NORMAL FORMS:

THE CENTRE x CENTRE x SADDLE CASE.

AUTHOR: PAU ROLDÁN

WORKING SEMINAR IN CELESTIAL MECHANICS

June 19 th, 2009.

Normal Forms : centre x centre x saddle

- Phase space $\mathbb{R}^{2n} = \mathbb{R}^6$.
- H smooth Hamiltonian, equilibrium pt at zero.

<u>Definition</u> The eq. point is centre x centre x saddle if there exists a canonical syst. of coords. (x,y) in which the Hamiltonian takes the form

$$(1) \qquad H(x,y) = H_2(x,y) + H_P(x,y)$$

where $\quad H_2 = \frac{\omega_1}{2}(x_1^2+y_1^2) + \frac{\omega_2}{2}(x_2^2+y_2^2) + \lambda x_3 y_3$

and $\quad H_P$ has a zero of order 3 at the origin.

<u>Remark</u> Equations of motion. Linear approximation

$$\left.\begin{aligned} \dot{x}_1 &= \omega_1 y_1 + \frac{\partial H_P}{\partial y_1} \\ \dot{y}_1 &= -\omega_1 x_1 - \frac{\partial H_P}{\partial x_1} \end{aligned}\right\} \longrightarrow \quad \ddot{x}_1 = -\omega_1^2 x_1$$

$$\vdots$$

$$\left.\begin{aligned} \dot{x}_3 &= \lambda x_3 + \frac{\partial H_P}{\partial y_3} \\ \dot{y}_3 &= -\lambda y_3 - \frac{\partial H_P}{\partial x_3} \end{aligned}\right\} \longrightarrow \quad \begin{aligned} x_3(t) &= x_3^0 e^{\lambda t} \\ y_3(t) &= y_3^0 e^{-\lambda t} \end{aligned}$$

Definition (semi-simple NF)

A system with an eq. point of type $c \times c \times s$ is in (semi-simple) normal form up to order $r \geq 3$ if

$$H^{(r)} = H_2 + Z^{(r)} + \mathcal{R}^{(r)}$$

where

- $H_2 = \frac{\omega_1}{2}(x_1^2 + y_1^2) + \frac{\omega_2}{2}(x_2^2 + y_2^2) + \lambda x_3 y_3$.
- $Z^{(r)}$ is a polyn. of degree r: $\{H_2, Z^{(r)}\} \equiv 0$.
- $\mathcal{R}^{(r)}$ is a small reminder:

$$|\mathcal{R}^{(r)}(z)| \leq C_r \|z\|^{r+1} \qquad \forall z \in \mathcal{U}^{(r)}$$

small neighborhood of the origin.

__Theorem__ For any integer $r \geq 2$, there exists a neighbourhood $\mathcal{U}^{(r)}$ of the origin and a canonical transf.

$$\mathcal{T}_r : \mathbb{R}^6 \supset \mathcal{U}^{(r)} \longrightarrow \mathbb{R}^6$$

which puts the system (1) in NF up to order r

$$H^{(r)} := H \circ \mathcal{T}_r = H_2 + Z^{(r)} + \mathcal{R}^{(r)} .$$

Moreover, $\mathcal{T}_r$ (and $\mathcal{T}_r^{-1}$) is close to the identity

$$|z - \mathcal{T}_r(z)| \leq C_r \|z\|^2 \qquad \forall z \in \mathcal{U}^{(r)} .$$

If the frequencies ω_1 and ω_2 are non-resonant to order r, the function $Z^{(r)}$ depends only on the basic invariants

$$I_1 = \frac{x_1^2 + y_1^2}{2} , \quad I_2 = \frac{x_2^2 + y_2^2}{2} \quad - \text{actions}$$

$$I_3 = x_3 y_3 .$$

Proof. (Lie transform method)

- Idea: construct a canonical transf. T_r that puts the system in a form that is as simple as possible.

More precisely, construct a canonical transf. ϕ_3 that normalizes terms of order 3, followed by a canonical transf. ϕ_4 that normalizes order 4, etc.

- Each transf. ϕ_j is constructed as the time-1 flow of a suitable auxiliary Hamiltonian G_j (generating Hamiltonian).

Definition $\mathcal{H}_j$ – the set of real-valued homogeneous polynomials of degree j.

