Effective computation of the dynamics around a two-dimensional torus of a Hamiltonian system*

F. Gabern[†] and À. Jorba
Departament de Matemàtica Aplicada i Anàlisi
Universitat de Barcelona
Gran Via 585, 08007 Barcelona, Spain
Emails: <gabern@mat.ub.es>, <angel@maia.ub.es>

Abstract

The purpose of this paper is to make an explicit analysis of the nonlinear dynamics around a two-dimensional invariant torus of an analytic Hamiltonian system. The study is based on normal form techniques and the computation of an approximated first integral around the torus. One of the main novel aspects of the current work is the implementation of the symplectic reducibility of the quasi-periodic time-dependent variational equations of the torus. We illustrate the techniques in a particular example that is a quasi-periodic perturbation of the well-known Restricted Three Body Problem. The results are useful to study the neighbourhood of the triangular points of the Sun-Jupiter system.

Keywords: Lower dimensional tori, quasi-periodic Floquet theory, reducibility.

*Work supported by the MCyT/FEDER Grant BFM2003-07521-C02-01, the CIRIT grant 2001SGR–70 and DURSI.

[†]Present Address: Control and Dynamical Systems, California Institute of Technology, CDS 107-81, 1200 East California Blvd, Pasadena, CA 91125, USA.

Contents

1 Introduction

Let us consider a quasi-periodically perturbed Hamiltonian system with two external frequencies,

$$H = H_0(x, y) + \varepsilon H_1(x, y, \theta), \tag{1}$$

where (x, y) is the configuration-momenta pair, $\theta = (\theta_1, \theta_2) = (\omega_1 t + \theta_1^{(0)}, \omega_2 t + \theta_2^{(0)})$ and $\omega_{1,2}$ are the external frequencies.

Let us suppose that the unperturbed Hamiltonian H_0 has an elliptic equilibrium point. If the perturbation ε is small enough and under some quite generic assumptions this elliptic fixed point is substituted, in the perturbed Hamiltonian, by an invariant 2-D torus with the same frequencies as the perturbation ([JS96, JV97, BHJ$^+$03]).

In this paper, we obtain the nonlinear dynamics around this 2-D invariant torus in a practical way, by means of perturbative methods. Once the torus has been computed, we construct a high-order normal form of the Hamiltonian around this quasi-periodic solution to describe the nonlinear dynamics nearby. The procedure is divided in the following steps: First, we compute the invariant torus as a Fourier series with numerical coefficients ([Jor00, CJ00]). The linear transformation that reduces the linear variational flow around the torus to constant coefficients (the so-called Floquet change) is also obtained as numerical Fourier series. Then, the Hamiltonian is expanded around the torus, in the coordinates given by the Floquet transformation. This implies that the expansion does not contain terms of degree 1 (because the torus is invariant) and the coefficients of the terms of degree 2 do not depend on time (because of the Floquet coordinates). Then, by means of the Lie series method, we construct the normal form of the Hamiltonian up to high order. Finally, we compute the changes of variables that send points from the initial phase space to the normal form coordinates, and vice versa. All these computations are performed by a specific algebraic manipulator, written in C++. This software is based on the code given in [Jor99].

Using the normal form we can describe the dynamics around the torus, and the changes of variables allow to send this information to the initial system. In particular, it is possible to compute invariant tori of dimensions 3, 4 and 5 around the initial 2-D torus. Finally, we also compute an approximate first integral of the system to estimate the diffusion around the invariant torus. This allows to derive a zone of effective stability (also known as Nekhoroshev stability) by computing a bound of the drift of this approximate integral.

We want to note that the computation of a normal form around a 2-D torus of an autonomous Hamiltonian system is different from the computations presented here. The main difference is that, in the autonomous case, the situation is far from being perturbative: the actions conjugated to the angles on the torus have to be defined in a neighbourhood of the torus, and this introduces a "semi-global" component in the construction. The same situation occurs for periodic orbits; for more details compare [JV98] with [SGJM95] or [GJ01].

1.1 A model example

We will illustrate the techniques in a particular case, the so-called Tricircular Coherent Problem (TCCP). This is a model for the motion of a particle under the gravitational attraction of Sun, Jupiter, Saturn and Uranus. The model is based on a quasi-periodic solution for the planar Four-Body problem given by these bodies, and can be written as a quasi-periodic time-dependent perturbation, with two basic frequencies, of the Sun-Jupiter Restricted Three-Body Problem (for more details on this model, see [GJ04]). Its Hamiltonian is

$$\begin{aligned} H_{TCCP} &= \frac{1}{2}\alpha_1(\theta_1,\theta_2)(p_x^2+p_y^2+p_z^2)+\alpha_2(\theta_1,\theta_2)(xp_x+yp_y+zp_z) \\ &\quad +\alpha_3(\theta_1,\theta_2)(yp_x-xp_y)+\alpha_4(\theta_1,\theta_2)x+\alpha_5(\theta_1,\theta_2)y \\ &\quad -\alpha_6(\theta_1,\theta_2)\left[\frac{1-\mu}{q_S}+\frac{\mu}{q_J}+\frac{m_{sat}}{q_{sat}}+\frac{m_{ura}}{q_{ura}}\right], \end{aligned} \tag{2}$$

where $q_S^2=(x-\mu)^2+y^2+z^2$, $q_J^2=(x-\mu+1)^2+y^2+z^2$, $q_{sat}^2=(x-\alpha_7(\theta_1,\theta_2))^2+(y-\alpha_8(\theta_1,\theta_2))^2+z^2$, $q_{ura}^2=(x-\alpha_9(\theta_1,\theta_2))^2+(y-\alpha_{10}(\theta_1,\theta_2))^2+z^2$, $\theta_1=\omega_{sat}t+\theta_1^0$ and $\theta_2=\omega_{ura}t+\theta_2^0$. The concrete values of the mass parameters are $\mu=9.538753600\times10^{-4}$, $m_{sat}=2.855150174\times10^{-4}$ and $m_{ura}=4.361228581\times10^{-5}$.

The functions $\alpha_i(\theta_1,\theta_2)_{\{i=1\div10\}}$ are auxiliary quasi-periodic functions that are computed by a Fourier analysis of the solution of the Four Body Problem (Sun + 3 planets). The concrete values of the frequencies are $\omega_{sat}=$ `0.597039074021947` and $\omega_{ura}=$ `0.858425538978989`. For a description on the construction of this model, as well as the concrete values of the $\alpha_i(\cdot)$ functions, see [GJ04] (a file with the numerical values of the Fourier coefficients can be downloaded from `http://www.maia.ub.es/~gabern/`).

2 Normal form around an invariant torus

This Section discusses the details of the computation of the Floquet transformation for the torus, and the effective computation of the normal form.

In what follows, we will use Taylor-Fourier expansions, with floating point coefficients, to represent the functions involved in the computations. For the examples here, the Taylor expansions are taken up to degree 16 and the truncation of the Fourier series has been selected so that the representation error is of the order 10^{-9}. More concretely, let us write a generic Taylor-Fourier polynomial as

$$P(q,p,\theta_1,\theta_2)=\sum_{r=0}^{N}\sum_{|k|=r}\sum_{j_1=-N_{f_1}}^{N_{f_1}}\sum_{j_2=\max(j_1-N_{f_1},-N_{f_2})}^{\min(N_{f_1}-j_1,N_{f_2})}P_{r,j}^k e^{i(j_1\theta_1+j_2\theta_2)}q^{k^1}p^{k^2},$$

where $P_{r,j}^k\in\mathbb{C}$, $j=(j_1,j_2)\in\mathbb{Z}^2$ and $k=(k^1,k^2)\in\mathbb{Z}^3\times\mathbb{Z}^3$ is a multi-index. Then, we have used the following truncation values: $N=16$, $N_{f_1}=20$ and $N_{f_2}=10$. In some places, we have used higher accuracy as it is explicitly mentioned in the text.

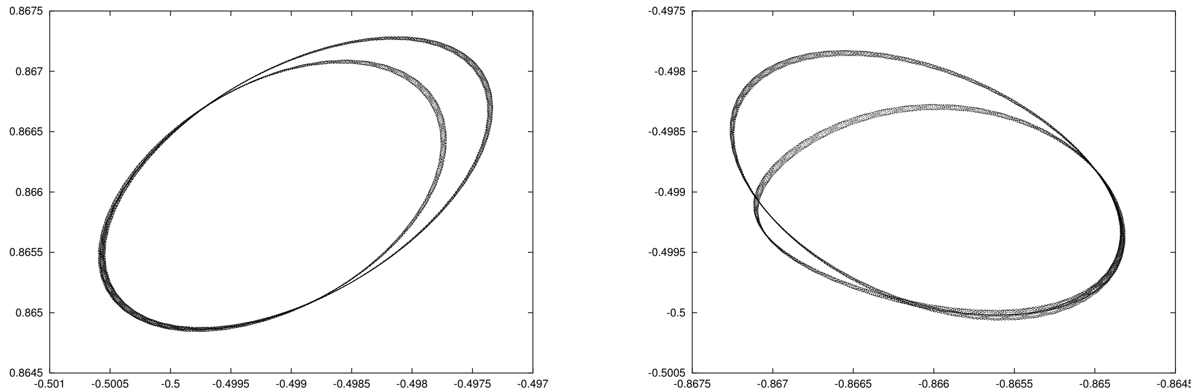

Figure 1: Planar projections of the 2-D invariant torus that replaces L_5: T_5. Left: (x, y)-projection. Right: (p_x, p_y)-projection.

2.1 The 2-D invariant torus that replaces the elliptic fixed point

It is known that, under general conditions, the equilibrium points of (1) for $\varepsilon = 0$ become quasi-periodic solutions with the same frequencies as the perturbation if ε belongs to a set of positive measure (for details, see [JS96, JV97, BHJ$^+$03]). This implies that, if the previous hypotheses hold, the triangular points $L_{4,5}$ of the RTBP are replaced by 2-D tori in the TCCP. For most of the values of ε, these tori are normally elliptic ([BHJ$^+$03]).

To compute this 2-D invariant torus, we have used the method described in [CJ00] adapted to the non-autonomous case. We have taken the section $\theta_1 = 0 (\mathrm{mod}\, 2\pi)$ to introduce the map

$$\left.\begin{array}{rcl} \bar{x} &=& f(x, \theta), \\ \bar{\theta} &=& \theta + \omega, \end{array}\right\} \tag{3}$$

where $\omega = 2\pi \left(\frac{\omega_2}{\omega_1} - 1 \right)$ and f can be evaluated from a numerical integration of the flow associated to (2).

Using the methods described in [Jor00, CJ00], we have computed the invariant curve of (3) that corresponds to the 2-D invariant torus that replaces L_5 in the TCCP model, with an accuracy of 10^{-12}. The 2-D torus is easily reconstructed using numerical integrations starting on a mesh of points on the invariant curve. Finally, a Fourier transform allows to compute the Fourier coefficients of a parametrization with respect to the angles (θ_1, θ_2). The (x, y) and (p_x, p_y) projections of the resulting invariant torus are shown in Figure 1. From now on, we will call this 2-D torus T_5. Due to the symmetries of this problem, the same results hold for L_4 so we will only discuss the L_5 case.

Applying the techniques described in [Jor01], we can see that this torus is normally elliptic, and we can obtain the three normal modes of the invariant curve. The normal modes are the frequencies of the three harmonic oscillators that describe the normal linear motion around the invariant torus. In Section 2.2, we discuss in detail the computation of these normal modes. In Table 1, the linear normal modes around the invariant curve corresponding to T_5 are shown.

j	$\mathrm{Re}\,(\lambda_j)$	$\pm\mathrm{Im}\,(\lambda_j)$	$\lvert\lambda_j\rvert$	$\pm\mathrm{Arg}\,(\lambda_j)$
1	0.662315481969	0.749225067883	1.0	0.846891268646
2	-0.485204809265	0.874400533546	1.0	2.077393707459
3	-0.453781923686	0.891112768249	1.0	2.041801148412

Table 1: Linear normal modes around the 2-D invariant torus T_5 in the TCCP system.

2.2 Second order normal form

In the previous section, we have discussed the computation of the invariant object that substitutes the elliptic fixed point L_5 of the RTBP when the quasi-periodic perturbation is added. By means of a quasi-periodic time-dependent translation from the origin to this 2-D invariant torus, one can cancel the first order terms in the Hamiltonian. Now, we will derive a linear change of variables, that depends on time in a quasi-periodic way, that puts the second degree terms of the Hamiltonian into a more convenient form. This is, essentially, the quasi-periodic Floquet transformation for the variational flow along the quasi-periodic orbit, but taking into account the symplectic structure of the problem. To simplify further steps in the normalizing process, we also apply a complexifying change of variables that puts the second degree terms of the Hamiltonian in the so-called diagonal form.

2.2.1 The symplectic quasi-periodic Floquet change

The linear flow around the 2-D invariant torus (T_5 in the example) is described by a linear system of differential equations (the variational equations), that depends quasi-periodically on time:

$$\begin{aligned} \dot{z} &= Q(\theta_1, \theta_2) z, \\ \dot{\theta}_1 &= \omega_1, \\ \dot{\theta}_2 &= \omega_2, \end{aligned} \tag{4}$$

where $z \in \mathbb{R}^6$ and Q is a real 6×6 matrix.

Our final goal is to find a real, symplectic and quasi-periodic change of variables, $z = P^r(\theta_1, \theta_2)x$ reducing (4) to a constant system with real coefficients:

$$\dot{x} = Bx, \quad \frac{d}{dt} B \equiv 0. \tag{5}$$

We will proceed in two steps: First, we will see that such a change of variables exists in the complex domain. As in our case the obtained complex matrix admits a real form, the second step will be to build a real change from the complex one (see [Jor01] for a concrete example where this real matrix does not exist).

All the process is constructive, so the implementation in a computer program will follow easily from the explanation.

Reducibility in the Poincaré section Let us consider the $\theta_1 = 0 (\bmod 2\pi)$ section of the flow defined by (4). Then, we have the following linear quasi-periodic skew product:

$$\left.\begin{array}{rcl} \bar{z} & = & A(\theta)z, \\ \bar{\theta} & = & \theta + \omega, \end{array}\right\} \tag{6}$$

where $\theta \equiv \theta_2$ and $\omega = 2\pi\left(\frac{\omega_2}{\omega_1} - 1\right)$ (here $\omega = 2.75080755611202$) is the rotation number of the invariant curve defined by the slice $\theta_1 = 0 (\bmod 2\pi)$ of the 2-D invariant torus.

Assume that (6) can be reduced to an autonomous diagonal system

$$\bar{y} = \Lambda y, \quad \Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_6),$$

by means of a linear transformation $z = C(\theta)y$. Let us write $C(\theta)$ as $(\Psi_1(\theta), \ldots, \Psi_6(\theta))$, where $\Psi_j(\theta)$ are the columns of $C(\theta)$. Then, it is clear that the couples (λ_j, Ψ_j) can be obtained as the eigenvalues and eigenfunctions of the following problem,

$$A(\theta)\Psi_j(\theta) = \lambda_j \Psi_j(\theta + \omega), \quad j = 1, \ldots, 6. \tag{7}$$

This problem has been solved with an accuracy of 10^{-12}. See [Jor01] for more details on this computation.

Remark As $A(\theta)$ is a real matrix, if λ_j and $\Psi_j(\theta)$ satisfy (7), then λ_j^* and $\Psi_j^*(\theta)$ also satisfy (7) (λ_j^* and $\Psi_j^*(\theta)$ are the complex conjugates of λ_j and $\Psi_j(\theta)$, respectively). We construct the matrix $C(\theta)$ as $C(\theta) = (\Psi_1(\theta), \Psi_2(\theta), \Psi_3(\theta), \Psi_1^*(\theta), \Psi_2^*(\theta), \Psi_3^*(\theta))$, where the Ψ_j are column vectors. Then, the matrix Λ takes the form $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_1^*, \lambda_2^*, \lambda_3^*)$. The eigenfunctions $\Psi_j(\theta)$ are scaled in such a way that $\|\Psi_j\|_2 = 1$, where $\|V(\theta)\|_2^2 = \|\sum_{k=1}^{3}(v_k(\theta)v_{k+3}^*(\theta) - v_k^*(\theta)v_{k+3}(\theta))\|_2$, for $V(\theta) = (v_1(\theta), v_2(\theta), \ldots, v_6(\theta))^t$ and $\|\alpha(\theta)\|_2^2 = \sum_l |\alpha_l|^2$ for $\alpha(\theta) = \sum_l \alpha_l e^{il\theta}$, $\alpha_l \in \mathbb{C}$.

The change of variables for the flow The next goal is to compute a quasi-periodic change of variables $z = P^c(\theta_1, \theta_2)y$ that transforms the flow given by (4) into

$$\dot{y} = D_B y, \tag{8}$$

where $D_B = \mathrm{diag}(i\nu_1, i\nu_2, i\nu_3, -i\nu_1, -i\nu_2, -i\nu_3)$ and ν_j is such that $\lambda_j = \exp(i\nu_j T_1)$, where T_1 is the period related to the first frequency $T_1 = \frac{2\pi}{\omega_1}$. Note that ν_j is defined modulus integer multiples of $\frac{2\pi}{T_1}$. In general, we select a special value of $k_j \in \mathbb{Z}$ for each $j = 1, 2, 3$ such that the values of ν_j are as close as possible to the ones of the RTBP. This is the natural choice from a perturbative point of view and it also allows to obtain a symplectic transformation.

Proposition 2.1 *The solution of*

$$\begin{array}{rcl} \dot{P}^c(\theta_1, \theta_2) & = & Q(\theta_1, \theta_2)P^c(\theta_1, \theta_2) - P^c(\theta_1, \theta_2)D_B, \\ \dot{\theta}_1 & = & \omega_1, \\ \dot{\theta}_2 & = & \omega_2, \end{array}$$

with initial conditions

$$\begin{aligned} P^c(0) &= C(\theta_2^{(0)}), \\ \theta_1(0) &= 0, \\ \theta_2(0) &= \theta_2^{(0)}, \end{aligned}$$

is the linear change of variables with complex quasi-periodic coefficients that transforms system (4) into system (8).

Proof If we insert the change $z = P^c(\theta_1, \theta_2)y$ into equation (4) and if we ask that (8) is satisfied, then P^c is such that:

$$\dot{P^c}(\theta_1, \theta_2) = Q(\theta_1, \theta_2)P^c(\theta_1, \theta_2) - P^c(\theta_1, \theta_2)D_B. \tag{9}$$

On the other hand, if we integrate equation (4) from $t = 0$ to $t = T_1$ with the initial condition $\left[z(0) = \Psi_j(\theta_2^{(0)}), \theta_1(0) = 0, \theta_2(0) = \theta_2^{(0)}\right]$ (note that this is equivalent to apply $A(\theta_2^{(0)})$ to the vector $\Psi_j(\theta_2^{(0)})$), the solution (that we denote by $\bar{x}(t)$) accomplishes

$$\bar{x}(T_1) = A(\theta_2^{(0)})\Psi_j(\theta_2^{(0)}) = \lambda_j \Psi_j(\theta_2^{(0)} + \omega),$$

where we have used equation (7).

An elementary result in the theory of ordinary differential equations states that if $\tilde{x}_1(t)$ is a solution of $\dot{x}_1 = Q(t)x_1$, then $\tilde{x}_2(t) = \exp(at)\tilde{x}_1(t)$ is a solution of $\dot{x}_1 = (Q(t) + aI)x_1$, being a any complex number. Thus, $\hat{x}(t) = \exp(-i\nu_j t)\bar{x}(t)$ is a solution of

$$\dot{P}_j^c = (Q - i\nu_j I_6)P_j^c,$$

(note that it corresponds to the first three columns of equation (9)) with the same initial condition ($\hat{x}(0) = \Psi_j(\theta_2^{(0)})$) and it satisfies the following relation:

$$\hat{x}(T_1) = \exp(-i\nu_j T_n u)\bar{x}(T_1) = (\lambda_j)^{-1}\lambda_j \Psi_j(\theta_2^{(0)} + \omega) = \Psi_j(\theta_2^{(0)} + \omega).$$

q.e.d.

Realification In order to actually implement the Floquet change, we are interested in computing the real change of variables.

Proposition 2.2 *Let us define the (real) matrix R by taking the real and imaginary parts of the columns of matrix C (recall that, due to the particular construction of C, the last three columns are the conjugate values of the first three ones),*

$$R(\theta) = \frac{1}{2}C(\theta)\left(\begin{array}{c|c} I_3 & -iI_3 \\ \hline I_3 & iI_3 \end{array}\right).$$

Then, the solution of

$$\begin{array}{rcl} \dot{P}^r(\theta_1,\theta_2) & = & Q(\theta_1,\theta_2)P^r(\theta_1,\theta_2) - P^r(\theta_1,\theta_2)B, \\ \dot{\theta}_1 & = & \omega_1, \\ \dot{\theta}_2 & = & \omega_2, \end{array} \tag{10}$$

with initial conditions

$$\begin{array}{rcl} P^r(0) & = & R(\theta_2^{(0)}), \\ \theta_1(0) & = & 0, \\ \theta_2(0) & = & \theta_2^{(0)}, \end{array}$$

defines a (real) linear quasi-periodic change of variables ($z = P^r(\theta_1,\theta_2)x$) that transforms system (4) into system (5). Moreover, this change of variables is canonical.

The (real) matrix B is defined as $B = R^{-1}CD_BC^{-1}R$ and takes the form

$$B = \left(\begin{array}{ccc|ccc} 0 & 0 & 0 & \nu_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \nu_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \nu_3 \\ \hline -\nu_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\nu_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\nu_3 & 0 & 0 & 0 \end{array}\right).$$

Proof Let us define the matrix P^r in the following way,

$$P^r(\theta_1,\theta_2) = P^c(\theta_1,\theta_2)C^{-1}(\theta_1,\theta_2)R(\theta_1,\theta_2), \tag{11}$$

where $R(\theta_1,\theta_2)$ and $C(\theta_1,\theta_2)$ are, respectively, the extensions of the matrices $R(\theta)$ and $C(\theta)$. Then, we have:

- $P^r(0)$ is a real matrix: $P^r(0) = P^c(0)C^{-1}(\theta_2^{(0)})R(\theta_2^{(0)}) = C(\theta_2^{(0)})C^{-1}(\theta_2^{(0)})R(\theta_2^{(0)}) = R(\theta_2^{(0)})$.

- If we integrate $\dot{P}^r = QP^r - P^rB$ with $P^r(0) = R(\theta_2^{(0)})$ as initial condition, then $P^r(\theta_1,\theta_2)$ is real $\forall(\theta_1,\theta_2) \in \mathbb{T}^2$.

- If we insert the relation (11) into the differential equation (10), we obtain

$$\dot{P}^cC^{-1}R + P^c(\dot{C^{-1}})R + P^cC^{-1}\dot{R} = QP^cC^{-1}R - P^cC^{-1}RB.$$

By multiplying this equation (in the right hand side) by the matrix $R^{-1}C$, we get

$$\dot{P}^c + P(\dot{C^{-1}})C + P^cC^{-1}\dot{R}R^{-1}C = QP^c - P^cD_B.$$

This equation holds (it corresponds to (9)) provided that

$$P^c(\dot{C^{-1}})C + P^cC^{-1}\dot{R}R^{-1}C = 0.$$

It is easy to see, by using the definition of R, that this equality is true:

$$\begin{aligned} P^c(\dot{C^{-1}})C + P^c C^{-1}\dot{R}R^{-1}C &= 0 \iff \\ (\dot{C^{-1}})R + C^{-1}\dot{R} &= 0 \iff \\ \frac{d}{dt}(C^{-1}R) &= 0 \end{aligned}$$

Finally, to ensure that the transformation is canonical, we only need to check that $P^r(\theta_1, \theta_2)$ is a symplectic matrix. This can be proved (see [GJMS01b]) by extending the matrix P^r to the phase space of the autonomous Hamiltonian

$$H_{\text{ext}}(x, y, \theta_1, \theta_2, p_{\theta_1}, p_{\theta_2}) = \omega_1 p_{\theta_1} + \omega_2 p_{\theta_2} + H(x, y, \theta_1, \theta_2),$$

where $(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3$, p_{θ_k} is the conjugate momenta of θ_k and $H(\cdot)$ is given by equation (1).

q.e.d.

In our example, to check the correctness of the software, we have tested numerically that $P^r(\theta_1, \theta_2)$ is symplectic on a mesh of values of (θ_1, θ_2), with an agreement of the order of the truncation of the Fourier series.

If we apply this quasi-periodic change of variables, the second degree terms of the Hamiltonian become:

$$H_2^r(x, y) = \frac{1}{2}\nu_1(x_1^2 + y_1^2) + \frac{1}{2}\nu_2(x_2^2 + y_2^2) + \frac{1}{2}\nu_3(x_3^2 + y_3^2), \tag{12}$$

where the frequencies ν_j are the normal frequencies of the torus T_5. In the TCCP system, they take the values: $\nu_1 = -0.080473064872369$, $\nu_2 = 0.996680625156409$ and $\nu_3 = 1.00006269133083$.

2.2.2 Complexification

As it is usual in these kind of computations, we use a complexifying change of variables to bring (12) into a diagonal form. The equations of this linear and symplectic transformation are

$$x_j = \frac{q_j + ip_j}{\sqrt{2}}, \quad y_j = \frac{iq_j + p_j}{\sqrt{2}}, \quad j = 1, 2, 3.$$

Thus, after composing the three linear symplectic changes of variables (translation of the origin to the 2-D invariant torus, quasi-periodic symplectic transformation and complexification), the second order of the Hamiltonian takes the form:

$$H_2(q, p) = H_2^c(q, p) = i\nu_1 q_1 p_1 + i\nu_2 q_2 p_2 + i\nu_3 q_3 p_3, \quad (q, p) \in \mathbb{C}^6. \tag{13}$$

2.3 Expansion of the Hamiltonian

To proceed with the algorithm that constructs the normal form, we need to produce a convergent Taylor-Fourier expansion of Hamiltonian (1), in the complex coordinates used to derive (13),

$$H = \sum_{n=2}^{N} H_n(q, p, \theta) + R_{N+1}(q, p, \theta),$$

where H_n, $n \geq 2$, denotes an homogeneous polynomial of degree n in the variables q and p, $H_2(q, p, \theta) = H_2(q, p)$ is given by (13) and $\theta = (\theta_1, \theta_2) \in \mathbb{T}^2$.

To produce the expansion in our particular example, we only need to expand the terms of the potential of the TCCP Hamiltonian (2). They are of the form

$$\frac{1}{s_l} = \frac{1}{\sqrt{(x - x_l(\theta))^2 + (y - y_l(\theta))^2 + z^2}},$$

where $(x, y, z) \in \mathbb{R}^3$ and l stands for S (the Sun), J (Jupiter), *sat* (Saturn) or *ura* (Uranus). It is possible to write these terms as

$$\frac{1}{s_l} = \sum_{n \geq 0} A_n^l(x, y, z, \theta),$$

where A_n^l denotes an homogeneous polynomial of degree n, whose coefficients are quasi-periodic functions of $\theta = (\theta_1, \theta_2) \in \mathbb{T}^2$, and can be recurrently computed using

$$\begin{aligned} A_{n+1}^l = \frac{1}{x_l^2(\theta) + y_l^2(\theta)} \quad & \left[\frac{2n+1}{n+1} (x_l(\theta)x + y_l(\theta)y) A_n^l \right. \\ & \left. - \frac{n}{n+1} (x^2 + y^2 + z^2) A_{n-1}^l \right], \end{aligned} \tag{14}$$

for $n \geq 1$. The recurrence can be started using the values

$$A_0^l = \frac{1}{\sqrt{x_l^2(\theta) + y_l^2(\theta)}}, \qquad A_1^l = \frac{x_l(\theta)x + y_l(\theta)y}{(x_l^2(\theta) + y_l^2(\theta))^{3/2}},$$

and can be derived easily from the recurrence of the Legendre polynomials.

Then, the expansion of the Hamiltonian is implemented as follows: First, the translation to the torus T_5 is composed with the Floquet transformation and the resulting affine transformation is substituted in (14). Then, it is not difficult to use these recurrences to obtain the expansion up to a given order. The remaining terms of the Hamiltonian (monomials of degrees 1 and 2 in (2)) are easily added by simply inserting the above-mentioned affine transformation This strategy for the expansions has already been used in several places (see, for instance, [SGJM95, GJMS01a, GJMS01b, GJ01]).

On the other hand, it is also convenient to add the momenta corresponding to the angular variables. If we denote it as $p_\theta = (p_{\theta_1}, p_{\theta_2}) \in \mathbb{C}^2$, it is possible to write the expanded Hamiltonian (in complex variables) as

$$H(q, p, \theta, p_\theta) = \langle \varpi, p_\theta \rangle + H_2(q, p) + \sum_{n \geq 3} H_n(q, p, \theta), \tag{15}$$

where $(q,p) \in \mathbb{C}^6$, $\theta \in \mathbb{T}^2$, $H_2(q,p)$ is given by (13), $\varpi = (\omega_1, \omega_2)$ and $\langle \cdot, \cdot \rangle$ is the Euclidean scalar product.

2.4 Normal form of order higher than 2

The previous expansion has been obtained in coordinates such that the Hamiltonian starts at degree 2 for the spatial variables, and that degree 2 is already in complex normal form (13).

The goal of the normalizing transformation is to eliminate the maximum number of terms of the expansion of the Hamiltonian. We use, basically, the Lie series method implemented as described in [Jor99], but introducing the necessary modifications in order to deal with quasi-periodic coefficients.

For completeness, we describe one step of the normalizing process. Let us suppose that the Hamiltonian is already in normal form up to degree $r-1$,

$$H = \langle \varpi, p_\theta \rangle + H_2(q,p) + \sum_{j=3}^{r-1} H_j(q,p) + H_r(q,p,\theta) + H_{r+1}(q,p,\theta) + \cdots$$

where $H_r(q,p,\theta) = \sum_{|k|=r} h_r^k(\theta_1,\theta_2) q^{k^1} p^{k^2}$, $h_r^k(\theta_1,\theta_2) = \sum_{j=(j_1,j_2)} h_{r,j}^k e^{i(j_1\theta_1 + j_2\theta_2)}$ and $k = (k^1,k^2) \in \mathbb{Z}^3 \times \mathbb{Z}^3$ is a multi-index.

We will make a change of variables that suppress the maximum number of monomials and removes the dependence in θ_1 and in θ_2 in the terms of order r, H_r, of the Hamiltonian expansion. The canonical transformation that accomplishes this purpose is given by the following generating function:

$$G_r = G_r(q,p,\theta) = \sum_{|k|=r} g_r^k(\theta_1,\theta_2) q^{k^1} p^{k^2},$$

where the coefficients are given by

$$g_r^k(\theta_1,\theta_2) = \begin{cases} \displaystyle\sum_{j=(j_1,j_2)} \frac{h_{r,j}^k e^{i(j_1\theta_1+j_2\theta_2)}}{i(j_1\omega_1 + j_2\omega_2 - \langle \nu, k^2 - k^1 \rangle)} & \text{if} \quad k^1 \neq k^2, \\ \displaystyle\sum_{j=(j_1,j_2)\neq(0,0)} \frac{h_{r,j}^k e^{i(j_1\theta_1+j_2\theta_2)}}{i(j_1\omega_1 + j_2\omega_2)} & \text{if} \quad k^1 = k^2, \end{cases}$$

In general, one should check that all the frequencies $\nu_1, \nu_2, \nu_3, \omega_1, \omega_2$ are not in resonance up to the order of the computations. Otherwise, we will have a zero divisor that implies that this (resonant) term cannot be eliminated.

In our example, the frequencies of the normal linear oscillations around the 2-D invariant torus T_5, $\nu = (\nu_1, \nu_2, \nu_3)$, and the intrinsic frequencies of the system, $\omega_1 = \omega_{sat}$ and $\omega_2 = \omega_{ura}$, are not in resonance up to order N. That is, we check at every step of the process that $j_1\omega_1 + j_2\omega_2 - \langle \nu, k \rangle \neq 0$, $j = (j_1,j_2) \in \mathbb{Z}^2$, $k \in \mathbb{Z}^3 \backslash \{0\}$, with $|j|$ and $|k|$ up to the orders we have worked with.

