrm{Alt}(T^\ell(E))$. As usual, the tensor product is denoted by $\otimes$, the symmetric product is $\cdot$ and the alternate product is $\wedge$. Recall that for $e_1,\dots,e_\ell \in E_\theta \simeq E_\theta^{**} = L(E_\theta^*;\mathbb{C})$, $e_1\otimes\cdots\otimes e_\ell \in T^\ell(E)$ is defined by

$$e_1\otimes\cdots\otimes e_\ell\ (v^1,\dots,v^\ell) = \sum_{i,j} v^i(e_j) \ ,$$

where $v^1,\dots,v^\ell \in E_\theta^* = L(E_\theta;\mathbb{C})$. Recall also that

$$e_1\cdot\cdots\cdot e_\ell = \mathrm{Sym}(e_1\otimes\cdots\otimes e_\ell) \ , \ e_1\wedge\cdots\wedge e_\ell = \mathrm{Alt}(e_1\otimes\cdots\otimes e_\ell) \ .$$

Similar definitions and constructions work for covariant tensor bundles.

A vector bundle automorphism $M : E \to E$ over a homeomorphism f induces a push-forward vector bundle automorphism on $T^\ell(E)$ over f by

$$M(\theta)w_\theta(e^1,\dots,e^\ell) = w_\theta(e^1\circ M(\theta),\dots,e^\ell\circ M(\theta)) \ ,$$

for $e^1,\dots,e^\ell \in E_{f(\theta)}^*$, and a pull-back vector bundle automorphism on $T_\ell(E)$ over f^{-1} by

$$M(\theta)^* w_\theta^*(e_1,\dots,e_\ell) = w_\theta^*(M(f^{-1}(\theta))e_1,\dots,M(f^{-1}(\theta))e_\ell) \ ,$$

for $e_1,\dots,e_\ell \in E_{f^{-1}(\theta)}$. Notice that the symmetrization and the alternation maps commute with push-forward and pull-back vector bundles maps.

A notation that will be useful is the product of sets:

Definition 11.1. *Given sets $\Sigma_1,\Sigma_2,\dots,\Sigma_\ell \subset \mathbb{C}$, we denote*

$$\Sigma_1\cdot\Sigma_2\cdots\Sigma_\ell = \{z = z_1\cdot z_2\cdots z_\ell \in \mathbb{C} \mid z_1\in\Sigma_1,\dots,z_\ell\in\Sigma_\ell\} \ .$$

Given a set $\Sigma \subset \mathbb{C}$*, we also write*

$$\Sigma^{\ell} = \{z^{\ell} \mid z \in \Sigma\} \ ,$$

for $l \in \mathbb{Z}$*.*

Given a set $\Sigma \subset \mathbb{C}$*, a partition* $A = \{\mathcal{A}_i\}_{i=1}^{k}$ *of* Σ *in* k *subsets, and a sequence* $\nu = (n_1, \dots, n_k)$ *of natural numbers, we denote*

$$\Sigma^{\ell;A,\nu} = \{z = z_1^{s_1} \cdots z_k^{s_k} \mid z_i \in \mathcal{A}_i, \sum_{i=1}^{k} s_i = \ell, s_i \le n_i\} \subset \Sigma^{\ell} \ .$$

The motivation of this notation is given by the following elementary result of linear algebra, that we will extend to bundles.

Proposition 11.2. *Let* $M : E \to E$ *be an endomorphism in an* n*-dimensional complex vector space* E*. Let* $\mathrm{Eig}(M, E) = \{\lambda_1, \dots, \lambda_k\}$ *be the set of eigenvalues of* M *acting on* E*, the eigenvalue* λ_i *having multiplicity* n_i*. We consider the push-forward* M *on* $T^{\ell}(E)$ *and the pull-back* M^* *on* $T_{\ell}(E)$*. Then:*

a) *On* $T^{\ell}(E)$*,* $M(e_1 \otimes \cdots \otimes e_{\ell}) = Me_1 \otimes \cdots \otimes Me_{\ell}$*, and*

$$Eig(M, T^{\ell}(E)) = (\mathrm{Eig}(M, E))^{\ell} \ ,$$

and the multiplicity of $\lambda = \lambda_1^{s_1} \dots \lambda_k^{s_k}$ *is* $n_1^{s_1} \dots n_k^{s_k}$*;*

b) *On* $S^{\ell}(E)$*,* $M(e_1 \cdot \cdots \cdot e_{\ell}) = Me_1 \cdot \cdots \cdot Me_{\ell}$*, and*

$$Eig(M, S^{\ell}(E)) = (\mathrm{Eig}(M, E))^{\ell} \ ,$$

and the multiplicity of $\lambda = \lambda_1^{s_1} \dots \lambda_k^{s_k}$ *is* $\binom{n_1+s_1-1}{s_1} \cdots \binom{n_k+s_k-1}{s_k}$.

c) *On* $\bigwedge^{\ell}(E)$*,* $M(e_1 \wedge \cdots \wedge e_{\ell}) = Me_1 \wedge \cdots \wedge Me_{\ell}$*, and*

$$Eig(M, \textstyle\bigwedge^{\ell}(E)) = \{\lambda = \lambda_1^{s_1} \dots \lambda_k^{s_k} \mid \displaystyle\sum_{i=1}^{k} s_i = \ell, s_i \le n_i\} \ ,$$

and the multiplicity of $\lambda = \lambda_1^{s_1} \dots \lambda_k^{s_k}$ *is* $\binom{n_1}{s_1} \cdots \binom{n_k}{s_k}$.

Similar equalities work for the pull-back M^* *on* $T_{\ell}(E)$*,* $S_{\ell}(E)$ *and* $\bigwedge_{\ell}(E)$*.*

Remark 11.3. Notice that $Eig(M, \bigwedge^{\ell}(E)) = (\mathrm{Eig}(M, E))^{l;A,\nu}$, where $A = \{\{\lambda_i\}\}_{i=1}^{k}$ and $\nu = (n_1, \dots, n_k)$.

The following objects are very useful in the construction of invariant manifolds (whiskers) attached to a compact invariant manifold. The case of quasi-periodic systems is undertaken in [**HdlLb, HdlL04, HdlL05a**].

Given vector bundle automorphisms E, F over the same base manifold $\mathcal{P}$, we consider the bundle of multilinear maps $L^{\ell}(F; E)$, for which the fiber in θ is just the space of ℓ-multilinear maps from F_{θ} to E_{θ}. Given two vector bundle automorphisms $M_f : E \to E$ and $N_f : F \to F$ over the same homeomorphism $f : \mathcal{P} \to \mathcal{P}$, we construct a vector bundle automorphism $S_f : L^\ell(F;E) \to L^\ell(F;E)$, by

$$(S(\theta)W_\theta)(v_1,\ldots,v_\ell) = M(\theta)W_\theta(N(\theta)^{-1}v_1,\ldots,N(\theta)^{-1}v_\ell)\ ,$$

where $W_\theta \in L^\ell(F_\theta;E_\theta)$, and $v_1,\ldots,v_\ell \in F_{f(\theta)}$. We will refer to this action as the Sylvester vector bundle automorphism (this nomenclature comes from [**BK98, CFdlL03a, CFdlL03b**]).

In this section we will analyze the spectrum of push-forward, pull-back and Sylvester transfer operators.

11.2. Spectrum of push-forward and pull-back transfer operators

The results on the spectrum of transfer operators acting on contravariant and covariant tensor sections are summarized in the following theorem.

THEOREM 11.4. *Let $M_f : E \to E$ be vector bundle automorphism. Let*

$$\mathcal{A}\mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E)) = \bigcup_{i=1}^k \mathcal{A}_i$$

be the annular hull of the spectrum, where each spectral annulus $A_i = \mathcal{A}_{\lambda_i^-,\lambda_i^+}$ has multiplicity n_i. Let $A = \{A_i\}_{i=1}^k$ and $\nu = (n_1,\ldots,n_k)$. Then, the spectra of the push-forward and pull-back transfer operators satisfy:

a) $\mathcal{A}\mathrm{Spec}(\mathcal{M}_f, \Gamma_B(T^\ell(E))) = \mathcal{A}\mathrm{Spec}(\mathcal{M}_f^*, \Gamma_B(T_\ell(E))) \subset (\mathcal{A}\mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E)))^\ell$;

b) $\mathcal{A}\mathrm{Spec}(\mathcal{M}_f, \Gamma_B(S^\ell(E))) = \mathcal{A}\mathrm{Spec}(\mathcal{M}_f^*, \Gamma_B(S_\ell(E))) \subset (\mathcal{A}\mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E)))^\ell$;

c) $\mathcal{A}\mathrm{Spec}(\mathcal{M}_f, \Gamma_B(\bigwedge^\ell(E))) = \mathcal{A}\mathrm{Spec}(\mathcal{M}_f^*, \Gamma_B(\bigwedge_\ell(E))) \subset (\mathcal{A}\mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E)))^{\ell;A,\nu}$;

d) $\mathcal{A}\mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E)) = \mathcal{A}\mathrm{Spec}(\mathcal{M}_f^*, \Gamma_B(E^*))$.

REMARK 11.5. If f is APD, all the spectrum involved in Theorem 11.4 is rotationally invariant, and the $\mathcal{A}$ can drop.

REMARK 11.6. If f is chain-recurrent, the previous inclusions also work for Weyl spectrum. In fact, the inclusions in this case also work in general (see Section).

REMARK 11.7. The above result implies similar results for spaces of continuous and L^p tensor sections (under the assumption that the invariant measure is topological). The proofs are easier on spaces of bounded tensor sections.

Proof: Let

$$E = \bigoplus_{i=1}^k E^i\ . \tag{11.1}$$

be the corresponding spectral splitting. The rank of the each subbundle E^i is n_i, and it is characterized, according to Theorem 2.18, by the rates of growth under iteration:

$$v_\theta \in E^i_\theta \iff \forall m \geq 0 \begin{cases} |M(\theta,m)v_\theta| \leq C_\varepsilon(\lambda_i^+ + \varepsilon)^m |v_\theta| \ , \\ |M(\theta,-m)v_\theta| \leq C_\varepsilon(\lambda_i^- - \varepsilon)^{-m} |v_\theta| \ , \end{cases} \tag{11.2}$$

where $\varepsilon > 0$ is small enough.

The decomposition (11.1) induces another splitting in $T^\ell(E)$

$$\bigotimes_{i=1}^{\ell} E = \bigoplus_{i_1,\cdots,i_\ell \in \{1,\cdots,k\}} E^{i_1} \otimes \cdots \otimes E^{i_\ell} \ . \tag{11.3}$$

The splitting (11.3) is clearly invariant under the push-forward $\mathcal{M}_f$. It is also very easy to see that if $w_\theta = w_1 \otimes \cdots \otimes w_\ell \in E^{i_1}_\theta \otimes \cdots \otimes E^{i_\ell}_\theta$, then for all $m \geq 0$:

$$\begin{aligned} |M(\theta,m)w_\theta| &\leq C^\ell_\varepsilon(\lambda^+_{i_1} + \varepsilon)^m \ldots (\lambda^+_{i_\ell} + \varepsilon)^m |w_\theta| \ , \\ |M(\theta,-m)w_\theta| &\leq C^\ell_\varepsilon(\lambda^-_{i_1} - \varepsilon)^{-m} \ldots (\lambda^-_{i_\ell} - \varepsilon)^{-m} |w_\theta| \ . \end{aligned}$$

We note that

$$\mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E^{i_1} \otimes \cdots \otimes E^{i_\ell})) \subset \mathcal{A}_{\lambda^-,\lambda^+} \ , \tag{11.4}$$

where $\lambda^- = \lambda^-_{i_1} \ldots \lambda^-_{i_\ell}$ and $\lambda^+ = \lambda^+_{i_1} \ldots \lambda^+_{i_\ell}$. Therefore, from (11.3),(11.4) we obtain that

$$\mathcal{A}\mathrm{Spec}(\mathcal{M}_f, \Gamma_B(T^\ell(E))) \subset (\mathcal{A}\mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E)))^\ell \ .$$

To prove the corresponding inclusion for the pull-back operator, notice that (11.8) induces again another splitting in $T_\ell(E)$

$$\bigotimes_{i=1}^{\ell} E^* = \bigoplus_{i_1,\cdots,i_\ell \in \{1,\cdots,k\}} E^{i_1 *} \otimes \cdots \otimes E^{i_\ell *} \ . \tag{11.5}$$

In this case, $E^{i_1 *} \otimes \cdots \otimes E^{i_\ell *} \simeq L^\ell(E^{i_1}, \ldots, E^{i_\ell}; \mathbb{C})$ is invariant under M^*_f, and if $w^*_\theta \in L^\ell(E^{i_1}, \ldots, E^{i_\ell}; \mathbb{C})$, then for all $m \geq 0$:

- for $v_1 \in E^{i_1}_{f^{-m}(\theta)}, \ldots, v_\ell \in E^{i_\ell}_{f^{-m}(\theta)}$ s.t. $|v_1| = \cdots = |v_\ell| = 1$:

$$\begin{aligned} &|M^*(\theta,m)w^*_\theta(v_1,\ldots,v_\ell)| \\ &\quad = |w^*_\theta(M(f^{-m}(\theta),m)v_1,\ldots,M(f^{-m}(\theta),m)v_\ell)| \\ &\quad \leq |w^*_\theta| \cdot C^\ell_\varepsilon \cdot (\lambda^+_{i_1} + \varepsilon)^m \ldots (\lambda^+_{i_\ell} + \varepsilon)^m \ ; \end{aligned}$$

- for $v_1 \in E^{i_1}_{f^m(\theta)}, \ldots, v_\ell \in E^{i_\ell}_{f^m(\theta)}$ s.t. $|v_1| = \cdots = |v_\ell| = 1$:

$$\begin{aligned} &|M^*(\theta,-m)w^*_\theta(v_1,\ldots,v_\ell| \\ &\quad = |w^*_\theta(M(f^m(\theta),-m)v_1,\ldots,M(f^m(\theta),-m)v_\ell)| \\ &\quad \leq |w^*_\theta| \cdot C^\ell_\varepsilon \cdot (\lambda^-_{i_1} - \varepsilon)^{-m} \ldots (\lambda^-_{i_\ell} - \varepsilon)^{-m} \ . \end{aligned}$$

The rest of the proof of the inclusion for the pull-back follows the same lines of that for the push-forward. So, a) is proved.

Notice that d) follows immediately from the fact that $E^{**} \simeq E$, $M_f^{**} = M_f$ and $E^* = T_1(E)$. We have just to apply a) twice.

We will prove the inclusions b) and c) for the push-forward operator, because the other inclusions follow from duality and d).

Since the push-forward M_f commutes with the projections Sym and Alt, then we apply Theorem A.9 to obtain the inclusions

$$\operatorname{Spec}(\mathcal{M}_f, \Gamma_B(S^\ell(E))) \subset \operatorname{Spec}(\mathcal{M}_f, \Gamma_B(T^\ell(E))) \tag{11.6}$$

and

$$\operatorname{Spec}(\mathcal{M}_f, \Gamma_B(\textstyle\bigwedge^\ell(E))) \subset \operatorname{Spec}(\mathcal{M}_f, \Gamma_B(T^\ell(E))) \ . \tag{11.7}$$

This proves b), but c) has to be proved with a more specific argument.

Point c) follows directly from the invariant splitting

$$\bigwedge^\ell(E) = \bigoplus_{\substack{s_1+\ldots+s_k = l \\ \forall i \ s_i \leq n_i}} \bigwedge^{s_1}(E^1) \wedge \cdots \wedge \bigwedge^{s_\ell}(E^\ell)$$

and (11.7) applied to each $\bigwedge^{s_i}(E^i)$. □

Remark 11.8. From Functional Analysis (Theorem A.11) we have the equality

$$\operatorname{Spec}((\mathcal{M}_f)^*, (\Gamma_B(E))^*) = \operatorname{Spec}(\mathcal{M}_f, \Gamma_B(E)) \ .$$

Nevertheless, notice that the dual space of bounded sections $(\Gamma_B(E))^*$ is a huge Banach space that has nothing to do with $\Gamma_B(E^*)$, and in this case $(\mathcal{M}_f)^*$ means the dual linear operator of $\mathcal{M}_f$. However, there are spaces in which both definitions (functional and dynamical) coincide: these are the spaces of L^2-sections. This is an immediate consequence of Riesz's representation theorem.

A purely functional proof of Theorem 11.4 (d) works if f preserves a topological measure μ:

$$\begin{aligned} \mathcal{A}\operatorname{Spec}(\mathcal{M}_f, \Gamma_B(E)) &= \mathcal{A}\operatorname{Spec}(M_f, \Gamma_{L^2(\mu)}(E)) \\ &= \mathcal{A}\operatorname{Spec}(\mathcal{M}_f^*, \Gamma_{L^2(\mu)}(E)) \\ &= \mathcal{A}\operatorname{Spec}(\mathcal{M}_f^*, \Gamma_B(E)) \ . \end{aligned}$$

Notice also that the condition on f implies that f is chain recurrent, and then the previous equalities also work for the Weyl spectrum.

11.3. Spectrum of Sylvester transfer operators

The result we obtain for the spectrum of Sylvester transfer operators is the following.

THEOREM 11.9. *Let $M_f : E \to E$, $N_f : F \to F$ two vector bundle automorphisms. Let S_f be the corresponding Sylvester vector bundle automorphism on $L^\ell(F;E)$. Then:*

$$\begin{aligned}\mathcal{A}\mathrm{Spec}(\mathcal{S}_f, \Gamma_B(L^\ell_s(F;E))) &\subset \mathcal{A}\mathrm{Spec}(\mathcal{S}_f, \Gamma_B(L^\ell(F;E))) \\ &\subset \mathcal{A}\mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E)) \cdot (\mathcal{A}\mathrm{Spec}(\mathcal{N}_f, \Gamma_B(F)))^{-l} \ .\end{aligned}$$

Proof: The first inclusion is a direct consequence of the fact that Sym commutes with the Sylvester vector bundle automorphism. Of course, the same inclusion works for alternate multilinear maps.

For the second inclusion, we consider the spectral decompositions

$$\mathcal{A}\mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E)) = \bigcup_{i=1}^k A_i \ , \ \mathcal{A}\mathrm{Spec}(\mathcal{N}_f, \Gamma_B(F)) = \bigcup_{j=1}^h A'_j \ ,$$

in spectral annuli $A_i = \mathcal{A}_{\lambda_i^-, \lambda_i^+}$, $A'_j = \mathcal{A}_{\mu_j^-, \mu_j^+}$, and the corresponding bundle splittings

$$E = \bigoplus_{i=1}^k E^i \ , \ F = \bigoplus_{j=1}^h F^j \ , \tag{11.8}$$

that are characterized by rates of growth like (11.2).

These splittings induce the splitting

$$L^\ell(F;E) = \bigoplus_{i=1}^k \bigoplus_{j_1,\dots,j_\ell \in \{1,\dots,h\}} L(F^{j_1}, \dots, F^{j_\ell}; E^i) \ ,$$

that is obviously invariant under S_f. We analyze now each subbundle of multilinear maps.

For each $w_\theta \in L(F^{j_1}, \dots, F^{j_\ell}; E^i)$, for $m \geq 0$, we have:

- for $v^1 \in F^{j_1}_{f^m(\theta)}, \dots, v^\ell \in F^{j_\ell}_{f^m(\theta)}$ s.t. $|v^1| = \ldots = |v^\ell| = 1$:

$$\begin{aligned}&|S(\theta, m) w_\theta(v^1, \dots, v^\ell)| \\ &\quad = |M(\theta, m) w_\theta(N(f^m(\theta), -m)v^1, \dots, N(f^m(\theta), -m)v^\ell)| \\ &\quad \leq C_\varepsilon \cdot (\lambda_i^+ + \varepsilon)^m \cdot |w_\theta| \cdot C_\varepsilon^\ell \cdot (\mu_{j_1}^- - \varepsilon)^{-m} \ldots (\mu_{j_\ell}^- - \varepsilon)^{-m} \ ;\end{aligned}$$

- for $v^1 \in F^{j_1}_{f^{-m}(\theta)}, \dots, v^\ell \in F^{j_\ell}_{f^{-m}(\theta)}$ s.t. $|v^1| = \ldots = |v^\ell| = 1$:

$$\begin{aligned}&|S(\theta, -m) w_\theta(v^1, \dots, v^\ell)| \\ &\quad = |M(\theta, -m) w_\theta(N(f^{-m}(\theta), m)v^1, \dots, N(f^{-m}(\theta), m)v^\ell)| \\ &\quad \leq C_\varepsilon \cdot (\lambda_i^- - \varepsilon)^{-m} \cdot |w_\theta| \cdot C_\varepsilon^\ell \cdot (\mu_{j_1}^+ + \varepsilon)^m \ldots (\mu_{j_\ell}^+ + \varepsilon)^m \ .\end{aligned}$$

From these rates of growth, $\mathrm{Spec}(\mathcal{S}_f, \Gamma_B(L(F^{j_1}, \dots, F^{j_\ell}; E^i)) \subset \mathcal{A}_{\rho^-, \rho^+}$, with

$$\rho^- = \frac{\lambda_i^-}{\mu_{j_1}^+ \ldots \mu_{j_\ell}^+} \ , \ \rho^+ = \frac{\lambda_i^+}{\mu_{j_1}^- \ldots \mu_{j_\ell}^-} \ ,$$

and the proof of Theorem 11.9 follows immediately. □

CHAPTER 12

Spectrum, symmetry and geometry

In this section we obtain a few results on the spectrum of a transfer operator whose generator preserves a given geometric structure on the bundle. As we will see, the spectrum inherits some properties. In particular, we consider here symplectic and volume structures, that appear very often in the literature, specially in applications. We also consider the case in which the cocycle is reversible.

There are reasons why transfer operators which preserve the above geometric structures appear in applications. We just note that these operators appear often as the linearization of vector fields that preserve the corresponding geometric structure. Reversibility is a property of many equations of Physics, roughly all physical systems without dissipation are reversible. For example, all equations describing circuits ignoring resistance are reversible. The Hamilton's canonical equations of mechanics preserve a symplectic structure and, as a consequence, conserve a volume. The motion of particles in an incompressible fluid, preserve the volume.

12.1. Reversible transfer operators

Along this section, $\pi : E \to \mathcal{P}$ is a vector bundle over the manifold $\mathcal{P}$.

DEFINITION 12.1. *A vector bundle automorphism $I_g : E \to E$ is an* involution *iff $I \circ I = \mathrm{Id}_{|E}$, that is to say, for all $\theta \in \mathcal{P}$*

$$I(g(\theta)) \circ I(\theta) = \mathrm{Id}_{E_\theta} \ .$$

In particular, g is an involution in $\mathcal{P}$: $g \circ g = \mathrm{Id}_{\mathcal{P}}$.

DEFINITION 12.2. *Given an involution $I_g : E \to E$ over $g : \mathcal{P} \to \mathcal{P}$, a vector bundle automorphism $M_f : E \to E$ over $f : \mathcal{P} \to \mathcal{P}$ is said to be I_g-reversible iff $I \circ M \circ I = M^{-1}$, that is to say, for all $\theta \in \mathcal{P}$*

$$I(f \circ g(\theta)) \circ M(g(\theta)) \circ I(\theta) = M(f^{-1}(\theta))^{-1} \ .$$

In particular, f is g-reversible in $\mathcal{P}$: $g \circ f \circ g = f^{-1}$.

REMARK 12.3. If $f : \mathcal{P} \to \mathcal{P}$ is a g-reversible diffeomorphism, where $g : \mathcal{P} \to \mathcal{P}$ is an involutive diffeomorphism, then the push forward $f_* : T\mathcal{P} \to T\mathcal{P}$ is a reversible vector bundle automorphism with respect to $g_* : T\mathcal{P} \to T\mathcal{P}$, and the pull back $f^* : T\mathcal{P} \to T\mathcal{P}$ is a reversible vector bundle automorphism with respect to $g^* : T^*\mathcal{P} \to T^*\mathcal{P}$.

For a I_g-reversible vector bundle automorphism $M_f : E \to E$, the action on sections of the transfer operator is also reversible

$$\mathrm{I}_g^2 = \mathrm{Id}\ , \mathrm{I}_g \circ \mathcal{M}_f \circ \mathrm{I}_g = \mathcal{M}_f^{-1}\ .$$

As a result, we have the following.

PROPOSITION 12.4. *Let $M_f : E \to E$ be a I_g-reversible vector bundle automorphism. Then:*

- $z \in \mathrm{Spec}_W(\mathcal{M}_f, \Gamma_B(E)) \Rightarrow \frac{1}{z} \in \mathrm{Spec}_W(\mathcal{M}_f, \Gamma_B(E))$.
- $z \in \mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E)) \Rightarrow \frac{1}{z} \in \mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E))$.

Proof: Is an immediate consequence of the fact that $\mathcal{M}_f$ and $\mathcal{M}_f^{-1}$ are conjugated with respect to I_g: $\mathrm{I}_g^{-1} \circ \mathcal{M}_f \circ \mathrm{I}_g = \mathcal{M}_f^{-1}$. □

The following result is also straightforward.

PROPOSITION 12.5. *Let $M_f : E \to E$ be a I_g-reversible vector bundle automorphism. Let $F \subset E$ be an invariant subbundle. Then, the subbundle $I_g F$ is also invariant.*

Proof: Let $v_\theta \in F_\theta$ and $w_{g(\theta)} \in I(\theta)F_\theta$. Then

$$\begin{aligned} M(g(\theta))w_{g(\theta)} &= M(g(\theta))I(\theta)v_\theta = I(f \circ g(\theta))^{-1} M(f^{-1}(\theta))^{-1} v_\theta \\ &= I(f^{-1}(\theta))M(f^{-1}(\theta))^{-1} v_\theta\ . \end{aligned}$$

Since $M(f^{-1}(\theta))^{-1}v_\theta \in F_{f^{-1}(\theta)}$, then $M(g(\theta))w_{g(\theta)} \in I((f^{-1}(\theta))F_{f^{-1}(\theta)}$. That is to say, since $M_f F = F$ then $M_f I_g F = I_g F$. □

As a corollary we obtain the structure of the spectrum of a reversible transfer operator.

THEOREM 12.6. *Let $M_f : E \to E$ be a I_g-reversible vector bundle automorphism. Then, the annular hull of the spectrum is like*

$$\mathcal{A}\mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E)) = \bigcup_{i=1}^{l} \mathcal{A}[\lambda_i, \mu_i] \cup \mathcal{A}[\rho, \rho^{-1}] \cup \bigcup_{i=1}^{l} \mathcal{A}[\mu_i^{-1}, \lambda_i^{-1}]$$

where $l \geq 0$, $\lambda_1 \leq \mu_1 < \lambda_2 \leq \mu_2 < \cdots < \lambda_l \leq \mu_l < 1$, and $\mu_l < \rho$ (if $\rho > 1$ we assume $\mathcal{A}[\rho, \frac{1}{\rho}] = \emptyset$). Let

$$E = \bigoplus_{i=1}^{l} E^{s,i} \oplus E^c \oplus \bigoplus_{i=1}^{l} E^{u,i}$$

be the corresponding spectral decomposition. Then

- *for each $i = 1, \ldots, l$ $E^{u,i} = IE^{s,i}$;*
- *$E^c = IE^c$.*

Proof: We have to prove that if a vector v_θ satisfies a rate of growth in positive (negative) time, then its symmetric satisfies a rate of growth in negative (positive) time.

