Two quadratic first integrals ⇒ integrability?

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Abstract

A dynamics defined by a vector field on a n-dimensional manifold is often studied through a Poincaré map on a $n-1$ dimensional section. The vector field may be changed into a direction field without changing the Poincaré map. If the vector field preserves an n-form (i.e. is volume preserving), the direction field is given as the kernel of a closed $(n-1)$-form.

Appell showed that a system of the form $\ddot{q} = f(q)$, $q \in R^m$, $f : R^m \to R^m$, may be transformed in infinitely many other systems *of the same form* by a central projection and a change of time. Indeed if the target of the projection is another flat R^m the equation keeps exactly the same form. If it is a sphere or anything else, a reaction term is added. Behind such a class of systems there is a unique system defined by a closed $(2m-1)$-form on some $2m$ dimensional manifold (defined in [1]).

Appell's projection transforms a linear first integral into a linear first integral, a quadratic first integral into a quadratic first integral, etc. But if the system is a natural Hamiltonian system the Hamiltonian is transformed into a quadratic first integral which is not the Hamiltonian of the transformed system. Possibly a quadratic first integral gives after the transformation a new Hamiltonian. We will give many examples among the classical systems: two fixed centers problem, geodesics of the ellipsoid, Calogero-Moser.

Appell's transformation gives the geometric interpretation to Lundmark's "cofactor type" property. Lundmark showed that two quadratic first integral forming a "cofactor pair" are sufficient to integrate the system $\ddot{q} = f(q)$.

References

[1] A. Albouy, Projective dynamics, http://arxiv.org/abs/math-ph/0501026

[2] H. Lundmark, thesis http://www.mai.liu.se/~halun/papers/hlthesis.pdf