Now, the H' obtained with such a generating function,

$$H' = H + \{H, G_r\} + \frac{1}{2!}\{\{H, G_r\}, G_r\} + \cdots,$$

does not depend on the variables θ_1 and θ_2 up to degree r (here, $\{\cdot, \cdot\}$ denotes the canonical Poisson bracket), that is, H' is in normal form up to degree r,

$$H' = \langle \varpi, p_\theta \rangle + H_2(q,p) + \sum_{j=3}^{r-1} H_j(q,p) + H'_r(q,p) + H'_{r+1}(q,p,\theta) + \cdots.$$

After performing all this changes up to a suitable degree $n = N$, the Hamiltonian takes the form

$$H = \langle \varpi, p_\theta \rangle + \mathcal{N}(q_1p_1, q_2p_2, q_3p_3) + \mathcal{R}(q_1, q_2, q_3, p_1, p_2, p_3, \theta_1, \theta_2), \tag{16}$$

where $\mathcal{N}$ denotes the normal form (that only depends on the products q_jp_j) and $\mathcal{R}$ is the remainder (of order greater than N).

Finally, we write the normal form $\mathcal{N}$ in real action-angle coordinates. This can be easily achieved by using the (canonical) transformation,

$$q_j = I_j^{1/2}\exp(i\varphi_j), \quad p_j = -iI_j^{1/2}\exp(-i\varphi_j), \quad j = 1, 2, 3.$$

It is not difficult to see that $\mathcal{N}$, in these coordinates, does not depend on the angles φ_j but only on the actions I_j,

$$\mathcal{N} = \sum_{|k|=1}^{[N/2]} h_k I_1^{k_1} I_2^{k_2} I_3^{k_3}, \qquad k \in \mathbb{Z}^3, \quad h_k \in \mathbb{R}. \tag{17}$$

Values for the coefficients h_k up to order $N = 6$ for our particular case of the TCCP system can be found in Table 2. As it has been mentioned before, these computations have been performed up to order $N = 16$.

2.5 Changes of variables

We have also computed explicit expressions for the transformation from the initial variables of (1) to the normal form variables and its inverse one. As usual, these changes of variables can be written as truncated Taylor-Fourier series, with the same truncation values as the Hamiltonian. They will be used to send information from the normal form coordinates to the initial ones, and vice versa.

2.6 Local nonlinear dynamics

If we are close enough to the 2-D invariant torus that replaces the equilibrium point, the (nonlinear) dynamics can be described accurately by the truncated normal form Hamiltonian (17). As this is an integrable normal form, the dynamics is very simple: the phase

k_1	k_2	k_3	$\mathrm{Re}\,(h_k)$	$\mathrm{Im}\,(h_k)$
1	0	0	-8.0473064872368966e-02	0.0000000000000000e+00
0	1	0	9.9668062515640865e-01	0.0000000000000000e+00
0	0	1	1.0000626913308270e+00	0.0000000000000000e+00
2	0	0	5.6008074695424814e-01	9.9022635266146223e-14
1	1	0	-1.5539627415430354e-01	1.9737284347219547e-14
0	2	0	5.5093985824138381e-03	-3.4515403004164990e-16
1	0	1	5.4161903856716140e-02	2.6837558280952768e-15
0	1	1	6.6103538676104013e-03	-2.3704452239135327e-16
0	0	2	-3.4144388415478051e-04	1.3980906058550924e-20
3	0	0	1.7078141909448842e+01	7.0030369427282057e-09
2	1	0	2.5316327595194634e+00	5.6348897124143457e-09
1	2	0	1.2040309679987733e+00	2.4234795726771734e-10
0	3	0	-1.7159208395247699e-03	8.2745567480971449e-12
2	0	1	-2.0884357984263224e-01	2.7145888846396389e-10
1	1	1	1.3097591687221137e+00	8.5687025776165656e-11
0	2	1	-8.7878491487452266e-03	1.1230163355821964e-12
1	0	2	-3.8394291301547680e-02	4.9155879462758908e-12
0	1	2	-8.1852662066825860e-03	2.9493958070531223e-13
0	0	3	4.8084373364571027e-04	4.1626694220605631e-14

Table 2: Coefficients of the normal form, up to degree 3 in the actions for the TCCP case. The first three columns contain the exponents of the actions, and the fourth and fifth columns are the real and imaginary parts of the coefficients. Imaginary parts must be zero, but they are not due to the different accumulation errors (basically, the one that comes from the truncation of the Fourier series).

space is completely foliated by a 3-parametric family of invariant tori, parameterized by the actions I. On each torus $I = I_0$, there is a linear flow with a given frequency $\Omega(I_0)$. If these frequencies are linearly independent over the rationals then the torus $I = I_0$ is filled densely by any trajectory starting on it. If the frequencies are linearly dependent over the rationals, then the orbits on this torus are not dense: if there are ℓ_i independent frequencies, the torus $I = I_0$ contains a $(3 - \ell_i)$-parametric family of ℓ_i dimensional tori, being each one densely filled by any trajectory starting on it. These tori of dimension ℓ_i are the lower dimensional tori, while the tori of dimension 3 are the maximal dimensional ones.

The effect of the remainder on these tori has been widely studied in the literature so we will skip further discussions on this topic. Here, as we want to use the tori of the normal form as approximations to invariant tori for the complete system, we need a procedure to estimate their accuracy. A possibility is to estimate the size of the remainder (see, for instance, [JV98] or [GJ01]), but here we have chosen a more straightforward approach (see Section 4.7 in [Jor99]): given a torus on the normal form, we can tabulate an orbit on it, send this table to the coordinates of the initial Hamiltonian (2) and check if each point is obtained from a numerical integration of the previous one. For tori sufficiently close to the origin of the normal form, this test is passed within an accuracy of the same order as the truncation of the Fourier series. All the tori displayed in this section have passed this test.

Therefore, we can easily compute lower and maximal invariant tori using the truncated normal form and send them, via the change of variables, to the initial coordinates of the system. This change of variables adds two additional frequencies (the system's intrinsic frequencies, ω_1 and ω_2) to the invariant tori. Thus, the invariant tori seen in the initial phase space are of dimensions three, four and five.

Figures 2, 3, 4, 5, 6 and 7 are examples of these computations for the TCCP system. More concretely, Figures 2 and 3 are obtained by setting $I_1 = I_2 = 0$ and $I_3 = I_3^{(0)}$ in (17), for some (small) value $I_3^{(0)} > 0$. This is a periodic Lyapunov orbit of the autonomous normal form $\mathcal{N}$ in (16), that corresponds to a three-dimensional torus for the initial Hamiltonian (2). This 3-D torus belongs to the Lyapunov family of the 2-D torus T_5 (see [JV97]), and their normal frequencies are $\frac{\partial \mathcal{N}}{\partial I_j}(0, 0, I_3^{(0)})$, $j = 1, 2$, where $\mathcal{N}$ is taken from (17). Figures 4 and 5 have been obtained in a similar way, but setting $I_2 = I_3 = 0$, $I_1 = I_1^{(0)}$ and $I_1 = I_3 = 0$, $I_2 = I_2^{(0)}$ respectively. In the coordinates of the initial Hamiltonian (2), these two tori are contained in the plane $z = p_z = 0$. Figure 6 displays two projections of a four-dimensional invariant torus near T_5. Finally, in Figure 7 two different projections of a five-dimensional invariant torus are shown. All these graphics have been obtained computing 10,000 points on a single orbit on the torus, with a time step of 0.1 units. See the captions for more details.

We note that, in this way, it is also possible to compute quasi-periodic orbits with a prescribed set of frequencies Ω_0, provided that Ω_0 belongs to the domain where the normal form is accurate. The procedure is based on solving the equation $\nabla\mathcal{N}(I) = \Omega_0$ by means of, for instance, a Newton method.

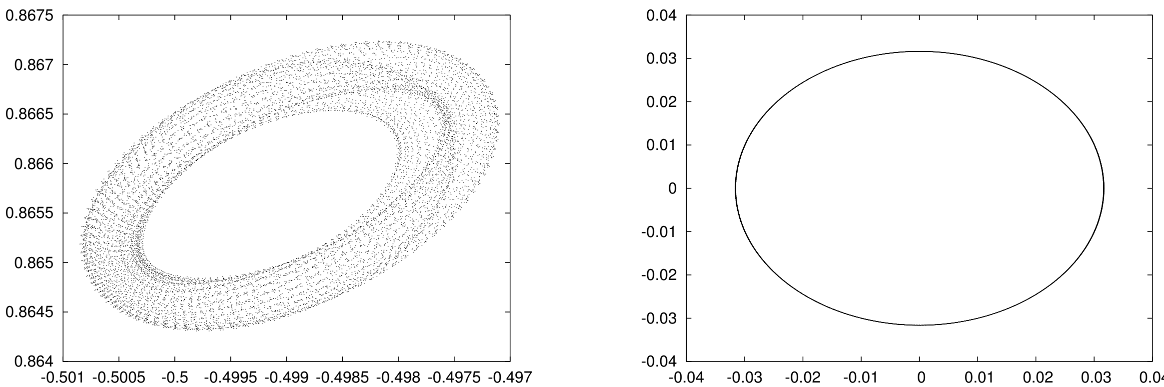

Figure 2: Projection on the (x, y) (left) and on the (z, p_z) (right) planes of an elliptic three-dimensional invariant torus near T_5. The intrinsic frequencies are ω_{sat}, ω_{ura} and $\nu_3 = 1.000062350$. The normal ones are $\nu_1 = -0.08044599352$ and $\nu_2 = 0.9966839283$.

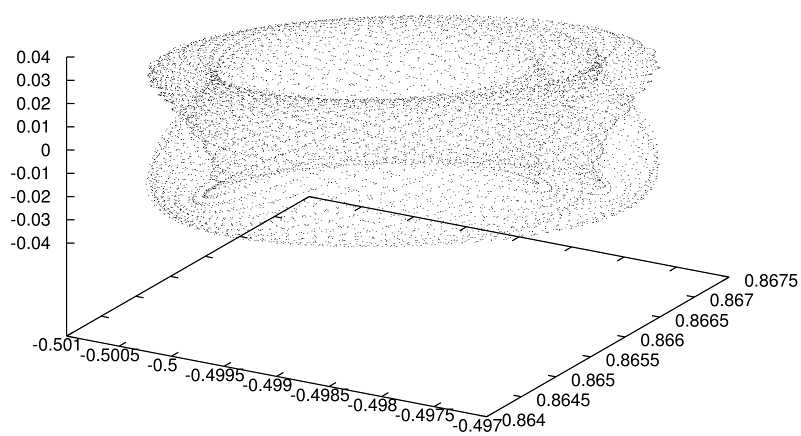

Figure 3: Projection on the configuration space of the three-dimensional invariant torus shown in Figure 2.

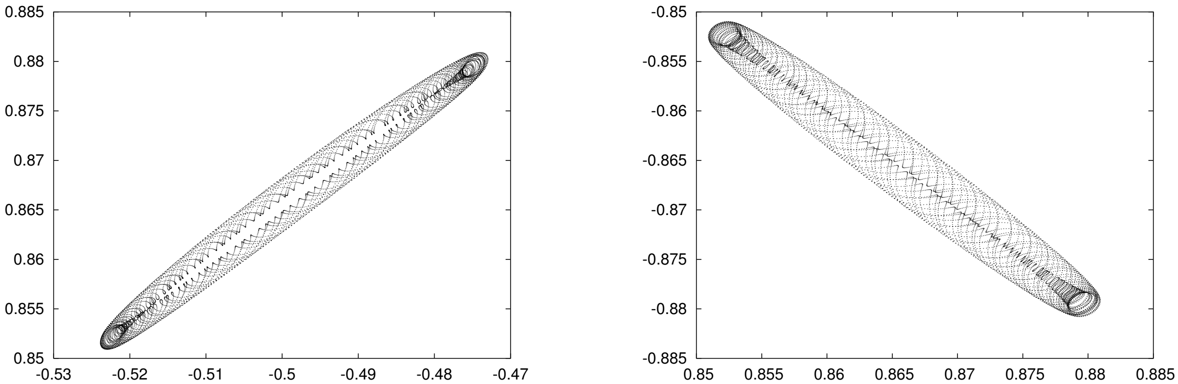

Figure 4: Projections on the (x, y) (left) and (y, p_x) (right) planes of an elliptic three-dimensional invariant torus. The intrinsic frequencies are ω_{sat}, ω_{ura} and $\nu_1 = -0.08046185813$, and the normal ones are $\nu_2 = 0.9966790714$ and $\nu_3 = 1.000063233$.

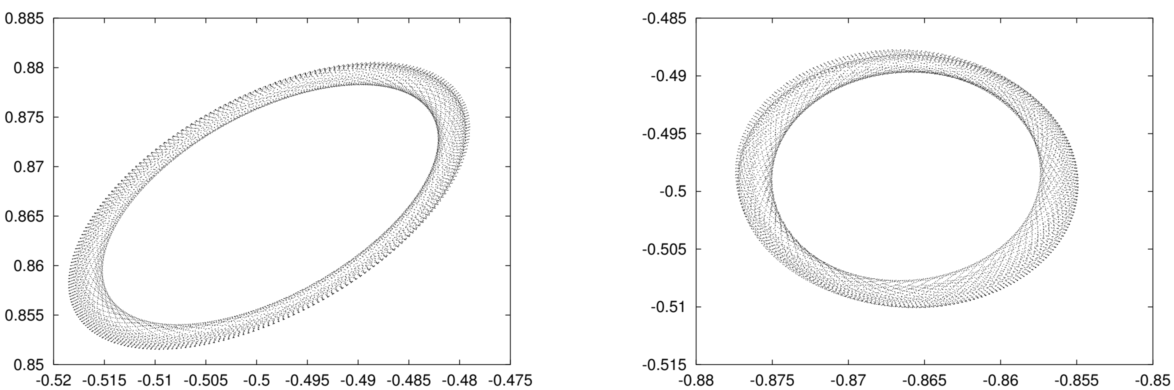

Figure 5: Projections on the (x, y) (left) and (p_x, p_y) (right) planes of an elliptic three-dimensional torus. The intrinsic frequencies are ω_{sat}, ω_{ura} and $\nu_2 = 0.9966811761$, and the normal ones are $\nu_1 = -0.08048083167$ and $\nu_3 = 1.000063022$.

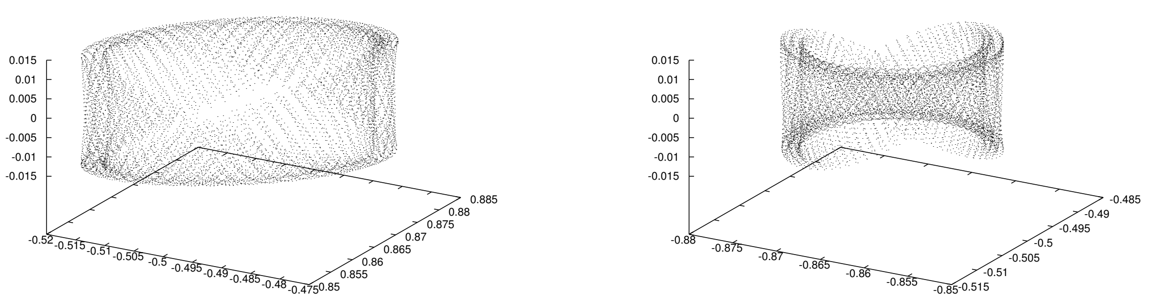

Figure 6: Projections on the (x, y, z) (left) and (p_x, p_y, p_z) (right) spaces of a four-dimensional torus near T_5. The intrinsic frequencies are ω_{sat}, ω_{ura}, $\nu_2 = 0.9966811761$ and $\nu_3 = 1.000063022$, and the normal one is $\nu_1 = -0.08048083168$.

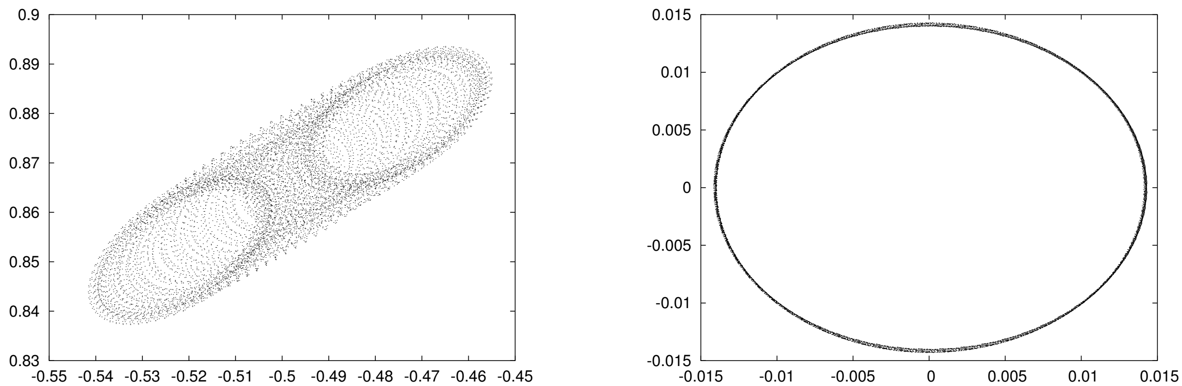

Figure 7: Projections on the (x, y) (left) and (z, p_z) (right) planes of a five-dimensional torus near T_5. The frequencies are ω_{sat}, ω_{ura}, $\nu_1 = -0.08046420047$, $\nu_2 = 0.9966802858$ and $\nu_3 = 1.000063496$.

3 Approximate first integral of the initial Hamiltonian

A first integral of a Hamiltonian system $H(q, p, \theta)$ is a function $F(q, p, \theta)$ that is constant on each orbit of the system. Functions having a small drift along the orbits are usually called approximate first integrals or quasi-integrals ([GG78, Mar80]). One of the main applications of approximate first integrals is to bound the rate of diffusion on certain regions of the phase space ([CG91, GJ01]).

3.1 Computing a quasi-integral

It is not difficult to see that, if $F(q, p, \theta)$ is a first integral, then

$$\{H, F\} = 0. \tag{18}$$

Here we will try to solve this equation by expanding H and F in Taylor-Fourier series, in the same coordinates used to obtain (15). That is, we suppose that H is expanded in complex coordinates, as in normal form up to degree 2. Let us write F as a truncated Fourier-Taylor expansion,

$$F(q, p, \theta_1, \theta_2) = \sum_{n=2}^{N} F_n(q, p, \theta_1, \theta_2),$$

where, as usual, F_n stands for an homogeneous polynomial of degree n in the variables (q, p), with coefficients that are (truncated) Fourier series in the angles θ_1 and θ_2:

$$F_n(q, p, \theta_1, \theta_2) = \sum_{|k|=n} \sum_{j=(j_1,j_2)} \left(f_{n,j}^{k} e^{i(j_1\theta_1 + j_2\theta_2)}\right) q^{k^1} p^{k^2}.$$

To compute the coefficients of this expansion, $f^k_{n,j} \in \mathbb{C}$, we solve equation (18) order by order. It is easy to see that there is some freedom while selecting the degree 2 of F, F_2. As our final goal is to bound the diffusion using this quasi-integral, a good choice is (see [GJ01])

$$F_2 = i \sum_{j=1}^{3} q_j p_j.$$

For what concerns to higher degrees, $n > 2$, it is possible to obtain $f^k_{n,j}$ recurrently,

$$f^k_{n,j} = \frac{i c^k_{n,j}}{j_1\omega_1 + j_2\omega_2 - \langle k^2 - k^1, \nu\rangle},$$

where $\nu = (\nu_1, \nu_2, \nu_3)$ and $c^k_{n,j}$ can be computed from the expansion of the Hamiltonian and the previously computed coefficients of F.

During the computations, two conditions must be verified at every step of the process:

a) ω_1, ω_2 and ν must satisfy that

$$j_1\omega_1 + j_2\omega_2 - \langle k, \nu\rangle \neq 0,$$

$$\forall\, (j_1, j_2) \in \mathbb{Z}^2, \quad \forall\, k \in \mathbb{Z}^3 \text{ such that } |j| + |k| \neq 0,$$

b) if $j_1 = j_2 = 0$ and $k^1 = k^2$, the value $c^k_{n,j}$ must vanish.

The first is the same non-resonance condition needed for the normal form computation and it only depends on the normal and internal frequencies of the torus. The second condition have to be checked before the computation of each F_j. In our example, this second condition is satisfied in all the cases. For a discussion on condition b), see [CG91].

In the model example, we have used the recurrence to compute the approximate first integral truncated at order $N = 16$.

3.2 Bounding the diffusion

As F is not an exact first integral, the variation of the values of F on a given trajectory of the Hamiltonian is not exactly zero. Its variation can be written, in terms of the Hamiltonian expanded in real coordinates $H(x, y, \theta, p_\theta)$ and in terms of the realified quasi-integral $F(x, y, \theta)$, as

$$\dot{F} = \{F(x, y, \theta), H(x, y, \theta, p_\theta)\},$$

where $(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3$ and, as usual, $p_\theta = (p_{\theta_1}, p_{\theta_2})$ are the momenta corresponding to the angles $\theta = (\theta_1, \theta_2) \in \mathbb{T}^2$. Then, it is easy to see that the diffusion can be estimated by bounding the following expansion:

$$\dot{F} = \sum_{n>N} \sum_{l=3}^{N} \{F_l, H_{n-l+2}\} + \sum_{n>N} \{F_2, H_n\}.$$

We will use the same procedure as in [GJ01]. Thus, we use Lemmas in [GJ01] and [CG91] (the norms should be modified in order to deal with two angles, but the lemmas are still valid) to estimate the size of the terms of the Hamiltonian that have not been numerically computed (those homogeneous polynomials with degree greater than N). A bound for the drift of the formal first integral F is obtained by means of the following

Lemma 3.1 *Let N and $\widehat{N}$ be integers such that $3 \leq N \leq \widehat{N}$ and*

$$\begin{array}{rcll} \|H_k\| & \leq & S_k & 3 \leq k \leq \widehat{N}, \\ \|H_k\| & \leq & h^{k-\widehat{N}+1} E & k > \widehat{N}, \\ \|F_k\| & \leq & Q_k & 3 \leq k \leq N. \end{array}$$

Then, if $h\rho < 1$,

$$\|\dot{F}\|_\rho \leq \mathcal{R}(\rho),$$

where

$$\begin{aligned} \mathcal{R}(\rho) \;=\; & \sum_{j=1}^{N-2} (j+2)\rho^j Q_{j+2} \sum_{N-j<l\leq \widehat{N}} l\rho^l S_l \\ & + \sum_{j=1}^{N-2} (j+2)\rho^j Q_{j+2} \frac{E}{h^{\widehat{N}}} h \frac{(\widehat{N}+1)(h\rho)^{\widehat{N}+1} - \widehat{N}(h\rho)^{\widehat{N}+2}}{(1-h\rho)^2} + \\ & + \sum_{N<l\leq \widehat{N}} l\rho^l S_l + \frac{E}{h^{\widehat{N}}} h \frac{(\widehat{N}+1)(h\rho)^{\widehat{N}+1} - \widehat{N}(h\rho)^{\widehat{N}+2}}{(1-h\rho)^2}. \end{aligned}$$

Proof See [CG91] and [GJ01].

Then, in order to find a region of effective stability, we define the following compact domain of the phase space:

$$D_\rho = \{(x,y) \in \mathbb{R}^6 \,;\; (x_1^2+y_1^2) + (x_2^2+y_2^2) + (x_3^2+y_3^2) \leq \rho^2\}.$$

Assume that we have an initial condition $(x(0), y(0))$ inside the domain D_{ρ_0}. We are interested in values $\rho > \rho_0$ such that the orbit $(x(t), y(t))$ is contained in D_ρ for all $t \in [0, T_S]$, where T_S is the (finite) lifetime of the considered physical system. A sufficient condition to achieve this is that

$$|F_2(x(t), y(t)) - F_2(x(0), y(0))| \leq \frac{1}{2}(\rho^2 - \rho_0^2), \quad 0 \leq t \leq T_S, \tag{19}$$

where $F_2(x,y) = \frac{1}{2}\sum_{j=1}^{3}(x_j^2 + y_j^2)$.

Let us now define

$$\Delta_N(\rho_0, \rho) = \frac{1}{2}(\rho^2 - \rho_0^2) - \sum_{j=3}^{N} \|F_j\|(\rho^j + \rho_0^j)$$

It is easy to see that, if $\Delta_N(\rho_0, \rho) \geq 0$ and

$$|F(x(t), y(t), \theta(t)) - F(x(0), y(0), \theta(0))| \leq \Delta_N(\rho_0, \rho),$$

then, (19) holds. The function $\Delta_N(\rho_0, \rho)$ is used to bound the maximum variation of $F(x, y, \theta)$ on the interval of time $[0, T_S]$ because (19) is a sufficient condition for the trajectory to be inside the domain D_ρ. Note that, due to the particular form of Δ_N, the values ρ_0 and ρ have to be sufficiently small to achieve $\Delta_N(\rho_0, \rho) \geq 0$ but, on the other hand, we want Δ_N to be as large as possible, to allow a large variation of F with a controlled variation of ρ. Hence, we will carefully select the values ρ_0 and ρ to obtain the largest region of effective stability for time T_S.

So, if $\Delta_N(\rho_0, \rho) \geq 0$, we can bound the escaping time as a function of the initial radius,

$$T(\rho_0) = \sup_{\rho} \frac{\Delta_N(\rho_0, \rho)}{\mathcal{R}(\rho)}, \tag{20}$$

where $\mathcal{R}(\rho)$ is given in Lemma 3.1.

In the TCCP system, we have solved equation (20) for different ρ_0's and, using inverse interpolation we have found the initial radius ρ_0 for which an orbit does not leave the domain D_ρ in a time span of length $T_S = 3 \times 10^9$ (the estimated age of the Solar System in adimensional units). The result obtained here is $\rho_0 = 1.7565 \times 10^{-4}$ and the maximal final radius is $\rho = 4.9666 \times 10^{-4}$. We want to mention that this region D_{ρ_0} (seen in the physical space of the TCCP) is qualitatively different from (it is not contained neither contains) the effective stability zones obtained with other models, such as the RTBP one (for example, in [Sim89, CG91, SD00]), the BCCP in [GJ01] and the ERTBP in [Gab03]. It can be seen as the subset of the phase space when a ball of radius ρ_0 "travels" along the invariant torus T_5. The projection of this region of effective stability into the Jupiter's plane of motion is shown in Figure 8.

References

References

[BHJ+03] H.W. Broer, H. Hanßmann, À. Jorba, J. Villanueva, and F.O.O. Wagener. Normal-internal resonances in quasiperiodically forced oscillators: a conservative approach. *Nonlinearity*, 16:1751–1791, 2003.

[CG91] A. Celletti and A. Giorgilli. On the stability of the Lagrangian points in the spatial Restricted Three Body Problem. *Celestial Mech.*, 50(1):31–58, 1991.

[CJ00] E. Castellà and À. Jorba. On the vertical families of two-dimensional tori near the triangular points of the Bicircular problem. *Celestial Mech.*, 76(1):35–54, 2000.

[Gab03] F. Gabern. *On the dynamics of the Trojan asteroids.* PhD thesis, Univ. de Barcelona, 2003.

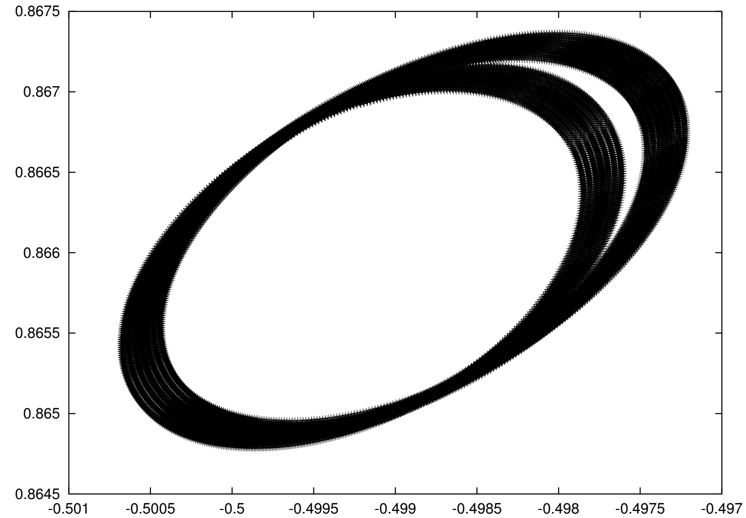

Figure 8: Projection on the Jupiter's plane of motion, (x, y), of the region of effective stability around the 2-D invariant torus T_5.

[GG78] A. Giorgilli and L. Galgani. Formal integrals for an autonomous Hamiltonian system near an equilibrium point. *Celestial Mech.*, 17:267–280, 1978.

[GJ01] F. Gabern and À. Jorba. A restricted four-body model for the dynamics near the Lagrangian points of the Sun-Jupiter system. *Discrete Contin. Dyn. Syst. Ser. B*, 1(2):143–182, 2001.

[GJ04] F. Gabern and À. Jorba. Generalizing the Restricted Three-Body Problem. The Bianular and Tricircular Coherent Problems. *Astron. Astrophys.*, 2004. To appear.

[GJMS01a] G. Gómez, À. Jorba, J. Masdemont, and C. Simó. *Dynamics and mission design near libration points. Vol. III, Advanced methods for collinear points*, volume 4 of *World Scientific Monograph Series in Mathematics*. World Scientific Publishing Co. Inc., 2001.

[GJMS01b] G. Gómez, À. Jorba, J. Masdemont, and C. Simó. *Dynamics and mission design near libration points. Vol. IV, Advanced methods for triangular points*, volume 5 of *World Scientific Monograph Series in Mathematics*. World Scientific Publishing Co. Inc., 2001.

[Jor99] À. Jorba. A methodology for the numerical computation of normal forms, centre manifolds and first integrals of Hamiltonian systems. *Exp. Math.*, 8(2):155–195, 1999.

[Jor00] À. Jorba. On practical stability regions for the motion of a small particle close to the equilateral points of the real Earth–Moon system. In J. Delgado, E.A. Lacomba, E. Pérez-Chavela, and J. Llibre, editors, *Hamiltonian Systems and*

Celestial Mechanics (HAMSYS-98), volume 6 of *World Scientific Monograph Series in Mathematics*, pages 197–213. World Scientific, 2000.

[Jor01] À. Jorba. Numerical computation of the normal behaviour of invariant curves of n-dimensional maps. *Nonlinearity*, 14(5):943–976, 2001.

[JS96] À. Jorba and C. Simó. On quasiperiodic perturbations of elliptic equilibrium points. *SIAM J. Math. Anal.*, 27(6):1704–1737, 1996.

[JV97] À. Jorba and J. Villanueva. On the persistence of lower dimensional invariant tori under quasi-periodic perturbations. *J. Nonlinear Sci.*, 7:427–473, 1997.

[JV98] À. Jorba and J. Villanueva. Numerical computation of normal forms around some periodic orbits of the Restricted Three Body Problem. *Phys. D*, 114:197–229, 1998.

[Mar80] C. Marchal. The quasi integrals. *Celestial Mech.*, 21:183–191, 1980.

[SD00] C. Skokos and A. Dokoumetzidis. Effective stability of the Trojan asteroids. *Astron. Astrophys.*, 367:729–736, 2000.

[SGJM95] C. Simó, G. Gómez, À. Jorba, and J. Masdemont. The Bicircular model near the triangular libration points of the RTBP. In A.E. Roy and B.A. Steves, editors, *From Newton to Chaos*, pages 343–370, New York, 1995. Plenum Press.

[Sim89] C. Simó. Estabilitat de sistemes Hamiltonians. *Mem. Real Acad. Cienc. Artes Barcelona*, 48(7):303–348, 1989.