Continuous averaging

Heuristic arguments:

Let $\hat{a}(z)$ be an analytic vector field and consider

$$\dot{z} = \hat{a}(z) \qquad z \in M \quad (m \text{ dim.})$$

We want a c.o.v. $z \longmapsto w(z, \hat{\delta})$; $\hat{\delta} > 0$

defined as a shift along solutions of

$$\frac{\partial w}{\partial \delta} = F(w, \delta) \qquad w(z, 0) = z$$
$$0 \leq \delta \leq \hat{\delta}$$

thus we get

$$\dot{w} = a(w, \delta)$$

and we want an easy way to get a simpler "a"

$$\partial a / \partial \delta = -[F, a]$$

where $[v_1, v_2] = \partial_{v_2} v_1 - \partial_{v_1} v_2$.

If the system we are considering is non-autonomous,

$$\frac{\partial a}{\partial \delta} = \frac{\partial F}{\partial t} - [F, a]$$

F is "more or less" arbitrary, but a good choice (Trescher's choice) is $F = \Sigma a$ with Σ linear (operator).

As a particular case, and the one he considers, suppose $\hat{a} = \varepsilon \hat{v}$, $a = \varepsilon v$. with

$$v(z, t, \varepsilon, \delta) = v^0 + \sum_{k>0} v^k(z, \varepsilon, \delta) e^{ikt} + \sum_{k<0} v^k($$

$$= v^0 + v^+ + v^-$$

Then we could ~~take~~ choose $\Sigma v = i(v^+ - v^-)$

Why??

③ (2)

$$V_\delta = (\xi V)_t - \varepsilon [\xi v, \cancel{v}] \qquad v(z,t,\varepsilon,0) = \hat{v}(z,t,$$

Expanding as Fourier series we get:

$$v^k_\delta = -|k| v^k + i\varepsilon \, \mathrm{sgn}(k) [v^0, v^k] + \varepsilon [\xi v, v^k] \quad k \in$$

$$v^k(z,\varepsilon,0) = \hat{v}^k(z,\varepsilon) \quad k \in \mathbb{Z} \qquad (\mathrm{sgn}(0) = 0)$$

if we neglect ~~ε-order~~ $\mathcal{O}(\varepsilon)$ terms and the initial system as initial condition, we get

$$v^k(z,\varepsilon,\delta) = e^{-|k|\delta} \hat{v}^k(z)$$

Now if δ is increasing, high-freq. terms decrease.

If we just neglect $\varepsilon[\xi v, v^k]$, we can also solve,

$$v^k(z,\varepsilon,\delta) = e^{-|k|\delta} \hat{V}^k \circ g^{i\varepsilon\delta \, \mathrm{sgn}(k)}(z)$$

$g^{(\cdot)}$ flow of $\dot{z} = v^0(z)$.

Here we have to work with complex time (that's why we asked for analyticity).

δ can be as big as $\delta \sim \alpha/\varepsilon$ with

$\alpha = \mathrm{Im}$ (closest singularity of g^t to the real axis)

Trexler's theorem:

Under some non-singular, non-degenerate conditions, there is a c. of v. $f(z,t,\varepsilon)$ $\mathcal{C}^\infty$ in ε

s.t.:

(i) $f(z,t,\varepsilon) = z + \mathcal{O}(\varepsilon)$

(ii) $\varepsilon \hat{v} \xmapsto{f} \varepsilon \hat{v}^0(z) + \varepsilon^2 \hat{v}_*(z,\varepsilon) + \tilde{v}(z,t,\varepsilon)$

and there is a $C_0 > 0$ s.t. "$t \in \Sigma_\rho$, $z \in V$"

$|\tilde{v}(z,t,\varepsilon)| \leq C_0\, e^{-\alpha/\varepsilon}$

where α is related to an analyticity domain

Some ideas of the proof

The "heuristic" part required ~~the~~ dropping of terms of high order.

For the complete proof:

1. Expand as Fourier series
2. You get an infinite system of coupled pde's
3. Use the method of majorants to prove existence of a solution for the <u>whole</u> system

3'. MOREOVER, you should be able to sum the terms you get

4. These bounds need to be very sharp, in very concrete domains

5. Finally you compare the final system with the original system.

Trescher's averaging and splitting of separatrices

Trescher studied the case of

$$\dot{x} = \frac{\partial \hat{H}}{\partial y} \quad , \quad \dot{y} = -\frac{\partial \hat{H}}{\partial x}$$

$$\hat{H} = \hat{H}^{0}(x,y) + e^{it/\varepsilon}\, \hat{H}^{(+1)}(x,y) + e^{-it/\varepsilon}\, \hat{H}^{(-1)}(x,y)$$

$$\hat{H}^{0} = y^2/2 - \cos x - 1 \qquad \hat{H}^{\pm 1} = B^{\pm}_{+} e^{ix} + B^{\pm}_{-} e^{-ix}$$

$$\left(\overline{B^{+}_{+}} = B^{-}_{-} \quad , \quad \overline{B^{+}_{-}} = B^{-}_{+} \right)$$

i.e. pendulum with rapidly osc. susp. point

System close to

$$\dot{x} = \frac{\partial \hat{H}_0}{\partial y} \qquad \text{for} \quad \varepsilon << 1$$
$$\dot{y} = -\frac{\partial \hat{H}_0}{\partial x}$$

PENDULUM.

Applying (i.e. solving the systems of pde's) the averaging method we get

$$\mathcal{A} = 4\sigma\varepsilon\, |P^{+}(\varepsilon)| + \mathcal{O}(\sigma^2)$$

where $P^{+}(0) \neq 0$, P^{+} is a "Melnikov integral" and

$$\sigma \approx e^{-\pi/2\varepsilon}$$

References : (D.V. Trexhov) (Google, preprints in his wpage)

* Sp. of exp. for a pendulum w. rap. osc. susp. point (preprint in English) (Russ. J. M.P. 1998

* Cont. av. in Ham. systems (S'Agaró, 1995)

* The method of c.a. in the problem of sep. of fast & slow motions (1992)