Remark

$$f \in \mathcal{H}_i,\ g \in \mathcal{H}_j \implies \{f, g\} \in \mathcal{H}_{i+j-2}.$$

Lie transform

Given $G \in \mathcal{H}_j$, let $\dot{z} = X_G(z)$ be the associated Hamiltonian eq, and define

$$\phi := \phi^t\big|_{t=1} \quad \text{the time-1 map}$$

(the Lie transform generated by G).

It is well-known that ϕ is canonical.

Lemma Let H be a polynomial, and $G \in \mathcal{H}_j$ $(j \geq 3)$, ϕ as before.

The Taylor expansion of $H \circ \phi$ is

$$K := H \circ \phi = H + \{H, G\} + \frac{1}{2!}\{\{H, G\}, G\} + \cdots$$

Suppose that our Hamiltonian H has already been normalized up to order $j-1$, and we hit it with the Lie transform ϕ_j generated by $G_j \in \mathcal{H}_j$ (unknown).

The new Hamiltonian will temporarily be called

$$K = K_2 + K_3 + K_4 + \cdots, \quad K_i \in \mathcal{H}_i.$$

Then we have

$$K_i = H_i \qquad , i<j$$

$$\boxed{K_j = H_j + \{H_2, G_j\}} \quad , i=j$$

$$K_i = H_i + \{\ldots\} + \{\{\ldots\}\} + \cdots \quad , \quad i>j.$$

Homological equation

We want to construct G_j such that

$$K_j = H_j + \{H_2, G_j\} \iff \{G_j, H_2\} = H_j - K_j .$$

is as simple as possible.

Definition (Homological operator)

$$\mathcal{L} : \mathcal{H}_j \to \mathcal{H}_j$$
$$G \to \mathcal{L}G := \{G, H_2\} .$$

We want to solve

$$\mathcal{L} G_j = H_j - K_j$$

with K_j as simple as possible.

- For instance, if $H_j \in \operatorname{im} \mathcal{L}$ then we can find G_j such that $\mathcal{L}G_j = H_j \Rightarrow K_j = 0$:)

- In general $H_j \notin \mathrm{im}\,\mathcal{L}$, but given a complement $\mathcal{N}_j$ to the space ~~$\mathcal{H}_j$~~ im $\mathcal{L}$,

$$\mathcal{H}_j = \mathrm{im}\,\mathcal{L} \oplus \mathcal{N}_j ,$$

we can make K_j lie in this complement:

$$\begin{array}{ccccc} \mathcal{L}G_j & = & H_j & - & K_j \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \mathrm{im}\,\mathcal{L} & & \mathcal{H}_j & & \mathcal{N}_j . \end{array}$$

- $\mathcal{N}_j$ is the <u>normal form space</u>.
- It is easy to see that the homological op. $\mathcal{L}$ diagonalizes in any one of the spaces $\mathcal{H}_j$.

To this end, introduce complex variables

$$q_1 = \frac{1}{\sqrt{2}}(x_1 + i y_1), \quad p_1 = \frac{1}{\sqrt{2}}(x_1 - i y_1)$$

$$q_2 = \frac{1}{\sqrt{2}}(x_2 + i y_2), \quad p_2 = \frac{1}{\sqrt{2}}(x_2 - i y_2)$$

$$q_3 = x_3, \quad p_3 = y_3.$$

<u>Remark</u> This change is not the same as [Jorba].
The <u>symplectic form</u> is <u>modified</u> to

$$i(dq_1 \wedge dp_1 + dq_2 \wedge dp_2) + dq_3 \wedge dp_3.$$

· In these complex variables, the invariants become

$$I_1 = \frac{x_1^2 + y_1^2}{2} = q_1 p_1, \quad I_2 = \frac{x_2^2 + y_2^2}{2} = q_2 p_2, \quad \in \mathbb{R}.$$

$$I_3 = x_3 y_3 = q_3 p_3,$$

and

$$H_2(q,p) = \omega_1 q_1 p_1 + \omega_2 q_2 p_2 + \lambda q_3 p_3.$$

· <u>Reality condition</u>: $H(q,p) = \sum h_{\alpha\beta}\, q^\alpha p^\beta$

$$h_{\alpha\beta} = \overline{h_{\beta\alpha}}.$$

Lemma

$$\mathcal{Z}\, q^{l} p^{m} = \left[i(l_1 - m_1)\omega_1 + i(l_2 - m_2)\omega_2 + (l_3 - m_3)\lambda \right] \cdot q^{l} p^{m} .$$

Corollary The homological op. $\mathcal{Z}$ is semisimple $\Longrightarrow$

$$\mathcal{H}_j = \operatorname{im} \mathcal{Z} \oplus \ker \mathcal{Z} .$$

Hence we can choose

$$\boxed{\mathcal{N}_j = \ker \mathcal{Z}}$$

as the normal form spaces. $\square$.

