THE POLYA-TCHEBOTAREV PROBLEM FOR 4 POINTS

Recall the following theorem ([Lav34]).

Theorem 1 (Laurentiev). *Given a finite number of points $E = \{a_1, \cdots, a_n, a_{n+1} = \infty\} \subset \mathbb{C}_\infty$, there exists a unique extremal domain $\Omega = f(\mathbb{D})$ for the problem 2 and it is characterized by the following properties:*

1. *Each point of the plane belongs to either Ω or $\Gamma := \partial\Omega$.*
2. *The boundary Γ consists of finitely many simple arcs of analytic curves. The points a_i are endpoints of n distinct arcs. Every point of Γ different from the a_i $(i = 1, \cdots, n+1)$ either belongs to a unique arc and is a regular point of Γ, or is the common end of at least threee arcs.*
3. *If k distinct analytic arcs belonging to Γ emanate from some point of Γ, then two adjacent curves form an angle of $2\pi/k$.*
4. *To any arc $\alpha\beta$ consisting of regular points of Γ there correspond under the conformal mapping f^{-1} two arcs of the same length on the unit circle.*

Laurentiev also proved that the extremal function $f : \mathbb{D} \to \Omega$ must satisfy the following differential equation

$$\left(\frac{zf'(z)}{f(z)}\right)^2 = C\frac{\prod_{i=1}^{n}(f(z) - a_i)}{\prod_{j=1}^{n-1}(f(z) - b_j)}$$

where the parameters b_j are unknown (they can get repeated) and C is the following constant

$$C = \frac{\prod_{j=1}^{n-1}(-b_j)}{\prod_{i=1}^{n}(-a_i)}.$$

Moreover, $\Omega \setminus E$ is the closure of the critical orbits of

$$\left(\frac{f'(z)}{f(z)}\right)^2 = -C\frac{\prod_{i=1}^{n}(f(z) - a_i)}{\prod_{j=1}^{n-1}(f(z) - b_j)}.$$

Remark. If some point b_i is a common end of m arcs, then the term $f(z) - b_i$ will appear exactly $m - 2$ times in the differential equations.

In the case of 4 points, the extremal domain can be of two types. If two of the points are symmetric respect to the line through the other two points (there is an explicit solution given by Fedorov in [Fed84]), the extremal domain has the structure shown in figure 1. In a more general case the extremal domain is like in figure 2.

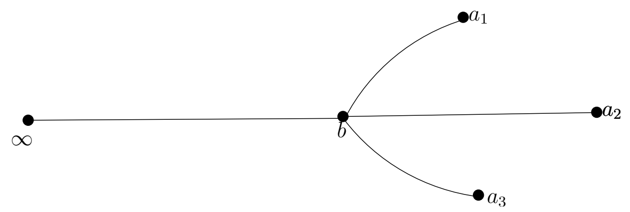

Figure 1: Structure of the extremal domain in the Fedorov case

Infact, in this last case, we have two type of configurations (partitions) of the unit

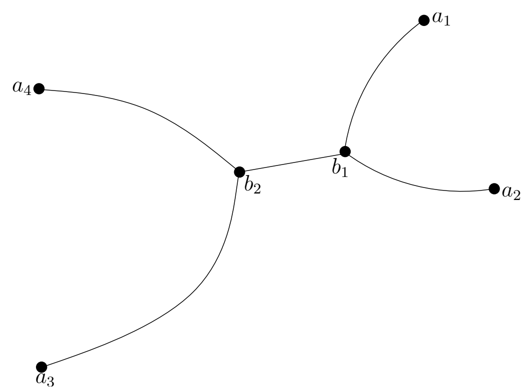

Figure 2: Structure of the extremal domain for 4 points

circle:

1. $0\ \beta_1^1\ \alpha_1\ \beta_2^1\ \beta_1^2\ \alpha_2\ \beta_2^2\ \alpha_3\ \beta_3^2\ \beta_3^1\ 2\pi$
2. $0\ \beta_1^1\ \beta_1^2\ \alpha_1\ \beta_2^2\ \alpha_2\ \beta_3^2\ \beta_2^1\ \alpha_3\ \beta_3^1\ 2\pi$

where $f(e^{i\beta_k^j}) = b_j$ for $j = 1, 2$ and $k = 1, 2, 3$, $f(e^{i\alpha_i}) = a_i$ for $i = 1, 2, 3$ and $f(1) = \infty$. Note that this two configurations are essentially the same (one is just a rotation of the other) but topologically they give us two different types of extremal domain. So now, we can use the same idea to solve the problem numerically. We will assume that $a_4 = \infty$ and $a_i \neq 0$. We have now the following differential equation

$$f'(z)^2 = C\frac{(f(z)-a_1)(f(z)-a_2)(f(z)-a_3)}{(f(z)-b_1)(f(z)-b_2)}\frac{f(z)^2}{z^2}. \tag{1}$$