For instance, assume that $v_\theta \in W^{\leq\lambda}$ (see Definition 2.1), and let $w_{g(\theta)} = I(\theta)v_\theta$ be its symmetric image. We will see that $w_{g(\theta)} \in W^{\geq\lambda^{-1}}$. Since $v_\theta \in W^{\leq\lambda}$, there exist a positive constant C such that for all $m \geq 0$ we have $|M(\theta,m)v_\theta| \leq C\lambda^m|v_\theta|$. Then, for all $m \leq 0$:

$$\begin{aligned}|M(g(\theta),m)w_{g(\theta)}| &= |M(g(\theta),m)I(\theta)v_\theta| = |I(f^{-m}(\theta))M(\theta,-m)v_\theta| \\ &\leq \|I\|_\infty C\lambda^{-m}|v_\theta| \ .\end{aligned}$$

So we are done with the prove of Theorem 12.6. □

Henceforth, for reversible systems we have the following situation:

- Each stable subbundle has a symmetric unstable subbundle;
- The central subbundle, if exists, is symmetric.

Remark 12.7. Reversible vector bundle automorphisms in vector bundles with odd rank are not hyperbolic.

12.2. Symplectic transfer operators

Now, we start to discuss the effect of a symplectic structure on the spectrum of transfer operators.

12.2.1. Some standard constructions in symplectic geometry. First, we collect some standard definitions on symplectic vector bundles (see, for instance, [**Wei77, LM87, MS95**]).

Definition 12.8. *A* symplectic vector bundle *is a real vector bundle E over a manifold $\mathcal{P}$ equipped with a section $\Omega : \mathcal{P} \to L^2_a(E;\mathbb{R})$ of the bundle of skew-symmetric bilinear forms non degenerate on each fiber E_θ.*

It will be important to note that, in order to have a non-degenerate 2-form, the rank of E is even. Henceforth, we will write $n = 2d$.

We also note that, since the 2-form Ω is nondegenerate, it induces a *musical* isomorphism $\Omega^\flat$ of E onto E^* by

$$(\Omega^\flat_\theta v_\theta)w_\theta = -\Omega_\theta(v_\theta, w_\theta) \ ,$$

whose inverse is denoted by $\Omega^\sharp$.

Example 12.9. In the trivial bundle $E = \mathcal{P} \times \mathbb{R}^{2d}$ we can define a standard symplectic structure using the standard symplectic matrix

$$J = \begin{pmatrix} 0 & \mathrm{Id}_d \\ -\mathrm{Id}_d & 0 \end{pmatrix} \ .$$

That is, the symplectic form is defined for $v_\theta = (\theta, v), w_\theta = (\theta, w) \in \mathcal{P} \times \mathbb{R}^{2n}$ by

$$\Omega_\theta(v_\theta, w_\theta) = v^\top J w \ .$$

As it is well known, Darboux theorem implies that given any vector bundle and a sufficiently small neighborhood, there is a transformation that maps the given symplectic structure into the standard one in Example 12.9. See, any of the references [**Wei77, LM87, MS95**]) for more details.

EXAMPLE 12.10. If $\mathcal{P}$ is a manifold, a symplectic vector bundle structure on $T\mathcal{P}$ is a non degenerate 2-form Ω is sometimes called *almost symplectic structure* on $\mathcal{P}$.

If Ω is closed, we say that the manifold $\mathcal{P}$ is symplectic and that Ω is a *Symplectic form on the manifold* or that it defines a *Symplectic structure on* $\mathcal{P}$

As we will see in Chapter 13, there will be important differences in the spectral theory of almost symplectic structures and symplectic structures.

The central object in our discussion are the vector bundle automorphisms that preserve a symplectic structure on the bundle. We will see that the preservation of this symplectic structure has important consequences on the spectral theory.

DEFINITION 12.11. *A vector bundle automorphism* $M_f : E \to E$ *is symplectic iff*

$$M(\theta)^*\Omega_{f(\theta)} = \Omega_\theta \ ,$$

that is:

$$\forall v_\theta, w_\theta \in L_\theta, \qquad \Omega_{f(\theta)}(M(\theta)v_\theta, M(\theta)w_\theta) = \Omega_\theta(v_\theta, w_\theta) \ .$$

The main goal of this section is to analyze the spectrum of the transfer operator associated to a symplectic vector bundle automorphism M_f. Of course, even if the geometric objects we consider are real valued, but we will apply the complexification trick described in Remark 1.2 and study the spectrum in the complex vector space obtained by complexifying the real bundle.

The first result we discuss is the following:

PROPOSITION 12.12. *Let* $M_f : E \to E$ *be a symplectic vector bundle automorphism. Then:*

$$z \in \mathcal{A}\mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E)) \Leftrightarrow \frac{1}{z} \in \mathcal{A}\mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E)) \ . \tag{12.1}$$

Proof: The proof is an immediate consequence of

$$\Omega_\theta^\sharp M(\theta)^*\Omega_{f(\theta)}^\flat = M^{-1}(\theta) \ ,$$

and Proposition 11.4. □

Our next goal is to clarify the geometrical properties of the invariant subbundles associated to spectral projections.

First, we recall some standard definitions in symplectic geometry.

DEFINITION 12.13. *Let* E *be a symplectic vector bundle of rank* $n = 2d$.

A vector subbundle L *is said to be* isotropic *if every fiber* L_θ *has that property relative to the symplectic form* Ω_θ. *More explicily,*

$$\forall v_\theta, w_\theta \in L_\theta \ \Omega_\theta(v_\theta, w_\theta) = 0 \ .$$

We note that a consequence of the definition is that if L is an isotropic subbundle, the rank of L is not bigger than d. If L is isotropic and its rank is d, then L is to said to be Lagrangian .

A vector subbundle S is said to be symplectic *if every fiber S_θ has the same property relative to the symplectic form Ω_θ.*

That is, Ω_θ defines a nondegenerate skew-symmetric 2-form in S_θ.

Remark 12.14. Although a symplectic vector space admits Lagrangian subspaces, a symplectic vector bundle may fail to admit Lagrangian subbundles. For example, the tangent bundle to the sphere S^2, endowed with the area form, contains no vector subbundle of rank 1 and therefore no Lagrangian subbundle. Besides this elementary obstruction (which only uses that a Lagrangian subbundle is a subbundle), there are more subtle obstructions which indeed use the symplectic geometry [**LM87**].

We will also use the concept of symplectic orthogonality.

Definition 12.15. *Let E be a symplectic vector bundle of rank $n = 2d$. Two vectors $v_\theta, w_\theta \in E$ are Ω-orthogonal if $\Omega_\theta(v_\theta, w_\theta) = 0$.*
If $F \subset E$ is a subbundle, we define the symplectic orthogonal subbundle

$$F^\Omega = \{w_\theta \in E \mid \Omega_\theta(v_\theta, w_\theta) = 0 \text{ for all } v_\theta \in F\} \ .$$

Remark 12.16. Notice that $\dim F^\Omega = 2d - \dim F$, but it is not true in general that $E = F \oplus F^\Omega$. The symplectic orthogonal F^Ω complements F if and only if F is a symplectic vector subbundle. Notice that L is an isotropic subbundle if and only if $L \subset L^\Omega$. Coisotropic subbundles are those subbundles L for which $L^\Omega \subset L$.

12.2.2. Invariant subbundles. Now, we start the charaterization of invariant subbundles under symplectic transfer operators. The main result of this section will be Theorem 12.21.

The following result shows that given an invariant subbundle, its symplectic orthogonal is also invariant. It also gives a relation between the linear mappings. These properties are very important for the parameterization method for invariant manifolds, specially in the cases that the dynamics on the manifold is particularly easy (e.g. rotations). See for example [**HdlLb, HdlL04, HdlL05a**]. It is also important in KAM theory [**dlLGJV05, dlL01b**]

Proposition 12.17. *Let $M_f : E \to E$ be a symplectic vector bundle automorphism. Let $F \subset E$ be an invariant subbundle. Then, the symplectic orthogonal subbundle F^Ω is also invariant. In particular:*

a) *If L is an invariant isotropic subbundle, then L^Ω is an invariant coisotropic subbundle (and $L \subset L^\Omega$).*
b) *If L is an invariant coisotropic subbundle, then L^Ω is an invariant isotropic subbundle (and $L^\Omega \subset L$).*
c) *If F is an invariant symplectic subbundle, then F^Ω is an invariant symplectic subbundle (and $E = F \oplus F^\Omega$).*

Proof: Let F be an invariant subbundle. Given $w_\theta \in F_\theta^\Omega$, we have to see that $M(\theta)w_\theta \in F_{f(\theta)}^\Omega$. For all $\bar{v}_{f(\theta)} \in F_{f(\theta)}$, there exists an unique $v_\theta \in F_\theta$ such that $\bar{v}_{f(\theta)} = M(\theta)v_\theta$, so

$$\Omega(\bar{v}_{f(\theta)}, M(\theta)w_\theta) = \Omega(M(\theta)v_\theta, M(\theta)w_\theta) = \Omega(v_\theta, w_\theta) = 0 \ .$$

Given the above considerations, items a), b) and c) are just an immediate consequence of the definitions. □

Proposition 12.17 does not assume that the invariant spaces considered are spectral subbundles. The next result makes this more precise in the case that the invariant spaces are spectral and that the transfer operator is symplectic.

PROPOSITION 12.18. *Let $M_f : E \to E$ be a symplectic vector bundle automorphism. Suppose that the spectrum* $\mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E))$ *has a gap in the circle of radius $\lambda < 1$. Then, there is another gap in the circle of radius $\lambda^{-1} > 1$. Let $E^{ss} = E^{<\lambda}$, $E^{uu} = E^{>\lambda^{-1}}$ and $E^{cc} = E^{>\lambda} \cap E^{<\lambda^{-1}}$ be the corresponding spectral subbundles (some of the above spaces in the assumption may be just the trivial zero section). Then:*

a) *E^{cc} and $E^{ss} \oplus E^{uu}$ are symplectic subbundles, and*

$$(E^{cc})^\Omega = E^{ss} \oplus E^{uu} \ .$$

b) *E^{ss} and E^{uu} are isotropic subbundles of the same rank.*

c) *In particular, if $\mathcal{A}\mathrm{Spec}(\mathcal{M}_f, , \Gamma_B(E)) \cap \mathcal{A}[\lambda, \lambda^{-1}]$ does not have gaps (it could be empty), then $E^{ss} = E^s$ is the stable subbundle, $E^{uu} = E^u$ is the unstable subbundle and $E^{cc} = E^c$ is the central subbundle (it could be E_0).*

Proof: a) Since $\dim(E^{ss} \oplus E^{uu}) = 2d - \dim E^c = \dim(E^{cc})^\Omega$, we have to prove that $E^{ss} \oplus E^{uu} \subset (E^{cc})^\Omega$. We will see that $E^{ss} \subset (E^{cc})^\Omega$, and an analogous proof works for $E^{uu} \subset (E^{cc})^\Omega$.

For $v_\theta \in E^{cc}$ and $w_\theta \in E^{ss}$, and $m > 0$,

$$\begin{aligned} |\Omega_\theta(v_\theta, w_\theta)| &\leq |\Omega_{f^m(\theta)}||M(\theta,m)v_\theta||M(\theta,m)w_\theta| \\ &\leq C_\varepsilon^2 \|\Omega\|_\infty (\lambda^{-1} - \varepsilon)^m (\lambda - \varepsilon)^m |v_\theta||w_\theta| \\ &= C_\varepsilon^2 \|\Omega\|_\infty (1 - (\lambda^{-1} + \lambda)\varepsilon + \varepsilon^2)^m |v_\theta||w_\theta| \ , \end{aligned}$$

and the last term tends to zero when $m \to +\infty$ if ε is small enough. So $\Omega_\theta(v_\theta, w_\theta) = 0$.

The above argument proves that $E^{ss} \oplus E^{uu} = (E^{cc})^\Omega$. It also proves that both subbundles $E^{ss} \oplus E^{uu}$ and $(E^{cc})^\Omega$ are symplectic.

b) The proof of the fact that E^{ss} and E^{uu} are isotropic follows similar lines.

For $v_\theta, w_\theta \in E^{ss}$, and $m > 0$,

$$|\Omega_\theta(v_\theta, w_\theta)| \leq (\lambda - \varepsilon)^{2m} |v_\theta||w_\theta|,$$

and the right hand side tends to zero because $\lambda < 1$. So E^{ss} is isotropic, and a similar proof works for E^{uu}.

Notice that $\mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E^{ss} \oplus E^{uu}))$ does not intersect the unit circle, that is to say $(M_f)_{|E^{ss}\oplus E^{uu}}$ is hyperbolic. Notice also that $\Omega_{|E^{ss}\oplus E^{uu}}$ is symplectic, and then both subbundles E^{ss} and E^{uu} are isotropic in $E^{ss} \oplus E^{uu}$. Since the dimension of an isotropic bundle is not bigger than half the dimension of the symplectic bundle (in this case, $E^{ss} \oplus E^{uu}$), then both E^{ss} and E^{uu} are Lagrangian in $E^{ss} \oplus E^{uu}$, so they have the same rank.

c) is and immediate consequence of the previous arguments. □

Using the same arguments, we an establish the following result:

PROPOSITION 12.19. *Let M_f be a symplectic transfer operator. Let E be a an invariant subbundle – not necessarily a spectral subbundle. Assume that the spectrum of M_f restricted to E is strictly inside (or outside) of the unit circle.*

Then, E is isotropic.

REMARK 12.20. As a corollary, we obtain that a symplectic vector bundle without Lagrangian subbundles does not admit hyperbolic symplectic vector bundles automorphisms, that is to say, the spectrum of the corresponding transfer operator contain points of the unit circle. This provides with a method of excluding the existence of Anosov symplectic systems on some manifolds.

Using induction from the previous results, we obtain the following characterization of the spectral subbundles of a symplectic vector bundle automorphism.

THEOREM 12.21. *Let $M_f : E \to E$ be a symplectic vector bundle automorphism. Then, the annular hull of the spectrum is of the form:*

$$\mathcal{A}\mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E)) = \bigcup_{i=1}^{l} \mathcal{A}[\lambda_i, \mu_i] \cup \mathcal{A}[\rho, \rho^{-1}] \cup \bigcup_{i=1}^{l} \mathcal{A}[\mu_i^{-1}, \lambda_i^{-1}]$$

where $l \geq 0$, $\lambda_1 \leq \mu_1 < \lambda_2 \leq \mu_2 < \cdots < \lambda_l \leq \mu_l < 1$, and $\mu_l < \rho$ (if $\rho > 1$ we understand that $\mathcal{A}[\rho, \rho^{-1}] = \emptyset$). Let

$$E = \bigoplus_{i=1}^{l} E^{s,i} \oplus E^c \oplus \bigoplus_{i=1}^{l} E^{u,i}$$

be the corresponding spectral decomposition. Then

- *for each $i = 1, \ldots, l$ $\dim E^{s,i} = \dim E^{u,i}$, and $E^{s,i} \oplus E^{u,i}$ is an invariant symplectic subbundle;*
- *$E^s = \bigoplus_{i=1}^{l} E^{s,i}$ and $E^u = \bigoplus_{i=1}^{l} E^{u,i}$ are the stable and unstable subbundles, respectively;*
- *E^c is an invariant symplectic subbundle, whose orthogonal is $(E^c)^\Omega = E^s \oplus E^u$.*

12.2.3. Complex structures. To analyze further geometrical properties of invariant subbundles we will use some objects (Lagrangian complements, complex structures), which we now recall.

If a symplectic vector bundle E admits a Lagrangian subbundle L, then there exists a Lagrangian complement $\hat{L}$, such that $E = L \oplus \hat{L}$. This can be proved by using this property for symplectic vector spaces and partitions of the unity. A cleaner construction is using adapted complex vector bundle structures, which we now recall.

DEFINITION 12.22. *An adapted complex vector bundle structure on a real symplectic vector bundle E is given by an automorphism over the identity $J : E \to E$, satisfying the following properties:*

- *J is symplectic. That is*

$$\Omega(u, v) = \Omega(Ju, Jv) \quad \forall u, v$$

- $J_\theta \circ J_\theta = -\mathrm{Id}_{|E_\theta}$*;*
- *The section $G : \mathcal{P} \to L^2_s(E; \mathbb{R})$ of the bundle of symmetric bilinear forms, defined by*

$$G_\theta(v_\theta, w_\theta) = \Omega_\theta(v_\theta, J_\theta w_\theta) \ ,$$

  defines a Riemannian structure on the bundle E (that is to say, G_θ is positive definite, and defines a scalar product on each fiber).

Every real symplectic vector bundle admits an adapted complex vector bundle structure (see [**Wei77, LM87, MS95**]*). The relation between the metric G and the complex structure J is often described as saying that the metric G tames the symplectic structure. The metric, complex structure, and symplectic form are often called Khaler structures .*

Remarkable geometric applications of the complex structures to symplectic geometry can be found in [**MS98, HZ94**].

We just sketch the construction and refer to the above references for more details. A similar construction in more general circumstances occurs also in [**AM78**, p. 178].

We will just construct the J and G fiberwise and check that the argument depends smoothly on all the elements and can be extended to the whole bundle by using partitions of unity etc.

Given a metric $\hat{G}$, there is a unique automorphism A defined by

$$\hat{G}(Au, v) = \Omega(u, v)$$

Because Ω is skew-symmetric and $\hat{G}$ is symmetric, we obtain

$$\begin{aligned} \hat{G}(Au, v) &= \Omega(u, v) = -\Omega(v, u) = -\hat{G}(Av, u) \\ &= -\hat{G}(u, Av). \end{aligned}$$

Therefore,

$$\hat{G}(A^2u, v) = -\hat{G}(Au, Av) = \hat{G}(u, A^2v)$$

We conclude that A^2 *is symmetric. Taking* $u = v$ *above and using that the metric* $\hat{G}$ *is positive definite, we obtain that* A^2 *is negative definite for the inner product given by* $\hat{G}$. *We can find a unique matrix, which we denote by* $\sqrt{-A^2}$ *which is positive definite for the inner product given by* $\hat{G}$ *and such that* $(\sqrt{-A^2})^2 = -A^2$.

The desired complex structure is

$$J = (\sqrt{-A^2})^{-1}A = A(\sqrt{-A^2})^{-1}$$

and the desired metric G *is*

$$G(u, v) = \Omega(u, Jv)$$

Clearly $J^2 = -\mathrm{Id}$. *Also, using that* A^2 *is self-adjoint and commutes with* $\sqrt{-A^2}$ *we have:*

$$\begin{aligned}
\Omega(Ju, Jv) &= \Omega(A(\sqrt{-A^2})^{-1}u, A(\sqrt{-A^2})^{-1}v) \\
&= \hat{G}(A^2(\sqrt{-A^2})^{-1}u, A(\sqrt{-A^2})^{-1}v) \\
&= \hat{G}(u, (\sqrt{-A^2})^{-1}A^3(\sqrt{-A^2})^{-1}v) \\
&= \hat{G}(u, Av) \\
&= \Omega(u, v)
\end{aligned}$$

The only thing that remains to be checked is that G *is positive definite, and hence, it is a metric. Note that*

$$\begin{aligned}
G(u, u) = \omega(u, Ju) &= -\hat{G}(u, AJu) \\
&= \hat{G}(u, \sqrt{(} - A^2)u
\end{aligned}$$

□

Remark 12.23. Notice that, of course, once we fix two elements of the structure, the third is determined uniquely. On the other hand, the problem is somewhat flexible. If we change the metric a small amount, we still can find a J.

Since the construction is so explicit we note that J depends smoothly on the metric and on the symplectic structure Ω.

Therefore, given a metric $\hat{G}$ in the bundle and a symplectic structure in the bundle which are smooth, if we perform the construction fiberwise, we obtain a metric G and a J which also depend smoothly on the fibers.

Since the 2-form G *is nondegenerate, it induces also musical isomorphisms* $G^\flat : E \to E^*$, *given by*

$$(G^\flat_\theta v_\theta)w_\theta = G_\theta(v_\theta, w_\theta) \ ,$$

and $G^\sharp = (G^\flat)^{-1}$.

Remark 12.24. From the construction, notice that J non only preserves the symplectic form but also the scalar product:

$$\Omega_\theta(J_\theta v_\theta, J_\theta w_\theta) = \Omega_\theta(v_\theta, w_\theta) \ , \ G_\theta(I_\theta v_\theta, I_\theta w_\theta) = G_\theta(v_\theta, w_\theta) \ ,$$

for all $v_\theta, w_\theta \in E$.

Using the metric G we have also a notion of metric orthogonality.

DEFINITION 12.25. *Let E be a symplectic vector bundle of rank $n = 2d$. Two vectors $v_\theta, w_\theta \in E$ are G-orthogonal if $G_\theta(v_\theta, w_\theta) = 0$. If $F \subset E$ is a subbundle, we define the orthogonal subbundle*

$$F^G = \{w_\theta \in E \mid G_\theta(v_\theta, w_\theta) = 0 \text{ for all } v_\theta \in F\} .$$

REMARK 12.26. Notice that $\dim F^G = 2d - \dim F$, and $E = F \oplus F^G$. The relation between both notions of symplectic and metric orthogonality is summarized by the formula

$$F^\Omega = (JF)^G \tag{12.2}$$

The G-orthogonal projection on the subbundle F is the morphism $P_F^G : E \to F$ given by

$$P_F^G = (\nu_F^* G)^\sharp \; \nu_F^* \; G^b .$$

where $\nu_F : F \to E$ is the inclusion of F in E.

In coordinates, if $\dim F = m < \dim E = n$, and F is the $n \times m$ matrix whose columns are given by a basis in F, then for a vector $e \in \mathbb{R}^n \simeq E$ its projection in F, given by $\bar{f} \in \mathbb{R}^m \simeq F$, is:

$$\bar{f} = (F^\top G F)^{-1} F^\top G e ,$$

where G is the positive definite symmetric matrix given the scalar product. This are just the well known normal equations.

The following result demonstrates the usefulness of the adapted complex structure for the study of subbundles.

PROPOSITION 12.27. *Let $L \subset E$ be an isotropic subbundle of the symplectic bundle E. Then:*

a) *$\hat{L} = JL$ is an isotropic subbundle such that $L \cap \hat{L} = E_0$;*
b) *$F = L \oplus \hat{L}$ is a symplectic subbundle. It admits a complementary symplectic subbundle given by*

$$\begin{aligned} F^\Omega &= (L \oplus \hat{L})^\Omega = (L \oplus \hat{L})^G = F^G \\ &= L^\Omega \cap \hat{L}^\Omega = L^G \cap \hat{L}^G ; \end{aligned}$$

c) *$L^\Omega = L \oplus F^\Omega = \hat{L}^G$ and $\hat{L}^\Omega = F^\Omega \oplus \hat{L} = L^G$ are coisotropic subbundles;*
d) *$E = L \oplus F^\Omega \oplus \hat{L}$.*
e) *In particular, if L is a Lagrangian subbundle, then $\hat{L} = JL$ is a complementary Lagrangian subbundle.*

Proof: Since the objects are defined on each fiber, we will suppress the subindex θ that fixes the fiber.

Notice that $\hat{L} = IL$ is isotropic because L is an isotropic subbundle and I is symplectic.

Let $v \in L \cap \hat{L}$, and $w \in L$ such that $v = Iw$. Then

$$G(v,v) = \Omega(v, Iv) = \Omega(Iw, Iv) = \Omega(w, v) = 0 \ ,$$

because L is isotropic. Since G is definite positive, $v = 0$. This proves $L \cap \hat{L} = E_0$.

We are going now to analyze the symplectic product in $F = L \oplus \hat{L}$. For $v + Iw, \bar{v} + I\bar{w} \in F$,

$$\Omega(v + Iw, \bar{v} + I\bar{w}) = G(v, \bar{w}) - G(\bar{v}, w) \ .$$

From this,

$$\Omega(v + Iw, Iv) = G(v, v) \ , \Omega(v + Iw, w) = -G(w, w) \ ,$$

and as a result Ω is non degenerate in F.

The rest of the proof is straightforward. □

Coming back to our main problem of spectral properties, we use the geometric structures above to obtain properties of the symplectic vector bundle automorphism from the existence of an an invariant isotropic vector subbundle. This result is important for the properties on perturbations of invariant manifolds and for KAM theory. The paper [**dlLGJV05**] uses a coordinate version of these properties in KAM theory. Similar properties occur in the study of perturbation of invariant manifolds in [**Sau01, LMS03**].

Proposition 12.28. *Let $M_f : E \to E$ be a symplectic vector bundle automorphism. Let $L \subset E$ be an invariant isotropic subbundle. Then, M_f is upper triangular with respect to the splitting $E = L \oplus F^G \oplus IL$, where $F = L \cap JL$.*

Let

$$M(\theta) = \begin{pmatrix} \Lambda(\theta) & B(\theta) & C(\theta) \\ 0 & M^{F^G}(\theta) & D(\theta) \\ 0 & 0 & \hat{\Lambda}(\theta) \end{pmatrix}$$

be the block representation of M_f with respect to such a splitting.

That is to say:

- $\Lambda_f : L \to L$ *satisfies* $M_f \circ \nu_L = \nu_L \circ \Lambda_f$;
- $M_f^{F^G} = P^G_{F^G} \circ M_f \circ \nu_{F^G}$ *is a vector bundle automorphism in* F^G;
- $\hat{\Lambda}_f = P^G_{IL} \circ M_f \circ \nu_{IL}$ *is a vector bundle automorphism in* $\hat{L} = IL$.

Then:

a) $M_f^{F^G}$ *is a symplectic vector bundle automorphism, so*

$$\mathcal{A}\mathrm{Spec}(\mathcal{M}_f^{F^G}, \Gamma_B(F^G)) = \Big(\mathcal{A}\mathrm{Spec}(\mathcal{M}_f^{F^G}, \Gamma_B(F^G))\Big)^{-1} \ ;$$

b) $\hat{\Lambda}_f$ *is conjugated to* $(\Lambda_f^{-1})^*$, *so*

$$\mathcal{A}\mathrm{Spec}(\hat{\Lambda}_f, \Gamma_B(IL)) = (\mathcal{A}\mathrm{Spec}(\Lambda_f, \Gamma_B(L)))^{-1} \ .$$

Proof: Since L is invariant, L^Ω is invariant. Since L is isotropic then $L \subset L^\Omega$. Notice also that $L^\Omega = L \oplus (L^\Omega \cap (IL)^\Omega) = L \oplus (L \oplus IL)^\Omega = L \oplus F^G$.

Let $v, \bar{v} \in F^G_\theta = (L_\theta \oplus IL_\theta)^G \subset L^\Omega_\theta$. Notice that $Mv, M\bar{v} \in L^\Omega_{f(\theta)}$. Moreover, for $w \in L^\Omega_\theta$, $P^G_F w = P^G_L w \in L_\theta$. Then, $P^G_F Mv, P^G_F M\bar{v} \in L_{f(\theta)}$. From these observations, we see that $\Omega_{f(\theta)}(M^{F^G}(\theta)v, M^{F^G}(\theta)\bar{v}) = \Omega_\theta(v, w)$. Indeed:

$$\begin{aligned}\Omega(M^{F^G}v, M^{F^G}\bar{v}) &= \Omega(P^G_{F^G}Mv, P^G_{F^G}M\bar{v}) = \Omega(Mv - P^G_F Mv, M\bar{v} - P^G_F M\bar{v})\\ &= \Omega(Mv, M\bar{v}) + \Omega(P^G_F Mv, P^G_F M\bar{v})\\ &\quad - \Omega(Mv, P^G_F M\bar{v}) - \Omega(P^G_F Mv, M\bar{v})\\ &= \Omega(v, \bar{v}) \ .\end{aligned}$$

This proves that $M^{F^G}_f$ is symplectic in F^G.