- Notice that

$$K_j \in \ker \mathcal{Z} \iff$$
$$\mathcal{Z} K_j = 0 \iff$$
$$\{K_j, H_2\} = 0 .$$

Corollary 2. (Description problem)

The space $\ker \mathcal{Z}$ is spanned by the monomials $q^{\ell} p^{m}$ such that

$$(\ell_1 - m_1)\omega_1 + (\ell_2 - m_2)\omega_2 = 0 \quad \underline{\text{and}}$$

$$\ell_3 = m_3 . \quad \square .$$

• Thus if the frequencies ω_1, ω_2 are nonresonant to degree r,

$$k_1\omega_1 + k_2\omega_2 \neq 0 \quad \forall (k_1, k_2) \in \mathbb{Z}^2, \quad 0 < |k_1 + k_2| \leq r,$$

the normal form space is spanned by the monomials

$$(q_1 p_1)^{\ell_1} (q_2 p_2)^{\ell_2} (q_3 p_3)^{\ell_3} \implies$$

the normal form is a polynomial of deg. r that depends only on I_1, I_2, I_3.

Remark The ~~space~~ normal form space is a ring of invariants

$$\frac{d}{dt} f(e^{At}) = 0 \iff \{f, H_2\} = 0 .$$

In the litterature there are other algorithms to compute the NF, see e.g. the book [Murdock].

① 'Direct' algoritms: Consider the system of diff. eqs.

$$\dot{x} = X_H(x) = Ax + a_2(x) + a_3(x) + \cdots, \quad a_j \in \mathcal{V}_j^{2n}$$

and construct the canonical transformations ϕ_j directly.

- Homological operator

$$L : \mathcal{V}_j^{2n} \longrightarrow \mathcal{V}_j^{2n}$$

$$v(x) \longmapsto v'(x)Ax - Av(x)$$

- $\mathcal{N}_j = \ker L$ is a module over the ring of invariants:

$$\dot{x} = Ax + \underbrace{\varphi_1(I_1, I_2, I_3)}_{\text{real invariant}} v_1 + \varphi_2(I_1, I_2, I_3) v_2 + \cdots + \varphi_6(\quad) v_6 .$$

basic equivariants.

② 'Recursive' algorithms [Deprit, Giorgilli-Galgani, etc.]

Construct only one canonical transf. $\mathcal{T}_r$ in a recursive way.

Spatial circular RTBP

- System centered at libration point L_1:

$$H = \frac{1}{2}(p_x^2+p_y^2+p_z^2) + y p_x - x p_y - \sum_{n\geq 2} c_n(\mu)\rho^n P_n\left(\frac{x}{\rho}\right).$$

One can find symplectic coords. in which

$$H = H_2 + H_P, \quad \text{with}$$

$$H_2 = \frac{\omega_1}{2}(x_1^2+y_1^2) + \frac{\omega_2}{2}(x_2^2+y_2^2) + \lambda x_3 y_3.$$

By our theorem,

- there exists a neighb. $\mathcal{U}^{(r)}$ and a canonical tr. $\mathcal{T}_r$ which puts the system in NF of order r:

$$H^{(r)} = H \circ \mathcal{T}_r = H_2 + Z^{(r)} + \mathcal{R}^{(r)}.$$

- If the freqs ω_1, ω_2 are non-resonant to order r (this is checked numerically), the func. $Z^{(r)}$ depends only on I_1, I_2, I_3.

Geometrical structures in NF's [Murdock, ch. 5].

- What does the NF tell us about dynamics?

Consider $\dot{x} = a(x)$ – full system

$\dot{x} = \hat{a}(x)$ – truncated system in NF.