In this case we have more unknown parameters. Let's take in mind the first configuration. Using the last property of theorem 1 we can reduce the number of unknown parameters: $\beta_3^1 = 2\pi - \beta_1^1$, $\beta_3^2 = \beta_3^1 - (\beta_1^2 - \beta_2^1)$. So the unknown values are $f'(0), b_1, b_2, \beta_1^1, \beta_2^1, \beta_1^2, \beta_2^2$, that is we need a system of 10 real equations which can be the following

$$\begin{cases} f(e^{i\beta_1^1/2}) = f(e^{-i\beta_1^1/2}) \\ f(e^{i(\alpha_1+\beta_1^1)/2}) = f(e^{i(\alpha_1+\beta_2^1)/2}) \\ f(e^{i(\beta_2^1+\beta_1^2)/2}) = f(e^{i(\beta_3^2+\beta_3^1)/2}) \\ f(e^{i(\alpha_2+\beta_1^2)/2}) = f(e^{i(\alpha_2+\beta_2^2)/2}) \\ f(e^{i(\alpha_3+\beta_2^2)/2}) = f(e^{i(\alpha_3+\beta_3^2)/2}) \end{cases}$$

Note that these are 5 complex equations. This has been implemmented and here is one of the pictures for some set of four points (see figure 3). It represents the conformal map $g : \mathbb{D}^c \to \Omega$ such that $g(\infty) = \infty$ (we can see the correspondence of the arcs of the unit circle with the ones of the boundary of Ω, this also gives us the behaviour of the harmonic measure of the arcs contained in $\partial\Omega$).

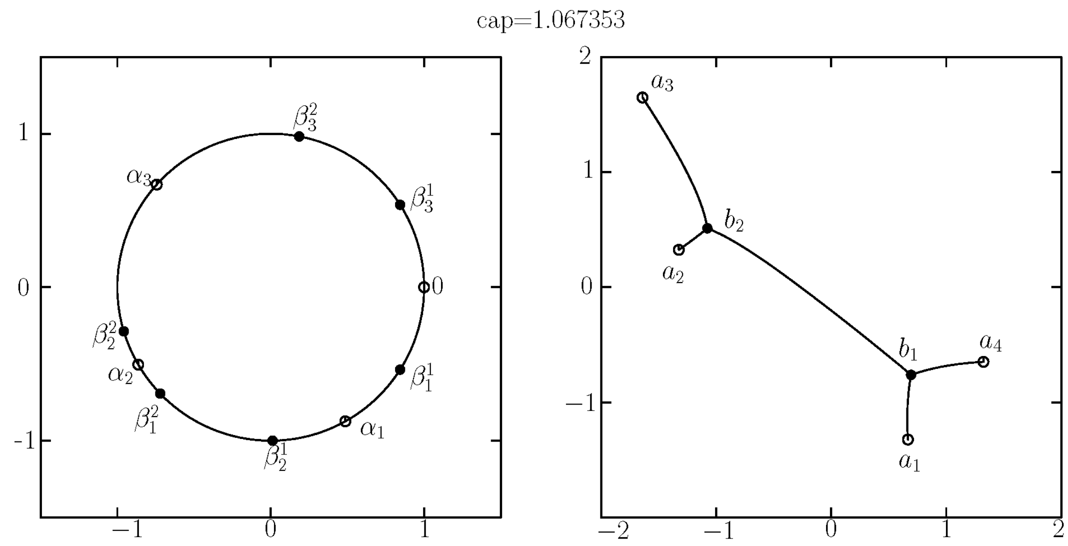

Figure 3: Extremal domain for $n = 4$

Remark 1. *Note that the solution of the problem depend continuously on the parameters a_i, so if we have one solution for some given four points we can do continuation to reach to any other set of four points. This has been used and we did the classic continuation (i.e. for the new set of points we take as a initial condition the solution of the last four points).*

References

[Fed84] S.I. Fedorov. Chebotarev's variational problem in the theory of the capacity of plane sets, and covering theorems for univalent conformal mappings. *Mat. Sb. (N.S.)*, 124(166)(1):121-139, 1984.

[Lav34] M. Lavrentieff. On the theory of conformal mappings. *Trudy Fiz.-Mat. Inst. Steklov. Otdel. Mat.*, 5:159–245, 1934.