We are going to compute now $\hat{\Lambda}_f : IL \to IL$. We denote $i = I_{|L} : L \to IL$, the restriction of I to L, and $-i : IL \to L$. So, $\nu_{IL} \circ i = I \circ \nu_L$. Then,

$$\begin{aligned}-i\circ\hat{\Lambda}_f\circ i &= -i\circ P^G_{IL}\circ M_f\circ\nu_{IL}\circ i\\ &= -i\circ(\nu^*_{IL}G)^\sharp\circ\nu^*_{IL}\circ G^b\circ M_f\circ\nu_{IL}\circ i\\ &= i\circ(\nu^*_{IL}G)^\sharp\circ i^*\circ\nu^*_L\circ I^*\circ G^b\circ M_f\circ I\circ\nu_L\end{aligned} \tag{12.3}$$

defines a vector bundle automorphism in L, over f. We split $-i\circ\hat{\Lambda}_f\circ i = B\circ A_f$, where

$$A_f = \nu^*_L\circ I^*\circ G^\flat\circ M_f\circ I\circ\nu_L : L \to L^* \ , \ B = i\circ(\nu^*_{IL}G)^\sharp\circ i^* : L^* \to L \ .$$

Firstly, we analyze $B : L^* \to L$. For $\varphi \in L^*_\theta$ and $l \in L_\theta$,

$$\begin{aligned}B(\varphi) = l &\Leftrightarrow (\nu^*_{IL}G)^\sharp\circ i^*(\varphi) = -i(l)\\ &\Leftrightarrow i^*\varphi = -(\nu^*_{IL}G)^\flat i(l)\\ &\Leftrightarrow \forall\bar{l} \in L \ \varphi(\bar{l}) = -i^*\varphi(i\bar{l}) = ((\nu^*_{IL}G)^\flat i(l))(i\bar{l}) = G(i(l), i(\bar{l})) = G(l, \bar{l}) \ .\end{aligned}$$

So,

$$B = i\circ(\nu^*_{IL}G)^\sharp\circ i^* = (\nu^*_L G)^\sharp \ . \tag{12.4}$$

Secondly, we compute $A_f : L \to L^*$. For all $l_\theta \in L_\theta$ and $\bar{l}_{f(\theta)} \in L_{f(\theta)}$,

$$\begin{aligned}(A_f l)(\bar{l}) &= (\nu^*_L\circ I^*\circ G^\flat\circ M_f\circ I\circ\nu_L(l))(\bar{l}) = G(M_f I\nu_L(l), I\nu_L(\bar{l}))\\ &= -\Omega(M_f I\nu_L(l), \nu_L(\bar{l})) = -\Omega(I\nu_L(l), M^{-1}_f\nu_L(\bar{l}))\\ &= -\Omega(I\nu_L(l), \nu_L\Lambda^{-1}_f(\bar{l})) = G(\nu_L(l), \nu_L\Lambda^{-1}_f(\bar{l}))\\ &= \left(\left(\Lambda^{-1}_f\right)^*\circ(\nu^*_L G)^\flat(l)\right)(\bar{l}) \ .\end{aligned}$$

Then,

$$A_f = \left(\Lambda^{-1}_f\right)^*\circ(\nu^*_L G)^\flat \ . \tag{12.5}$$

From (12.5) and (12.4) we obtain that (12.3) is

$$-i\circ\hat{\Lambda}_f\circ i = (\nu_L^* G)^\sharp\circ \left(\Lambda_f^{-1}\right)^* \circ(\nu_L^* G)^\flat \ ,$$

and with this formula we finish the proof of Proposition 12.28. □

12.3. Conformally symplectic vector bundle automorphisms

Many of the results of this section can be generalized to conformally symplectic vector bundle automorphisms in a complex symplectic vector bundle. The following is a brief summary of conformally symplectic vector automorphisms.

These automorphisms appear naturally in several models of statistical mechanics systems that at the same time are forced and dissipate. See, for example [**WL98**]. Indeed, the paper [**WL98**] observed that the existence of the conformal symplectic structure explained the so-called *Pairing rule for Lyapunov exponents* which had been observed empirically in [**DM96**]. This paring rule for Lyapunov exponents, can be generalized to other spectral properties as we will see.

Definition 12.29. *Let $\sigma : \mathcal{P} \to \mathbb{C}$ be a non-vanishing function. A vector bundle automorphism $M_f : E \to E$ is conformally symplectic with factor σ iff*

$$M(\theta)^{*2}\Omega_{f(\theta)} = \sigma(\theta)\Omega_\theta \ ,$$

that is to say,

$$\forall v_\theta, w_\theta \in E_\theta \ \Omega_{f(\theta)}(M(\theta)v_\theta, M(\theta)w_\theta) = \sigma(\theta)\Omega_\theta(v_\theta, w_\theta) \ .$$

We say that M_f is conformally symplectic. If $\sigma = 1$, we say that M_f is symplectic.

Remark 12.30. One of the natural places where these conformally symplectic vector bundles appear is when E is the tangent bundle of a symplectic manifold with symplectic form Ω, M is f^* and it satisfies $f^*\Omega = \sigma\Omega$ for some function σ.

This situation, appears in practical problems [**WL98**] such as the Gaussian thermostat, but it is quite rigid.

Note that

$$0 = f^* d\Omega = d\sigma\Omega = d\sigma \wedge \Omega$$

Hence, if the dimension of the manifold is larger than 2, we obtain $d\sigma = 0$. Hence, for dimensions greater than 2, the only conformal factors possible are constants.

We note that, besides the conformally symplectic systems in the sense mentioned above, in the literature, one can find the name conformal symplectic structure in different meanings, which we will not discuss here, even if they lead also to very interesting geometries [**Vai85, DO98, Ban02**].

For our purposes in this section, we will not be assuming that the bundle is a tangent bundle and that the structures on it correspond to structures on the manifold. Hence, these considerations do not play a role in this section and the results apply also to the linearization of the maps preserving some of the conformal structures mentioned above. The effect of assuming that the things come from the lift of structures on a manifold will be considered in Chapter 13.

We summarize now the results for the spectrum of conformally symplectic vector bundle automorphisms.

THEOREM 12.31. *Let $M_f : E \to E$ be a conformally symplectic vector bundle automorphism with factor σ. Assume one of the following hypotheses:*

- *The modulus of σ is constant, with $\hat{\sigma} = |\sigma(\theta)|$ for all $\theta \in \mathcal{P}$;*
- *f is uniquely ergodic and $\hat{\sigma} = \exp\left(\int \log|\sigma(\theta)|\, d\mu\right)$, where μ is the invariant measure.*

Then:

a) $z \in \mathcal{A}\mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E)) \Leftrightarrow \dfrac{\hat{\sigma}}{z} \in \mathcal{A}\mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E))$.

b) *The symplectic orthogonal subbundle F^Ω of an invariant subbundle $F \subset E$ is also invariant. (In fact, this works without assumptions on the factor σ).*

c) *Define $\gamma = \sqrt{\hat{\sigma}}$. Suppose that the spectrum $\mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E))$ has a gap in the circle of radius $\lambda < \gamma$. Then, there is another gap in the circle of radius $\mu = \hat{\sigma}\lambda^{-1} > \gamma$. Then, $E^{>\lambda} \cap E^{<\mu}$ and $E^{<\lambda} \oplus E^{>\mu}$ are complementary invariant symplectic subbundles, and $E^{<\lambda}$ and $E^{>\mu}$ are isotropic subbundles of the same rank.*

d) *The annular hull of the spectrum is of the form*

$$\mathcal{A}\mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E)) = \bigcup_{i=1}^{l} \mathcal{A}[\lambda_i, \mu_i] \cup \mathcal{A}[\rho, \rho^{-1}\hat{\sigma}] \cup \bigcup_{i=1}^{l} \mathcal{A}[\hat{\sigma}\mu_i^{-1}, \hat{\sigma}\lambda_i^{-1}]$$

where $l \geq 0$, $\lambda_1 \leq \mu_1 < \lambda_2 \leq \mu_2 < \cdots < \lambda_l \leq \mu_l < \gamma = \sqrt{\hat{\sigma}}$, and $\mu_l < \rho$ (if $\rho > \gamma$ we assume $\mathcal{A}[\rho, \hat{\sigma}\rho^{-1}] = \emptyset$). Let

$$E = \bigoplus_{i=1}^{l} E^{\gamma s,i} \oplus E^{\gamma c} \oplus \bigoplus_{i=1}^{l} E^{\gamma u,i}$$

be the corresponding spectral decomposition. Then

- *for each $i = 1, \ldots, l$ $\dim E^{\gamma s,i} = \dim E^{\gamma u,i}$, and $E^{\gamma s,i} \oplus E^{\gamma u,i}$ is an invariant symplectic subbundle;*
- *$E^{\gamma s} = \bigoplus_{i=1}^{l} E^{\gamma s,i}$ and $E^{\gamma u,i} = \bigoplus_{i=1}^{l} E^{\gamma u,i}$ are the γ-stable and γ-unstable subbundles $E^{<\gamma}$ and $E^{>\gamma}$, respectively;*
- *$E^{\gamma c}$ is an invariant symplectic subbundle, whose orthogonal is $(E^{\gamma c})^\Omega = E^{\gamma s} \oplus E^{\gamma u}$.*

Remark 12.32. Recall that under the assumption of the existence of an unique invariant measure the homeomorphism f is APD, and so the spectrum involved is rotationally invariant.

Proof: The proof of a) is an immediate consequence of

$$\Omega^\sharp_\theta M(\theta)^* \Omega^\flat_{f(\theta)} = \sigma(\theta) M^{-1}(\theta) \ ,$$

and Propositions 2.44,11.4.

To prove b), just notice that the relation of symplectic orthogonality is preserved by conformally symplectic vector bundle maps.

The proof of c) is again similar to the corresponding result in Proposition 12.18. For instance, for $v_\theta, w_\theta \in E^{<\lambda}$, and $m > 0$,

$$\begin{aligned}
|\Omega_\theta(v_\theta, w_\theta)| &\leq \frac{1}{|\sigma(\theta,m)|} |\Omega_{f^m(\theta)}||M(\theta,m)v_\theta||M(\theta,m)w_\theta| \\
&\leq C_\varepsilon^2 \|\Omega\|_\infty \frac{(\lambda-\varepsilon)^{2m}}{|\sigma(\theta,m)|} |v_\theta||w_\theta| \ ,
\end{aligned}$$

and the last term tends to zero when $m \to +\infty$ if ε is small enough, because $\lambda^2 < \hat{\sigma}$. So then, $\Omega_\theta(v_\theta, w_\theta) = 0$.

The rest of the proof follow similar lines. □

12.4. Volume preserving transfer operators

In this section we assume that M_f preserves a volume element defined on each fiber of the vector bundle E, whose rank is n.

Definition 12.33. *A* volume vector bundle *is a vector bundle E over a manifold $\mathcal{P}$ equipped with a section $\mathcal{V} : \mathcal{P} \to L^n_a(E;\mathbb{C})$ of the bundle of skew-symmetric n-linear forms that is non degenerate on each fiber E_θ.*

Definition 12.34. *A vector bundle automorphism $M_f : E \to E$ is volume preserving automorphism iff*

$$M(\theta)^{*n} \mathcal{V}_{f(\theta)} = \mathcal{V}_\theta \ ,$$

that is to say,

$$\forall v^1_\theta, \dots, v^n_\theta \in E_\theta \ \mathcal{V}_{f(\theta)}(M(\theta)v^1_\theta, \dots, M(\theta)v^n_\theta) = \mathcal{V}_\theta(v^1_\theta, \dots, v^n_\theta) \ .$$

Proposition 12.35. *Let $\mathcal{M}_f : E \to E$ be a volume preserving vector bundle automorphism. Let*

$$\mathcal{A}\mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E)) = \bigcup_{i=1}^k \mathcal{A}_i$$

be the annular hull of the spectrum, where each spectral annulus $A_i = \mathcal{A}_{\lambda_i,\mu_i}$ has multiplicity n_i. Then:

$$\lambda_1^{n_1} \dots \lambda_k^{n_k} \ \leq \ 1 \ \leq \ \mu_1^{n_1} \dots \mu_k^{n_k} \ .$$

Proof:

Let

$$E = \bigoplus_{i=1}^{k} E^i \ . \tag{12.6}$$

be the corresponding spectral splitting. The rank of the each subbundle E^i is n_i, and it is characterized, according to Theorem 2.18, by the rates of growth under iteration:

$$v_\theta \in E^i_\theta \iff \forall m \geq 0 \ \begin{cases} |M(\theta, m)v_\theta| \leq C_\varepsilon(\mu_i + \varepsilon)^m |v_\theta| \ , \\ |M(\theta, -m)v_\theta| \leq C_\varepsilon(\lambda_i - \varepsilon)^{-m} |v_\theta| \ , \end{cases} \tag{12.7}$$

where $\varepsilon > 0$ is small enough.

Fixed $\theta \in \mathcal{P}$, for each $i = 1, \dots, k$ we consider a vector base of E^i_θ given by $v^i_1, \dots, v^i_{n_i}$. Then, for $\varepsilon > 0$ small enough,

$$\begin{aligned} 0 &\neq |V_\theta(v^1_1, \dots, v^1_{n_i}, \dots, v^k_1, \dots, v^k_{n_k})| \\ &= |V_{f^m(\theta)}(M(\theta,m)v^1_1, \dots, M(\theta,m)v^1_{n_i}, \dots, M(\theta,m)v^k_1, \dots, M(\theta,m)v^k_{n_k})| \\ &\leq \|V\|_\infty C^n_\varepsilon (\mu_1 + \varepsilon)^{mn_1} \dots (\mu_k + \varepsilon)^{mn_k} \ , \end{aligned}$$

and then

$$\begin{aligned} 1 &= \lim_{m \to \infty} |V_\theta(v^1_1, \dots, v^1_{n_i}, \dots, v^k_1, \dots, v^k_{n_k})|^{\frac{1}{m}} \\ &\leq (\mu_1 + \varepsilon)^{n_1} \dots (\mu_k + \varepsilon)^{n_k} \ . \end{aligned}$$

Therefore,

$$1 \leq \mu_1^{n_1} \dots \mu_k^{n_k} \ .$$

The second inequality follows using similar arguments. □

Remark 12.36. If the annuli are in fact circles, i.e. $\lambda_i = \mu_i = \rho_i$, then we have the equality

$$1 = \rho_1^{n_1} \dots \rho_k^{n_k} \ .$$

CHAPTER 13

Spectrum in locally constrained spaces

13.1. Introduction

In several applications, it is natural to consider transfer operators acting on vector fields which satisfy some constraints of a local nature. The precise definitions will follow later, but we anticipate that *(locally) constrained spaces* are spaces of vector fields (or sections) whose derivatives have to satisfy some relations. Examples to keep in mind, are spaces of vector fields which have zero divergence or spaces of forms which are exact. The name *(locally) constrained* makes a reference to the fact that the nature of the constrain is a relation on derivatives. By examining an arbitrarily small enough region, we may conclude that the space does not satisfy the constraint.

There are many situations where these local considerations are important. Let us just anticipate two examples, which will be discussed in more detail in Section 13.2.

In hydrodynamics, it is natural to consider the evolution equations acting on volume preserving vector fields. In hydrodynamics parlance, the push forward on vector fields is just the *transport operator of vector quantities.*

It is well known that Euler equations are just the transportation of the velocity vector. So that the variational equations of Euler equations is just the transportation of the small perturbations.

The push-forward also appears naturally in the magneto-hydrodynamics of perfectly conducting fluids. The fact that the conductivity is perfect means that the magnetic fields generate currents, which in turn generate magnetic fields. If we ignore the effect of the currents on the velocities of the fluid (a natural assumption for small magnetic fields) it is a standard calculation, that the magnetic vector field is transported by the velocity field. Since, by Maxwell's equations the magnetic fields have zero divergence, it is natural to consider the push-forward acting on this space. For a reviews of the physical discussions of this motivation, we refer to [**RS92, Chi92, Mof95, FMN03, Gil03**]. Applications in hydrodynamics are in [**VF93, SV04, SF05**].

Another motivation from the field of pure mathematics is the study the effect on dynamical systems on topology (e.g. we want to study the effect in homology), it is natural to study the spectrum on closed forms. When studying regularity of solutions of invariance equations, it is natural to study the spectrum in spaces of jets.

We can consider spaces in which the divergence zero constrain is defined in a weak sense (the flux over every small enough ball vanishes). With this definition, the space of continuous zero divergence vector fields is a closed subspace of the space of continuous vector fields. Similarly, the space of forms such that the integral is zero over any small enough boundary, is a closed space of the set of forms. Of course, both subspaces are strict subspaces for non-trivial manifolds.

If $Y \subset X$ is a closed linear subspace and $A : X \to X$ is a linear operator such that $A(Y) = Y$, one can guess that $\mathrm{Spec}(A, Y) \subset \mathrm{Spec}(A, X)$. Indeed, such inclusion is true in finite dimensional spaces or in the case of self-adjoint or normal operators, which dominates the treatments of spectral theory in the literature. Generally speaking, the inclusion follows if there exists a projection P of X over Y that commutes with A, see Theorem A.9. As it turns out, we will see that in the case of transfer operators acting locally constrained spaces the opposite inclusion is true!

Indeed, one can be very precise about the way that the spectrum grows. The precise result is the *No Gaps Theorem*, see Theorem 13.11.

Of course, the reason why the spectrum grows is that, restricting the space, may generate residual spectrum. See Section A.1.0.1. Some simple examples of the growth of the spectrum by restricting the space can be found in [**HP69**].

This No Gaps Theorem was proved first in [**dlL93**] in the context of the dynamo problem. See also [**CLMS95, CLMS96**].

In this chapter, we will show that this phenomenon happens not only for zero divergence vector fields, but also for spaces of forms. As we will see this has several consequences for dynamical systems. In particular, we will find relations between this phenomenon and some global properties of dynamical systems, in particular, we will find a relation with a famous conjecture of R. Bowen about the homology of Anosov systems and we will also find some relations with regularity theory of homology equations and with the integrability of intermediate foliations.

13.2. Spaces of sections: vector fields, forms and jets with constrains

We will denote by $\chi^1_{C^0}$, χ^1_B, $\chi^{*1}_{C^0}$, χ^{*1}_B the Banach spaces obtained by complexifying the spaces of continuous vector fields, bounded vector fields, continuous forms, and bounded forms on a manifold $\mathcal{P}$, equipped with the supremum norm.

These are, of course, the same as taking sections of the complexification of the tangent and cotangent bundle.

Other well known examples are tensor fields of covariance (n, m) and we will denote by $\chi^{n,m}_{C^0}$, $\chi^{n,m}_B$ the Banach spaces of C^0, and bounded tensor fields. (Again, in this chapter we will always consider complexification.) Of course, n forms can be identified with antisymmetric tensor fields of covariance $(0, n)$. The space of n forms is a closed subspace of the space of tensor fields of covariance $(0, n)$

We will also consider spaces of sections in a jet bundle. There are many equivalent definitions of n-jets of functions at a point. The most common one is equivalence classes of C^∞ functions under the equivalence relation

$$\Phi \approx \tilde{\Phi} \Leftrightarrow D^i\Phi(x_0) = D^i\tilde{\Phi}(x_0) \qquad i \leq n \ .$$

We can think of them as the set of Taylor polynomials of order n after we take coordinates. (By Borel's theorem given any symmetric polynomial we can find a C^∞ function which has it a Taylor polynomial.) Of course, jets at a point are a linear space, which can be given a canonical norm if there is a metric in the tangent space – as we will assume is the case –. We will denote by $J^n_{C^0}$ and by J^n_B the spaces of continuous and bounded sections in the n-jet bundle equipped with the supremum norm. We note that if the function f is C^n, there is a natural push-forward in the space of jets which corresponds to considering associating to the function Φ the function $\Phi \circ f$. It is easy to note that if Φ and $\tilde{\Phi}$ agree to order n at x, then, $\Phi \circ f$ and $\tilde{\Phi} \circ f$ agree to order n at $f(x)$. Hence, the operation projects down to the space of jets.

Similarly, we will consider other spaces of jets and their corresponding spaces of sections. For example, we can consider the jets of sections of a vector bundle E, which we will denote by $J^n(E)$. Again, we will denote by a subindex the regularity we are considering.

Particular cases of this situation will be jets of vector fields and forms or, more generally jets of tensor fields of type k, l. We will denote by $J^{n,k,l}_{C^0}$ $J^{n,k,l}_B$ the spaces of sections of n-jets to tensor fields of type k, l, continuous and bounded, respectively. Of course, the jets of functions can be considered a particular case of those. The jets of forms can be considered as a closed subspace.

We note that if we have a vector bundle map $\mathcal{M}_f$ acting on sections of tensor fields $\chi^{k,l}$ and if $\mathcal{M}_f$ and f are C^n, it is possible to define an operator on $J^{n,k,l}$ associated to it. If Ψ and $\tilde{\Psi}$ agree to order n at x, then $\mathcal{T}\Psi$ and $\mathcal{T}\tilde{\Psi}$ will agree to order n at $f(x)$. Hence, the operation can be defined on jets. This is a very common construction when we try to consider regularity properties of invariant objects. We will refer to these operators as *derived operators*.

In geometry, it is also customary to think about jets of other objects such as submanifolds. Since the goal of this paper is to consider spectral properties of linear operators, we will need that the spaces of jets we consider have a linear structure. Hence, we will consider the jets motioned above. Nevertheless, we will point out that the analysis in the space of linear jets can be used as a tool for the study of non-linear objects via linearization, implicit function theorems, etc. Indeed we have included a section on applications to dynamical systems.

Note that since there is a natural way of considering J^n as containing in J^{n-1} we can consider $J^{=n} = J^n/J^{n-1}$ and therefore, $J^n = \bigoplus_{i=0}^n J^{=i}$. This corresponds to sections of Taylor homogeneous polynomials of order n, and each $J^{=i}$ can be identified with sections of symmetric i multilinear operators, see Chapter 11. Identical considerations can, of course, be made for the spaces of jets of tensor fields and derived operators.

An important assumption that we will make about operators acting on spaces of jets is that they are upper triangular with respect to the above decomposition on degrees. This assumption is satisfied by all the geometric operators that we will consider in our applications.

We call attention to the fact that a section of the space of jets means choosing a jet at each point. Note that a section of the jet bundle does not need to agree at each point with the jet of a single function. For example, the space of 1-jets in a manifold is just space of vector fields — any vector at a point can be the gradient at a function –. Nevertheless, to be the gradient at a function on an open set requires extra conditions such as zero curl and that the integral over a homologically non-trivial path vanishes. Moreover, if we consider also second jets of a single function, taking the derivative of the first jet, we can recover the second one. If we choose a section of the 2-jet bundle that does not preserve this constraint, it cannot be the second jet of a function. These observations will become very important later.

Besides the above spaces of sections of vector bundles we will be interested in some of their closed subspaces. For example we will consider spaces of continuous vector fields with zero divergence or spaces of 1-forms which are closed or exact as well as spaces of jets that are locally, or globally, the jet of a single function. As we mentioned before, one motivation for the study of zero divergence is magneto-hydrodynamics and a motivation for closed forms is the study of transfer operators on differentiable functions.

The definitions that we will find useful in our study for zero divergence vector fields and for closed and exact forms are in a weak sense.

Definition 13.1. *We will consider the following spaces:*

$$\begin{aligned}
\chi^1_{nd} &= \{v \in \chi^1_{C^0} \mid \mathit{Flux}_\Sigma(v) = 0 \ \ \forall\ \Sigma\ C^1 \textit{ boundariless surface}\},\\
\chi^1_{lnd} &= \{v \in \chi^1_{C^0} \mid \mathit{Flux}_\Sigma(v) = 0 \ \ \forall\ \Sigma,\ C^1 \textit{ surface } \Sigma = \partial V\\
&\qquad\qquad \textit{for } V \textit{ a region in space contractible to a point.}\},\\
\chi^{*1}_e &= \{\gamma \in \chi^{*1}_{C^0} \mid \int_\beta \gamma = 0\ ,\ \ \forall\ \beta\ C^1 \textit{ path.}\},\\
\chi^{*1}_c &= \{\gamma \in \chi^{*1}_{C^0} \mid \int_\beta \gamma = 0\ ,\ \forall\ \beta\ C^1 \textit{ path homologous to zero}\}.
\end{aligned}$$

Note that $\chi^1_{nd} \subset \chi^1_{lnd} \subset \chi^1_{C^0}$, $\chi^{*1}_e \subset \chi^{*1}_c \subset \chi^{*1}_{C^0}$ and the inclusions are, in general, non trivial (χ^{*1}_c/χ^{*1}_e is of course, the first cohomology of the manifold and the others are significantly subtler except in the case of one-dimensional manifolds). Moreover all the inclusions are closed subspaces since the definitions make it clear that the spaces are the intersection of closed conditions.

Remark 13.2. We point out that the definition of zero divergence as the zero flux is more natural not only from the mathematics point of view but also from the physical point of view. When we consider magnetic fields in a region that contains two media, the H magnetic field (independent of the media) is continuous, but the B magnetic field, which is affected by the permeability of the media may be discontinuous. Nevertheless, the condition of zero flux is always enforced.

We will also consider J^n_c, J^n_e the complexified Banach spaces of C^0 sections of n-jets which agree with the jet of a C^n function in a sufficiently small neighborhood of any point and in the whole manifold. respectively. We have $J^n_c \subset J^n_e$ and the quotient of the two spaces is the homology of the manifold. The spaces J^n_c and J^n_e are closed subspaces of $J^n_{C^0} \subset J^n_B$.

If we recall that a 1-jet is the value of the function and its differential and that if we fix the value of the function at one point and have given the differential function we can recover the function at any point, we see that we have a natural identification

$$\chi^{*1}_e \oplus \mathbb{C} \approx J^{=1}_e \oplus \mathbb{C} \approx J^1_e \approx C^1$$

We will add a subindex c to denote jets that are locally the jet of a function or tensor field and an e to denote those that are the jet of a function or tensor field defined globally. Similar identifications occur for functions with a higher number of derivatives or for the spaces of jets of tensor fields. When we consider geometrically natural operators such as the push forward, they act independently on the constant part and on the one form part. So that the decomposition of the spectrum is block diagonal. (In more common language, to know the derivatives of a transported function $\varphi \circ f$, we do not need to know the values of the function. It suffices to know the values of the derivatives.)

When we consider higher order jets similar phenomena occur. If we are reconstructing a function – or a tensor field – from its jet, it suffices to consider only the higher order jet and a finite dimensional space of initial data that allow us to perform the integrations. Hence, when discussing the effect of the local constraints, it makes sense to consider only the the components of maximal order. That is,

$$J^{=k}_e \oplus \mathbb{C}^{s(k,d)} \approx C^k$$

where $s(k, d)$ is a combinatorial number depending on the degree and the dimension of the manifold, that is the dimension of the space of jets at a point of degree smaller than k.

For $n \geq 2$ there are two complications with this identification. The first one is that it forces us to consider non-local operators. Even for the push forward, we note that $\mathrm{D}^2(\varphi \circ f) = (\mathrm{D}^2\varphi) \circ f\, \mathrm{D}f^{\otimes 2} + (\mathrm{D}\varphi \circ f)\mathrm{D}^2 f$. If we decide to express the RHS only in terms of $\mathrm{D}^2\varphi$ and the initial values, we have to express $\mathrm{D}\varphi$ in terms of $\mathrm{D}^2\varphi$ using an integral. The second complication is that the decomposition of the action on J^n by degree is not block diagonal but only upper triangular.

This considerations suggest that we consider $J^{=n}$ as the unconstrained space of integrable n-jets. Of course, a better justification will come from the statements of the theorems. In dealing with them we will have to take into account the peculiarities.

From now on, when dealing with exact forms, zero divergence filed, we will make the standing assumption that the manifold M we are considering has dimension greater or equal than two. In one dimensional manifolds, zero divergence vector fields are constants, all one forms are closed, and therefore the results we present here become trivially true or trivially false and it is quite easy to figure out which one is which and it would be space consuming to deal with this special case. We leave the case of the dimension equal to one to the exceedingly meticulous reader.

Definition 13.3. *We will refer to the spaces*

$$\chi^1_{nd}, \chi^1_{lnd}, \chi^{*1}_c, \chi^{*1}_e, J^{=n}_c, J^{=n}_e, J^{=n}_c(E), J^{=n}_e(E)$$

as "locally constrained" spaces, since they have to satisfy some relations around each point and we will refer to

$$\chi^1_B, \chi^1_B, \chi^{*1}_B, \chi^{*1}_B, J^{=n}_B, J^{=n}_B, J^{=n}_B(E), J^{=n}_B(E)$$

as the corresponding unconstrained spaces, respectively.

Remark 13.4. We can take the unconstrained spaces with C^0 regularity. The spectral implications are the same, since considering C^0 or bounded sections do not change the spectrum and Weyl spectrum of transfer operators.

Many operators – indeed most of the operators of geometric interest – have the property that, even if defined in the unconstrained space preserve the constrained space. Important examples of maps over a diffeomorphism are the geometric push forward and pull back. On vector fields, the push forward is defined by

$$(f_*v)(x) = (\mathrm{D}f(f^{-1}(x))v(f^{-1}(x))$$

and, on forms, the push forward is

$$(f_*u)(x) = (\mathrm{D}f^{-1}(x))^\top u(f^{-1}(x))$$

where $(\ )^\top$ denotes the adjoint with respect to the metric.

For diffeomorphisms the pull back f^* is just the inverse of the push forward. We remark that for maps that are not injective it is possible to define the pull-back for forms and the push-forward for vector fields. Some of the arguments that we develop here works in this generality. For example, the invariance of the spectrum under rotations, but other arguments that depend on the existence of complementary splittings with contraction in the future or in the past does no.

We also point out that if f is volume preserving map f_* preserves the spaces of zero divergence vector fields: $f_*\chi^1_{nd} = \chi^1_{nd}$. Any diffeomorphism preserves the space of closed and exact forms: $f_*\chi^{*1}_c = \chi^{*1}_c$, $f_*\chi^{*1}_e = \chi^{*1}_e$.

In all those cases, we will show that the Weyl spectrum agrees with that of the unconstrained spaces. (We call attention that we will not make any assumptions on the dynamics of the map, in contrast with the results in [**dlL93**] which assumed APD.) That is, we will show that

$$\operatorname{Spec}_W(f_*, \chi^1_{nd}) = \operatorname{Spec}_W(f_*, \chi^1_{C^0}), \tag{13.1}$$

and analogously for closed and exact 1-forms and for closed and exact n-jets. In fact,

$$\operatorname{Spec}_W(f_*, \chi^{*1}_e) = \operatorname{Spec}_W(f_*, \chi^{*1}_c) = \operatorname{Spec}_W(f_*, \chi^{*1}_{C^0}) \tag{13.2}$$

so that the Weyl spectra in (13.1) and (13.2) are identical. The results in spaces of n-jets with $n \geq 2$ are somewhat more complicated to state since they require the consideration of what is the appropriate local model. We will postpone a precise statement.

We will also show that the spectrum in all of those constrained spaces does not contain a gap which includes a circle centered at the origin. If the map is APD, and therefore the spectrum is rotationally invariant, we conclude that, for some $0 < \lambda_- \leq \lambda_+$

$$\operatorname{Spec}(f_*, \tilde{\chi}) = \mathcal{A}_{\lambda_-,\lambda_+},$$

where $\tilde{\chi}$ is any of the locally constrained spaces mentioned above. This has the corollary that in all these constrained spaces, the push forward has residual spectrum filling the gaps of the spectrum in an unconstrained space.

Even if we will not discuss it in this paper, we point out that there is a generalization of this theory for vector fields. From the point of view of many applications to rates of growth it suffices to take time one maps and apply the theory presented here. Nevertheless it is interesting to develop the theory of vector fields in its own. Many of the techniques we present here work, mutatis mutandis, but the unboundedness of the vector field makes it significantly more complicated since one has to take into consideration domains, the fact that applying the operator to an approximate eigenfunction may render it not an approximate eigenfunction etc. One interesting question is to prove that the spectrum of the time one map is the exponential of the flow and – much more difficult – to show that the spectrum of the flow is the full inverse image under the exponential of the spectrum of the time one map. Other questions of interest for dynamical systems include the study of the dependence of the spectrum on the map, and its stability with respect to random perturbations, the study of operators acting of Hölder sections, etc. We plan to come back to these questions, but we cannot include them here.

13.3. Mather theory in locally constrained spaces

13.3.1. Definition of locally constrained space of sections. In this section we will study the spectrum of bundle maps on constrained spaces of sections (see Definition 13.5 below), keeping in mind the spaces χ^1_{nd}, χ^{*1}_c, χ^{*1}_e, J^n_c, J^n_e, $J^{n,i,j}_e$ introduced in the previous section.

For all of them, we will establish several results that can be described informally as follows. Assume f is APD.

i) The Weyl spectrum of the operator $\mathcal{M}$ in a constrained space is the same as in the corresponding unconstrained space.
ii) Annular gaps in the spectrum of $\mathcal{M}$ acting in a locally constrained space exist also in the spectrum of $\mathcal{M}$ acting on the corresponding unconstrained space. Moreover, they correspond to local projections.

   The point of ii) is that there are no local projections that leave invariant the constrained spaces. So that ii) is only an intermediate result along the way to show that its hypothesis do not hold. Hence,
iii) There are no annular gaps on the spectrum of $\mathcal{M}$ in the locally constrained spaces.

Therefore, when the Weyl spectrum is invariant under rotation the locally constrained spaces can be obtained by “filling in” the gaps of the spectrum in the unconstrained spaces. Since the Weyl spectrum is the same for the locally constrained and the unconstrained spaces, we conclude that the regions that have been filled in are residual spectrum.

We point out that the arguments employed are rather general (as it is made plain by the fact that we can deal with 5 or 10 different constrained spaces). We only need that functions with a well defined direction can be approximated in the locally constrained space after changing the modulus — but not the direction — and that these spaces are not invariant under local projections.

This “no gap phenomenon” was discovered in [**dlL93**] for the space of zero divergence vector fields, an example motivated by applications to magneto-hydrodynamics. The arguments were streamlined and extended in [**CLMS95, CLMS96, CL99**].

It seems quite plausible that the arguments presented here can be adapted to other problems, e.g., spectra of differentiable tensor fields and may be to other regularity classes such as Sobolev spaces. This could be interesting for the hydrodynamic applications. Nevertheless, we did not consider it in this paper.

A formalization of the idea of locally constrained space of sections follows.

DEFINITION 13.5. *Let E be a Finslered vector bundle on a base manifold $\mathcal{P}$. Let $\Gamma \subset \Gamma_B(E)$ be a closed Banach subspace (with norm $\|\|_\infty$). We* *will say the the space of sections* Γ *is* locally constrained *iff the following properties hold:*

(a) *Fattening condition: vectors are well approximated by sections in* Γ*. That is, given* $v_0 = v_{\theta_0} \in E \setminus E_0$ *and a continuous extension* $v \in \Gamma_{C^0}(E)$ *with* $v(\theta_0) = v_0$*, for all* $\sigma > 0$ *there exist a section* $w \in \Gamma$ *and a function* $\rho \in B(\mathcal{P})$ *such that:*

(a.1) $\operatorname{supp} w \subset B_\sigma(\theta_0)$, $w(\theta_0) = v_0$;

(a.2) $\operatorname{supp} \rho \subset B_\sigma(\theta_0)$, $\rho(\theta_0) = 1$, $0 \le |\rho(\theta_0)| \le 1$;

(a.3) $|w(\theta) - \rho(\theta)v(\theta)| < \sigma|v_0|$.

(b) *Constraining condition:* Γ *is not stable by projection on proper subbundles. That is, for all* $F \subset E$ *proper subbundle, with projection* Π_F*, there exists a locally constrained section* $v \in \Gamma$ *whose projection on* F *is not locally constrained, i.e.* $\Pi_F v \notin \Gamma$*.*

Remark 13.6. The role played by the continuous extension in condition (a) above is just to capture the idea that w is "almost parallel" to v_0. In a trivialization, we can use the statement $|w(\theta) - \rho(\theta)v_0| < \sigma|v_0|$. Notice also that the result is independent of the extension we choose, because if v_1, v_2 are continuous extensions and w is close to ρv_1, then

$$\|w - \rho v_2\|_\infty \le \|w - \rho v_1\|_\infty + \|\rho(v_1 - v_2)\|_\infty \le \sigma|v_0| + \eta(\sigma) \ ,$$

where $\eta(\sigma)$ is the modulus of continuity of $v_1 - v_2$.

Instead of using trivializing neighborhoods to fattening up v_0 to v, we can also use a connector. This device define a parallel transport $\mathrm{T}_{\theta_0,\theta}$ between close fibers E_{θ_0} and E_θ (see [**HPPS70, HdlL03a**]). Then, we can define v in $B_\sigma(\theta_0)$ by $v(\theta) = \rho(\theta)\mathrm{T}_{\theta_0,\theta}v_0$.

13.3.2. Equality of Weyl spectra. In this subsection we show that in locally constrained spaces the Weyl spectrum is the same as in the non-constrained spaces. In this subsection we state the abstract result, that will be applied later to the locally constrained spaces considered in the previous section (zero divergence vector fields, closed forms, exact forms, integrable 1-jets) the Weyl spectrum is the same as in the non-constrained spaces. For the n-jet spaces we will also have some results, but they are more complicated to state – due mainly to the difficulty of selecting a local model –,hence, we will state them carefully in a subsequent subsection.

The result of equality of Weyl spectrum holds under the assumption that vectors are well approximated by locally constrained sections.

Theorem 13.7. *Let* $M_f : E \to E$ *be a vector bundle automorphism. Let* $\Gamma \subset \Gamma_B(E)$ *be a closed Banach subspace (with norm* $\|\|_\infty$*) satisfying the fattening condition (a) of Definition 13.5. Assume that* Γ *is invariant under* $\mathcal{M}_f$*. Then:*

$$\mathrm{Spec}_W(\mathcal{M}_f, \Gamma) = \mathrm{Spec}_W(\mathcal{M}_f, \Gamma_B(E)) \ .$$

Proof: Since $\mathrm{Spec}_W(\mathcal{M}_f, \Gamma) \subset \mathrm{Spec}_W(\mathcal{M}_f, \Gamma_B(E))$ follows from Functional Analysis, it suffices to establish the opposite inclusion to prove the first part of Theorem 13.7.

Let $z \in \mathrm{Spec}_W(\mathcal{M}_f, \Gamma_B(E))$. Given an approximate eigensection $v \in \Gamma_B(E)$ we will produce another one $w \in \Gamma$. Notice that we can assume that the bounded approximate eigensection is of a particularly simple kind, that is satisfying one of the set of properties (a) and (b) of Lemma 3.1. Then, we will fatten up this localized approximate eigensection to produce an approximate eigensection in Γ.

(a) In this case we have obtained a bounded approximate eigensection v supported in a finite segment of orbit $\{f^i(\theta_0)\}_{i=-N}^N$, with $f^i(\theta_0) \neq f^j(\theta_n)$ for $i \neq j$ with $|i| \leq N, |j| \leq N$, with $N = [1/\varepsilon]$. The error as approximate eigensection is smaller than $2\varepsilon|z|$.

Moreover, if we denote $v_i = v(f^i(\theta_0))$, we have chosen those vectors in such a way that

$$v_{i+1} = \frac{\gamma_i}{z} M(f^i(\theta_0)) v_i$$

for $i = -N, \ldots, N-1$. The γ's are constants in $\left[\frac{1}{2}, \frac{3}{2}\right]$, and satisfy

$$\left|\frac{1}{\gamma_{i-1}} - 1\right| |v_i| \leq 2\varepsilon \ , \tag{13.3}$$

that follows from the construction that makes v an approximate eigensection. We use the cocycle notation for $\gamma(i) = \gamma_{i-1} \ldots \gamma_1 \gamma_0$ if $i > 0$, $\gamma(0) = 1$ and $\gamma(i) = \gamma_i^{-1} \ldots \gamma_{-2}^{-1} \gamma_{-1}^{-1}$ if $i < 0$. Notice that

$$v_i = \frac{\gamma(i)}{z^i} M(\theta_0, i) v_0 \ ,$$

for $i = -N, \ldots, N$.

We will replace each of the vectors on the orbit by a function that has support in a small neighborhood chosen in such a way that it is still an approximate eigensection. Notice that we can pick coordinates around each of the points in the finite segment $\{f^i(\theta_0)\}_{i=-N}^N$ in such a way that the bundle is trivialized.

Let $\sigma > 0$ be small enough so that the neighborhoods $f^i(B_\sigma(\theta_0))$ with $i = -N, \ldots, N$ are disjoint and included in trivializing neighborhoods, and σ satisfies other smallness conditions that will be specified later. Then, we construct the section $w \in \Gamma$ and the function $\rho \in B(\mathcal{P})$ satisfying properties (a.1),(a.2),(a.3) of Definition 13.5. We denote $\delta(\theta) = w(\theta) - \rho(\theta) v_0$, that satisfies $\|\delta\|_\infty < \sigma |v_0|$.

We extend the definition of $w = w_0$ in $B_\sigma(\theta_0)$ around the points of the finite segment of orbit. So, if $\theta \in f^i(B_\sigma(\theta_0))$ with $i = -N, \ldots, N$ we set

$$w(\theta) = w_i(\theta) = \frac{\gamma(i)}{z^i} M(f^{-i}(\theta), i) w_0(f^{-i}(\theta)) \ ,$$

and $w(\theta) = 0$ otherwise.

Then, for $\theta \in f^i(B_\sigma(\theta_0))$ and working in coordinates,

$$\begin{aligned}
&|M(f^{-1}(\theta))w(f^{-1}(\theta)) - zw(\theta)| \\
&= \frac{1}{|z|^{i-1}} \left|(\gamma(i-1) - \gamma(i))M(f^{-i}(\theta), i)w_0(f^{-i}(\theta_0))\right| \\
&\le \frac{|(\gamma(i-1) - \gamma(i))|}{|z|^{i-1}} \left(\|M(\cdot, i)\|_\infty \|\delta\|_\infty + |\rho(f^{-i}(\theta))M(f^{-i}(\theta), i)v_0|\right) \\
&\le \frac{|(\gamma(i-1) - \gamma(i))|}{|z|^{i-1}} \left(\sigma\|M(\cdot, i)\|_\infty |v_0| + |\rho(f^{-i}(\theta))||M(\theta_0, i)v_0|\right. \\
&\quad \left. + |\rho(f^{-i}(\theta))||(M(f^{-i}(\theta), i) - M(\theta_0, i))v_0|\right) \\
&\le \frac{|(\gamma(i-1) - \gamma(i))|}{|z|^{i-1}} \left(\sigma\|M(\cdot, i)\|_\infty + \eta_i(\sigma) + \left|\frac{z^i}{\gamma(i)} v_i\right|\right) \\
&\le \frac{|(\gamma(i-1) - \gamma(i))|}{|z|^{i-1}} (\sigma\|M(\cdot, i)\|_\infty + \eta_i(\sigma)) + 2\varepsilon|z| \ ,
\end{aligned}$$

where η_i is the modulus of continuity of $M(\cdot, i)$. Notice that by taking σ small we make the first two terms arbitrarily small. Clearly, outside of the neighborhoods the difference is zero. So, we can make w an approximate eigensection in Γ, and we are able of fattening up localized approximate bounded eigensections under the alternative (a).

(b) The second alternative is that the approximate eigensections are supported on periodic orbits of minimal period N. Moreover, the points around the periodic point θ_0 of the approximate eigensection are also periodic, with period a multiple of N.

Much more, the approximate eigensection that is given by the vectors $v(f^i(\theta_0)) = v_i$, satisfies

$$v_{i+1} = \frac{1}{z_0} M(f^i(\theta_0))v_i$$

for all $i \in \mathbb{Z}$ (that means z_0^N is an eigenvalue of $M(\theta_0, N)$ with eigenvector v_0), and $|z - z_0| < \varepsilon$.

Proceeding analogously to the analysis of the previous alternative, we define for $\theta \in f^i(B_\sigma(\theta_0))$ with $i = 0, \ldots, N-1$

$$w(\theta) = w_i(\theta) = \frac{1}{z_0^i} M(f^{-i}(\theta), i)w_0(f^{-i}(\theta)) \ ,$$

and $w(\theta) = 0$ otherwise. The section is well defined because the points around θ_0 have period multiple of N. The rest of the analysis follows the lines of the alternative (a). □

REMARK 13.8. We call attention to the fact that Theorem 13.7 does not include any hypothesis on the dynamics of f.

REMARK 13.9. Notice that the fattening condition (a) in Definition 13.5 implies that evaluation of locally constrained sections at points are well defined operations, see hypothesis (b) in Theorem 3.14. Moreover, the equality of Weyl spectrum in Theorem 13.7 implies the inclusion (c) in Theorem 3.14. As a result, we have the inclusion

$$\mathcal{A}\mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E)) \subset \mathcal{A}\mathrm{Spec}(\mathcal{M}_f, \Gamma)\ . \tag{13.4}$$

REMARK 13.10. If f is APD then all the spectrum involved is rotationally invariant and the $\mathcal{A}$ can drop from (13.4). We have just to apply the equality of Weyl spectrum of Theorem 13.7 and the invariance of the spectrum under rotations of Theorem 3.11.

We point out, however, that the invariance under rotations could be done directly using the methods used in the proof of Theorem13.7, based strongly in the Mather localization arguments of Lemma 3.1. This is what was done in [**dlL93**].

13.3.3. Absence of gaps in the spectrum in locally constrained spaces. Now we can establish the *No Gaps Theorem*

THEOREM 13.11. *Let $M_f : E \to E$ be a vector bundle automorphism. Let $\Gamma \subset \Gamma_B(E)$ a locally constrained space of sections, invariant under $\mathcal{M}_f$. Then, there are no annular gaps in* $\mathrm{Spec}(\mathcal{M}_f, \Gamma)$*. That is, it is impossible to find $\rho > 0$ such that we have*

$$\begin{aligned} &\{z \mid |z| = \rho\} \cap \mathrm{Spec}(\mathcal{M}_f, \Gamma) = \emptyset\ ;\\ &\{z \mid |z| > \rho\} \cap \mathrm{Spec}(\mathcal{M}_f, \Gamma) \neq \emptyset\ ;\\ &\{z \mid |z| < \rho\} \cap \mathrm{Spec}(\mathcal{M}_f, \Gamma) \neq \emptyset\ . \end{aligned} \tag{13.5}$$

Proof: If ρ as in (13.5) existed, we could find nontrivial spectral projections $P^{<\rho}$, $P^{>\rho}$. Since by 13.7 we know that

$$\mathrm{Spec}_P(\mathcal{M}_f, \Gamma_B) \subset \mathrm{Spec}_W(\mathcal{M}_f, \Gamma_B) = \mathrm{Spec}_W(\mathcal{M}_f, \Gamma) \subset \mathrm{Spec}(\mathcal{M}_f, \Gamma),$$

so we can apply Theorem 3.14 and conclude that there should be a non-trivial splitting of the bundle $E = E^{<\rho} \oplus E^{>\rho}$ that corresponds to the spectral projections $P^{<\rho}$, $P^{>\rho}$. That is if $\Pi^{<\rho}$,$\Pi^{>\rho}$ are the bundle projections, then there are functional-geometrical identities

$$(P^{<\rho}v)(\theta) = \Pi_\theta^{<\rho} v(\theta)\ ,\ (P^{>\rho}v)(\theta) = \Pi_\theta^{>\rho} v(\theta)\ .$$

for all $v \in \Gamma$.

On the other hand, we note that for a given a non-trivial subbundle it is always possible to find a section in Γ whose projection is not in Γ.

We conclude that no such projections $P^{<\rho}$, $P^{>\rho}$ can exist and, therefore, that a ρ satisfying (13.5) cannot exist. □

As an easy corollary we obtain

COROLLARY 13.12. *Let $M_f : E \to E$ be a vector bundle automorphism over an APD homeomorphism. Let $\Gamma \subset \Gamma_B(E)$ a locally constrained space of sections, invariant under $\mathcal{M}_f$. Then, for some $0 < \rho_- \le \rho_+$ we have*

$$\mathrm{Spec}(\mathcal{M}_f, \Gamma) = \mathcal{A}_{\rho_-,\rho_+}$$

The proof consists in observing that, since f is APD, by Theorem 3.11, the only gaps in the spectrum that could occur are precisely annular gaps that are excluded by 13.11.

Note that the inner and outer boundaries $\{z \mid |z| = \rho_-, \rho_+\}$ are in the Weyl spectrum and hence in the spectrum of $\mathcal{M}_f$ in the unconstrained space $\Gamma_B(E)$. Conversely, the inner and outer boundaries of the spectrum in the unconstrained space are also in the spectrum in the constrained space.

Hence for all APD maps, $\mathrm{Spec}(\mathcal{M}_f, \Gamma)$ is obtained just by filling in the gaps of $\mathrm{Spec}(\mathcal{M}_f, \Gamma_B(E))$, the spectrum in the corresponding unconstrained space. Since the Weyl spectrum cannot increase, we conclude that the filling is in residual spectrum (see A.1.0.1).

13.4. Examples of locally constrained spaces of sections

In this section we will prove that the locally constrained spaces of sections χ^1_{nd}, χ^{*1}_e satisfy indeed Definition 13.5.

LEMMA 13.13. *The spaces χ^1_{nd}, χ^{*1}_e are locally constrained, that is, they satisfy the fattening condition and the constraining condition of Definition 13.5.*

Proof: The proof of Lemma 13.13 will have to be different for each of the constrained spaces considered in it. Notice that they satisfy the constraining condition of Definition 13.5, so we have to prove the fattening condition.

The case of zero-divergence vector fields was already considered in [**dlL93**] where a procedure to construct the approximate eigenfunctions was described in detail. Independently, and roughly at the same time, the paper [**Núñ94**] contained explicit formulas for approximate eigenfunctions of volume preserving vector fields, even if it did not draw conclusions about the spectrum. The work [**CLMS95, CL99**] contains very explicit formulas for the constructions sketched in [**dlL93**].

We will not repeat all the details here and we refer to the above literature for more details. Nevertheless, we will describe informally the idea. We consider two thin tubes of flow which widen at the ends. The wide sides start to turn around till they connect. A divergence free flow of a physical fluid in such a configuration will have a high velocity in the thin part of the tubes and its direction will be, very closely, in direction of the tubes. As the tubes are getting wider the flow is slowing so that when it starts to turn around its magnitude is very small.

Now we consider the case of χ^{*1}_e. Even if this is a particular case of the case J^n_e and could be done by the same methods, it is instructive to give an independent proof.

Working in a trivialization it suffices to produce a function with compact support and whose gradient is roughly in the direction of x_1. (We can without loss of generality assume that in the trivialization chosen the vector ν lies along the x_1 axis.)

The idea behind the construction is to obtain functions whose dependence in the x_1 variable is much stronger than that in the other variables. This can be easily achieved but taking a regular cut-off function and scaling it by very different factors.

If φ is a C^∞ cut-off function on the real line $\varphi(t) = 1$, $t \in [-1/2, 1/2]$; $\varphi(t) = 0$, $|t| > 1$; $0 \le \varphi \le 1$; $|\varphi'| \le 4$, we set:

$$u(x) = \int_{-\infty}^{x_1} \varphi(s^2/\varepsilon^4 + (x_2^2 + \cdots + x_d^2)/\varepsilon^2)\, ds. \tag{13.6}$$

We denote $\rho(x) = \varphi(s^2/\varepsilon^4 + (x_2^2 + \cdots + x_d^2)/\varepsilon^2)$ the integrand in (13.6). From the fact that $0 \le \varphi \le 1$ and that the support of ρ is in $|x_1| \le \varepsilon^2$ we obtain that $0 \le u(x) \le 2\varepsilon^2$. Note that in fact the support of the integrand is in $|x_1| \le \varepsilon^2$, $|x_2|^2 + \cdots + |x_d|^2 \le \varepsilon^2$ (and then in $|x_j| \le \varepsilon$ for $j \ne 1$), and that the dependence on the $x_2, \ldots, x_d$ variables is much smaller than on the x_1 variable.

Note that

$$\begin{aligned} \frac{\partial u}{\partial x_1}(x) &= \varphi(x_1^2/\varepsilon^4 + (x_2^2 + \cdots + x_d^2)/\varepsilon^2) \\ \frac{\partial u}{\partial x_j}(x) &= \int_{-\infty}^{x_1} 2x_j/\varepsilon^2\ \varphi'(s^2/\varepsilon^4 + (x_2^2 + \cdots + x_d^2)/\varepsilon^2)\, ds, \qquad j \ne 1, \end{aligned} \tag{13.7}$$

from where we obtain

$$\left| \frac{\partial u}{\partial x_j}(x) \right| \le 16\varepsilon, \qquad j \ne 1.$$

Notice that $w(x) = \nabla u(x)$ satisfy all the desired properties in the fattening condition of Definition 13.5, except that its support is not small. Unfortunately, working in a trivialization we need to ensure that we work in a sufficiently small system of coordinates.

We just set

$$\tilde{u}(x) = \varphi((x_1^2 + \cdots + x_d^2)/\varepsilon^2)\ u(x) \tag{13.8}$$

and $w = \nabla \tilde{u}(x)$. Its support is in $|x_j| \le \varepsilon$, for $j = 1, \ldots, d$. Then

$$\frac{\partial \tilde{u}}{\partial x_j}(x) = \varphi(|x|^2/\varepsilon^2) \frac{\partial u}{\partial x_j}(x) + 2x_j/\varepsilon^2\ \varphi'(|x|^2/\varepsilon^2) u(x)\ . \tag{13.9}$$

Since we had already established $|u(x)| \le 2\varepsilon^2$, we have that the last term in the RHS of (13.9) can be bounded by 16ε. Hence

$$\begin{aligned} \frac{\partial \tilde{u}}{\partial x_1}(x) &= \varphi(|x|^2/\varepsilon^2)\rho(x) + O(\varepsilon) \\ \left| \frac{\partial \tilde{u}}{\partial x_j}(x) \right| &= O(\varepsilon), \qquad j \ne 1. \end{aligned} \tag{13.10}$$

The function $\tilde{u}$ has arbitrarily small compact support and hence, it is possible to lift it to the manifold in such a way that the properties are not disturbed. This finishes the proof of 13.13 for χ_e^{*1}. □

As a corollary, we obtain the following theorem.

THEOREM 13.14. *(a) Let $\mathcal{T}_f$ be a geometrically natural operator on χ_B^1 leaving χ_{nd}^1,χ_{lnd}^1 invariant. Then:*

$$\mathrm{Spec}_W(\mathcal{T}_f, \chi_{nd}^1) = \mathrm{Spec}_W(\mathcal{T}_f, \chi_{lnd}^1) = \mathrm{Spec}_W(\mathcal{T}_f, \chi_B^1).$$

Moreover, the spectra $\mathrm{Spec}(\mathcal{T}_f, \chi_{nd}^1)$, $\mathrm{Spec}(\mathcal{T}_f, \chi_{lnd}^1)$ *have no gaps.*

(b) Let $\mathcal{T}_f$ be a geometrically natural operator on χ_B^{*1} leaving χ_e^{*1},χ_c^{*1} invariant. Then

$$\mathrm{Spec}_W(\mathcal{T}_f, \chi_e^{*1}) = \mathrm{Spec}_W(\mathcal{T}_f, \chi_c^{*1}) = \mathrm{Spec}_W(\mathcal{T}_f, \chi_B^{*1}).$$

Moreover, the spectra $\mathrm{Spec}(\mathcal{T}_f, \chi_e^{*1})$,$\mathrm{Spec}(\mathcal{T}_f, \chi_c^{*1})$ *have no gaps.*

REMARK 13.15. Notice that from the inclusions $\chi_{nd}^1 \subset \chi_{lnd}^1 \subset \chi_B^1$ we obtain the corresponding inclusions for the Weyl spectra. Notice also that Theorem 13.7 applied to χ_{nd}^1 shows the equality of Weyl spectra in χ_{nd}^1 and in χ_B^1, incidentally proving the equality for χ_{lnd}^1.

The same remark works for $\chi_e^{*1} \subset \chi_c^{*1} \subset \chi_B^{*1}$.

Part 5

Applications

CHAPTER 14

Structural stability and shadowing

14.1. Introduction

In this chapter we apply the previous results to prove some results in dynamical systems.

We will prove several variants of results on structural stability and in shadowing. The common theme of these results is that they amount to showing that some functional equation has a solution. The transfer operators which we have considered in this chapter will appear as the linearized version. Hence, the desired results will follow from an application of appropriate implicit function theorems. One important side effect of this approach is that we will obtain differentiability with respect to parameters.

Also, by considering weighted spaces, we will obtain results on the localization of perturbations. These results are important in the theory of Livsic on solutions of cohomology equations.

Finally, a very similar approach to that in this chapter is applied to some results on invariant manifolds, based on the recent developments on the parameterization method, which relies heavily on the spectral properties [**CFdlL03a, CFdlL03b, CFdlL05**].

The first result we present is the celebrated result on structural stability of Anosov (or Axiom A) systems. This result was the main motivation for [**Mat68**].

We recall that structural stability for Anosov systems shows that given f Anosov diffeomorphism (several characterizations of Anosov are shown to be equivalent in Proposition 14.1) then, for all g C^1 close, there exists a homeomorphism h in such a way that

$$h \circ f = g \circ h. \tag{14.1}$$

Structural stability for Anosov systems was first proved in [**Ano69**]. The functional analysis proof (initiated in [**Mos69**]) is very straightforward after the material we have developed. Indeed, historically, this proof was the main motivation for [**Mat68**], which was the basis of the theory.

In this chapter, we will follow and approach different from that of [**Mos69**]. Following [**dlLMM86**] we will use an implicit function theorem. This approach gives easily that, when we consider f fixed, the map $g \to h$ is C^r if the g are given the C^r topology and the h are given the C^r topology. The regularity of the dependence of h on f is significantly more delicate and we will discuss it later using a more sophisticated argument. A functional analysis approach using the implicit function theorem was suggested in [**Mat97**]. We will discuss [**Mat97**] in Remark 14.12.

Actually, it was remarked by one of us that one could also obtain that the map $g \to h$ is C^{r-1} when r is given the C^r topology and h is given the Hölder C^α topology for $0 < \alpha \le \alpha_0$ for some α_0 sufficiently small. An analogous result for flows implies smooth dependence of the topological entropy of Anosov flows [**KKPW89, KKPW90, KKW91**].

In Theorem 14.2 we present the result on structural stability with dependence on parameters and Hölder dependence on the space.

From the smooth dependence of the conjugating diffeomorphism on the perturbation we can easily deduce a result of [**Mañ90**] which establishes the smooth dependence of stable bundles on the corresponding points of structural stability. See Theorem 14.3.

Since thermodynamic formalism relates several ergodic theoretic properties to properties of the map with respect to a potential formed out of the stable and unstable Jacobians, this result tells us that these ergodic properties can be considered as properties of a fixed dynamics when one changes the potential. This is a much easier problem than the original one since the dependence on the potential is very easy to study. Hence, following this route one can obtain results on smooth dependence on several quantities in smooth ergodic theory. Several implementations of these strategy occur in [**Mañ90, Wei92, Con95, Rue97, Rue03**].

One advantage of the functional analysis approach to structural stability is that it adapts to coupled map lattices with fast enough decay in the couplings.

In [**JdlL00**], there is a treatment of smooth dependence on structural stability and other ergodic properties applying the functional analysis approach to the whole couple map lattice without using finite approximation. In [**JdlL04**],there is is a treatment of the approximations by systems of finite sites.

In Section 14.3 we consider the shadowing property.

Functional analysis proofs of shadowing for hyperbolic orbits appear in [**Shu78, MS87**].

In Theorem 14.16, we present a proof of the shadowing theorem which is specially geared towards numerical applications. We recall that a good numerical computation produces just a sequence of points that are a pseudo-orbit with a small error.

The theorems which are most useful from the point of view of validating numerics are those which make only assumptions on the pseudo-orbit, which very often is the only piece of information accessible. For example, the assumption that the pseudo-orbit is close to a hyperbolic set – which is very common in more theoretical applications – is not very easy to verify for a numerical application. Certainly, it does not follow from the numerical pseudo-orbit. Results similar to the ones we consider here as well as applications to problems in celestial mechanics are found in [**KS04**].

In the case that we consider a hyperbolic orbit and that the perturbation depends smoothly on parameters, we also obtain the smooth dependence on parameters.

This allows us to recover the smooth dependence on parameters for structural stability.

One advantage of this approach is that it also applies to non-uniformly hyperbolic systems and, therefore we obtain a shadowing theorem for non-uniformly hyperbolic systems without zero Lyapunov exponents. Of course, the allowed strength for the perturbations around each orbit is very non-uniform.

14.2. Structural stability

We start by recalling the following characterization of Anosov diffeomorphisms. This was started in [**Mat68**] but it includes some refinements developed in the previous chapters. Recall that $\chi^1_b = \Gamma_B(TM)$ and $\chi^1_{C^0} = \Gamma_{C^0}(TM)$ (see Chapter 13).

PROPOSITION 14.1. *Let M be a compact manifold. Let $f : M \to M$ be a C^1 diffeomorphism on M. The following are equivalent*

i) $1 \notin \operatorname{Spec}(f_*, \chi^1_{C^0})$
ii) $1 \notin \operatorname{Spec}(f_*, \chi^1_B)$
iii) *There exists $\delta > 0$,*

$$\mathcal{A}_{1-\delta,\,1+\delta} \cap \operatorname{Spec}(f_*, \chi^1_{C^0}) = \emptyset$$

iv) *There exist $C > 0$, $\lambda < 1$ so that for all $x \in M$, we can write*

$$T_xM = E^s_x \oplus E^u_x$$

where

$$v \in E^s_x \Leftrightarrow |Df^n(x)v| \le C\lambda^{|n|}|v| \qquad n \ge 0$$

$$v \in E^u_x \Leftrightarrow |Df^n(x)v| \le C\lambda^{|n|}|v| \qquad n \le 0$$

v) *Same as iv) but the splitting are continuous (i.e., the mapping $x \to E^s_x$, $x \to E^u_x$ are continuous).*

In the case that $f \in C^{1+\beta}$ with $\beta > 0$, we have furthermore the equivalence with the following properties:

vi) *Same as iv) but the splitting is C^α for some $\alpha > 0$.*
vii) *There exists $\tilde\delta, \alpha^* > 0$ such that for $0 < \alpha < \alpha^*$*

$$\mathcal{A}_{1-\tilde\delta,\,1+\tilde\delta} \cap \operatorname{Spec}(f_*, \chi^1_{C^\alpha}) = \emptyset.$$

When one—and hence all— of the above properties hold, we say f is an Anosov diffeomorphism.

Proof: We have already shown in Theorem 3.25 that

$$\mathrm{Spec}(f_*, \chi^0_B) = \mathrm{Spec}(f_*, \chi^0_{C^0})$$

hence, i) $\Leftrightarrow$ ii).

We have also shown in Proposition 4.3 that if x is such that $f^n(x) = x$ and λ is an eigenvalue of $Df^n(x)$, then any $z \in \mathbb{C}$ such that $z^n = \lambda$, belongs to $\mathrm{Spec}(f_*, \chi^0_B)$. Therefore i) implies that if $f^n(x) = x$, then $1 \notin \mathrm{Eig} Df^n(x)$. By the implicit function theorem, this implies that for a fixed n, $\{x \mid f^n(x) = x\}$ is nowhere dense.

By the Baire category theorem, we conclude that the set of aperiodic points is dense. Using Theorem 3.11, we conclude that the spectrum is invariant under rotations. Hence i) $\Rightarrow$ iii). (The converse is, of course, trivial.)

The fact that the bundles are continuous was proved in Theorem 2.4 and the fact that they are C^α is a consequence of the Invariant Section Theorem.

From the fact that the bundles are C^α, we obtain that there is a spectral gap. □

The following is our first result towards the full result of structural stability. We will show the existence of a semiconjugacy with smooth dependence on parameters.

Later, we will show that the mapping is a conjugacy and we will show the persistence of the Anosov property under C^1 perturbations.

These different results have different regularity assumptions on the maps considered and obtain different regularity conclusions. Hence, it is better to prove them in different stages.

THEOREM 14.2. *Let f be a $C^{r+\alpha}$ diffeomorphism $r \geq 1, \alpha \geq 0$.*

Assume that $1 \notin \mathrm{Spec}(f_*, \chi^1_{C^\alpha})$. Then, there exists a $C^{r+\alpha}$ neighborhood U of f and a C^α neighborhood V of the identity such that given $g \in U$, there exists a unique $h \in V$ such that

$$g \circ h = h \circ f. \tag{14.2}$$

Moreover, the mapping that to g applies the corresponding h is C^r when we give g the $C^{r+\alpha}$ topology and h the C^α topology.

We will write $h(g, f)$ for the map that satisfies (14.2) given by Theorem 14.2.

Proof: We will assume, without loss of generality that the manifold is endowed with an analytic Riemannian metric.

We recall that a neighborhood of the identity in the space mappings from a manifold to itself can be identified with the set of vector fields by associating to a map h the unique section v such that

$$h(x) = \exp_x v(x) \ \forall x \in M,$$

where exp denotes the Riemannian metric exponential mapping. It is an easy consequence of the implicit function theorem that all h can be written in this way. If we assume that the metric is sufficiently regular – as we will assume – the implicit function theorem establishes that the identification of maps has the same regularity. Hence, the procedure above produces an identification of a neighborhood of the identity in $C^0, C^\alpha, C^r, C^{r+\alpha}$ maps of a manifold with a neighborhood of the zero section in the spaces of $C^0, C^\alpha, C^r, C^{r+\alpha}$ sections.

Since an open ball the space of C^α (resp. C^0) section is an open set in a Banach space, we can talk about derivatives etc. with respect to functions.

The tangent space of C^α (resp. C^0) mappings on the identity mapping can be identified with the space of sections

$$T_{\mathrm{Id}} C^\alpha(M, M) = \chi^1_{C^\alpha(M)}.$$

We emphasize that, so far, we have not tried to give the space of C^α mappings the structure of a manifold. We are just using a coordinate patch near the identity. Describing neighborhoods requires that the composition mapping is somewhat regular.

A $C^{r+\alpha}$ neighborhood of f can be parameterized by sections by associating to g the section v given by

$$g = \exp_{f(x)} v(x) \ \forall x \in M.$$

With these identifications, we can transfer the functional equations between mappings to functional equations among sections so that all the tools from calculus in Banach space, notably implicit function theorems, are available.

We define the operator $\mathcal{T} : C^{r+\alpha} \times C^\alpha \to C^\alpha$ defined by

$$\mathcal{T}(g, h) = g \circ h \circ f^{-1} \tag{14.3}$$

so that the equation (14.2) is just the fixed point equation $\mathcal{T}(g, h) = h$.

Since

$$\mathcal{T}(f, \mathrm{Id}) = \mathrm{Id}\ ,$$

it is natural to try to obtain Theorem 14.2 from the implicit function theorem in Banach spaces. A version that we have found useful is Theorem 14.13.

The fact that the implicit function theorem does apply follows from the following considerations.

The fact that $\mathcal{T}$ is a C^r function in the indicated topologies follows from the results of [**dlLO99**]. The derivative $D_2\mathcal{T}(f, \mathrm{Id})$ is not difficult to guess heuristically by writing $h = \mathrm{Id} + \Delta$ and expanding the expression for $\mathcal{T}$ in (14.3). The results of [**dlLO99**] justify that indeed

$$D_2\mathcal{T}(f, \mathrm{Id}) = f_*.$$

Since we are assuming that $1 \notin \mathrm{Spec}(D_2\mathcal{T}(f, \mathrm{Id}), C^\alpha)$ the desired result follows from the standard implicit function theorem in Banach spaces. □

So far, we have not argued whether h is a homeomorphism. In the next paragraphs, we will argue that indeed h is a homeomorphism. This turns out to be closely related to showing that, under the assumptions of Theorem 14.2, we have that g is also Anosov.

We first observe that $h(M) = M$. This follows when M is a manifold from index arguments and that h is a small perturbation of the identity.

We note however, that the results we will present, will just show that $h(M)$ is a hyperbolic set. This applies also to the case that M is a perfect set and f is an axiom-A diffeomorphism.

The following result shows that the perturbed maps g are also hyperbolic and also to get some regularity on the dependence on the map.

The key observation, which appeared in [**Mañ90**] is that hyperbolicity of g is related to the cocycle of linear operator Dg covering the map g, but if we study it at a carefully chosen point, it will become easy to analyze.

That is, we consider

$$M_g(x,n) = Dg \circ g^{n-1}(x) \cdots Dg \circ g(x) Dg(x)$$

Using (14.2) we note that

$$\begin{aligned} M_g(h(x),n) &= Dg \circ g^{n-1} \circ h(x) \cdots Dg \circ g \circ h(x) Dg \circ h(x) \\ &= Dg \circ h \circ f^{n-1}(x) \cdots Dg \circ h \circ f(x) Dg \circ h(x) \end{aligned} \tag{14.4}$$

Therefore $M_g(h(x),n)$ is a cocycle over f.

Therefore, we can study the spectral properties of the cocycle $M_g(h(x),n)$ by studying the cocycle over f generated by the bundle automorphism $Dg \circ h$.

This observation is very useful because we have already studied in Section 2.5 the dependence of the invariant splittings on the matrix when the dynamics on the base is left fixed.

Note that when we give g the C^r topology, h — given the C^0 topology — depends in a C^r fashion.

When we give g the C^r topology, h the C^0 topology and $Dg \circ h$ the C^0 topology, we obtain that

$$\begin{aligned} &g \longrightarrow h \ \text{ is } \ C^r \\ &g \longrightarrow Dg \ \text{ is } \ C^r \\ &(Dg,h) \longrightarrow Dg \circ h \ \text{ is } \ C^{r-1} \ . \end{aligned}$$

Since the dependence of the spectral bundles on the transfer operator is analytic, we obtain:

THEOREM 14.3. *In the conditions of Theorem 14.2 we have*

i) *the mappings*

$$\begin{aligned} g &\longrightarrow E^s_{h(x)} \\ g &\longrightarrow E^u_{h(x)} \end{aligned} \tag{14.5}$$

are C^{r-1} when the stable and unstable bundles are given the C^0 topology.

ii) *The mappings in* (14.5) *are $C^{r-q-\alpha}$ when the bundles are given the C^α topology.*

ii) *The spectral radii of the spectral gaps (i.e. the radii of the spectral annuli) in C^0-sections are C^{r-1} as a function of g.*

As a consequence, we obtain that

COROLLARY 14.4. *There is a C^1 neighborhood V of f for which all the diffeomorphism $g \in V$ are Anosov.*

Out of Theorem 14.3 we obtain

LEMMA 14.5. *In the conditions of Theorem 14.2 there is an open set V and $\alpha_0 > 0$, $C > 0$ such that if $\tilde{f} \in V$, $\alpha \leq \alpha_0$, then*

$$\|(\tilde{f}_* - \mathrm{Id})^{-1}\|_{C^\alpha} \leq C$$

From Lemma 14.5 we obtain that if we have g is sufficiently small neighborhood of f we can apply the Theorem 14.2 and obtain

$$g \circ \hat{h} = \hat{h} \circ f \tag{14.6}$$

Out of Theorem 14.3 we obtain

LEMMA 14.6. *Let $\beta \geq 0$. Assume that $f \in C^{1+\alpha}$ is Anosov.*

Then, there is α_0 depending on the C^1 properties of f such that there is an open set in $C^{1+\alpha}$ around f for which $(g_* - \mathrm{Id})^{-1}$ is uniformly bounded as an operator from C^α to C^β, $0 \leq \beta \leq \min(\alpha_0, \alpha)$.

Observing that the invertibility of $(g_* - \mathrm{Id})$ is one of the characterization of Anosov systems, we obtain that, as a consequence of Lemma 14.6 we have:

COROLLARY 14.7. *If f is Anosov, there is an open C^1 neighborhood of f such that all the maps in this neighborhood are Anosov.*

We note that, provided that f, g, k are C^1 diffeomorphisms so that Theorem 14.2 applies to all their pairs, we have the cocycle property

$$h(f, g) \circ h(g, k) = h(f, k) \tag{14.7}$$

In particular, taking $f = g$ and noting that $h(f, f) = \mathrm{Id}$, we obtain that

$$h(f, g) = h^{-1}(g, f) \tag{14.8}$$

Now, we are ready to tackle the dependence of $h(f, g)$ solving (14.1) on f.

The following example will be very illustrative. It will show that the dependence cannot be expected to be even Lipschitz no matter what topology we impose on the f, g. It will also show that the regularity on the dependence on f is tied up to the regularity of the conjugacy, something that will show up in the proof.

EXAMPLE 14.8. We denote by T_α the translation by α in $\mathbb{T}^2$, by $A = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$.

We fix close enough to A but such that the fixed point has eigenvalues different to those of A.

We denote by h the solution close to the identity of

$$g \circ h = h \circ A$$

produced in Theorem 14.2.

Then, denoting by $T_\xi(x) = x + \xi$ we have that $h(g, T_\xi A)$ is only Hölder as a function of ξ.

Proof: We note that

$$g \circ h \circ T_\xi = h \circ T_\xi \circ T_\xi^{-1} \circ A \circ T_\xi$$

Hence $h \circ T_\xi$ is the conjugacy corresponding to

$$T_\alpha^{-1} A \circ T_\xi = T_{A\xi - \xi} A$$

We see that this depends in an analytic way on ξ even in an analytic topology.

On the other hand, the map $\mathbb{R}^2 \to C^0$ given by $\xi \to h \circ T_\xi$ cannot be more than C^α since the regularity of $\xi \to h \circ T_\xi$ implies regularity of h.

□

REMARK 14.9. If we give h Hölder topologies the dependence may be worse.

If we take α to be the largest α such that $h \in C^\alpha$, the dependence of h on parameters may be discontinuous. For example, if $\varphi(x) = |x|^\alpha$ we have that $\|\varphi \circ T_\xi - \varphi\|_{C^\alpha} = 2$ no matter how small is ξ.

For smaller α, we may obtain that the map h depends on f on a Hölder fashion but not any better. As the example shows, this does not improve if we take a finer topology in f.

The previous Example 14.8 shows that there is a very close relation between the modulus of continuity of the conjugacy and the dependence on parameters of the conjugacy.

Denote by Ω_f, the modulus of continuity of all $h(f, g)$ for g in a $C^{r+\alpha}$ neighborhood of f.

So far, we have shown $\Omega(t) = Ct^\beta$ for some $\beta > 0$ when $\alpha > 0$.

Following ideas in [**Ano69**], it is possible to show that indeed we have $\Omega(t) = t^\beta$ for some explicit $\beta > 0$ related to the spectrum of f. This β is bounded away from 0 uniformly even as α approaches 0. This result does not follow straightforwardly from the spectral results we have discussed and it requires a deeper geometric insight. We have, therefore, stated our main dependence result in a conditional manner that will allow to graft easily geometric arguments in functional analysis conclusions.

THEOREM 14.10. *In the conditions and the notations of Theorem 14.2, denote by $h(g, f)$ the mapping that to the f, g associates the unique h close enough to the identity solving* (14.2) *produced in Theorem14.2.*

Let $\Omega(t)$ be the modulus of continuity of $h(g, f)$ as g ranges over a $C^{r+\alpha}$ neighborhood of f.

Then, we have that $h(g, f)$ has modulus of continuity Ω considered as a function of f.

Note that Example 14.8 shows that it is impossible to obtain a substantially better modulus of continuity since by conjugating f with translations, we can transfer statements about dependence on the function f to regularity of the conjugating function.

Proof: The proof we present is inspired by the proof presented in [**Mos69**].

We note that we have already established as a corollary of the smooth dependence on parameters, Theorem 14.2

$$d_{C^0}(h(f,g),h(k,g)) \le C d_{C^1}(f,k)$$

Because of (14.7), we have that $h(g,f) = h(g,k) \circ h(k,f)$.

Hence,

$$\begin{aligned} d_{C^0}(h(g,f),h(g,k)) &= d_{C^0}(h(g,k) \circ h(k,f),h(g,k)) \\ &\le \Omega_{h(g,k)}(d_{C^0}(h(k,f),\mathrm{Id})) \\ &\le \Omega_{h(g,k)}(d_{C^1}(f,k)) \end{aligned}$$

where $\Omega_{h(g,k)}$ denotes the modulus of continuity of $\Omega_{h(g,k)}$. □

REMARK 14.11. We point out that the proof presented in [**Mos69**] contains a mistake in p. 420.

In the line 4, it is used that

$$d_{C^0}(u,u_0) = d_{C^0}(u^{-1} \circ u_0, \mathrm{Id})$$

This leads to the conclusion that the modulus of continuity of $h(g,f)$ with respect to f is Lipschitz, which, as we have seen in Example 14.8 cannot happen.

The correct inequality is, of course

$$d_{C^0}(u,u_0) = \Omega(d_{C^0}(u^{-1} \circ u_0, \mathrm{Id}))$$

where Ω is the modulus of continuity of u. This is what we have used in the above proof.

REMARK 14.12. The work [**Mat97**] proposes to obtain a result similar to Theorem 14.2 and Theorem 14.10 by applying an implicit function theorem to

$$\mathcal{A}(g,h,t) = g \circ h - h \circ f$$

(We abuse slightly the notation to use additive.) Really what we mean is

$$\exp^{-1}_{f(x)} g \circ h(x) - \exp^{-1}_{f(x)} h \circ f(x)$$

One observes that

$$D_2\mathcal{A}(gh,f) = g' \circ h - h \circ f$$

and

$$D_2\mathcal{A}(f,\mathrm{Id}f)\Delta = \Delta - \Delta \circ f$$

Hence $D_2\mathcal{A}(f,\mathrm{Id}f)$ is invertible when f is Anosov.

Unfortunately, the derivative $D_2\mathcal{A}$ is discontinuous as a function of its arguments when we give it the topology of linear operators from $C^0 \to C^0_0$.

If $f, \tilde{f}$ are different, we can find Δ such that $\|\Delta\|_{C^0} = 1$ and

$$\|\Delta \circ f - \Delta \circ \tilde{f}\|_{C^0} \geq \frac{1/2}{\|}\Delta\|_{C^0} \ .$$

Hence, the operators of composition on the right are highly discontinuous, since $\|f_* - \tilde{f}_*\| \geq 1/2$ when $f \neq \tilde{f}$, irrespective of how close they are.

We also note that the discontinuity of the linear operator given by composition in the left translates into the discontinuity of the operator f_* acting on C^0 sections. This is the reason why the results on continuity of the spectrum for transfer operators developed in Section 2 required somewhat elaborate arguments.

As we have shown before, following [**dlLMM86**], if one considers f fixed, there is no dependence of the derivative in g is indeed regular and one can obtain smoothness of h with respect to g.

We are grateful to J. Mather, who pointed out to one of use in private conversations the gap in [**Mat97**].

14.2.1. An abstract implicit function theorem. We recall the following result from functional analysis, which is a slightly sharpened version from the most standard implicit function theorem one finds in all textbooks, e.g., [**Nir01, Sch69**].

The proof is exactly the same as that of the more standard theorem.

THEOREM 14.13. *Let X, Y be Banach spaces, U a neighborhood of* $(0,0)$ *on $X \times Y$, $F : U \to X$ be a continuous function.*

Assume

H1

$$F(0,0) = 0$$

H2 $D_2F(x,y) - \mathrm{Id}$ *is invertible for all x, y in a neighborhood and the bound is uniform.*

We note that the hypothesis H2 above is implied by

H2'.1 $D_2F(0,0) - \mathrm{Id}$ *is invertible.*

H2'.2 $D_2F(x,y)$ *is continuous at* $(0,0)$.

Then, there exist open sets $o \in \tilde{U} \subset X$, $o \in \tilde{V} \subset Y$ such that for every $y \in \tilde{V}$, there is one and only one $x \in \tilde{U}$ such that

$$F(x,y) = 0$$

Moreover, the mapping that to y sends x is continuous.

Proof: Let M denote $(D_2F(0,0))^{-1}$ and consider for fixed y, the following fixed point problem

$$x = x - MF(x,y) \equiv \mathcal{T}_y(x) \tag{14.9}$$

We observe that

$$D\mathcal{T}_y(x) = \mathrm{Id} - MD_2F(x,y)$$

Since $D_2F(x,y)$ is continuous, there is a neighborhood $\hat{V}$ of $(0,0)$ in such a way that $\|D\mathcal{T}_y(x)\| \le 1/2$.

By the continuity properties in the assumption, we can find $\rho, \delta > 0$ such that

$$\||y| \le \rho,\ |x| \le 5\delta\} \subset \hat{V}$$
$$|y| \le \rho \Longrightarrow |\tau_y(0)| \le \delta$$

Then, for any $|y| < \rho$, the ball $B_{3\delta}(\tau_y(0))$ of radius 3δ around $\mathcal{T}_y(0)$ satisfies

$$B_{3\delta}(\mathcal{T}_y(0)) \subset \hat{V}$$

and, therefore

$$\mathcal{T}_y(B_{3\delta}(\mathcal{T}_y(0))) \subset B_{3\delta}(\mathcal{T}_y(0)) \subset \hat{V}$$

The contraction mapping principle implies that there is a unique fixed point of $\mathcal{T}_y$ in $B_{3\delta}(\mathcal{T}_y(0))$.

Moreover, the fact that $\mathcal{T}_y$ is continuous in y implies that the fixed point depends continuously on y.

□

14.3. Shadowing theorems

We recall that "shadowing" is jargon originating in detective stories. The detective follows a person so closely and so reliably that he becomes a shadow of that person.

Then "shadow" came mean staying close for a long time.

Hence, the shadowing theorems always say that there is an orbit close to another one, no matter what the suspect tries to do to avoid it.

We recall the following definition

DEFINITION 14.14. *We say that $\{x_n\}_{n\in\mathbb{Z}}$ is a δ-pseudo-orbit for the map f when*

$$d(f(x_n), x_{n+1}) \le \delta \quad \forall n \in \mathbb{Z}. \tag{14.10}$$

In particular, when $\delta = 0$, it is an orbit. We will say that a sequence **y** *ε-shadows another sequence* **x** *when*

$$d(x_n, y_n) \le \varepsilon\ \forall\ n \in \mathbb{Z} \tag{14.11}$$

Shadowing theorems always say that, under certain extra hypothesis, the δ-pseudo-orbits are ε shadowed.

In this chapter we will discuss shadowing theorems that follow from hyperbolicity properties. We should mention that, besides shadowing theorems based on hyperbolicity there are important shadowing theorems based on topological methods (well aligned windows, Conley index etc.) [**Eas75, EM79, Eas89, ZG04, GZ04**] or in variational methods [**Ang90, Mat93, Sli99**].

Shadowing theorems have many applications. We have already mentioned that one application of the sharp shadowing theorems imply structural stability. Often shadowing theorems can be used to produce interesting orbits, taking as an intermediate step the — somewhat easier — construction of pseudo-orbits.

For example, much of the work in Arnold diffusion is based on constructing approximate orbits, using invariant objects and then using different techniques to construct real orbits. Unfortunately, in many of these cases, one does not have full hyperbolicity and one has to supplement the results with topological arguments [**DdlLS04, DdlLS03, FM00, FM01, FM03**]. Methods based on the broken geodesics method appear in [**Bes96, BBB03, BCV01**] among others.

One application which has attracted considerable interest is that shadowing theorems can serve as justification of numerical results.

Note that a computer only produces δ-pseudo-orbits, where δ is a result of the roundoff and different truncation errors of the computer. If there is a good shadowing theorem available for the system we are studying, we can be sure that the orbits produced by the computer have real counterparts.

We refer to [**Pal00, Pil99**] for surveys on shadowing methods based on hyperbolicity as well as applications.

The most standard hyperbolic shadowing result is the following

THEOREM 14.15. *Let f be an Anosov system. Then, given $\varepsilon > 0$, $\exists\, \delta > 0$ such that if $\mathbf{x}$ is a δ-pseudo-orbit, then, there exists an orbit $\mathbf{y}$ it ε-shadows $\mathbf{x}$. Moreover, if f is C^2 we can choose $\delta = K\varepsilon$.*

We will deduce Theorem 14.15 from a more flexible Theorem 14.16. The argument that goes from Theorem 14.16 to 14.15 will also lead to applications in numerical analysis.

In order to formulate the results for manifolds, we will use the connectors of Definition 1.23. We just indicate that, if the dynamical system is defined in a subset of $\mathbb{R}^d$, they are not needed.

THEOREM 14.16. *Let M be a compact manifold, $f : M \to M$ be a C^1 map. Let $x \in M^{\mathbb{Z}}$ be a sequence.*

Assume

1) $\mathbf{x}$ *is a δ pseudo-orbit for f*
2) *δ is small enough so that the connectors $E_{x,y}$ can be defined for points x, y at a distance 2δ.*
3) *Denote*

$$A_n = E_{f(x_n),x_{n+1}} Df(x_n) \tag{14.12}$$

Assume that the cocycle

$$M^n = \begin{cases} A_n \cdots A_0 & n \geq 0 \\ A_n^{-1} \cdots A_{-1}^{-1} & n \leq 0 \end{cases}$$

is hyperbolic. That is, we can write

$$T_{x_n}M = E_n^s \oplus E_n^u$$

in such a way that

a) *The angle between E_n^s, E_n^u is bounded uniformly away from zero. Equivalently, denoting by Π_n^s, Π_n^u the projections, we have*

$$\|\Pi_n^s\|, \|\Pi_n^u\| \le \alpha$$

b) $\exists\ C > 0$, $\lambda < 1$ *such that*

$$v \in E_n^s \Leftrightarrow |A_{n+k} \cdots A_n v| \le C\lambda^k \qquad k \ge 0$$

$$v \in E_n^u \Leftrightarrow |A_{n-k}^{-1} \cdots A_{n-1}^{-1} A_n^{-1} v| \le C\lambda^{|k|} \qquad k \ge 0$$

4) *Assume that δ is small enough depending on the constants α, C, λ and the modulus of continuity of f. Then, there exists an orbit $\mathbf{y}$ which $K\delta$ shadows $\mathbf{x}$.*

The main conceptual difference between Theorem 14.15 and Theorem 14.16 is that the former assumes hyperbolicity properties in all the exact orbits. Whereas the second assumes hyperbolicity properties only on the approximate solution.

Proof: Following [**Shu78, MS87, Kat, Lan85**] we formulate the problem as a fixed point problem in space of sequences.

If we fix y_n we can consider the space $X = \prod_{n\in\mathbb{Z}} T_{y_n}M$ endowed with the supremum norm. We define $\mathcal{T}_y : X \to X$ by

$$(\mathcal{T}_y\Delta)_{n+1} = \exp_{y_{n+1}}^{-1} f(\exp_{y_n} \Delta_n)$$

Heuristically

$$\mathcal{T}(\Delta)_{n+1} = f(y_n + \Delta_n) - y_{n+1}$$

Note that Δ_n is a fixed point of $\mathcal{T}$ if and only if $x_n = \exp_{y_n} \Delta_n$ (heuristically $x_n = y_n + \Delta_n$) is an orbit of the dynamical system.

We also observe that $\mathcal{T}$ is continuously differentiable and, moreover, we have

$$E_{y_n+\Delta_n,\, y_{n+1}} Df(y_n + \Delta_n)\sigma_n - \sigma_{n+1} \tag{14.13}$$

where E is, as before the connector, which was a linear operator.

Similarly there is a $C > 0$ such that $x_n = \exp_{y_n} \Delta_n$ is a δ-pseudo-orbit if

$$|\mathcal{T}(\Delta_n)| \le C\delta$$

and if

$$|\mathcal{T}(\Delta_n)| \le C^{-1}\delta$$

then x_n as above is a δ-pseudo-orbit.

A consequence of (14.13) is that $D\mathcal{T}$ is uniformly continuous and we can estimate the modules of continuity of $\mathcal{T}$ by the modules of continuity of f.

In particular if f is C^2, we obtain that $D\mathcal{T}$ is Lipschitz.

Finally, we observe that hypothesis 1) of Theorem 9.1 implies (actually we showed that it was equivalent to)

$$\|D\mathcal{T}(0) - \mathrm{Id})^{-1}\| < \infty \; .$$

Here, we will just go quickly over the arguments to show that indeed one can estimate $\|(D\mathcal{T}(0) - \mathrm{Id})^{-1}\|$ in terms of the geometric quantities in 2).

We note that

$$(D\mathcal{T}(0) - \mathrm{Id})\sigma = \eta$$

amount to the infinite set of equation

$$A_n\sigma_n - \sigma_{n+1} = \eta_n \tag{14.14}$$

Using the invariance under A_n of the splitting $E_n^s \oplus E_n^u$ in 2) we obtain that (14.14) is equivalent to:

$$\begin{aligned} A_n(\Pi_n^s\sigma_n) - \Pi_{n+1}^s\sigma_{n+1} &= \Pi_{n+1}^s\eta_{n+1} \\ A_n(\Pi_n^u\sigma_n) - \Pi_{n+1}^u\sigma_{n+1} &= \Pi_{n+1}^u\eta_{n+1} \end{aligned} \tag{14.15}$$

The equations (14.15) about the explicit solution

$$\begin{aligned} \Pi_{n+1}^s\sigma_{n+1} = \Pi_{n+1}^s\eta_{n+1} + A_n\Pi_n^s\eta_n + A_nA_{n-1}\Pi_{n-1}^s\eta_{n-1} + \cdots + \\ A_nA_{n-1}\cdots A_{n-k}\Pi_{n-1}^s\eta_{n-k} + \cdots \\ \Pi_n^u\sigma_n = A_n^{-1}\Pi_{n+1}^u\eta_{n+1} + A_n^{-1}A_{n+1}^{-1}\Pi_{n+2}^u\eta_{n+2} + \cdots + \\ A_n^{-1}A_{n+1}^{-1}\cdots A_{n+k}^{-1}\Pi_{n+k+1}^{-1}\Pi_{n+k+1}^u\eta_{n+k+1} \end{aligned} \tag{14.16}$$

The right hand side of (14.16) can be readily estimated by a geometric series yielding

$$\|(D\mathcal{T}(0) - \mathrm{Id})^{-1}\| \le \alpha C(1-\lambda)^{-1} \tag{14.17}$$

Since the hypothesis give us bounds on $\|\mathcal{T}(0) - 0\|$, $\|(D\mathcal{T}(0) - \mathrm{Id})^{-1}\|$ and the modulus of continuity of $\mathcal{T}$, the standard Newton-Kantorovich theorem produces the existence of a fixed point close to zero, which using the remarks is precisely the shadowing theorem we want. □

CHAPTER 15

Invariant tori and their whiskers in quasi periodic maps

In this chapter we present several results on persistence of invariant tori and their asymptotic invariant manifolds, commonly referred to as the whiskers, in quasi periodic systems. The proofs are based on the parameterization method of [**CFdlL03a, CFdlL03b**], and can be found in [**HdlLb**].

The results we obtain on regularity of the invariant tori and their whiskers are very optimal, in the sense that the objects we find are as smooth as the system. These regularity results come from the spectral theory of transfer operators over rotations of Part 3.

The results for invariant manifolds include as particular cases the usual (strong) stable and (strong) unstable manifolds, but also include other non-resonant manifolds. The non-resonant conditions to construct the whiskers are based on the spectral theory of Sylvester transfer operators of Section 11.3.

All these results take advantage of the special skew-product structure of the systems at hand, and the quasi-periodicity.

The method lends itself to numerical implementations whose analysis and implementation is studied in [**HdlL04, HdlL05a**]. The theorems can also be used to validate the results of numerical computation, since they are stated as *a posteriori* results.

The approach can be extended to cover the theory of existence and persistence of normally hyperbolic manifolds and laminations [**HdlL05b**], and the theory of hyperbolic invariant sections of general bundle maps.

15.1. Quasi periodic maps and their invariant manifolds

The systems we consider are *quasi periodic maps*

$$\begin{aligned} \bar{x} &= F(x,\theta) \ , \\ \bar{\theta} &= \theta + \omega \ , \end{aligned} \tag{15.1}$$

where $x \in \mathbb{R}^n$ and $\theta \in \mathbb{T}^d$ are variables, and $\omega \in \mathbb{R}^d$ is the *rotation vector*.

Systems of the form (15.1) are called skew-products in the mathematical literature, and are bundle maps over rotations.

In applications they appear when one forces a system with a quasi-periodic external perturbation. That is, the system (15.1) has the form

$$F(x,\theta) = F_0(x) + F_1(x,\theta) \ , \tag{15.2}$$

where F_1 is small. For $F_1 \equiv 0$ the dynamics of x and θ are uncoupled and for a fixed point x_0 of F_0 the torus

$$\mathcal{K}_0 = \{x_0\} \times \mathbb{T}^d$$

is invariant for the whole system (15.1) given by (15.2). It is then natural to look for an invariant torus close to $\mathcal{K}_0$ in the perturbed system. If x_0 had invariant manifolds, we can consider whether there are corresponding objects for the quasi-periodically excited problem.

We will consider two types of problems for quasi-periodic systems (15.1):

a) Existence of *invariant tori* (of dimension d) which are (normally) hyperbolic, and persistence of such tori under perturbations;
b) Existence of *asymptotic invariant manifolds* attached to an invariant torus.

REMARK 15.1. We consider here the problems a) and b) for discrete time systems. For a discussion for continuous time systems, see [**HdlLb**].

Our approach differs from the classical approach of normal hyperbolicity [**Fen72, Fen74, Fen77, HP69, HPS77**], and thanks to the special structure of the system, it also has some mathematical advantages over the general theory.

a) The invariant tori are as smooth as the system (including analytic) and that they depend smoothly (including analytic) on parameters. Such results are false for more general systems (see e.g. [**dlL01b**] for explicit examples).
b) The asymptotic manifolds associated to non-resonant parts of the linearization include as particular cases the strong stable and the strong unstable manifolds, but it also include other cases. For instance, we can consider *slow manifolds* that correspond to the slowest directions, that are important in applications because they dominate the asymptotic behavior of the systems. (See [**dlL97, ElB01, CFdlL03a, CFdlL03b, dlL03**] for non-resonant invariant manifolds for fixed points.)

The proofs we present are based on the parameterization method, in which it is formulated a functional equation for both the parameterization of the invariant manifold and the dynamics on it. The fact that the motion in the angle variables variables is a rotation with the frequency of the external perturbation simplifies substantially the functional equations considered and eliminates the main source of difficulties in the analysis considered in [**CFdlL03a, CFdlL03b**], namely, the existence of unknown functions that appear as composition in the right.

The functional equations give rise to differentiable operators in C^r spaces that can be studied with the regular implicit function theorem in Banach spaces. These regularity properties of the functional equations are better than those that appear in the graph transform method, and they seem to translate into good stability properties of the numerical methods [**HdlL04, HdlL05a**].

15.2. Existence and persistence of invariant tori

The existence and persistence of invariant tori for (15.1) is based on the equation

$$F(K(\theta-\omega),\theta-\omega)-K(\theta)=0\ , \tag{15.3}$$

where $F:\mathbb{R}^n\times\mathbb{T}^d\to\mathbb{R}^n$ and $\omega\in\mathbb{R}^d$ are given and we are supposed to find $K:\mathbb{T}^d\to\mathbb{R}^n$. Notice that for a solution K of (15.3) the set of points

$$\mathcal{K}=\{\mathcal{K}_\theta=(K(\theta),\theta)\mid\theta\in\mathbb{T}^d\} \tag{15.4}$$

is invariant under the dynamical system (15.1). Indeed, K is a parameterization of a torus in which the dynamics is a rotation.

It is quite important to notice that provided $F\in C^{r+1}(\mathbb{R}^n\times\mathbb{T}^d,\mathbb{R}^n)$, the operator $\mathcal{T}:C^r(\mathbb{T}^d,\mathbb{R}^n)\to C^r(\mathbb{T}^d,\mathbb{R}^n)$ defined by

$$\mathcal{T}(K)(\theta)\equiv F(K(\theta-\omega),\theta-\omega)-K(\theta) \tag{15.5}$$

is a differentiable operator when $C^r(\mathbb{T}^d,\mathbb{R}^n)$ is given the C^r topology. See [**dlLO99**]. Hence we can study (15.3) using standard implicit function theorems in C^r spaces in the case that the invariant torus is normally hyperbolic.

Note that a formal calculation – which is justified in [**dlLO99**] – gives

$$\mathrm{D}\mathcal{T}(K)\Delta(\theta)=\mathrm{D}_xF(K(\theta-\omega),\theta-\omega)\Delta(\theta-\omega)-\Delta(\theta)\ . \tag{15.6}$$

Hence, once we show that $\mathrm{D}\mathcal{T}$ is invertible as an operator in C^r, it is clear by the Implicit Function Theorem that the existence of approximate solutions implies existence of true solutions.

In particular, if we have a true solution for a certain F, for which $\mathrm{D}\mathcal{T}$ is invertible, it will be an approximate solution if we modify F slightly and, hence, we have a true solution for the modified F.

For an invariant torus, the invertibility of $\mathrm{D}\mathcal{T}$ in C^0 is closely related to the fact that the torus is normally hyperbolic. Notice also that the invertibility of $\mathrm{D}\mathcal{T}$ comes from properties of the spectrum of the transfer operator $\mathcal{M}_\omega$ associated to the monodromy matrix $M(\theta)=\mathrm{D}_xF(K(\theta),\theta)$, which gives a vector bundle map M_ω over the rotation ω. Much more, one of the main developments in Part 3 is that invertibility of $\mathrm{D}\mathcal{T}$ in C^0 spaces is equivalent to invertibility of $\mathrm{D}\mathcal{T}$ in C^r spaces, $r\in\mathbb{N}\cup\{\infty,a\}$ (where a means analytic). This is not true in systems which are not of the form (15.1).

From this circle of ideas we obtain the following result (see [**HdlLb**]).

THEOREM 15.2. *Let $U\subset\mathbb{R}^n$ be an open set. Let $F:U\times\mathbb{T}^d\subset\mathbb{R}^n\times\mathbb{T}^d\to\mathbb{R}^n$ be a map of class C^{r+1}, with $r\geq 0$ – including $C^{r+1}=C^a$ in the analytic case $r=a$ –, such that for all $\theta\in\mathbb{T}^d$ the map $F(\cdot,\theta):U\to\mathbb{R}^n$ is a local diffeomorphism. Let $\omega\in\mathbb{R}^d$ be a rotation.*

We consider the skew product

$$\bar{x} = F(x, \theta) \ ,$$
$$\bar{\theta} = \theta + \omega \ ,$$

that is a bundle map on the bundle $E = \mathbb{R}^n \times \mathbb{T}^d$.

Let $K : \mathbb{T}^d \to U \subset \mathbb{R}^n$ *be a* C^r *map such that:*

- $\mathcal{K}$ *is an approximate invariant torus, that is*

$$\|F(K(\theta), \theta) - K(\theta + \omega)\|_{C^r} < \varepsilon \ . \tag{15.7}$$

- *For the* C^r *matrix valued map* $M : \mathbb{T}^d \to \mathrm{GL}_n(\mathbb{R})$, *defined by*

  $$M(\theta) = \mathrm{D}_x F(K(\theta), \theta) \ ,$$

  the corresponding transfer operator $\mathcal{M}_\omega$ *satisfies the spectral gap condition*

$$\mathrm{Spec}(\mathcal{M}_\omega, \Gamma_b(E)) \cap \{z \in \mathbb{C} \mid |z| = 1\} = \emptyset \ . \tag{15.8}$$

Then:

- *If* ε *is small enough, there exists a* C^r *map* $K_F : \mathbb{T}^d \to U \subset \mathbb{R}^n$ *such that*

$$F(K_F(\theta), \theta) = K_F(\theta + \omega) \ , \tag{15.9}$$

  and $\|K_F - K\|_{C^r} = \mathrm{O}(\varepsilon)$.
- *The solution* K_F *above is the only* C^0 *solution of* (15.9) *in a* C^0 *neighborhood of* K.
- *The torus* K_F *is normally hyperbolic.*

Moreover, the map $F \to K_F$ *is* C^1 *when* F *is given the* C^{r+1} *topology and* K_F *the* C^r *topology.*

Remark 15.3. Notice that the spectral gap assumption (15.8) for an invariant torus implies normal hyperbolicity ([**Mn78, HPS77, Swa83**]), because the motion on the manifold is a rotation, which has zero Lyapunov exponents.

Remark 15.4. Notice that the spectral gap assumption (15.8) is formulated in the space of bounded sections – not necessarily continuous –. Using the results in Chapter 3 about the spectrum of transfer operators over rotations, we obtain that the spectrum over bounded sections is the same as that over C^r sections. This is what allows to obtain C^r regularity in the conclusions. Of course, these results depend very heavily on the fact that the motion on the torus is a rotation.

Remark 15.5. It is clear from the proof of Theorem 15.2 that one derivative of the system F is lost in the result K. So, if F is C^{r+1} then K is C^r. Notice also that Theorem 15.2 gives also some control of the errors in solving the functional equations in the C^r norms. But it is also proved in [**HdlLb**] that the torus is in fact C^{r+1}!

Much more, there is a bootstrap on the regularity of the torus, in such a way that a C^0 hyperbolic invariant torus is in fact as smooth as the system, i.e. C^{r+1} in Theorem 15.2. Notice, however, that in such a result we loose the control on derivatives.

REMARK 15.6. One can state an optimal hypothesis in which the system F is differentiable with respect to x, and the differential is C^r with respect to both x and θ.

In particular, one can obtain continuous invariant tori from systems that are C^1 with respect to x but only continuous with respect to θ. It does not follow from the general theory of normally hyperbolic manifolds, for which some smoothness is necessary.

The key point is that there is a natural transversal bundle to any torus defined by a section (which in the general theory is a normal bundle complementary to the tangent bundle).

REMARK 15.7. The formulation we have presented of Theorem 15.2, implies the more commonly formulated result on the persistence of normally hyperbolic invariant tori.

If K_F is a parameterization of a torus invariant under a map F, it will be smooth and it will satisfy (15.7) for all the maps G close to F. Furthermore, if the torus is normally hyperbolic for F, then, the operator $M(\theta) = \mathrm{D}_x F(K(\theta), \theta)$ is hyperbolic. By the stability of the spectrum under perturbations, we will obtain that (15.8) will be satisfied for G close to F.

Hence, we have verified that, given a normally hyperbolic invariant torus, if we perturb the map slightly, we have all the assumptions of Theorem 15.2 for the perturbed map and the original invariant torus. The conclusions of Theorem 15.2 give the persistence of the invariant torus.

REMARK 15.8. We call attention to the fact that the proof works for ω resonant or non-resonant (ergodic). For ω irrational, the spectral gap condition is equivalent to $1 \notin \mathrm{Spec}(\mathcal{M}_\omega, \Gamma_b(E))$, since in such a case the spectrum is rotationally invariant.

Hence, we can obtain rather easily results on smooth dependence on parameters, just using the trick of considering such parameters as new variables and adding extra equations to the system (15.1).

REMARK 15.9. The formulation of Theorem 15.2 is very similar to the a-posteriori estimates of numerical analysis. A numerical method can produce an approximate solution that satisfies (15.7) up to a few units of round–off error. It is also possible to verify the other hypothesis of Theorem 15.2 on the computed solution. These issues of numerical analysis, as well and results of implementations are discussed in more detail in [**HdlL04, HdlL05a**].

REMARK 15.10. One can obtain a similar theorem in the generality of hyperbolic invariant sections of bundle maps $F : E \to E$ over $f : \mathcal{P} \to \mathcal{P}$. A section $K : \mathcal{P} \to E$ is invariant if $F \circ K = K \circ f$. The hyperbolicity property is just a spectral condition

$$\text{Spec}(\mathcal{M}_f, \Gamma_b(N\mathcal{K})) \cap \{z \in \mathbb{C} \mid |z| = 1\} = \emptyset \ , \tag{15.10}$$

where $N\mathcal{K} = VT_{\mathcal{K}}E$ is the vertical tangent bundle on the manifold $\mathcal{K} = \{K(\theta) \mid \theta \in \mathcal{P}\}$, and $M_f = \mathrm{D}F_{|N\mathcal{K}}$. Notice that the hyperbolic assumption involves only the dynamics along the transversal directions, while in the general theory of normally hyperbolic manifolds involves also the dynamics along the tangential directions, namely that the growth in the transversal directions dominate that on tangential directions.

This formulation gives a theory of existence and persistence of hyperbolic continuous sections of bundle maps. The persistence has to do with perturbations that preserve the skew product structure of the bundle maps, and if this structure is lost in the perturbation, say tangent and transversal directions are fully coupled, then the invariant manifolds can be destroyed.

Notice also that the regularity of the invariant section produced depends heavily on the dynamics f on the base manifold $\mathcal{P}$, which implies spectral inclusions between spaces of sections with different regularities.

REMARK 15.11. Equation (15.3) can also be used to find invariant tori in some cases where $\mathcal{K}$ is not normally hyperbolic.

Notably, in the case of Hamiltonian systems, equations very similar to (15.3) has been used to compute KAM tori or lower dimensional tori, including also their existence under quasi-periodic perturbations.

A version of KAM theory related to the theory developed here is found in [**Rüs76, CC97, JdlLZ99, dlL01b, GJdlLV00**] (see also [**JS92**] for perturbative results in the context of lower dimensional tori).

15.3. Asymptotic invariant manifolds

In this section, given an invariant torus, we consider the existence of other invariant manifolds so that the motion on them converges to the torus. The presentation includes all the essential ideas of the problem but avoids some of the technical complications. The discussion will be informal and we will not keep track of what are the differentiability assumptions, etc. This is left to the statement of Theorem 15.14, whose complete proof appears in [**HdlLb**].

The main geometric requirement is that there exists an invariant transversal bundle around the torus such that the spectrum of the transfer operator restricted to this bundle is contractive and satisfies some finite non-resonance assumptions with respect to the transfer operator on the whole transversal bundle. Then, we can find an invariant manifold tangent to this bundle. In the case that the bundle is a spectral bundle associated to the most contractive sectors – in such a case the non-resonance assumptions are satisfied automatically, we recover the classical strong stable manifold theorem. As indicated by the theory developed here, a smooth slow manifold may exist or not depending on whether the resonance conditions are met.

15.3.1. The invariance equation. Let $\mathcal{K}$ be an invariant torus of (15.1) parameterized by $K : \mathbb{T}^d \to \mathbb{R}^n$. We note that the perturbations in the dynamic variables propagate by the variational equations of (15.1) on the torus $\mathcal{K}$:

$$\begin{aligned} \bar{v} &= M(\theta)v \ , \\ \bar{\theta} &= \theta + \omega \ , \end{aligned} \tag{15.11}$$

where $v \in \mathbb{R}^n$ and $\theta \in \mathbb{T}^d$, and $M(\theta) = \mathrm{D}F(K(\theta), \theta)$. In the language of global differential geometry, (15.11) is a vector bundle map on the bundle $N\mathcal{K} \simeq \mathbb{R}^n \times \mathcal{K} \simeq \mathbb{R}^n \times \mathbb{T}^d$, whose fibbers are $\mathbb{R}^n$ and whose base points are the points in $\mathcal{K} \simeq \mathbb{T}^d$.

We show in Theorem 15.14 that given a subbundle E_1 of $N\mathcal{K}$ invariant under the variational equations (15.11) and such that the spectrum of the linearization restricted to it satisfies certain non-resonance conditions, then there is an invariant manifold tangent to this subbundle which is invariant under the map. This manifold is referred to as a *whisker* of the invariant torus.

Our study of whiskers is based on the study of the equation

$$F(W(\eta, \theta), \theta) = W(\Lambda(\eta, \theta), \theta + \omega) \ , \tag{15.12}$$

where we are supposed to find $W : \mathbb{R}^{n_1} \times \mathbb{T}^d \to \mathbb{R}^n$ and $\Lambda : \mathbb{R}^{n_1} \times \mathbb{T}^d \to \mathbb{R}^{n_1}$, where $n_1 \leq n$. The equation (15.12) implies that

$$\mathcal{W} = \{\mathcal{W}_\theta(\eta) = (W(\eta, \theta), \theta) \mid \theta \in \mathbb{T}^d \ , \eta \in \mathbb{R}^{n_1}\} \ , \tag{15.13}$$

is invariant under F, and Λ is the induced dynamics on the manifold.

Moreover, $W(0, \theta) = K(\theta)$ is the parameterization of the invariant torus and $\Lambda(0, \theta) = 0$. That is, $\mathcal{W}$ extends the invariant manifold found solving (15.3).

Notice also that

$$\mathrm{D}_x F(K(\theta), \theta) W^1(\theta) = W^1(\theta + \omega) \Lambda^1(\theta) \ ,$$

where $W^1(\theta) = \mathrm{D}_\eta W(0, \theta)$, $\Lambda^1(\theta) = \mathrm{D}_\eta \Lambda(0, \theta)$. This says that the bundle E_1 spanned by the columns of W^1 is invariant under the variational equations (15.11). Notice that in this formulation such a bundle is trivial, but all the procedure works even if this is not the case. Notice, however, that even if the bundles could be non-trivial, using a device in [**HP69**] one can augment the bundles so that they become trivial.

Notice that W and Λ are not uniquely defined. Nevertheless, as we will see, it is possible to chose normalizations that make them unique. We will try to find simple expressions for Λ, in particular, polynomial expressions.

15.3.2. Finding the dynamics on the manifold. In this section we show that, under suitable non-resonance hypotheses, we can solve the invariance equation (15.30) up to order L, that is there exists a polynomial bundle map $W^{\leq} : E_1 \to E$ over the identity and a polynomial bundle map $\Lambda : E_1 \to E_1$ over ω, both of them of degree L, such that

$$F(W^{\leq}(\eta,\theta),\theta) = W^{\leq}(\Lambda(\eta,\theta),\theta+\omega) + o(|\eta|^L) \ .$$

We write

$$W^{\leq}(\eta,\theta) = \sum_{k=0}^{L} W^k(\eta,\theta) \ , \ \Lambda(\eta,\theta) = \sum_{k=1}^{L} \Lambda^k(\eta,\theta) \tag{15.14}$$

where Λ^k and W^k are homogeneous polynomials in η or degree k. Substituting (15.14) into the invariance equation (15.30) and matching terms of the same degree, we obtain that (15.14) is equivalent to a sequence of equations for the Λ^k and W^k, which we now study recursively.

We have already found the zero-order and first order term of $W^{\leq}$ and Λ. In particular, W^1 parameterizes a vector bundle E_1 on the torus $W^0 = K$, whose linearized dynamics is given by Λ^1. We assume we choose a complementary bundle E_2 to E_1. We emphasize that this bundle E_2 is not needed to be invariant under the linearization M. Notice that M is then an upper triangular vector bundle map with respect to the splitting $E^1 \oplus E^2$, so we write

$$M(\theta) = \begin{pmatrix} M_1(\theta) & B(\theta) \\ 0 & M_2(\theta) \end{pmatrix} \ . \tag{15.15}$$

Notice also that, if $\mathcal{A} = \mathcal{A}\mathrm{Spec}(\mathcal{M}_\omega, \Gamma_b(E))$, $\mathcal{A}_i = \mathcal{A}\mathrm{Spec}(\mathcal{M}_{i,\omega}, \Gamma_b(E_i))$ i=1,2, are the angular hulls of the spectra, we have $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$, see Section 3.10. (If the rotation ω is irrational the spectra involved is rotationally invariant.)

