On the dynamics of the Lyness 3-D recurrence

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Abstract

For any real $a > 0$, consider the dynamical system generated by the map

$$F(x, y, z) = \left(y, z, \frac{a + y + z}{x} \right),$$

defined in $O^+ := \{(x, y, z) \in \mathbf{R}^3 \, : \, x > 0 \, , y > 0 \, , z > 0\}$. The iterates of $F(x, y, z)$ give the behavior of the three order difference equation

$$x_{k+3} = \frac{a + x_{k+1} + x_{k+2}}{x_k}, \quad \text{with} \quad x_1 > 0 \, , x_2 > 0 \, , x_3 > 0,$$

also known as Lyness 3-D recurrence. For $a = 1$ it is easy to see that all points in O^+ are 8-periodic by F. We prove:

Theorem. Assume that $a > 0$. Define the two subsets of O^+ :

$$\mathcal{G} = \{(x, y, z) \, : \, -y^3 - (x+z+a+1)y^2 - (x+z+a)y + xz(x+1)(z+1) = 0\} \cap O^+.$$

and $\mathcal{L} = \{(x, (x + a)/(x - 1), x) \, : \, x > 1\} \cap O^+$. Then the following holds:

- The point $p := \mathcal{L} \cap \mathcal{G}$ is a fix point of F and the set $\mathcal{L} \setminus \{p\}$ is formed by two periodic points of F.
- The set $\mathcal{G}$ is invariant by F. Furthermore $\mathcal{G} \setminus \{p\}$ is filled of invariant closed curves, each one of them being diffeomorphic to a circle and such that the dynamical system generated by F over it is conjugated to a rotation.
- The subset $O^+ \setminus \mathcal{L}$ is filled of closed curves, each one of them being invariant by $F \circ F$, diffeomorphic to a circle and such that the dynamical system generated by $F \circ F$ over it is conjugated to a rotation.