Part I: Numerical Fourier analysis of quasi–periodic functions

I.1 The procedure

$$f(t) \sim Q_f(t; A, \omega)$$

$$\mathrm{DFT}(f) = \mathrm{DFT}(Q_f)$$

I.2 Error estimation

- *Applying the above procedure, the system to be solved can be written as*

$$g(y + \Delta y) = b + \Delta b.$$

- *The exact frequencies and amplitudes would be obtained if* $\Delta b = 0$.
- *A bound for the error is given by*

$$\|\Delta y\| \leq \|Dg(y)^{-1}\| \|\Delta b\|.$$

- *We will bound* $\|Dg(y)^{-1}\|$ *and* $\|\Delta b\|$.

I.3 Applications

- *Academic example*, to show the procedure and the accuracy of the error estimation
- *Solar System models:*

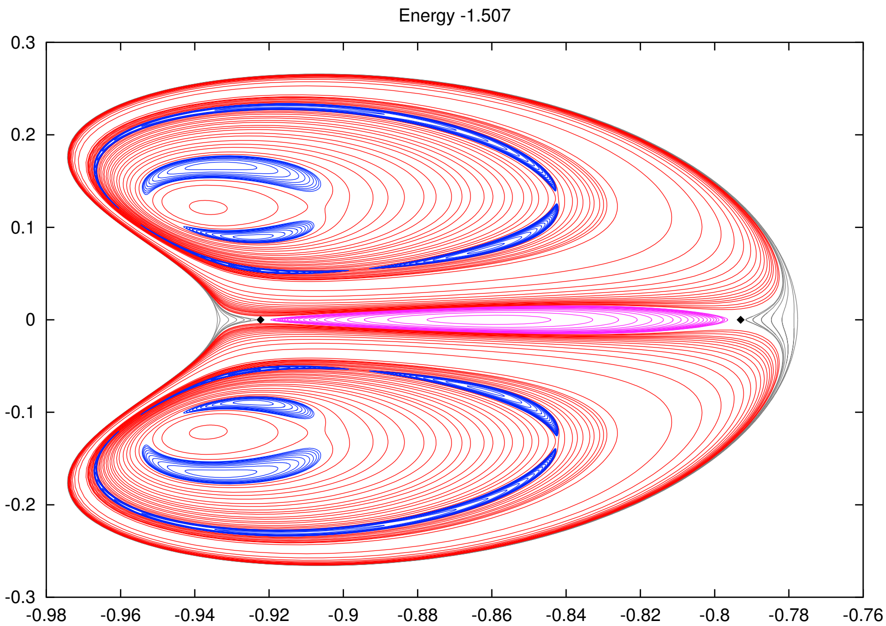
$$\mathrm{SS} = \mathrm{RTBP} + \mathrm{Perturbations}$$

$$\Downarrow \quad \textit{(Fourier analysis)}$$

$$\mathrm{SS} = \mathrm{RTBP} + \mathrm{Perturbations}(\omega_1, \omega_2, \ldots)$$

Part II: The neighborhood of the collinear equilibrium points in the RTBP

Final goal: *To get Poincaré section representations of the flow around* $L_{1,2,3,}$.



Tools: *Parallel algorithms for the computation of different kinds of families of periodic orbits and invariant tori.*

I.1 Refined Fourier analysis procedure

Given a **real analytic quasi–periodic function**, $f(t)$, at N equally–spaced points on an interval $[0, T]$,

$$\{f(t_l)\}_{l=0}^{N-1}, \quad t_l = l\frac{T}{N},$$

the goal is to compute a trigonometric approximation

$$Q_f(t) = A_0^c + \sum_{l=1}^{N_f} (A_l^c \cos(2\pi \frac{\nu_l}{T} t) + A_l^s \sin(2\pi \frac{\nu_l}{T} t).$$

The Discrete Fourier Transform (DFT) of f is defined through

$$f(t_l) = \frac{1}{2}(c_{f,T,N}(0) + c_{f,T,N}(\frac{N}{2}) \cos(\frac{2\pi \frac{N}{2} t_l}{T}))+$$
$$\sum_{j=1}^{N/2-1} (c_{f,T,N}(j) \cos(\frac{2\pi j t_l}{T}) + s_{f,T,N}(j) \sin(\frac{2\pi j t_l}{T})).$$

where

$$c_{f,T,N}(j) = \frac{2}{N} \sum_{l=0}^{N-1} f(t_l) \cos(2\pi \frac{j}{N} l), \quad j = 0, ..., \frac{N}{2},$$
$$s_{f,T,N}(j) = \frac{2}{N} \sum_{l=0}^{N-1} f(t_l) \sin(2\pi \frac{j}{N} l), \quad j = 1, ..., \frac{N}{2} - 1.$$

Algorithm

1. Set an starting **thresold** for collecting **peaks of the modulus of the DFT** of f(t).

2. Find initial **approximations of the frequencies**, starting from the peaks of the DFT greater than the thresold.

3. Find the **amplitudes** of the frequencies found in the previous step (DFT(Q_f) = DFT(f)).

4. Simultaneously **refine** ALL the **frequencies and amplitudes** of the current quasi–periodic approximation of f.

5. Perform a **DFT of the input signal minus the current quasi–periodic approximation** obtained in step 4, **decrease** the **thresold** and go **back to step 2**.

Equations of step 3

We ask $\mathrm{DFT}(Q_f) = \mathrm{DFT}(f)$, being

$$Q_f(t) = A_0^c + \sum_{l=1}^{N_f} (A_l^c \cos(2\pi \frac{\nu_l}{T} t) + A_l^s \sin(2\pi \frac{\nu_l}{T} t).$$

The system of equations to be solved is **linear** and $(1 + 2N_f) \times (1 + 2N_f)$:

$$\begin{aligned}
A_0^c c_{1,T,N}^{n_h}(0) + \sum_{l=1}^{N_f} (A_l^c \overline{c}_{\nu_l,N}^{n_h}(0) + A_l^s \widetilde{c}_{\nu_l,N}^{n_h}(0)) &= c_{f,T,N}^{n_h}(0) \\
A_0^c c_{1,T,N}^{n_h}(j) + \sum_{l=1}^{N_f} (A_l^c \overline{c}_{\nu_l,N}^{n_h}(j) + A_l^s \widetilde{c}_{\nu_l,N}^{n_h}(j)) &= c_{f,T,N}^{n_h}(j) \\
\sum_{l=1}^{N_f} (A_l^c \overline{s}_{\nu_l,T}^{n_h}(j) + A_l^c \widetilde{s}_{\nu_l,T}^{n_h}(j)) &= s_{f,T,N}^{n_h}(j)
\end{aligned}$$

where $j = [\nu_l + 0.5]$, $l = 1 \div N_f$, and

$$\begin{aligned}
c_1^{n_h}(j) &= c_{1,T,N}^{n_h}(j), \\
\overline{c}_{\nu_l,N}^{n_h}(j) &= c_{\cos(\frac{2\pi\nu_l}{T}),T,N}^{n_h}(j), \quad & \overline{s}_{\nu_l,N}^{n_h}(j) &= s_{\cos(\frac{2\pi\nu_l}{T}),T,N}^{n_h}(j), \\
\widetilde{c}_{\nu_l,N}^{n_h}(j) &= c_{\sin(\frac{2\pi\nu_l}{T}),T,N}^{n_h}(j), \quad & \widetilde{s}_{\nu_l,N}^{n_h}(j) &= s_{\sin(\frac{2\pi\nu_l}{T}),T,N}^{n_h}(j).
\end{aligned}$$

Equations for step 4

The system of equations is a $(1+3N_f)\times(1+3N_f)$ **non–linear system** which is solved by **Newton's method**:

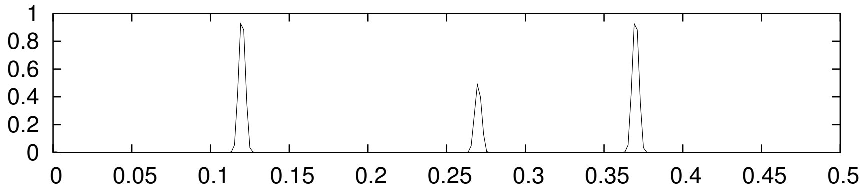
$$
\begin{aligned}
A_0^c c_{1,T,N}^{n_h}(0) + \sum_{l=1}^{N_f}(A_l^c \overline{c}_{\nu_l,N}^{n_h}(0) + A_l^s \widetilde{c}_{\nu_l,N}^{n_h}(0)) &= c_{f,T,N}^{n_h}(0) \\
A_0^c c_{1,T,N}^{n_h}(j_i) + \sum_{l=1}^{N_f}(A_l^c \overline{c}_{\nu_l,N}^{n_h}(j_i) + A_l^s \widetilde{c}_{\nu_l,N}^{n_h}(j_i)) &= c_{f,T,N}^{n_h}(j_i) \\
\sum_{l=1}^{N_f}(A_l^c \overline{s}_{\nu_l,N}^{n_h}(j_i) + A_l^s \widetilde{s}_{\nu_l,N}^{n_h}(j_i)) &= s_{f,T,N}^{n_h}(j_i) \\
A_0^c cs_{1,T,N}^{n_h}(j_i^+) + \sum_{l=1}^{N_f}(A_l^c \overline{cs}_{\nu_l,N}^{n_h}(j_i^+) + A_l^s \widetilde{cs}_{\nu_l,N}^{n_h}(j_i^+)) &= cs_{f,T,N}^{n_h}(j_i^+)
\end{aligned}
$$

being $j_i = [\nu_i + 0.5]$, $j_i^+ \neq j_i, |j_i^+ - j_i| = 1$.

An example:

$$f(t) = \cos(2\pi 0.23t) - \frac{1}{2}\sin(2\pi 0.27t) + \sin(2\pi 0.37t).$$

1. Starting thresold: 0.8
modulus of the DFT of the input data:



$\Rightarrow$ peaks $j = 61$, $j = 189$.