The subsequent equations matching terms of order $k = 2, \dots L$ are to be considered equations for W^k,Λ^k, assuming that $W^1, \dots, W^{k-1}$, $\Lambda^1, \dots, \Lambda^{k-1}$ are known. More concretely, the equation for the order k is

$$M(\theta)W^k(\eta,\theta) = W^1(\theta+\omega)\Lambda^k(\eta,\theta) + W^k(\Lambda^1(\theta)\eta,\theta+\omega) + R^k(\eta,\theta) \ , \tag{15.16}$$

where R^k is a homogeneous polynomial of degree k over the rotation ω, depending polynomially on $W^1, \dots, W^{k-1}$ and $\Lambda^1, \dots, \Lambda^{k-1}$. We rewrite equation (15.16) as:

$$\begin{aligned} M(\theta-\omega)W^k(\Lambda^1(\theta-\omega)^{-1}\eta,\theta-\omega) - W^1(\theta)\hat\Lambda^k(\eta,\theta) - W^k(\eta,\theta) \\ = \hat R^k(\eta,\theta) \ , \end{aligned} \tag{15.17}$$

where $\hat R^k(\theta) \in L^k_s(E_1;E)$ is known, defined by

$$\hat R^k(\theta)(\eta) = \hat R^k(\eta,\theta) = R^k(\Lambda^1(\theta-\omega)^{-1}\eta,\theta-\omega) \ ,$$

and the unknown terms are $\hat\Lambda^k(\theta) = \Lambda^k(\theta-\omega)\circ\Lambda^1(\theta-\omega)^{-1} \in L^k_s(E_1;E_1)$ and $W^k(\theta) \in L^k_s(E_1;E)$.

We now indicate how to solve (15.17).

Taking projections over E_1, E_2 in equation (15.17), and taking into account that $W^1 = I_1$, we obtain:

$$\begin{aligned}(15.18)\quad \hat{R}_1^k(\eta,\theta) &= M_1(\theta-\omega)W_1^k(M_1(\theta-\omega)^{-1}\eta,\theta-\omega) - W_1^k(\eta,\theta)\\ &\quad + B(\theta-\omega)W_2^k(M_1(\theta-\omega)^{-1}\eta,\theta-\omega) - \hat{\Lambda}^k(\eta,\theta)\ ,\\ (15.19)\quad \hat{R}_2^k(\eta,\theta) &= M_2(\theta-\omega)W_2^k(M_1(\theta-\omega)^{-1}\eta,\theta-\omega) - W_2^k(\eta,\theta)\ .\end{aligned}$$

(15.18) is an equation for $W_1^k(\theta), \hat{\Lambda}_1^k(\theta) \in L_s^k(E_1; E_1)$ and and (15.19) is an equation for $W_2^k(\theta) \in L_s^k(E_1; E_2)$. We solve first (15.19), and then solve (15.18). The operators that appear in both equation are a generalization of the Sylvester operators in [**BK98, dlLW95, CFdlL03a**], and have been studied in great detail in Chapter 11, see Theorem 11.9.

Hence, introducing the Sylvester vector bundle maps $S_1^k = S_{\omega,M_1,M_1}^k$, $S_2^k = S_{\omega,M_2,M_1}^k$ and $S_B^k = S_{\omega,B,M_1}^k$, (15.18) and (15.19) can be rewritten as

$$(15.20)\qquad \mathcal{S}_1^k W_1^k - W_1^k - \hat{\Lambda}^k = \hat{R}_1^k - S_B^k W_2^k\ ,$$

$$(15.21)\qquad \mathcal{S}_2^k W_2^k - W_2^k = \hat{R}_2^k\ ,$$

respectively.

Equation 15.21 can be solved under the assumption that

$$1 \notin \mathcal{A}\mathrm{Spec}(\mathcal{S}_2^k, \Gamma_b(L_s^k(E_2; E_1)))\ ,$$

which, since by Theorem 11.9

$$\mathcal{A}\mathrm{Spec}(\mathcal{S}_2^k, \Gamma_b(L_s^k(E_2; E_1))) \subset \mathcal{A}_2 \cdot \mathcal{A}_1^{-k},$$

it is satisfied if $\mathcal{A}_1^k \cap \mathcal{A}_2 = \emptyset$ for $k = 0, \ldots, L$ (see H.3. of Theorem 15.14). Notice that the spectrum does not depend on the regularity of the sections, and so it coincides with that on acting on smooth sections. Hence, the regularity of $\hat{R}_2^k$ is inherited by the unique solution W_2^k of (15.21).

We solve (15.20) as follows. If $\mathcal{A}_1 \cap \mathcal{A}_1^k = \emptyset$ we conclude that

$$1 \notin \mathrm{Spec}(\mathcal{S}_1^k, \Gamma_b(L_s^k(E_1; E_1)))\ ,$$

and we take $\Lambda^k = 0$ and W_1^k solving $\mathcal{S}_1^k W_1^k - W_1^k = \hat{R}_1^k - S_B^k W_2^k$. Otherwise we will choose $W_1^k = 0$ and $\hat{\Lambda}^k = -\hat{R}_1^k + S_B^k W_2^k$.

The construction above let us to find the parameterization of the manifold up to a certain order, and a polynomial expression for the dynamics on it (see claim b of Theorem 15.14). It remains to find the higher order terms of the parameterization.

REMARK 15.12. Notice that solutions of (15.20) are not unique. We could choose $W_1^k = 0$ and $\hat{\Lambda}^k = -\hat{R}_1^k + S_B^k W_2^k$ (and then we compute Λ^k), for which we do not need non-resonance condition such as $\mathcal{A}_1 \cap \mathcal{A}_1^k = \emptyset$. Notice that this election corresponds to find the invariant manifold as a graph (over E_1).

REMARK 15.13. When

$$|\mathrm{D}_\eta\Lambda(0,\theta)| \le \lambda < 1$$

or, more generally, that for some $m \in \mathbb{N}$,

$$|\mathrm{D}_\eta\Lambda(0,\theta+(m-1)\omega)\cdots \mathrm{D}_\eta\Lambda(0,\theta)| \le \lambda < 1$$

we obtain that the points of $\mathcal{W}$ close to $\mathcal{K}$ converge to $\mathcal{K}$ upon iteration of the map. In other words, when $\Lambda(\cdot,\theta)$ is a contraction for all θ, all η sufficiently small, the manifolds that we obtain are submanifolds of the usual stable manifold.

15.3.3. The equation for the higher order terms. Once we have obtained the polynomial vector bundle map Λ over the rotation ω and the L-order approximation $W^{\le}$ of the invariant manifold W, we have to find the higher order terms of the parameterization of the invariant manifold, $W^{>}$. We will write

$$W = W^{\le} + W^{>},$$

where $W^{>} : E_1 \to E$ is a bundle map over the identity such that $\mathrm{D}^j_\eta W^{>}(0,\theta) = 0$ for every $j \le L$.

The invariance equation (15.30) is reformulated in terms of $W^{>}$ as

$$\begin{aligned} &M(\theta)\left[W^{\le}(\eta,\theta) + W^{>}(\eta,\theta)\right] + N\left(W^{\le}(\eta,\theta) + W^{>}(\eta,\theta),\theta\right)\\ &= \ W^{\le}(\Lambda(\eta,\theta),\theta+\omega) + W^{>}(\Lambda(\eta,\theta),\theta+\omega)\end{aligned}$$

or, with more compact notation,

$$\begin{aligned} W^{>}_\theta - M_\theta^{-1}\cdot W^{>}_{\theta+\omega}\circ\Lambda_\theta = &-\left(W^{\le}_\theta - M_\theta^{-1}\cdot W^{\le}_{\theta+\omega}\circ\Lambda_\theta\right)\\ &- M_\theta^{-1}\cdot N_\theta\circ(W^{\le}_\theta + W^{>}_\theta)\ . \end{aligned} \tag{15.22}$$

(15.22) is an equation for $W^{>}$ to be solved in a suitable space of bundle maps from E_1 to E, over the identity, whose L first vertical derivatives vanish on the zero section of E_1.

Notice that the way we have constructed $W^{\le}$ and Λ ensures that, if $W^{>}$ satisfies the conditions above, then the right hand side of (15.22) satisfies also the same conditions.

If we define the operator $\mathcal{S}$ by

$$(\mathcal{S}H)_\theta = H_\theta - M_\theta^{-1}H_{\theta+\omega}\circ\Lambda_\theta\ , \tag{15.23}$$

then (15.22) reduces to the fixed point equation

$$W^{>}_\theta = -\mathcal{S}^{-1}(W^{\le}_\theta - M_\theta^{-1}\cdot W^{\le}_{\theta+\omega}\circ\Lambda_\theta + M_\theta^{-1}\cdot N_\theta\circ(W^{\le}_\theta + W^{>}_\theta))\ , \tag{15.24}$$

provided that $\mathcal{S}^{-1}$ exists and it is continuous in suitable spaces.

The existence of $\mathcal{S}^{-1}$ is equivalent to solve the linearized equation

$$(\mathcal{S}H)_\theta = H_\theta - M_\theta^{-1}\cdot H_{\theta+\omega}\circ\Lambda_\theta = R_\theta\ . \tag{15.25}$$

Formally, the solution of (15.25) is

$$H_\theta = \sum_{k=0}^{\infty} M^{-k}_{\theta+k\omega} R_{\theta+k\omega}\circ\Lambda^k_\theta\ . \tag{15.26}$$

To prove the existence of $\mathcal{S}^{-1}$, it is analyzed the convergence of (15.26) is suitable spaces. Then one shows that (15.24) defines a solution $W^{>}$. Notice that in this procedure one derivative is lost, but there is a bootstraps for which the regularity of the solution $W^{>}$ is optimal.

15.3.4. Statement of results. In order to obtain sharp results on regularity of the invariant manifolds, it will be very important for us to distinguish the regularities of the functions with respect to the horizontal variables (θ) and the vertical variables (x, η), because the angle variables parameterizing the torus and the real variables used to parameterize the stable directions enter very differently in the functional equations. Hence, when one is interested in optimal regularity it is natural to introduce spaces in which the regularity along these two variables is not the same.

In particular, we consider $C^{\Sigma_{r,s}}$ classes of maps $F = F(x,\theta)$, for which $D_\theta^i D_x^j F(x,\theta)$ exists and it is continuous for (i,j) such that $i \le r$ and $i+j \le r+s$. Notice, however, that results also will work for $C^{r+s} \subset C^{\Sigma_{r,s}}$.

We can also consider maps in the analytic category, in which the results we present have simpler proofs. For the sake of simplicity, we will consider functions that are analytic in both horizontal and vertical variables.

The result we obtain for existence and uniqueness of whiskers is the following (see [**HdlLb**]).

THEOREM 15.14. *Let $U \subset \mathbb{R}^n$ be an open set. Let $F : U \times \mathbb{T}^d \subset \mathbb{R}^n \times \mathbb{T}^d \to \mathbb{R}^n$ be a map of class $C^{\Sigma_{r,s}}$, with $r \ge 0$ and $s \ge 2$ – including $C^{\Sigma_{r,s}} = C^a$ in the analytic case $r = a$ –, such that for all $\theta \in \mathbb{T}^d$ the map $F(\cdot,\theta) : U \to \mathbb{R}^n$ is a local diffeomorphism. Let $\omega \in \mathbb{R}^d$ be a rotation. Let $\mathcal{K}$ be an invariant torus whose parameterization is given by $K \in C^r(\mathbb{T}^d, U)$.*

Let $\mathcal{M}_\omega$ be the transfer operator defined from $M(\theta) = \mathrm{D}_x F(K(\theta),\theta)$.

Assume that there is a decomposition

$$N\mathcal{K} = E_1 \oplus E_2 \tag{15.27}$$

into C^r subbundles such that E_1 is invariant under M. Equivalently, if we take a representation of the transfer operator in a frame associated to the decomposition (15.27) *we have*

$$M(\theta) = \begin{pmatrix} M_1(\theta) & B(\theta) \\ 0 & M_2(\theta) \end{pmatrix} . \tag{15.28}$$

We denote by $\mathcal{M}_{1,\omega}$, $\mathcal{M}_{2,\omega}$ the transfer operators acting on sections of E_1, E_2, respectively associated to M_1, M_2. The annular hull of the spectrum

$$\mathcal{A} = \mathcal{A}\mathrm{Spec}(\mathcal{M}_\omega, \Gamma_b) = \{ z e^{i\alpha} \mid z \in \mathrm{Spec}(\mathcal{M}_\omega, \Gamma_b), \alpha \in \mathbb{R} \} ,$$

is then the union of the annular hulls of each $\mathcal{M}_\omega^1$, $\mathcal{M}_\omega^2$:

$$\mathcal{A}_1 = \mathcal{A}\mathrm{Spec}(\mathcal{M}_{1,\omega}, \Gamma_b(E_1)) \ , \ \mathcal{A}_2 = \mathcal{A}\mathrm{Spec}(\mathcal{M}_{2,\omega}, \Gamma_b(E_2)) ,$$

and $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$.

Assume that:

H.1 $\mathcal{A}_1 \subset \{z \in \mathbb{C} \mid |z| < 1\}$;
H.2 $\mathcal{A}_1^{L+1}\mathcal{A}^{-1} \subset \{z \in \mathbb{C} \mid |z| < 1\}$, *for a certain* $L \geq 1$.
H.3 $\mathcal{A}_1^i \cap \mathcal{A}_2 = \emptyset$ *for every* i *with* $2 \leq i \leq L$ *(in case that* $L \geq 2$*)*;
H.4 $L + 1 \leq s$.

Then:

a) *We can find a polynomial bundle map* $\Lambda : E_1 \to E_1$ *over the rotation* ω, *of degree not larger than* L *and of class* $C^{r,\infty}$ *with*

$$\Lambda(0,\theta) = 0 \ , \quad \mathrm{D}_\eta \Lambda(0,\theta) = M_1 \ , \tag{15.29}$$

and a $C^{\Sigma_{r,s}}$ *bundle map* $W : U_1 \subset E_1 \to X$ *over the identity, where* U_1 *is an open tubular neighborhood of the zero section of* E_1, *such that*

$$F(W(\eta,\theta),\theta) = W(\Lambda(\eta,\theta),\theta + \omega) \tag{15.30}$$

holds in U_1, *and*

$$\begin{aligned} &W(0,\theta) = K(\theta) \ , \\ &\Pi_1 \mathrm{D}_\eta W(0,\theta) = \mathrm{Id}_{E_1} \ , \quad \Pi_2 \mathrm{D}_\eta W(0,\theta) = 0 \ , \end{aligned} \tag{15.31}$$

for all $\theta \in \mathbb{T}^d$, *where* Π_1, Π_2 *are the projections on* E_1, E_2, *respectively.*

b) *In case that we further assume for* $\ell \geq 2$ *that*

$$\mathcal{A}_1^i \cap \mathcal{A}_1 = \emptyset \qquad \text{for every integer } i \text{ with } \ell \leq i \leq L \ , \tag{15.32}$$

then we can choose Λ *in a) above to be a polynomial of degree not larger than* $\ell - 1$.

In particular, if (15.32) *happens for* $\ell = 2$, *then*

$$\mathcal{A}_1^i \cap \mathcal{A}_1 = \emptyset \qquad \text{for every integer } i \text{ with } 2 \leq i \leq L \ , \tag{15.33}$$

and we can choose Λ *in a) above to be linear.*

c) *The* $C^{\Sigma_{r,s}}$ *manifold produced in a) is unique among the* $C^{\Sigma_{r,L+1}}$ *locally invariant manifolds tangent to* E_1 *at* $\mathcal{K}$. *That is, every two* $C^{\Sigma_{r,L+1}}$ *locally invariant manifolds will coincide in a neighborhood of* K *in* E.

REMARK 15.15. Note that we do not assume that $\mathcal{A}_1 \cap \mathcal{A}_2 = \emptyset$ since the assumption [H.3] only requires that the intersection is empty for powers bigger or equal that 2.

Also, E_2 is not assumed invariant.

One example of this situation occurs when the linearization is a Jordan block. The bundle E_1 corresponds to the eigenspace and the bundle E_2 corresponds to the generalized eigenvalues. Note that E_2 is not invariant, indeed, in this example, there are no invariant complementary bundles.

REMARK 15.16. Note that a consequence of (15.30) is that $\mathcal{W} = W(U_1)$ is a $C^{\Sigma_{r,s}}$ manifold invariant under F and tangent to E_1 at $\mathcal{K}$.

About the uniqueness statement c) of Theorem (15.14), note that the parameterization W and the map Λ need not be unique; it is the manifold $\mathcal{W} = W(U_1)$ which is unique.

REMARK 15.17. Theorem 15.14 includes the case in which the bundles E_1,E_2 are non trivial. As we will see in [**HdlL05a**], the fact that the bundles are non trivial happens very often near resonant situations.

REMARK 15.18. It follows from the results in Part 2 that the spectrum of a transfer operator changes by a small amount if the transfer operator changes by a small amount.

Therefore, the non-resonance conditions H.1 – H.4 in Theorem 15.14 hold for open sets of transfer operators.

In particular, in case that the torus is normally hyperbolic, applying Theorem 15.2 we obtain that the torus persists and that the linearization is close to the original one.

Hence, if the original torus has spectral spaces which satisfy the hypothesis H.1 – H.4, then, the perturbed tori will also have spectral subspaces satisfying these hypothesis and, hence, by Theorem 15.14, it will also have invariant manifolds associated to these spectral subspaces.

CHAPTER 16

Towards a proof of a conjecture of Bowen

In [**Bow78**, p. 21], R. Bowen proposed the following conjecture:

CONJECTURE 16.1. *Let f be an Anosov system on a compact manifold M. Let $f_\#$ be the action on cohomology induced by M. Then, $f_\#$ does not have any eigenvalue of modulus* 1*.*

The goal of this section is to present a functional analysis approach to the proof of Bowen's conjecture We will not succeed in proving the Conjecture 16.1. Nevertheless, the failure of the functional analysis approach presented here provides with a nice illustration of several of the subtleties about spectral properties of constrained spaces (e.g. the existence of residual spectrum, etc.)

The idea for the strategy is very simple. We consider the decomposition

$$\chi_c^{*1} = \chi_e^{*1} \oplus H \tag{16.1}$$

were, as in Definition 13.1, the spaces χ_e^{*1},χ_c^{*1} are the spaces of exact and closed forms and H is a finite dimensional space spanned by closed forms that are representatives of a basis in cohomology. We will assume that we have picked these representatives.

Since f_*, the push-forward by f, sends closed forms into closed forms and exact forms into exact forms, we have that we can express f_* as an upper triangular matrix with respect to the decomposition (16.1). Denoting by f_*^e, f_*^c the restrictions of the push-forward to χ_e^{*1} and χ_c^{*1} respectively, we have:

$$f_*^c = \begin{pmatrix} f_*^e & K \\ 0 & f_\# \end{pmatrix} \tag{16.2}$$

where K is just a mapping from the space H to the space of exact forms.

Now, we have by Theorem 13.14 that

$$\mathrm{Spec}_W(f_*^e, \chi_e^{*1}) = \mathrm{Spec}_W(f_*^c, \chi_c^{*1}) = \mathrm{Spec}_W(f_*, \chi_{C^0}^{*1}) \tag{16.3}$$

If Weyl spectrum behaved as the finite dimensional spectrum, Bowen's conjecture would follow from the observations above. In finite dimensions we would have from the upper triangular structure (16.2) that the spectrum would be the union of the spectrum of the diagonal blocks. Hence, from (16.3) we would obtain that the spectrum of $f_\#$ would be contained in

$\mathrm{Spec}_W(f_*, \chi^1_B)$ which by the characterization of Anosov systems in [**Mat69**] does not contain the unit circle.

Unfortunately, in Banach spaces, we do not have these identities. In general, the best that can be said from the upper triangular structure is that, applying Theorem 3.44, we have:

$$\mathrm{Eig}(f_\#, H) \cap \mathrm{Res}(f^e_*, \chi^{*1}_e) \subset \mathrm{Spec}_W(f^c_*, \chi^{*1}_c).$$

(Notice that H is finite dimensional, so the spectrum of $f_\#$ are eigenvalues.) Unfortunately, because of the No Gaps Theorem 13.11, $\mathrm{Res}(f^e_*, \chi^{*1}_e)$ lies outside an annulus obtained closing the gaps in $\mathrm{Spec}(f_*, \chi^{*1}_B)$. In particular, we cannot produce the desired result.

The fact that the Theorem 3.44 cannot be improved in our particular case can be clarified by explicit calculations for the problem at hand.

We first discuss the possibility that the eigenvalues of $f_\#$ are in the point spectrum of f^c_*.

Let λ be an eigenvalue of $f_\#$ and let $b \in H$ be an eigenvector. Showing that λ is an eigenvector for f^c_* amounts to finding an exact form $a \in \chi^{*1}_e$, $a \neq 0$ such that

$$f^E_* a - \lambda a = -Kb. \tag{16.4}$$

A form a in χ^{*1}_e has a primitive A – a complex valued function at M – defined as follows: We pick an origin x_0 and a number $c \in \mathbb{C}$ and we define

$$A(x) = c + \int_\gamma a, \tag{16.5}$$

where γ is a C^1 path that joins x_0 to x. Because of the definition of χ^{*1}_e, the integral of a over every closed loop is zero, hence, the integral in (16.5) is independent of the path γ chosen.

The primitives for a form differ only by a constant. Since the forms are C^0, the primitives will be C^1 functions. Moreover, the primitives behave well under transformations. It is easy to check that if A is a primitive for a, then $A \circ f$ is a primitive for $f_* a$.

Hence a necessary condition for (16.4) is that given KB a primitive for Kb we can find a function A that satisfies the equation

$$A \circ f - \lambda A = KB. \tag{16.6}$$

If $|\lambda| > 1$, the unique continuous solution of (16.6) is

$$A = -\sum_{i=0}^{\infty} \lambda^{-i-1} (KB) \circ f^i \tag{16.7}$$

Unless $|\lambda|$ is larger than the spectral radius of f_*, the sum in (16.7) is, in general, not C^1. This is a variant of the classical Weierstrass analysis for non-differentiable functions. A treatment precisely for the case at hand can be found in [**dlL92**] p. XX ff. (A heuristic guide for the lack of differentiability of (16.7) is to take derivatives term by term and observe that the general term may grow exponentially).

Since the function A is not C^1, it cannot be a primitive of a C^0 form. Hence, we conclude that the solution (16.4) does not have a solution. Hence, we conclude that λ is not an eigenvalue of f_*^c.

We also note that the obstructions for the differentiability of (16.7) discussed in [**dlL92**] are C^1 open for KB. Hence we have shown that the range of $f_*^e - \lambda$ is not dense when λ is as we have considered. Hence $\lambda \in \mathrm{Spec}_R(f_*^e, \chi_e^{*1})$ (the residual spectrum). This illustrates the no-gap phenomena stated in Theorem 13.11.

A similar analysis can be done in the case that $|\lambda| < 1$. In that situation, the only continuous solution is given by

$$A = \sum_{i=0}^{\infty} \lambda^i (KB) \circ f^{-i-1}. \tag{16.8}$$

If $|\lambda|$ is smaller than the inverse of the spectral radius of f_*^{-1}, the same analysis shows that (16.8) will not be differentiable for an C^1 open set of KB and that therefore, λ is not an eigenvalue. Indeed, what we have shown is that $f_*^e - \lambda$ does not have a dense range, which implies that $\lambda \in \mathrm{Spec}_R(f_*^e, \chi_e^{*1})$.

The case when $|\lambda| = 1$ is even easier to discuss since there are many more obstructions. For example, if $\lambda = 1$, we have the well known obstruction that the sum of KB over periodic orbits of f has to add to zero. (Note that the failure of this condition is C^0 open and, a fortiori, C^1 open).

The analysis above can be extended to show that from $\lambda \in \mathrm{Eig}(f_\#, H)$ we cannot conclude that $\lambda \in \mathrm{Spec}_W(f_*^c, \chi_c^{*1})$. If we wanted to construct an approximate eigenfunction we would be lead to finding a sequence of functions E_n such that $\lim_{n\to\infty} \|E_n\|_{C^1} = 0$ $A_n \in C^1$ such that, in analogy with (16.6) we have:

$$A_n \circ f - \lambda A_n = KB + E_n. \tag{16.9}$$

The impossibility of finding such A_n follows because the obstructions for differentiability of the solutions of (16.6) is C^1 open and, therefore, if the E_n are C^1 small enough, the obstructions for KB start to work.

APPENDIX A

A summary of Spectral Theory

In this section, we collect some definitions and results of spectral theory in Banach spaces, that will be important in the analysis of transfer operators, extending those given in Section 1.6. The reader can obtain much more information in any book on linear functional analysis (some references are [**Kat76, DS88a, DS88b, DS88c, RS80, RS78, Heu82, Con85**]). Unfortunately, most of the references in spectral theory concentrate on the much more extensive spectral theory for normal or self-adjoint operators.

Along this section, X is a complex Banach space with norm $\|\cdot\|$, and $L : X \to X$ denotes a bounded linear operator. When the inverse of L is needed, we will assume that it exists and it is bounded.

REMARK A.1. Recall that if $\tilde{X}$ is a real Banach space, then we can construct the complexification $X = \tilde{X} + \mathbf{i}\tilde{X}$, being X a complex Banach space. The norm in X is just $\|x + \mathbf{i}y\| = \sqrt{\|x\|^2 + \|y\|^2}$ for $x, y \in \tilde{X}$. We have a complex conjugation on X: $\overline{x + \mathbf{i}y} = x - \mathbf{i}y$.

A.1. The spectrum of a bounded linear operator

In this section, we recall some standard results and definitions.

DEFINITION A.2. $\mathrm{Res}(L, X)$ *denotes the* resolvent set *of L, this is the set of all* regular values *of L, that are the complex numbers λ for which $\lambda\mathrm{Id} - L$ is bijective. Banach's isomorphism theorem implies that the* resolvent operator $R_\lambda L = (\lambda\mathrm{Id} - L)^{-1}$ *is continuous.*

The complement of the resolvent is $\mathrm{Spec}(L, X)$*, the* spectrum *of L, that is the set of* spectral values *of L.*

REMARK A.3. If $\tilde{L}$ is a bounded operator on a real Banach space $\tilde{X}$, we can complexify it as acting on $X = \tilde{X} + \mathbf{i}\tilde{X}$: $L(x + \mathbf{i}y) = \tilde{L}x + \mathbf{i}\tilde{L}y$. We define the resolvent set of $\tilde{L}$ as the resolvent set of its complexification L. So, even if $\tilde{L}$ is real, we will consider complex regular values. Notice that, in such a case, if $\lambda \in \mathrm{Res}(L, X)$, then $\bar{\lambda} \in \mathrm{Res}(L, X)$. The same considerations we take for the spectrum.

All the spectral theory will be done for complex Banach spaces. If we are working with real Banach spaces – which appear naturally when we consider geometric problems – we will always understand the spectrum of the complexification. We will mention later that some constructions such as spectral projections of the complexification of a real operator, have a meaning in the original real Banach space.

Remark A.4. The next properties are straightforward, and generalize well known properties of finite dimensional matrices. For X, Y Banach spaces and $L : X \to X$ bounded linear operator:

a) If $\alpha \in \mathbb{C}^*$ and $\beta \in \mathbb{C}$, $\mathrm{Spec}(\alpha L - \beta \mathrm{Id}, X) = \alpha \mathrm{Spec}(L, X) - \beta$.
b) If $P : Y \to X$ is an isomorphism, $\mathrm{Spec}(P^{-1}LP, Y) = \mathrm{Spec}(L, X)$.
c) If L has bounded inverse, $\mathrm{Spec}(L^{-1}, X) = \mathrm{Spec}(L, X)^{-1}$.

A very useful result is the *Neumann series*, valid for $\|L\| < 1$:

$$(\mathrm{Id} - L)^{-1} = \sum_{k=0}^{\infty} L^k. \tag{A.1}$$

In particular (A.1) shows that $\mathrm{Res}(L, X) \supset \{\lambda \in \mathbb{C} \,\big|\, |\lambda| > \|L\|\}$ and hence, that $\mathrm{Spec}(L, X)$ is bounded.

The following result is the standard result on stability of the resolvent.

If $\lambda \in \mathrm{Res}(L, X)$ and A is a bounded linear operator, we can write

$$\lambda \mathrm{Id} - L - A = (\lambda \mathrm{Id} - L)(\mathrm{Id} - R_\lambda A) \tag{A.2}$$

Hence, when $\|A\| \, \|R_\lambda\| < 1$, we can apply (A.1) to obtain:

Proposition A.5. *Assume that* $\lambda \in \mathrm{Res}(L, X)$. *If* $\|A\| \, \|R_\lambda\| < 1$. *Then,* $\lambda \mathrm{Id} - L - A$ *is invertible, i.e.* $\lambda \in \mathrm{Res}(L + A, X)$.

Moreover,

$$(\lambda \mathrm{Id} - L - A)^{-1} = \left(\sum_{i=0}^{\infty} (R_\lambda A)^i \right) (\lambda \mathrm{Id} - L)^{-1} \tag{A.3}$$

As a consequence, we have:

- $\mathrm{Res}(L, X)$ *is open. In fact:*

$$\lambda \in \mathrm{Res}(L, X) \ , \ |\lambda - \mu| < \|(L - \lambda \mathrm{Id})^{-1}\|^{-1} \Rightarrow \mu \in \mathrm{Res}(L, X) \ . \tag{A.4}$$

- *Hence* $\mathrm{Spec}(L, X)$ *is closed.*
- R_λ *is an analytic function of* $\lambda \in \mathrm{Res}(L, X)$.
- $R_\lambda(L + A)$ *is an analytic function of* A.

A.1.0.1. *Classification of the spectral values.* The spectrum of a bounded linear operator L is a compact subset of $\mathbb{C}$, and it is decomposed in three parts:

$$\mathrm{Spec}(L, X) = \mathrm{Spec}_P(L, X) \cup \mathrm{Spec}_R(L, X) \cup \mathrm{Spec}_C(L, X) \ ,$$

where

- $\mathrm{Spec}_P(L, X)$ is the *point spectrum*, that is the set of all $\lambda \in \mathbb{C}$ for which $\lambda \mathrm{Id} - L$ is not injective (its elements are the *eigenvalues* of L);
- $\mathrm{Spec}_R(L, X)$ is the *residual spectrum*, that is the set of all $\lambda \in \mathbb{C}$ for which $\lambda \mathrm{Id} - L$ is injective but $(\lambda \mathrm{Id} - L)(X)$ is not dense in X;

- $\mathrm{Spec}_C(L,X)$ is the *continuous spectrum*, that is the set of all $\lambda \in \mathbb{C}$ for which $\lambda\mathrm{Id} - L$ is injective, $(\lambda\mathrm{Id} - L)(X)$ is dense but not closed in X (that is $(\lambda\mathrm{Id} - L)(X)$ is a proper subset of X).

Other useful subsets of the spectrum are:

- $\mathrm{Spec}_{NR}(L,X) = \mathrm{Spec}_P(L,X) \cup \mathrm{Spec}_C(L,X)$ is the *non residual spectrum* of L;
- $\mathrm{Spec}_W(L,X)$ is the *Weyl spectrum*, that is the set of all $\lambda \in \mathbb{C}$ for which $\lambda\mathrm{Id} - L$ is not injective or $(\lambda\mathrm{Id} - L)(X)$ is not closed in X.

Immediately from the definitions we follow that $\mathrm{Spec}_{NR}(L,X) \subset \mathrm{Spec}_W(L,X)$, but the Weyl spectrum can contain residual spectral values.

The spectral radius. We have seen that the spectrum is a compact subset of $\mathbb{C}$. Its bounds are given by the spectral radii.

Definition A.6. *The* spectral radius *is*

$$r_s(L,X) = \max\{|\lambda| \mid \lambda \in \mathrm{Spec}(L,X)\} \ .$$

We have: $r_s(L,X) \leq \|L\|$.

We define also

$$r_i(L,X) = \min\{|\lambda| \mid \lambda \in \mathrm{Spec}(L,X)\} \ .$$

We have: $r_i(L,X) \geq \|L^{-1}\|^{-1}$.

The following result is a simple version of the so called functional calculus.

Theorem A.7 (Spectral mapping theorem for polynomials). *Let p be a complex polynomial. Then:*

$$\mathrm{Spec}(p(L),X) = p(\mathrm{Spec}(L,X)) \ .$$

That is to say, the spectrum of $p(L)$ consists of those points μ such that $p(\lambda) = \mu$ for some $\lambda \in \mathrm{Spec}(L,X)$.

As a corollary one obtains the *spectral radius formula*.

Theorem A.8 (Spectral radius formula).

$$r_s(L,X) = \lim_{m\to+\infty} \sqrt[m]{\|L^m\|} \ , \ r_i(L,X) = \lim_{m\to-\infty} \sqrt[m]{\|L^m\|} \ . \tag{A.5}$$

A.1.1. Other results. It is well known that if $Y \subset X$ is a closed space invariant under L, then it is not necessarily true that $\mathrm{Spec}(Y,L) \subset \mathrm{Spec}(X,L)$. One case in which this inclusion works is the following.

Theorem A.9 (Spectrum on projections). *Let $L : X \to X$ a bounded linear operator. Let $P : X \to X$ a (continuous) projection, that is, a bounded linear operator such that $P^2 = P$. Let $Y = P(X)$ the closed subspace of points fixed by P. Suppose that L and P commute: $LP = PL$. Then, Y is invariant under L and*

$$\mathrm{Spec}(Y,L) \subset \mathrm{Spec}(X,L) \ .$$

An important property, inherited from the spectral theory on Banach Algebras (see [**Rud73**]) is the following. It is an extension of the previous result.

THEOREM A.10 (Spectrum of commuting bounded operators). *Let $A, B : X \to X$ be bounded linear operators such that $AB = BA$. Then,*

$$\mathrm{Spec}(A+B, X) \subset \mathrm{Spec}(A, X) + \mathrm{Spec}(B, X) \ ,$$
$$\mathrm{Spec}(AB, X) \subset \mathrm{Spec}(A, X)\mathrm{Spec}(B, X) \ .$$

We recall an important result on duality. The dual map $L^* : X^* \to X^*$ is defined on the dual space X^*, the set of the bounded linear functionals $x^* : X \to \mathbb{C}$, by the composition $L^*x^* = x^* \circ L$.

THEOREM A.11 (Duality and spectrum). *The spectrum of L^* is*

$$\mathrm{Spec}(L^*, X^*) = \mathrm{Spec}(L, X) \ .$$

A.2. Spectral sets and spectral projections

A very useful notion in our analysis is that of spectral set.

DEFINITION A.12. *A* spectral set *of L is a subset σ of* $\mathrm{Spec}(L, X)$ *such that σ and* $\mathrm{Spec}(L, X) \setminus \sigma$ *are closed.*

In the case that the space X is the complexification of a real space, we will also include in the definition of spectral set that σ is closed under complex conjugation. (i.e. $\lambda \in \sigma \implies \bar{\lambda} \in \sigma$).

Note that given the compactness of the spectrum, the fact that σ and $\mathrm{Spec}(L, X) \setminus \sigma$ are closed, implies that they are a finite distance apart. The following two theorems can be found in [**Kat76, Heu82**].

THEOREM A.13 (Spectral Decomposition). *Let $\sigma_1, \dots, \sigma_m$ be pairwise disjoint spectral sets of L such that* $\mathrm{Spec}(L, X) = \sigma_1 \cup \dots \cup \sigma_m$.

Then, to each spectral set σ_i we can associate a continuous projection $P_i = P_{\sigma_i} : X \to X$, known as the spectral projection *or* Riesz projection *associated to σ_i, such that:*

a) *$X_i = P_i(X)$ is invariant under L.*
b) $\mathrm{Spec}(L, X_i) = \sigma_i$,

Moreover:

c) $\mathrm{Id} = P_1 + \dots + P_m$ *(i.e. $X = X_1 + \dots + X_m$).*
d) $P_i P_j = 0$ *if $i \neq j$ (i.e. $X_i \cap X_j = \{0\}$ if $i \neq j$).*

That is to say: $X = X_1 \oplus \dots \oplus X_m$.

In case that X is a the complexification of a real space $\tilde{X}$ and that L is the complexification of an operator $\tilde{L}$ on $\tilde{X}$, since each of the σ_i is closed under complex conjugation, we conclude that the spaces X_i are also closed under complex conjugation. As a result, the spaces X_i are the complexification of spaces $\tilde{X}_i$ and we have $P_i\tilde{X} = \tilde{X}_i$ and $\tilde{X} = \tilde{X}_1 \oplus \dots \oplus \tilde{X}_m$.

The following theorem relates the spectrum and the norms of iterated transformations. In particular, it relates the existence of annular gaps in the spectrum with the existence of invariant subspaces. We will denote

$$\mathcal{A}_{\lambda,\mu} = \{z \in \mathbb{C} \mid \lambda \leq |z| \leq \mu\} \ .$$

as the annulus of radii $0 < \lambda \leq \mu$.

THEOREM A.14 (Characterization of spectral projections). *Suppose that the spectrum has a gap in the annulus $\mathcal{A}_{\lambda,\mu}$, i.e.*

$$\mathrm{Spec}(L, X) \cap \mathcal{A}_{\lambda,\mu} = \emptyset \ .$$

Let $\sigma_{<\lambda}$ be the maximal spectral set contained in $\{z \in \mathbb{C} \mid |z| < \lambda\}$, and $\sigma_{>\mu}$ be the maximal spectral set contained in $\{z \in \mathbb{C} \mid |z| > \mu\}$.

Then, the corresponding projections are given by

$$P^{<\lambda} = \lim_{m\to\infty} \left(\mathrm{Id} - (L/\lambda)^m\right)^{-1} \ , \ P^{>\mu} = \lim_{m\to\infty} \left(\mathrm{Id} - (L/\mu)^{-m}\right)^{-1} \ .$$

Moreover, the invariant subspaces $X^{<\lambda} = P^{<\lambda}(X)$ and $X^{>\mu} = P^{>\mu}(X)$ satisfy $X = X^{<\lambda} \oplus X^{>\mu}$ and they are characterized by

$$\text{(A.6)} \qquad \begin{aligned} x \in X^{<\lambda} \quad &\Leftrightarrow \quad \limsup_{m\to\infty} \|L^m x\|^{\frac{1}{m}} < \lambda \ , \\ x \in X^{>\mu} \quad &\Leftrightarrow \quad \limsup_{m\to\infty} \|L^{-m} x\|^{\frac{1}{m}} < \mu^{-1} \ . \end{aligned}$$

We will write $L^{<\lambda} = L_{|X^{<\lambda}}$ and $L^{>\mu} = L_{|X^{>\mu}}$. We rewrite the previous theorem in terms of rates of growth of the iterations of the vectors.

THEOREM A.15 (Rates of growth in the spectral subspaces). *Under the hypothesis of Theorem A.14, for all $\varepsilon > 0$ small enough there exists a constant $C_\varepsilon > 0$ such that the invariant splitting $X = X^{<\lambda} \oplus X^{>\mu}$ is characterized by*

$$\text{(A.7)} \qquad \begin{aligned} x \in X^{<\lambda} \quad &\Leftrightarrow \quad \forall m \geq 0 \ \|L^m x\| \leq C_\varepsilon (\lambda - \varepsilon)^m \|x\| \ , \\ x \in X^{>\mu} \quad &\Leftrightarrow \quad \forall m \geq 0 \ \|L^{-m} x\| \leq C_\varepsilon (\mu + \varepsilon)^{-m} \|x\| \ . \end{aligned}$$

Proof: Let $\varepsilon > 0$ be such that $\lambda - 2\varepsilon \geq r_s(L, X^{<\lambda})$ and $\mu + 2\varepsilon \leq r_i(L, X^{>\mu})$. First, define

$$C_\varepsilon^{<\lambda} = \sup_{m\geq 0} (\lambda - \varepsilon)^{-m} \|L^m_{|X^{<\lambda}}\| \ , \ C_\varepsilon^{>\mu} = \sup_{m\geq 0} (\mu + \varepsilon)^m \|L^{-m}_{|X^{>\mu}}\| \ ,$$

that are finite from the spectral radius formula applied to $L_{|X^{<\lambda}}$ and $L_{|X^{>\mu}}$, respectively. Finally, we have just to define

$$C_\varepsilon = \max\{C_\varepsilon^{<\lambda}, C_\varepsilon^{>\mu}\}$$

to obtain the rates of growth (A.7). □

This theorem will play an important role in the characterization of bundles by the asymptotic properties of cocycles.

The following result follows immediately from the previous theorems.

COROLLARY A.16. *Under the hypothesis of Theorem A.14, if $Y \subset X$ is an L-invariant closed subspace that is also invariant under the spectral projections $P^{<\lambda}$ and $P^{>\mu}$, then*

$$\mathrm{Spec}(L, Y) \cap \mathcal{A}_{\lambda,\mu} = \emptyset \ .$$

That is to say, the spectrum of L in Y has also a gap.

A.3. Adapted norms

From the spectral radius formula we obtain that $\mathrm{Spec}(L, X) \subset \{\lambda \in \mathbb{C} \mid |\lambda| \le \rho\}$ is equivalent to

$$\forall \varepsilon > 0 \ \exists c_\varepsilon > 0 \mid \forall m \ge 0 \ \|L^m\| \le c_\varepsilon (\rho + \varepsilon)^m \ . \tag{A.8}$$

By the uniform boundedness principle, this is equivalent to

$$\forall v \in X \ \forall \varepsilon > 0 \ \exists c_{\varepsilon,v} \mid \forall m \ge 0 \ \|L^m v\| \le c_{\varepsilon,v} (\rho + \varepsilon)^m \ . \tag{A.9}$$

In some applications it is convenient to be able to that c_ε in (A.8) equal to 1 by introducing an *adapted norm*, equivalent to the original one.

An explicit expression for an adapted norm for an operator.

$$\|x\|_\varepsilon^2 = \sum_{k=0}^{\infty} \|L^k x\|^2 (\rho + \varepsilon)^{-2k} \ .$$

It is easy to verify that the norms $\|\cdot\|$ and $\|\cdot\|_\varepsilon$ are equivalent, and $\|Lx\|_\varepsilon \le (\rho + \varepsilon)\|x\|_\varepsilon$.

Similarly, if the spectrum has a gap in the annulus $\mathcal{A}_{\lambda,\mu}$, it is possible to develop norms adapted to the splitting. Let $X^{<\lambda}$ and $X^{>\mu}$ the corresponding invariant subspaces. Then, for all $\varepsilon > 0$ small enough there exist a constant $C_\varepsilon > 0$ such that

$$x \in X^{<\lambda} \quad \Leftrightarrow \quad \forall m \ge 0 \ \|L^m x\| \le C_\varepsilon (\lambda - \varepsilon)^m \|x\| \ ,$$

$$x \in X^{>\mu} \quad \Leftrightarrow \quad \forall m \ge 0 \ \|L^{-m} x\| \le C_\varepsilon (\mu + \varepsilon)^{-m} \|x\| \ .$$

We can obtain again an adapted norm such that $C_\varepsilon = 1$. Given $x \in X$, let $x = x_{<\lambda} + x_{>\mu}$ be the corresponding decomposition, with $x_{<\lambda} \in X^{<\lambda}$ and $x_{>\mu} \in X^{>\mu}$, then we define:

$$\|x_{<\lambda}\|_\varepsilon^2 = \sum_{k=0}^{\infty} \|L^k x_{<\lambda}\|^2 (\lambda - \varepsilon)^{-2k} \ , \ \|x_{>\mu}\|_\varepsilon^2 = \sum_{k=0}^{\infty} \|L^{-k} x_{>\mu}\|^2 (\mu + \varepsilon)^{2k} \ ,$$

and

$$\|x\|_\varepsilon^2 = \|x_{<\lambda}\|_\varepsilon^2 + \|x_{>\mu}\|_\varepsilon^2 \ .$$

This norm is equivalent to the original one and satisfies $\|L_{|X^{<\lambda}}\|_\varepsilon \le \lambda - \varepsilon$, $\|L^{-1}_{|X^{>\mu}}\|_\varepsilon \le (\mu + \varepsilon)^{-1}$.

REMARK A.17. Notice that if the original norm comes from a scalar product, then the new one also comes from a scalar product.

A.4. Perturbation of the spectrum

The spectrum $\mathrm{Spec}(L, X)$ as a function of L has some continuity properties. Before stating the theorem, we recall some definitions. Given a metric space E, let $\mathcal{H}(E)$ be the set of non empty compact sets of E. Given $A, B \in \mathcal{H}(E)$ the distance from A to B is defined to be

$$d(A, B) = \max_{a \in A} \min_{b \in B} d(a, b) \ .$$

d does not define a distance in $\mathcal{H}(E)$. This is done by defining the *Hausdorff distance*

$$h(A, B) = \max(d(A, B), d(B, A)) \ .$$

Since $d(A, B) < \varepsilon$ is equivalent to $A \subset B + \varepsilon = \{x \in X \mid \exists b \in B : d(x, b) < \varepsilon\}$, then $h(A, B) < \varepsilon$ is equivalent to $A \subset B + \varepsilon, B \subset A + \varepsilon$. It is well know that $\mathcal{H}(E)$, endowed with the distance h, is complete (compact) if E is complete (compact).

The following result is classical.

THEOREM A.18 (Upper semi-continuity of the spectrum). *Let L_0 be a bounded linear operator in the Banach space X. Then, for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $\|L - L_0\| < \delta$ then $d(\mathrm{Spec}(L, X), \mathrm{Spec}(L_0, X)) < \varepsilon$ (that is to say, for all $\lambda_1 \in \mathrm{Spec}(L)$ there exists $\lambda \in \mathrm{Spec}(L_0)$ such that $|\lambda_1 - \lambda| < \varepsilon$).*

Proof: We refer for more details to [**Kat76**, p. 208].

The result is equivalent to showing that if Γ is a closed of the resolvent of L_0, then, Γ is contained in the resolvent of all the operators L such that $\|L - L_0\| \leq \delta$

We take

$$\delta^{-1} = \max\{\|(\lambda - L_0)^{-1}\| \mid \lambda \notin \mathrm{Spec}(L_0, X) + \varepsilon\} \ ,$$

Such δ exists because

$$\lim_{\lambda \to \infty} \|(\lambda - L_0)^{-1}\| = 0 \ .$$

Then, we can use the results on stability of the resolvent in Proposition A.5 to conclude the desired result. ☐

REMARK A.19. Theorem A.18 shows that if we have an operator, the spectrum in a neighborhood cannot be much bigger, but it can be significantly smaller.

It is interesting to study what happens when we have a family of operators converging to a limit.

We note that the neighborhoods produced in Theorem A.18 are very dependent on the operator, not just on the spectral properties. As the operator approaches a limit, it could well happen that these neighborhoods shrink and that therefore, the limit of the spectrum could be strictly smaller than the spectrum of the limit.

See [**Kat76**, p. 209] for an example of this phenomenon.

Another example of a related phenomenon, more motivated by dynamical systems is Example 4.18.

A finer result says that each separated part of $\operatorname{Spec}(L, X)$ is upper semicontinuous. For simplicity we state the result for the case in which the spectrum has a gap (see [**Kat76**], p. 212, for a more general statement of this theorem).

THEOREM A.20. *Let L_0 be a bounded linear operator in the Banach space X. Assume that*

$$\operatorname{Spec}(L_0, X) \cap \mathcal{A}_{\lambda,\mu} = \emptyset \ ,$$

where $0 < \lambda \le \mu$. Then there exists $\delta > 0$ with the following properties:

a) *If $\|L - L_0\| < \delta$ then*

$$\operatorname{Spec}(L, X) \cap \mathcal{A}_{\lambda,\mu} = \emptyset \ ,$$

b) *Let $X = X^{<\lambda}[L] + X^{>\mu}[L]$ be the associated decomposition. Then $X^{<\lambda}[L]$ and $X^{>\mu}[L]$ are respectively isomorphic with $X^{<\lambda}[L_0]$ and $X^{>\mu}[L_0]$.*

Moreover, the decomposition $X = X^{<\lambda}[L]+X^{>\mu}[L]$ is continuous in L in the sense that the corresponding projections $P^{<\lambda}[L]$ and $P^{>\mu}[L]$ tend respectively to $P^{<\lambda}[L_0]$ and $P^{>\mu}[L_0]$ in norm when $\|L - L_0\| \to 0$.

Proof: (Sketch) We refer to [**Kat76**, p. 212] for details.

The key ingredient is that there are explicit formulas for the spectral projections associated to a component of the spectrum in terms of the resolvent.

$$P^{<\lambda} = \frac{1}{2\pi \mathbf{i}} \int_{\mathbb{S}_\lambda} (L - z)^{-1} dz$$

The desired result follows from the continuity of the resolvent given by the Neumann formula.

We refer to [**Kat76**], among others, for the details. □

REMARK A.21. As in Remark A.19, we remark that the neighborhoods of stability depend very much on the operator.

As an operator approaches a limit, it could well happen that the neighborhoods shrink so that the limit has a spectrum which is much larger than the limit of the spectrum.

For example, it could well happen that, even if the spectrum does not change as we approach the limit, the norms of the spectral projections grow unbounded and this is what causes the neighborhoods of stability to shrink.

The paper [**HdlLa**] contains numerical evidence suggesting that this phenomenon could happen in the context of bundle maps automorphisms. In the examples presented in [**HdlLa**], the numerical calculations suggest that, for a family of operators, even if the spectrum remains more or less constant in an open set of parameters, the spectral bundles approach each other. This has the consequence that the spectral projections grow unbounded and that in the limit of the set of parameters considered, the spectrum is significantly larger.

In [**HdlLa**], it is conjectured, based on the numerical evidence that this is a mechanism for loss of normal hyperbolicity.

The paper [**HdlLa**] also reports some remarkable scaling properties found numerically. It is to be hoped that these numerical regularities will find a mathematical explanation.

A.5. The Weyl spectrum

A notion from general spectral theory that we will find useful is the notion of *Weyl spectrum*. From the definitions in Section A.1, it is obvious that the Weyl spectrum contains the point spectrum.

In fact, the Weyl spectrum is also known as *approximate point spectrum*, because

$$\begin{aligned} \lambda \in \operatorname{Spec}_W(L, X) \quad &\Leftrightarrow \quad \forall \varepsilon > 0 \ \exists x \in X \mid \|Lx - \lambda x\| < \varepsilon \|x\| \\ &\Leftrightarrow \quad \exists \{x_n\}_n \subset X \mid \forall n \ \|x_n\| = 1 \ , \|Lx_n - \lambda x_n\| \xrightarrow{n\to\infty} 0 \ . \end{aligned}$$

The sequence $\{x_n\}_n$ is called an approximate eigenvector of the approximate eigenvalue λ. This will be the property that we will use along this paper.

Remark A.22. If $\tilde{L}$ is a bounded operator on a real Banach space $\tilde{X}$, we state the previous definitions for the corresponding complexifications. Notice that if $\lambda \in \operatorname{Spec}_P(\tilde{L}, \tilde{X})$ then $\bar{\lambda} \in \operatorname{Spec}_P(\tilde{L}, \tilde{X})$. The same result holds for $\operatorname{Spec}_W(\tilde{L}, \tilde{X})$.

It is obvious from the definitions that $\operatorname{Spec}_W(L, X) \subset \operatorname{Spec}(L, X)$ and $\operatorname{Spec}_W(L, X)$ is closed. The opposite inclusion is not true in general. Nevertheless, the Weyl criterion (see [**RS80**] p. 237) asserts that for normal operators (i.e. operators which commute with their adjoint) the Weyl spectrum is the whole spectrum.

Remark A.23. It is straightforward that if $Y \subset X$ is an invariant closed subspace, $\operatorname{Spec}_W(L, Y) \subset \operatorname{Spec}_W(L, X)$.

We emphasize that this monotonicity with respect to the spaces is not true for the whole spectrum. Several examples of the phenomenon of growth of the spectrum when the space decreases were studied in [**dlL93**]. The examples in that paper are transfer operators similar to those considered here. The spaces are spaces of zero divergence vector fields. In Part 4 we will encounter more examples like that.

From the point of view of numerical analysis, the Weyl spectrum is the easiest to compute. This is a consequence of the following proposition.

Proposition A.24. *Let $\{L_n\}_n$ be a sequence of bounded linear operators in the Banach space X that is convergent to L. Let $\{\lambda_n\}_n$ be a sequence of* *complex numbers such that for all $n > 0$ $\lambda_n \in \mathrm{Spec}_W(L_n, X)$. Assume furthermore, λ_n converges to λ.*

Then, $\lambda \in \mathrm{Spec}_W(L, X)$.

REMARK A.25. Notice that any sequence $\{\lambda_n\}_n$ as in Proposition A.24 has a subsequence that is convergent, and so the limit is in $\lambda \in \mathrm{Spec}_W(L, X)$.

Henceforth, in the determination of the spectrum of a linear operator, one can produces a sequence of discretizations of the problem from where we can determine the spectrum using finite-dimensional techniques. Unfortunately, this procedure will locate only the Weyl spectrum, but will fail to locate the full spectrum.

Some information about the spectrum can be obtained from the Weyl spectrum. It is a standard result the points on the boundary of the spectrum are approximate eigenvalues (see e.g. [**Con85**], p. 215). This will be important for us since the edges of the spectrum, give control on the rates of growth.

PROPOSITION A.26. *We have*

$$\partial \mathrm{Spec}(L, X) \subset \mathrm{Spec}_W(L, X) \ .$$

In particular, the points in the spectrum with moduli the spectral radii are in the Weyl spectrum.

Proof: Let $\lambda^* \in \partial \mathrm{Spec}(L, X)$. We can find a sequence $\{\lambda_n\}_{n\in\mathbb{N}} \subset \mathrm{Res}(L, X)$ such that $\lambda_n \to \lambda^*$. From (A.4) we conclude that

$$\|(L - \lambda_n \mathrm{Id})^{-1}\| \geq |\lambda^* - \lambda_n|^{-1} \ .$$

Since $\|(L - \lambda_n \mathrm{Id})^{-1}\| \to \infty$, we can obtain a sequence of vectors $\{w_n\}_{n\in\mathbb{N}}$ satisfying

$$\|(L - \lambda_n \mathrm{Id})^{-1} w_n\| = 1, \ \|w_n\| \to 0 \ .$$

Therefore, since

$$(L - \lambda^* \mathrm{Id})(L - \lambda_n \mathrm{Id})^{-1} w_n = (\lambda_n - \lambda^*)(L - \lambda_n \mathrm{Id})^{-1} w_n + w_n$$

we see that the $(L - \lambda_n \mathrm{Id})^{-1} w_n$ are approximate eigenvectors of λ^*. □

Since we are going to be interested in rotational properties of the spectrum, we will need the following result, that follows directly from the previous proposition by using elementary point set topology.

COROLLARY A.27. *If $\mathrm{Spec}_W(L, X)$ is rotationally invariant, then $\mathrm{Spec}(L, X)$ is also rotationally invariant.*

Proof: We will show that it is impossible to have a circle $S_\rho = \{z \big| |z| = \rho\}$ that intersects both $\mathrm{Spec}(L, X)$ and its complement.

If there was such a circle, there would have to be a point $z \in S_\rho \cap \partial \mathrm{Spec}(L, X)$. Since the points in the frontier of the spectrum are Weyl, by assumption, we have that $S_\rho \subset \mathrm{Spec}_W(A, X) \subset \mathrm{Spec}(A, X)$. This is a contradiction with the fact that there are points in S_ρ both in the spectrum and its complement.

We conclude that a circle S_ρ is either contained in the spectrum or in its complement. $\square$

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