Thomson's Heptagon: a case of bifurcation at infinity.

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Vortex modeling has a long history. Descartes (1644) used it as a model for the solar systems. J. J. Thomsom (1883) used it as a model for the atom. We consider point-vortex systems, which can be regarded as "discrete" solutions of the Euler equation. Their dynamics is described by a Hamiltonian system of equations. We are interested in polygonal configurations and how their stability depends upon various dynamical variables. A bit of Celestial Mechanics' techniquess helped us to simplify a problem that has been studied during over a century.

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I. INTRODUCTION

In 1883, when searching a model for the atom, J.J. Thomson was brought to study the linear stability of polygonal configurations of N identical point-vortices in the plane. In his analysis he reached the conclusion that a ring of six or fewer vortices was stable while for seven vortices he erroneously concluded that the ring was slightly unstable [12]. Fifty years later, in 1931, T. Havelock [6] succeeded in solving the ring linear analysis in full generality and showed that Thomson's Heptagon was *neutrally* stable. In 1999, Cabral and Schmidt [4] performed the nonlinear stability analysis for polygonal configurations with a central vortex (see Figure 1.a)). Finally very recently, in 2003, L.G. Kurakin and V.I. Yudovich [8] showed that the heptagon is nonlinearly stable. Then the "biblical" question arises: why should $N = 7$ be any special? Why seven should be the border-line between stability and instability in the plane? What is happening for rings of vortices, say, on a sphere? In this article we show that the conclamated case of Thomson's Heptagon is actually a case of bifurcation at infinity! People were looking at the problem in a reduced parameter space – i.e. for a special value of an extra parameter at infinity. This is particularly clear when considering the problem of a ring of vortices on a sphere with two polar vortices of variable intensities, Γ_N and Γ_S, respectively at the North and South Pole.

II. EQUATIONS OF MOTION

Let us consider a non-rotating sphere of radius R. The position of a point-vortex on the surface of the sphere is specified by means of the usual spherical coordinates (ϕ, θ), where $\theta \in [0, \pi]$ is the co-latitude and $\phi \in [0, 2\pi]$ the longitude. It has already been shown in the literature (see for example [2, 7]) that on a sphere the dynamics of N point vortices of strengths $\Gamma_1, \ldots, \Gamma_N$ is given by the Hamiltonian system of equations

$$\dot{q}_\alpha = -\frac{\partial H}{\partial p_\alpha}, \quad \dot{p}_\alpha = \frac{\partial H}{\partial q_\alpha}, \qquad \alpha = 1, ..., N, \qquad (1)$$

where $q_\alpha = \phi_\alpha$ and $p_\alpha = \Gamma_\alpha R^2(\cos\theta_\alpha - 1)$ are the canonical variables associated to the αth vortex, and H is the autonomous Hamiltonian

$$H(q_1, ..., q_N, p_1, ..., p_N) = c \sum_{\alpha<\beta} \Gamma_\alpha \Gamma_\beta \ln(1 - d_{\alpha\beta}), \qquad (2)$$

with $c = -1/4\pi$ and $d_{\alpha\beta} = \left(1 + \frac{p_\alpha}{R^2\Gamma_\alpha}\right)\left(1 + \frac{p_\beta}{R^2\Gamma_\alpha}\right) + \sqrt{\frac{p_\alpha}{R^2\Gamma_\alpha}\frac{p_\beta}{R^2\Gamma_\beta}\left(2 + \frac{p_\alpha}{R^2\Gamma_\alpha}\right)\left(2 + \frac{p_\beta}{R^2\Gamma_\beta}\right)}\cos(q_\alpha - q_\beta)$.

III. A RING OF IDENTICAL VORTICES

Now let us focus our attention on the case of the unit sphere, $R = 1$, and a vortex configuration consisting of one latitudinal ring of N identical vortices, say of vorticity $\Gamma_1 = ... = \Gamma_N = 1$, i.e.

$$q_\alpha(0) = \frac{2\pi(\alpha - 1)}{N}, \qquad p_\alpha(0) = \cos\theta_o - 1 \qquad (3)$$

and two polar vortices respectively of vorticity $\Gamma_{\rm N}$ and $\Gamma_{\rm S}$, held fixed at each pole, as shown in Figure 1.c). The Hamiltonian of the vortex system is $H = H_o + H_{FP}$, where

$$H_{FP} = c\Gamma_{\rm N} \sum_{\beta=1}^{N} \ln(-p_\beta) + c\Gamma_{\rm S} \sum_{\beta=1}^{N} \ln(2 + p_\beta)$$

is the part describing the interaction of the polar vortices with each vortex in the ring, and H_o is as H in (2), the

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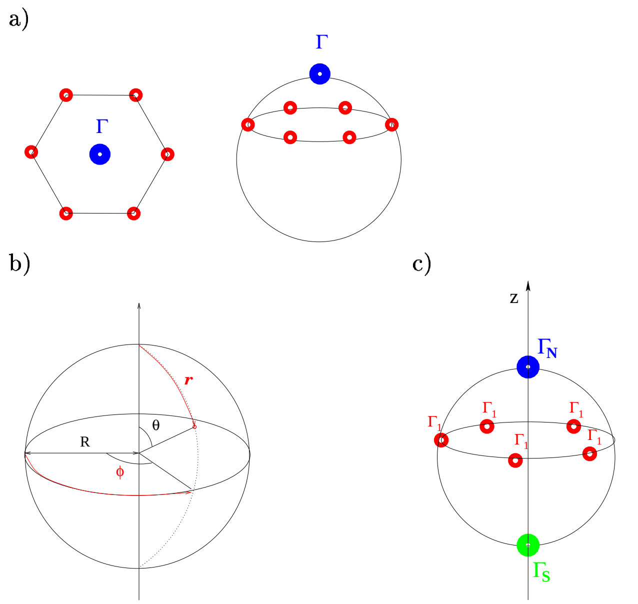

FIG. 1: a) In the plane and on a sphere, a configuration of a ring of identical vortices with a center vortex of vorticity Γ. b) A point on the sphere of radius R can be localized by specifying its longitude φ and its co-latitude θ. c) Configuration of a ring and two polar vortices on a sphere.

Hamiltonian describing the interaction of the vortices of the ring. It has been shown (see [3, 9]) that the dynamics of such a configuration is a rigid rotation – i.e. a *relative equilibrium* –

$$q_\alpha(t) = \nu t + q_\alpha(0), \quad p_\alpha(t) = z_o - 1, \tag{4}$$

where $z_o = \cos\theta_o$, $\nu = c\left[\frac{z_o}{\rho_o^2}(N-1) + \frac{\Gamma_{\rm N}}{1-z_o} - \frac{\Gamma_{\rm S}}{1+z_o}\right]$ is the rotational frequency deduced in [3] and $\rho_o = \sqrt{1-z_o^2}$. Now the natural question arises:

How does the stability (linear and non linear) of such configuration depend upon N, z_o, Γ_N e Γ_S ?

To tackle this question we begin by rewriting the system (1) as

$$\frac{dX}{dt} = J\nabla_X H,$$

where $X = (q_1, ..., q_N, p_1, ..., p_N)$, $J = \left(\begin{array}{c|c} O & -\mathbb{I} \\ \hline \mathbb{I} & O \end{array}\right)$ and $\nabla_X = \left(\frac{\partial}{\partial q_1}, ..., \frac{\partial}{\partial q_N}, \frac{\partial}{\partial p_1}, ..., \frac{\partial}{\partial p_N}\right)$. It is clear that exchanging the values of $\Gamma_{\rm N}$ and $\Gamma_{\rm S}$ is equivalent to exchange z_0 by $-z_0$. Hence, it will be enough to discuss the behaviour for fixed values of $\Gamma_{\rm S}$ and let vary $\Gamma_{\rm N}$ and z_0. Then we do the following (see for details Boatto and Simó [3]):

i) Change of reference frame: we view the dynamics in a frame corotating with the relative equilibrium configuration. In the corotating reference system, the Hamiltonian takes the form

$$\tilde{H} = H + \nu M,$$

where $M = N + \sum_{\alpha=1}^{N} p_\alpha$ is the momentum of the system – associated to the invariance under translations of the Hamiltonian H along parallels. The relative equilibrium becomes an equilibrium, X^*, in the new reference system, and the standard techniques can be used to study its stability.
The relevant equation to be studied is therefore

$$\frac{d\Delta X}{dt} = JS\,\Delta X \tag{5}$$

where $X = X^* + \Delta X$, and S is the Hessian of $\tilde{H}$ evaluated at the equilibrium X^*. For the *linear stability* we study the eigenvalues of the matrix JS (spectral stability), while for *nonlinear stability* we make use of Dirichlet's Criterion, i.e. we study the definiteness of the Hessian S [4, 5]:

Theorem III.1 (Dirichlet's Criterion)

Let X^* be an equilibrium of an autonomous system of ordinary differential equations

$$\frac{dX}{dt} = f(X), \qquad X \in \Omega \subset \mathbb{R}^{2N}, \tag{6}$$

that is, $f(X^*) = 0$. If there exists a positive (or negative) definite integral $\mathbf{\Psi}$ of the system (6) in a neighborhood of the equilibrium X^*, then X^* is stable.

Then let X^* be an equilibrium (3) and S the Hessian of $\tilde{H}$ evaluated at the equilibrium configuration. Since $\tilde{H}$ is a first integral of the system, Dirichlet's theorem implies that if at the equilibrium the quadratic form $\Delta X^T S \Delta X$ is positive (or negative) definite, then the equilibrium is stable. Notice that the Hessian S is a symmetric matrix and therefore it is diagonalizable, i.e. there exists an orthogonal matrix C such that $C^T SC = D$, where $D = diag(\lambda_1, ..., \lambda_{2N})$ is a diagonal matrix. Furthermore the matrix C can be chosen to leave invariant the symplectic form (equivalently $J = C^T JC$). Then by the canonical change of variables $Y = C^T X$ Eq. (5) becomes

$$\frac{d\Delta Y}{dt} = JD\,\Delta Y, \tag{7}$$

where $Y = (\tilde{q}_1, ..., \tilde{q}_N, \tilde{p}_1, ..., \tilde{p}_N)$ and $(\tilde{q}_j, \tilde{p}_j)$, $j = 1, ..., N$, are pairs of conjugate variables. Eq. (7) can be rewritten as

$$\frac{d^2\Delta\tilde{q}_j}{dt^2} = -\lambda_j\lambda_{j+N}\Delta\tilde{q}_j, \qquad j = 1, ..., N.$$

Then we have linear stability if

$$\Lambda_j = \lambda_j\lambda_{j+N} > 0 \tag{8}$$

for all $j = 1, ..., N$, with the exception of the zero eigenvalues due to symmetries of H.

ii) As deduced in [3] the Hessian S of $\tilde{H}$ has the structure

$$S = \left(\begin{array}{c|c} Q & \mathbf{O} \\ \hline \mathbf{O} & P \end{array}\right) \tag{9}$$

where the matrices Q and P are of the form

$$Q = c(-s\mathbb{I} + A), \qquad P = c\frac{1}{\rho_o^4}((t_1 - \tilde{t}_1)\mathbb{I} - A), \tag{10}$$
$$t_1 = s - (N-1)(1+z_o^2), \quad \tilde{t}_1 = \rho_o^4(\Gamma_N \eta^2 + \Gamma_S \tilde{\eta}^2),$$
$$\eta = \frac{1}{1-z_o}, \quad \tilde{\eta} = \frac{1}{1+z_o}, \quad s = \frac{N^2-1}{6},$$

and A is a symmetric circulant matrix with first row

$$a_1 = 0, \quad \text{and} \quad a_j = \frac{1}{1-\cos(2\pi(j-1)/N)} = a_{N-j+2},$$

$j = 1, \ldots, N$, and has minimum and maximum eigenvalues

$$\lambda_{A_{min}} = -\frac{1}{24}[2N^2 + 1 + 3(-1)^N], \quad \lambda_{A_{max}} = s. \tag{11}$$

It follows that Q has a zero eigenvalue, to be denoted as λ_{Q_1} and the other ones are positive (since $c < 0$).

iii) As Q and P are linear combinations of I and A, they diagonalize in the same basis as A. Let $\lambda_{Q_j}, \lambda_{P_j}$ be the respective eigenvalues. Hence, we can diagonalize JS and its eigenvalues are

$$\lambda_{JS_j} = \sqrt{\lambda_{Q_j}\lambda_{P_j}}, \quad \lambda_{JS_{j+N}} = -\sqrt{\lambda_{Q_j}\lambda_{P_j}},$$

$j = 1, \ldots, N$. Then the single zero eigenvalue λ_{Q_1} of S corresponds to double zero eigenvalue of JS, i.e. $\lambda_{JS_1} = \lambda_{JS_{N+1}} = 0$.

Then following the procedure described above (see [3] for details) we consider a symplectic change of variables which diagonalizes the Hessian. It is enough to use, both for the q and p variables the eigenbasis of A. As said above the nonzero eigenvalues of Q are positive. It follows from Eq. (10) and (11) that

$$\begin{aligned} \lambda_{P\,min} &= -\frac{c}{\rho_o^4}\left\{-\frac{1}{24}[2N^2+1+3(-1)^N] - \frac{N^2-1}{6}\right. \\ &+ \left.(N-1)(1+z_o^2) + \Gamma_N(1+z_o)^2 + \Gamma_S(1-z_o)^2\right\} \end{aligned} \tag{12}$$

Linear stability is assured when $\lambda_{P\,min} > 0$.

Then what about nonlinear stability?

It follows from the discussion above and from Dirichlet's Criterion, Theorem III.1, that nonlinear stability is assured when the minimum eigenvalue of P is positive [3], as well!

Theorem III.2 (spherical case)
The equilibrium X^* (3) is (linearly and nonlinearly) stable if

$$\lambda_{P\,min} > 0,$$

i.e. if

$$\begin{aligned} -(N-2)^2 - \delta &+ 4(N-1)z_0^2 \\ &+ 4(1+z_0)^2\Gamma_N + 4(1-z_0)^2\Gamma_S > 0, \end{aligned} \tag{13}$$

where $\delta = 0$ for N even, $\delta = 1$ for N odd. It is linearly unstable if the inequality in (13) is reversed.

Remarks:

1) From the Theorem above we guarantee stability if

$$\Gamma_N > \frac{(N-2)^2 + \delta - 4(N-1)z_0^2 - 4(1-z_0)^2\Gamma_S}{4(1+z_0)^2}$$

with δ defined as before.

2) For $\Gamma_N > 0$ and $\Gamma_S > 0$, notice the stabilizing influence of the polar vortices – i.e. of the factor $4(1+z_o)^2\Gamma_N + 4\Gamma_S(1-z_o)^2$ in the equations of the theorem above.

3) Concerning stability in the critical case $\lambda_{P\,min} = 0$ it is necessary to carry out a computation of higher order terms of the Normal Form of the Hamiltonian around the fixed point, as done in [4] for the planar case, $N = 7$.

4) Notice $\theta = \sqrt{K}r$ where r is the geodesic distance from the north pole and $K = 1/\sqrt{R}$ is the curvature of the sphere. Then fixing r, we recover the planar case (see Figure 1) by considering $K \to 0$ (as fully discussed in [2]), obtaining

$$\lim_{K\to 0} p_\alpha = -\frac{r_\alpha^2}{2}.$$

In an equivalent way we can recover the planar case by letting z_0 tend to 1. The previous theorem reduces to the case already studied by Cabral and Schmidt [4]

Corollary III.3 (planar limit)
The equilibrium $X^* = \left(0, \frac{2\pi}{N}, \ldots, \frac{2\pi(N-1)}{N}, -\frac{r_o^2}{2}, \ldots, -\frac{r_o^2}{2}\right)$ is (linearly and nonlinearly) stable if the vorticity strength of the center vortex verifies

$$\Gamma_N > \frac{(N-2)^2 + \delta - 4(N-1)}{16}, \tag{14}$$

where $\delta = 0$ for N even, $\delta = 1$ for N odd. It is linearly unstable if the inequality is reversed.

A complete discussion about the planar limit is given in [2].

5) *What is the relevance of having two polar vortices?*

As shown in Figures 2 and 3 when $\Gamma_S = 0$ the stability region for $N < 7$ is quite different from the one of the case $N \geq 7$. In the case $N < 7$ both polar cups exhibit a stability region, and the region grows with the increasing of the value of Γ_N (see figures 2.a)-c) and 3.a)). More specifically, for a give $N < 7$ there is a particular value of Γ_N above which the stability region is the whole sphere! The situation is quite different for $N \geq 7$. As Γ_N increases the stability region around the north polar cup increases (see figures 2.b)-c) and 3.b)), but never reaches the south pole!

Now when $\Gamma_S \neq 0$ things are quite different! Let us illustrate this with an example. Consider the case of a ring of eight vortices, i.e. $N = 8$, and let us see how the presence of a southern polar vortex could extend the assured stability region – and equivalently reduce the linear instability region. In Figure 4 the curves delimiting the stability

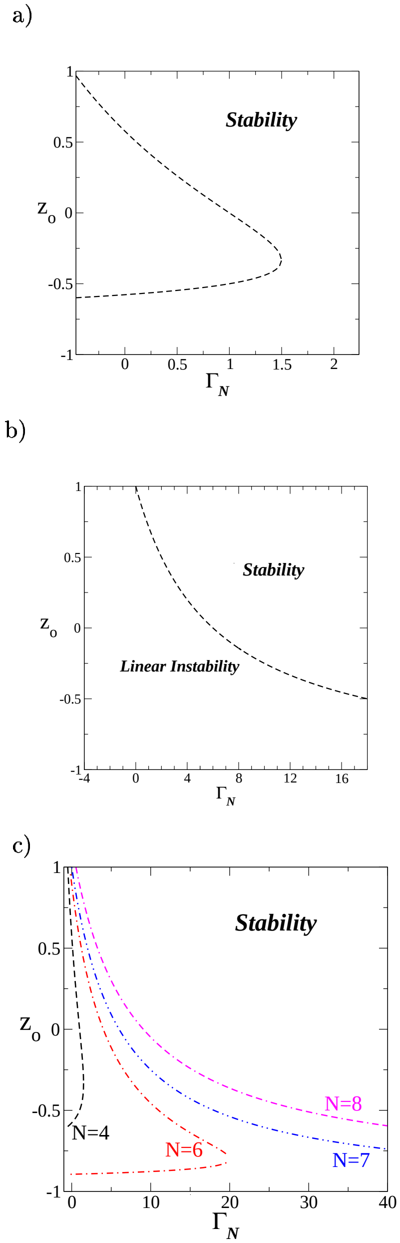

FIG. 2: Regions of stability and of linear instability for *a*) $N = 4$, *b*) $N = 7$ and *c*) $N = 4, 6, 7, 8$.

region are given for different values of the strength of the southern polar vortex, Γ_S. In particular, notice that if the value of Γ_S is above the critical one (i.e. $\Gamma_S > \Gamma_S^*$=1/2) the stability region can extend to the southern polar region for negative values of Γ_N, as well. Analogously, in the case of a ring of four vortices, i.e. $N = 4$, if the southern polar vortex has a strength $\Gamma_S < \Gamma_S^* = -1/2$ the assured stability region does not include a neighborhood of the south pole see Figure 5).

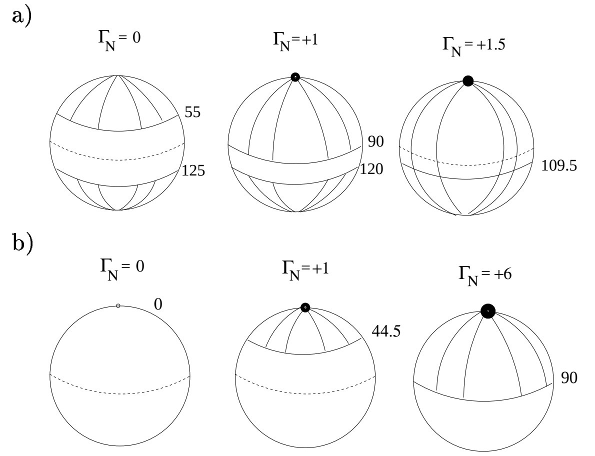

FIG. 3: $\Gamma_S = 0$. Stability region of a ring of N identical vortices with a vortex of vorticity Γ_N held fixed at the North Pole. Notice how the stability band increases with Γ_N. *a*) for $N = 4$; *b*) for $N = 7$.

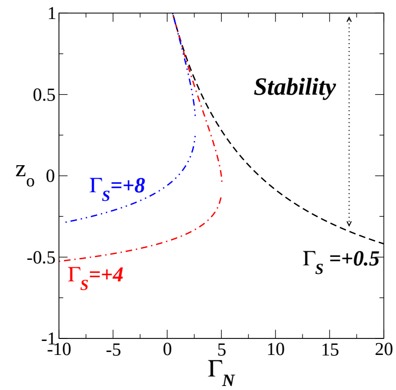

FIG. 4: For $N = 8$, different stability regions for different values of $\Gamma_S \geq \tilde{\Gamma}_S^*$.

Conclusions

On a sphere, we investigated the linear and nonlinear stability of a latitudinal polygonal ring of identical point-vortices, in the presence of two fixed polar vortices. The purpose of our study was to show the full symmetry of the stability problem. In fact as already widely discussed in the literature for over a century (see [4, 6, 8, 12]), in the planar case it would appear that a ring of seven vortices is a special case. When considering a ring of vortices with a central vortex (as in Figure 1.a)) the stability behaviour is quite different for rings with more or less than seven vortices. The Thomson Heptagon appears as a mysterious boundary case! In this article we showed that on the sphere– that can be thought as the plane plus the point at infinity– when letting Γ_S varying we are accordingly setting new boundary values for N. In particular our study gives the critical value of Γ_S^* for all N, i.e.

$$\Gamma_S^* \;=\; (N^2 - 8N + 8 - \delta)/16,$$

where $\delta = 0$ for N even, $\delta = 1$ for N odd. To obtain this critical value it is enough to set equality to zero in (13) and

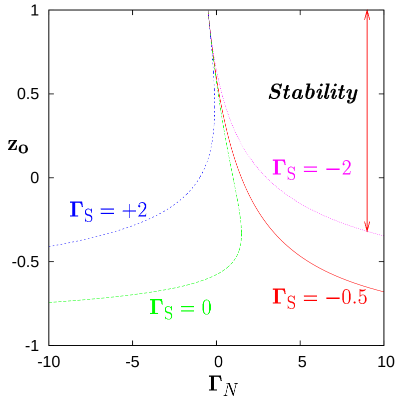

FIG. 5: For $N = 4$, different stability regions for different values of $\Gamma_S \geq \tilde{\Gamma}_S^*$.

let z_0 tend to -1. To illustrate this more clearly in Figure 4 we showed that for $\Gamma_S^* = 1/2$ the boundary value for N is 8. In other words, $N = 7$ is a special boundary case only for $\Gamma_S^* = 0$, and somehow the planar setting can be viewed as a case of bifurcation at infinity!

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[1] H. Aref, P. K. Newton, M. A. Stremler, T. Tokieda and D. L. Vainchtein. Vortex Crystals, Advances in Applied Mathematics, 39, 2003.

[2] S. Boatto, Curvature Perturbations and Stability of a Ring of Vortices, Discrete and Continuous Dynamical Systems, to appear (2007).

[3] S. Boatto and C. Simó, Stability of Latitudinal Vortex Rings with Polar Vortices, preprint (2004) (http://www.ma.utexas.edu/mp_arc/a/04-67).

[4] H. E. Cabral and D. S. Schmidt, Stability of relative equilibria in the problem of $N + 1$ vortices. SIAM J. Math. Anal., 31 no. 2 (1999/00), 231–250.

[5] G. Lejeune Dirichlet, Werke, Vol. 2, Berlin, pp 5-8, 1897.

[6] T. H. Havelock, The Stability of Motion of Rectilinear Vortices in Ring Formation, Philosophical Magazine S. 7, Vol. 11, No. 70 (1931), 617–633.

[7] Y. Kimura, Vortex Motion on Surfaces with Constant Curvature, Proc. R. Soc. Lond. A **455** (1999), 245-259.

[8] L. G. Kurakin and V. I. Yudovich. The Stability of Stationary Rotation of a Regular Vortex Polygon, Chaos, 12 (3), 574–595 (2002).

[9] C. Lim, J. Montaldi, and M. Roberts, Relative equilibria of point vortices on the sphere, Physica D, 148 (2001), 97-135..

[10] C. Marchioro and M. Pulvirenti, Vortices and Localization in Euler Flows, Commun. Math. Phys. 154 (1993), 49–61; Mathematical Theory of Incompressible Nonviscous Fluids, AMS 96, Springer-Verlag, New York, 1994.

[11] G.J. Mertz, Stability of body centered polygonal configurations of ideal vortices, Physics of Fluids, 21 (1978), 1092–1095.

[12] J.J. Thomson, A Treatise on the Motion of Vortex Rings, Macmillian, New York (Adam Prize Essay), 1883; Electricity and Matter, Westmister Archibald Constable & Co., Ltd, 1904.