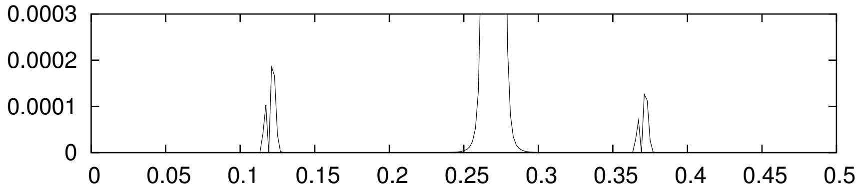
2. Approximation of frequencies (Laskar's method):

peak 61	$\Rightarrow$ frequency 0.11999948789
peak 189	$\Rightarrow$ frequency 0.36999965075

3. Computation of amplitudes from known frequencies:

Frequency	Cosine amplitude	Sine amplitude
0.119999487888	0.999907666367	-0.000823654552
0.369999650752	0.000561727398	0.999937098420

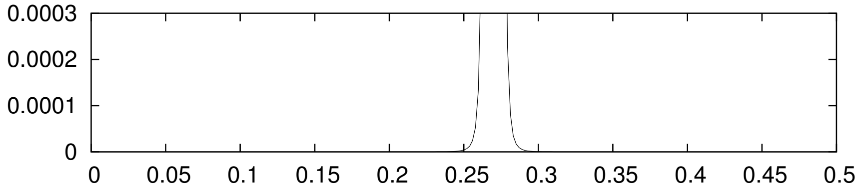
modulus of the DFT of the residual

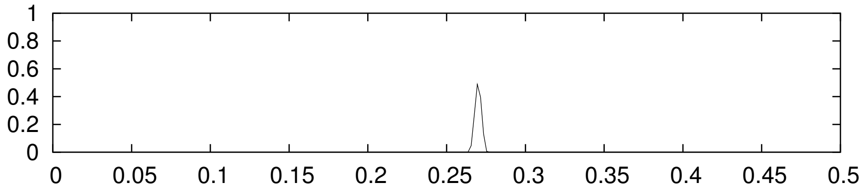


4. Iterative refinement:

Frequency	Cosine amplitude	Sine amplitude
0.120000000003	1.000000000686	0.000000005106
0.369999999995	0.000000006660	1.000000000297

5. modulus of the DFT of input signal minus step 4:





New threshold: 0.2

2. Approximation of frequencies (Laskar's method):

   peak 138 $\Rightarrow$ frequency 0.2699999999988849

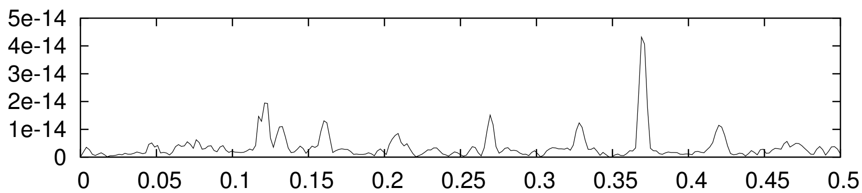
3. Amplitudes from known frequencies:

Frequency	Cosine amplitude	Sine amplitude
0.120000000003	1.000000000586	0.000000005245
0.369999999995	0.000000007480	0.999999999164
0.269999999999	-0.000000000897	-0.499999999946

4. Iterative refinement:

Frequency	Cosine amplitude	Sine amplitude
0.12000000000000	0.99999999999999	-0.00000000000008
0.37000000000000	-0.00000000000001	1.00000000000000
0.27000000000000	0.00000000000000	-0.50000000000000

modulus of the DFT of the residual:



I.2 Error estimation

We assume

- f is **real analytic** and **quasi–periodic**,

$$f(t) = A_0^c + \sum_{\substack{k\in\mathbb{Z}^m \\ k\omega>0}} (A_k^c \cos(2\pi k\omega t) + A_k^s \sin(2\pi k\omega t)),$$

  and its Fourier coefficients a_k satisfy the **Cauchy estimates**

$$\sqrt{(A_k^c)^2 + (A_k^s)^2} \le Ce^{-\delta|k|} \quad \forall k \in \mathbb{Z}^m.$$

- The frequency vector $\omega = (\omega_1, \ldots \omega_m)$ satisfies a **Diophantine condition**

$$|k\omega| \ge \frac{D}{|k|^\tau},$$

  with $D, \tau > 0$.

- We determine the frequencies $\{\nu_l\}_{l=1}^{N_f}$ of order $\le r_0 - 1$ ($\nu_l \approx Tk\omega,\ 1 \le |k| \le r_0 - 1$) and the corresponding amplitudes $\{A_l^c\}_{l=0}^{N_f}$, $\{A_l^s\}_{l=1}^{N_f}$.

Strategy

Let us denote,

$$f_{r_0}(t) = A_0^c + \sum_{\substack{|k|\le r_0-1\\ k\omega>0}} (A_k^c \cos(2\pi k\omega t) + A_k^s \sin(2\pi k\omega t)).$$

The system of equations used for simultaneous improvement of frequencies and amplitudes can be written as

$$\underbrace{\begin{aligned} A_0^c + \sum_{l=1}^{N_f} (A_l^c \overline{c}^{n_h}_{\nu_l,N}(0) + A_l^s \widetilde{c}^{n_h}_{\nu_l,N}(0)) \\ A_0^c c_1^{n_h}(j_i) + \sum_{l=1}^{N_f} (A_l^c \overline{c}^{n_h}_{\nu_l,N}(j_i) + A_l^s \widetilde{c}^{n_h}_{\nu_l,N}(j_i)) \\ \sum_{l=1}^{N_f} (A_l^c \overline{s}^{n_h}_{\nu_l,N}(j_i) + A_l^s \widetilde{s}^{n_h}_{\nu_l,N}(j_i)) \\ A_0^c cs_1^{n_h}(j_i^+) + \sum_{l=1}^{N_f} (A_l^c \overline{cs}^{n_h}_{\nu_l,N}(j_i^+) + A_l^s \widetilde{cs}^{n_h}_{\nu_l,N}(j_i^+)) \end{aligned}}_{g(y+\Delta y)} \begin{aligned} &= \\ &= \\ &= \\ &= \end{aligned} \begin{aligned} \underbrace{\begin{matrix} c^{n_h}_{f_{r_0},T,N}(0) \\ c^{n_h}_{f_{r_0},T,N}(j_i) \\ s^{n_h}_{f_{r_0},T,N}(j_i) \\ cs^{n_h}_{f_{r_0},T,N}(j_i^+) \end{matrix}}_{b} \begin{matrix} + \\ + \\ + \\ + \end{matrix} \underbrace{\begin{matrix} c^{n_h}_{f-f_{r_0},T,N}(0) \\ c^{n_h}_{f-f_{r_0},T,N}(j_i) \\ s^{n_h}_{f-f_{r_0},T,N}(j_i) \\ cs^{n_h}_{f-f_{r_0},T,N}(j_i^+) \end{matrix}}_{\Delta b} \end{aligned}.$$

We would get the **exact** frequencies and amplitudes **if** $\Delta b = 0$.

The **error in frequencies and amplitudes** is given, in the first order approximation, by

$$\|\Delta y\|_\infty \lesssim \|Dg(y)^{-1}\|_\infty \|\Delta b\|_\infty.$$

We will obtain **bounds for** $\|Dg(y)^{-1}\|_\infty$ **and** $\|\Delta b\|_\infty$.

Bound for $\|Dg(y)^{-1}\|_\infty$
Preliminaries

We can write

$$Dg(y) =: M = \begin{pmatrix} 2 & B_{0,1} & \dots & B_{0,N_f} \\ 0 & B_{1,1} & \dots & B_{1,N_f} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & B_{N_f,1} & \dots & B_{N_f,N_f} \end{pmatrix}.$$

We **split** $M = M_D + M_O$,

$$M = \begin{pmatrix} 2 & 0 & \dots & 0 \\ 0 & B_{1,1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & B_{N_f,N_f} \end{pmatrix} + \begin{pmatrix} 0 & B_{0,1} & \dots & B_{0,N_f} \\ 0 & 0 & \dots & B_{1,N_f} \\ 0 & \vdots & \ddots & \vdots \\ 0 & B_{N_f,1} & \dots & 0 \end{pmatrix}.$$

The idea is to obtain **bounds for** $\|M_D^{-1}\|$, $\|M_O\|$ and use

$$\|(M_D + M_O)^{-1}\| \leq \frac{\|M_D^{-1}\|}{1 - \|M_D^{-1}\|\|M_O\|}.$$

The components $B_{i,l}$ of M $(i \neq 0)$ have the following general aspect

$$\begin{pmatrix} A_l^c \partial \overline{c}_{\nu_l,N}^{n_h}(j_i) + A_l^s \partial \widetilde{c}_{\nu_l,N}^{n_h}(j_i) & \overline{c}_{\nu_l,N}^{n_h}(j_i) & \widetilde{c}_{\nu_l,N}^{n_h}(j_i) \\ A_l^c \partial \overline{s}_{\nu_l,N}^{n_h}(j_i) + A_l^s \partial \widetilde{s}_{\nu_l,N}^{n_h}(j_i) & \overline{s}_{\nu_l,N}^{n_h}(j_i) & \widetilde{s}_{\nu_l,N}^{n_h}(j_i) \\ A_l^c \partial \overline{cs}_{\nu_l,N}^{n_h}(j_i^+) + A_l^s \partial \widetilde{cs}_{\nu_l,N}^{n_h}(j_i^+) & \overline{cs}_{\nu_l,N}^{n_h}(j_i^+) & \widetilde{cs}_{\nu_l,N}^{n_h}(j_i^+) \end{pmatrix}.$$

First simplification: $M \to \mathcal{M}$

The first simplification consists in the use of the **Truncated Continuous Fourier Tansform** $(\mathcal{C}, \mathcal{S})$,

$$\frac{1}{T}\int_0^T H_T^{n_h}(t) f(t) e^{-i2\pi\frac{j}{T}t} =: \frac{1}{2}(\mathcal{C}_{f,T}^{n_h}(j) + i\mathcal{S}_{f,T}^{n_h}(j))$$

instead of the DFT (c, s).

We define $\widetilde{\widetilde{\mathcal{CS}}}_\nu^{n_h}(j)$ as in the discrete case.

In this way,

$$\begin{array}{rcl} M & = & M_D + M_O \\ & & \downarrow \text{(substitute DFT by TCFT)} \\ \mathcal{M} & = & \mathcal{M}_D + \mathcal{M}_O \end{array}$$

Remark 1: The difference between $\widetilde{\widetilde{cs}}$ and $\widetilde{\widetilde{\mathcal{CS}}}$ can be bounded from a **discrete** version of **Poisson's summation formula**:

$$c_{f,T,N}^{n_h}(j) = \sum_{l=-\infty}^{\infty} \mathcal{C}_{f,T}^{n_h}(\frac{j+lN}{T}), \ldots$$

Remark 2: Explicit formulae for the components of $\mathcal{M}$ can be obtained.

Formulae for the components of $\mathcal{M}$

The components of the matrix $\mathcal{M}$ are given by

$$\begin{array}{rcl}
\overline{\mathcal{C}}_{\nu}^{n_h}(j) & = & K_{n_h}\Big(\dfrac{\sin(2\pi(\nu-j))}{\psi_{n_h}(\nu-j)} + \dfrac{\sin(2\pi(-\nu-j))}{\psi_{n_h}(-\nu-j)} \Big), \\
\vdots & & \vdots \qquad\qquad\qquad \vdots \\
\partial\overline{\mathcal{C}}_{\nu}^{n_h}(j) & = & K_{n_h}\Big(\dfrac{h_r(\nu-j)}{\psi_{n_h}(\nu-j)} - \dfrac{h_r(-\nu-j)}{\psi_{n_h}(-\nu-j)} \Big), \\
\vdots & & \vdots \qquad\qquad\qquad \vdots
\end{array}$$

where

$$\begin{array}{rcl}
K_{n_h} & = & \dfrac{(-1)^{n_h}(n_h!)^2}{2\pi} \\
\psi_{n_h}(x) & = & \displaystyle\prod_{l=-n_h}^{n_h}(x+l) \\
h_r(x) & = & 2\pi\cos(2\pi x) - r_{n_h}(x)\sin(2\pi x), \\
h_i(x) & = & 2\pi\sin(2\pi x) - r_{n_h}(x)(1-\cos(2\pi x)), \\
r_{n_h}(x) & = & \displaystyle\sum_{l=-n_h}^{n_h}\frac{1}{x+l} = \frac{\psi'_{n_h}(x)}{\psi_{n_h}(x)}.
\end{array}$$

Second simplification: $\mathcal{M} \to \mathfrak{M}$

The second simplification consist in **eliminating the second summands** of the expressions of the components of $\mathcal{M}$ as given in the previous lemma.

In this way,

$$\begin{array}{rcl} \mathcal{M} & = & \mathcal{M}_D + \mathcal{M}_O \\ & & \downarrow \text{(remove the second summands)} \\ \mathfrak{M} & = & \mathfrak{M}_D + \mathfrak{M}_O \end{array}$$

Remark 1: The difference between the components of $\mathcal{M}$ and $\mathfrak{M}$ ($\widetilde{\mathcal{CS}}$ and $\widetilde{\mathfrak{cs}}$) can be bounded.

Remark 2: the expression for $\widetilde{\mathfrak{cs}}$ only depends on the difference $\nu - j$.

Bound for $\|\mathfrak{M}_D^{-1}\|_\infty$

Since $\mathfrak{M}_D$ is diagonal, to compute $\mathfrak{M}_D^{-1}$ we only need to invert its diagonal blocks $\mathfrak{B}_{i,i}$

$$\begin{pmatrix} A_i^c \partial \overline{\mathfrak{c}}^{n_h}_{\nu_i,N}(j_i) + A_i^s \partial \widetilde{\mathfrak{c}}^{n_h}_{\nu_i,N}(j_i) & \overline{\mathfrak{c}}^{n_h}_{\nu_i,N}(j_i) & \widetilde{\mathfrak{c}}^{n_h}_{\nu_i,N}(j_i) \\ A_i^c \partial \overline{\mathfrak{s}}^{n_h}_{\nu_i,N}(j_i) + A_i^s \partial \widetilde{\mathfrak{s}}^{n_h}_{\nu_i,N}(j_i) & \overline{\mathfrak{s}}^{n_h}_{\nu_i,N}(j_i) & \widetilde{\mathfrak{s}}^{n_h}_{\nu_i,N}(j_i) \\ A_i^c \partial \overline{\mathfrak{c}s}^{n_h}_{\nu_i,N}(j_i^+) + A_i^s \partial \widetilde{\mathfrak{c}s}^{n_h}_{\nu_i,N}(j_i^+) & \overline{\mathfrak{c}s}^{n_h}_{\nu_i,N}(j_i^+) & \widetilde{\mathfrak{c}s}^{n_h}_{\nu_i,N}(j_i^+) \end{pmatrix}.$$

For that, we first show that, if $(A_i^c, A_i^s) \neq (0,0)$, $\mathfrak{B}_{i,i}$ is invertible either setting $\mathfrak{c}s = \mathfrak{c}$ or $\mathfrak{c}s = \mathfrak{s}$.

The actual bounds of $\mathfrak{B}_{i,i}$ are computed **numerically** for each n_h, by first minimizing w.r.t. $\mathfrak{c}s = \mathfrak{c}, \mathfrak{s}$ and maximizing w.r.t. $\theta \in [0, 2\pi]$, $|\nu - j| \leq 1/2$ the supremum norm of

$$\begin{pmatrix} \partial \overline{\mathfrak{c}}^{n_h}_{\nu_i,N}(j_i) \cos\theta + \partial \widetilde{\mathfrak{c}}^{n_h}_{\nu_i,N}(j_i) \sin\theta & \overline{\mathfrak{c}}^{n_h}_{\nu_i,N}(j_i) & \widetilde{\mathfrak{c}}^{n_h}_{\nu_i,N}(j_i) \\ \partial \overline{\mathfrak{s}}^{n_h}_{\nu_i,N}(j_i) \cos\theta + \partial \widetilde{\mathfrak{s}}^{n_h}_{\nu_i,N}(j_i) \sin\theta & \overline{\mathfrak{s}}^{n_h}_{\nu_i,N}(j_i) & \widetilde{\mathfrak{s}}^{n_h}_{\nu_i,N}(j_i) \\ \partial \overline{\mathfrak{c}s}^{n_h}_{\nu_i,N}(j_i^+) \cos\theta + \partial \widetilde{\mathfrak{c}s}^{n_h}_{\nu_i,N}(j_i^+) \sin\theta & \overline{\mathfrak{c}s}^{n_h}_{\nu_i,N}(j_i^+) & \widetilde{\mathfrak{c}s}^{n_h}_{\nu_i,N}(j_i^+) \end{pmatrix}^{-1}.$$

We call G_{n_h} the obtained quantity. Some values are

n_h	0	1	2	3
G_{n_h}	4.84	8.83	13.3	17.7

In this way, we get

$$\|(\mathfrak{M}_D)^{-1}\| \leq \max(A_{min}^{-1}, 1) G_{n_h},$$

being

$$\begin{aligned} A_i &= ((A_i^c)^2 + (A_i^s)^2)^{1/2} \\ A_{min} &= \min_{i=1\div N_f} \{A_1, \dots, A_{N_f}\} \end{aligned}$$

Bounds for $\|M^{-1}\|_\infty$

The computation of the bound for $\|Dg(y)^{-1}\|$ is completed by following the following scheme:

$$\|\mathfrak{M}_D^{-1}\| \quad \leq \quad \max(A_{min}^{-1}, 1) G_{n_h}$$

$$\Downarrow$$

$$\|\mathcal{M}_D^{-1}\| \quad \leq \quad \frac{\|\mathfrak{M}_D^{-1}\|}{1 - \|\mathfrak{M}_D^{-1}\| \|\mathfrak{M}_D - \mathcal{M}_D\|}$$

$$\Downarrow$$

$$\|M_D^{-1}\| \quad \leq \quad \frac{\|\mathcal{M}_D^{-1}\|}{1 - \|\mathcal{M}_D^{-1}\| \|\mathcal{M}_D - M_D\|}$$

$$\Downarrow \|M_O\| \leq \|\mathfrak{M}_O\| + \|\mathfrak{M}_O - \mathcal{M}_O\| + \|\mathcal{M}_O - M_O\|$$

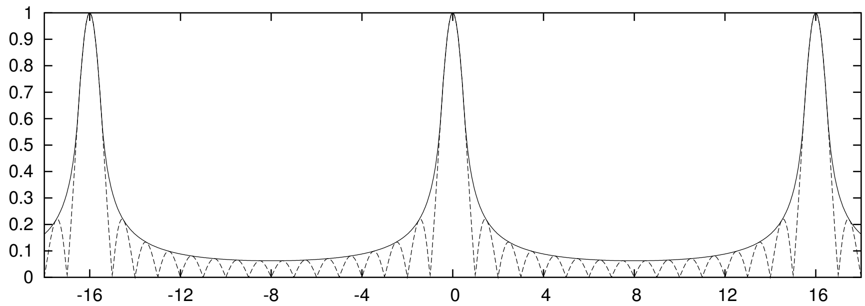
$$\|M^{-1}\| \quad \leq \quad \frac{\|M_D^{-1}\|}{1 - \|M_D^{-1}\| \|M_O\|}$$

Bound for $\|\Delta b\|_\infty$

We have

$$\|\Delta b\| \leq 2C \max_{j\in J} \sum_{|k|=r_0}^{\infty} e^{-\delta|k|}|\widetilde{h}_N^{n_h}(Tk\omega - j)|,$$

where $|\widetilde{h}_N^{n_h}|$ is the envelope displayed below ($N = 16$, $n_h = 0$).



The **Diophantine condition** gives a **lower bound for** $|Tk\omega - j|$:

$$|Tk\omega - j| \quad \geq \quad \frac{TD}{(|k| + |k_j|)^\tau} - 1.$$

For $|k|$ **small**, $|\widetilde{h}_N^{n_h}(Tk\omega - j)| \ll 1$.

After some order r_*, $|\widetilde{h}_N^{n_h}(Tk\omega - j)|$ may approach 1.

Therefore,

$$\|\Delta b\| \leq 2C(\max_{j\in J} \sum_{|k|=r_0}^{r_*-1} e^{-\delta|k|}|\widetilde{h}_N^{n_h}(Tk\omega - j)| + \max_{j\in J} \sum_{|k|=r_*}^{\infty} e^{-\delta|k|}),$$

where:

- The first term is bounded by **replacing the DFT by the TCFT**. This introduces an additional error term due to this approximation.
- All the sums are reduced to **sums of the form** $\sum_j j^\alpha e^{-\delta j}$, which are bounded by **incomplete Gamma functions**.

Final theorem

Under the stated hypothesis, the error in frequencies and amplitudes can be bounded as

$$\|\Delta y\| \lesssim \|M^{-1}\|\|\Delta b\|,$$

where

$$\|M^{-1}\| \leq \frac{\|M_D^{-1}\|}{1 - \|M_D^{-1}\|\|M_O\|}$$

and

$$\|M_O\| \leq \frac{(n_h!)^2}{\pi}\Bigg(\frac{\sqrt{2}(\sum_{l=1}^{N_f} A_l)(\pi + \ln(\frac{TD}{(2r_0-2)^\tau} - 1 + n_h) - \ln(\frac{TD}{(2r_0-2)^\tau} - 2 - n_h)) + 2N_f}{(\frac{TD}{(2r_0-2)^\tau} - 1 - n_h)^{1+2n_h}}$$

$$+ \frac{\sqrt{2}(\sum_{l=1}^{N_f} A_l)(\pi + \ln([\nu_{min}] + n_h) - \ln([\nu_{min}] - 1 - n_h)) + 2N_f}{([\nu_{min}] - n_h)^{1+2n_h}}$$

$$+ \frac{4\Big(\sqrt{2}(\sum_{l=1}^{N_f} A_l)(\pi + \ln(N - \Omega_0 + n_h) - \ln(N - \Omega_0 - 1 - n_h)) + 2N_f\Big)(1 + \frac{1}{2n_h})}{(N - \Omega_0 - n_h)^{1+2n_h}}\Bigg)$$

and

$$\|M_D^{-1}\| \leq \frac{\|\mathcal{M}_D^{-1}\|}{1 - \|\mathcal{M}_D^{-1}\|\varepsilon_1}, \quad \|\mathcal{M}_D^{-1}\| \leq \frac{\|\mathfrak{M}_D^{-1}\|}{1 - \|\mathfrak{M}_D^{-1}\|\varepsilon_2}, \quad \|\mathfrak{M}_D^{-1}\| \leq \frac{G_{n_h}}{\min(1, A_{min})},$$

being

$$\varepsilon_1 = \frac{4(n_h!)^2\Big(\sqrt{2}A_{max}(\pi + \ln(N - \Omega_0 + n_h) - \ln(N - \Omega_0 - 1 - n_h)) + 2\Big)(1 + \frac{1}{2n_h})}{\pi(N - \Omega_0 - n_h)^{1+2n_h}},$$

$$\varepsilon_2 = \frac{(n_h!)^2\Big(\sqrt{2}A_{max}(\pi + \ln(2[\nu_{min}] + n_h) - \ln(2[\nu_{min}] - n_h - 1)) + 2\Big)}{\pi(2[\nu_{min}] - n_h)^{1+2n_h}},$$

$$\Omega_0 = T(2r_0 - 2)\|\omega\|_\infty + 1,$$

Error estimates: Final theorem (cont.)

As for $\|\Delta b\|$,

$$
\begin{aligned}
\|\Delta b\| \;\le\; & \frac{2^{m+1}C}{(m-1)!}\Bigg(\\
& \chi_{\{r_*>r_0\}}\frac{(n_h!)^2 e^{\delta(r_0-1)}\sum_{l=0}^{m-1}\binom{m-1}{l}(\frac{m}{2}-r_0+1)^{m-1-l}G_f(2r_0-1, r_0+r_*-2, l+\tau(1+2n_h),\delta)}{E_*\pi(TD)^{1+2n_h}} \\
+ \;& \chi_{\{r_*>r_0\}}\frac{2(n_h!)^2 e^{\delta\frac{m}{2}}(1+\frac{1}{2n_h})G_f(r_0+\frac{m}{2}, r_*-1+\frac{m}{2}, m-1,\delta)}{\pi(N-\Omega-n_h)^{1+2n_h}} \\
+ \;& e^{\delta\frac{m}{2}}G_\infty(r_*+\frac{m}{2}, m-1,\delta)\Bigg),
\end{aligned}
$$

where

$$
\begin{aligned}
\Omega &= T(r_*+r_0-2)\|\omega\|_\infty+1 \\
r_* &= \max\Bigg(r_0, \min\Big(\Big[\Big(\frac{TD}{\max((\frac{(n_h!)^2}{\pi})^{\frac{1}{1+2n_h}}+1+n_h, 2(1+n_h))}\Big)^{\frac{1}{\tau}}-r_0+2\Big], \\
&\qquad \Big[\frac{N-1-n_h}{T\|\omega\|_\infty}-r_0+1\Big]\Big)\Bigg) \\
E_* &= \frac{(z_*-1-n_h)^{1+2n_h}}{z_*^{1+2n_h}}, \\
z_* &= \frac{TD}{(r_*+r_0-2)^\tau},
\end{aligned}
$$

In the above formulas, $\chi_{\{\mathit{condition}\}}$ equals 1 if *condition* is true and 0 otherwise.

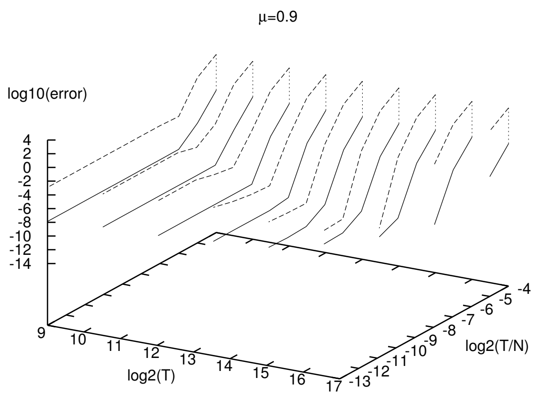
I.3 Applications
"Academic" example

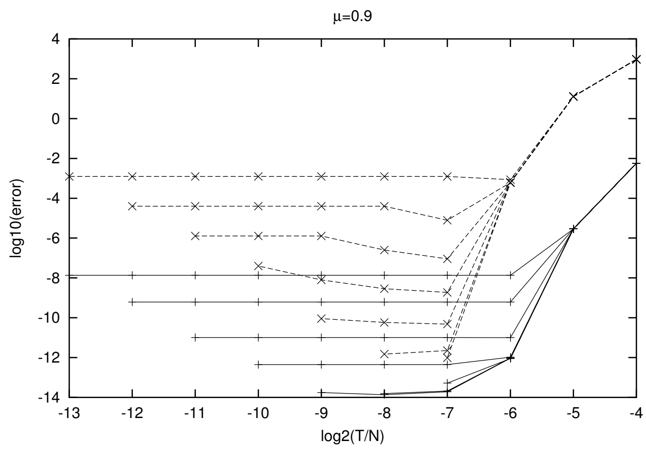
We consider the **quasi–periodic function**

$$f_{0.9}(t) = \frac{\sin(2\pi\omega_1 t + \varphi_1)}{1 - 0.9\cos(2\pi\omega_1 t + \varphi_1)} \cdot \frac{\sin(2\pi\omega_2 t + \varphi_2)}{1 - 0.9\cos(2\pi\omega_2 t + \varphi_2)}.$$

Explicit formulae for frequencies and amplitudes can be obtained, as well as the **Cauchy estimates** and the **Diophantine condition**.

We have performed **Fourier analysis** of this function for **several** T, N.



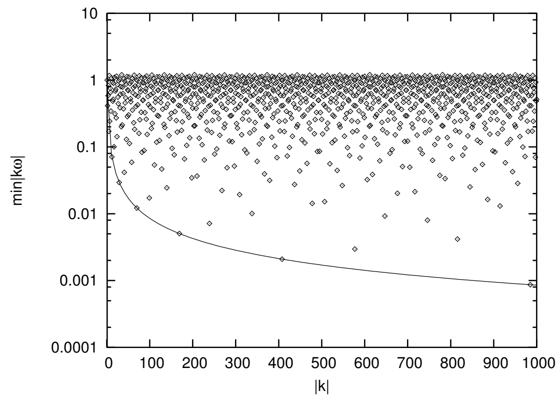


Solid line: actual error. Dashed line: estimated error.

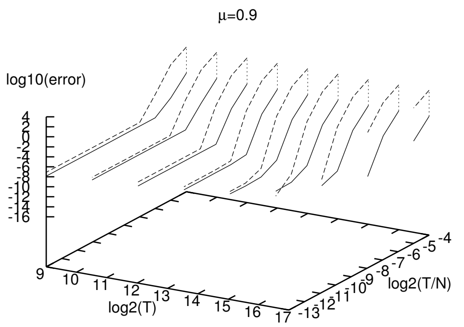
Numerical test of the bounds obtained: actual error vs. predicted error

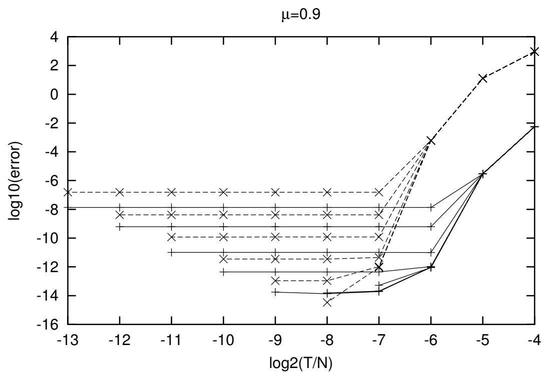
In the preceeding slide, the difference between the **error predicted** and the **actual error** is big because of the **Diophantine condition**, which is reached for for **very few orders** $|k|$.

The points in the plot below represent $\min_{|k|=\text{const.}} |k\omega|$ for $|k| = 1 \div 1000$. The curve represents the values of the Diophantine condition $0.85355/|k|$.



If we **substitute the first term of** $\|\Delta b\|$ by what is obtained just **before the use of the Diophantine condition**, the following figure is obtained.





Solid line: actual error. Dashed line: estimated error.