Theorem Assume $\dot{x} = \hat{a}(x)$ is in truncated NF.
Then the linear subspaces

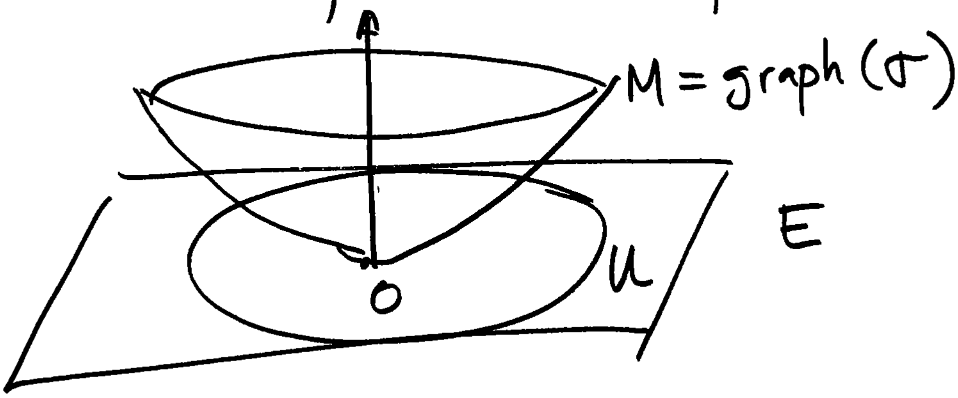
$$E^s, E^u, E^c, E^{cs}, E^{cu}, E^{ss}, E^{su}$$

defined in the usual way are invariant by the flow of the truncated system.

Theorem. Assume $\dot{x} = \hat{a}(x)$ is in truncated NF.
Then the stable/unstable fibrations of E^{cs}, E^{cu} are preserved by the flow of the truncaded syst.

- Other structures: periodic orbits, inv. tori.

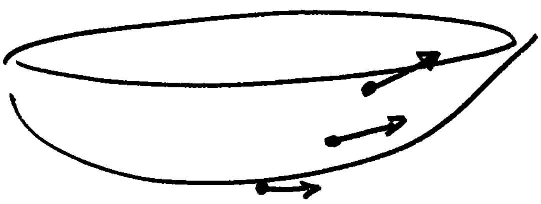
Definition A <u>local manifold</u> $M \subset \mathbb{R}^n$ is a <u>graph</u> over a linear subspace $E \subset \mathbb{R}^n$ if:



- M has k-th order contact with E if

$$\sigma(0) = D\sigma(0) = \cdots = D^k\sigma(0) = 0 .$$

- M is invariant under a flow if

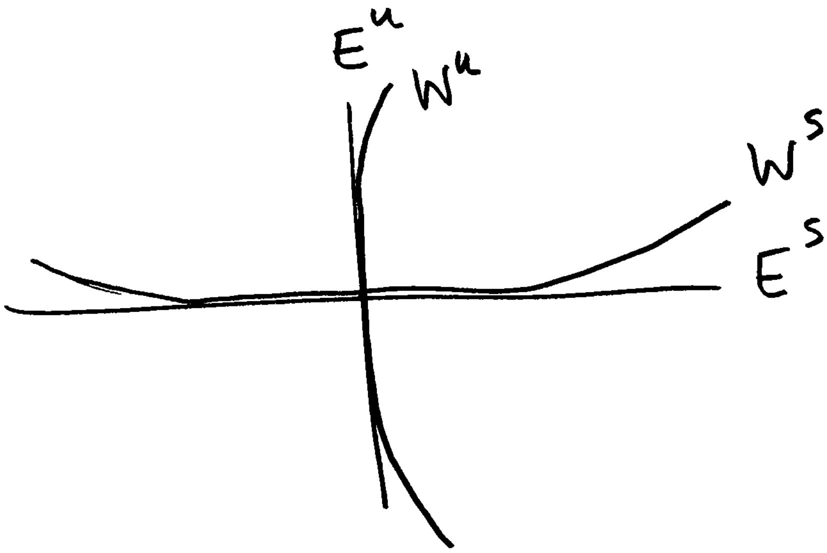


Theorem (Stable, unstable, center, etc. theorem)

In the full system $\dot{x} = a(x)$,

- There exist unique smooth local invariant stable/unstable manifolds W^s, W^u, expressible as graphs over E^s, E^u, having **k-th order contact** with E^s, E^u.

- For each k: $r \le k < \infty$, there exists a (not necessarily unique) local invariant C^k center manifold W^c, expressible as a graph over E^c, having **k-th order contact** with E^c.



Remark

The usual versions of the st, unst, centre theorem do not assume that the system is in NF.
In this case, the local manifolds have only 1st order contact (tangent) with E^s, E^u, E^c.

Remark

Computing NF to degree r automatically computes approximations to these manifolds: