Luca Biasco

Università *ROMA TRE*

Low-order resonances in weakly dissipative spin–orbit models

joint work with Luigi Chierchia – Università *Roma Tre*

download: http://www.mat.uniroma3.it/users/chierchia

December 1, 2008

Physical motivation:

Physical motivation: Spin-orbit resonances

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A familiar example:

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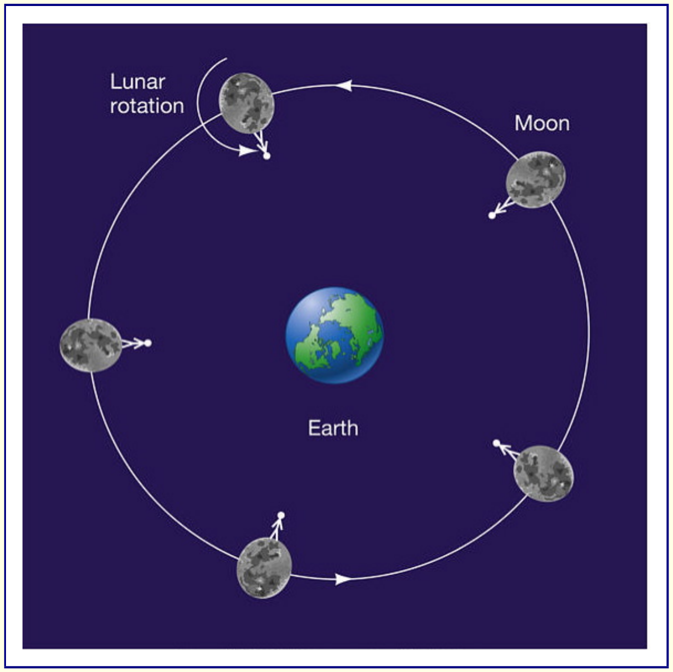

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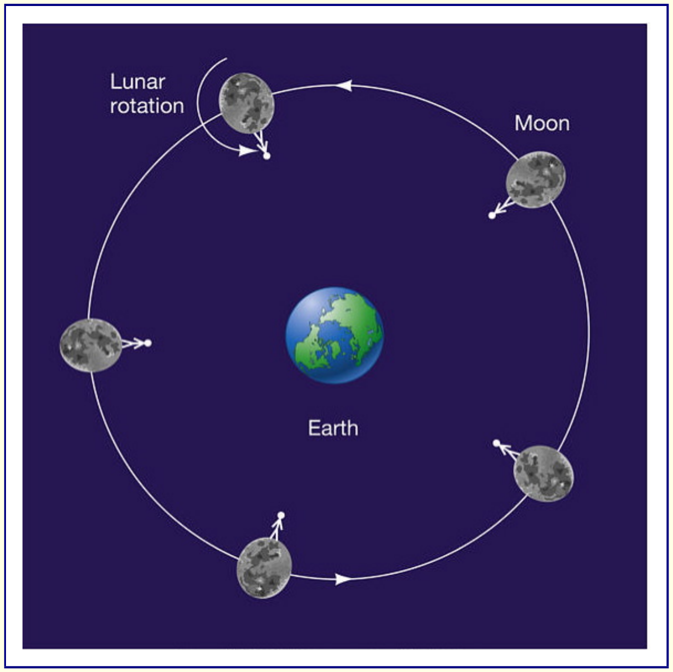

☞ Besides our Moon, in the Solar system, there are 22 satellites in 1-1 spin-orbit resonance:

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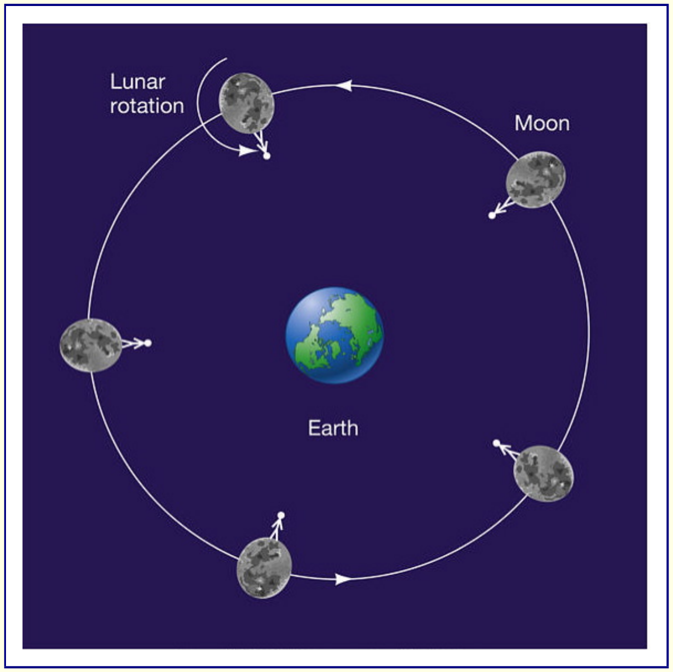

☞ Besides our Moon, in the Solar system, there are 22 satellites in 1-1 spin-orbit resonance: Phobos, Deimos [Mars]; Io, Europa, Ganymede, Callisto, Amalthea [Jupiter]; Mimas, Enceladus, Tethys, Dione, Rhea, Titan, Iapetus, Janus, Epimetheus [Saturn]; Ariel, Umbriel, Titania, Oberon, Miranda [Uranus;] Charon [Pluto].

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- ☞ There is only one more body in spin–orbit resonance:

  Mercury

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Mercury observed in a 3:2 resonance

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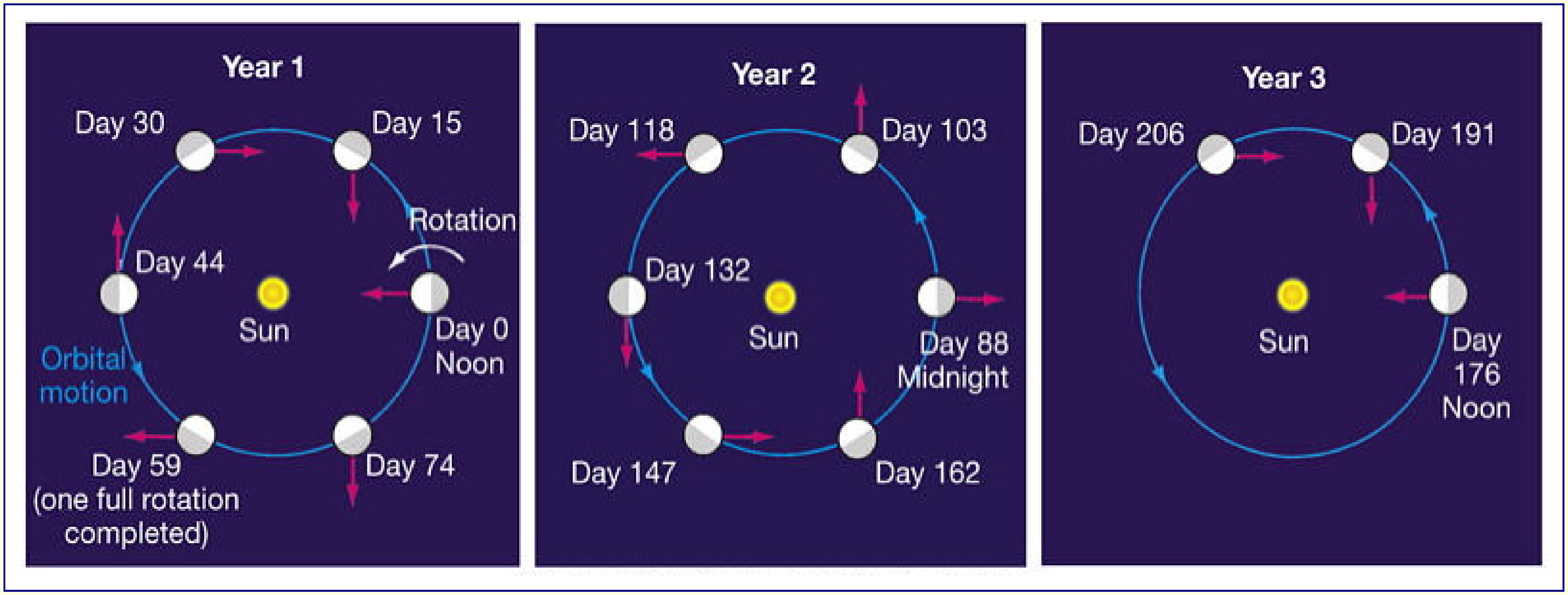

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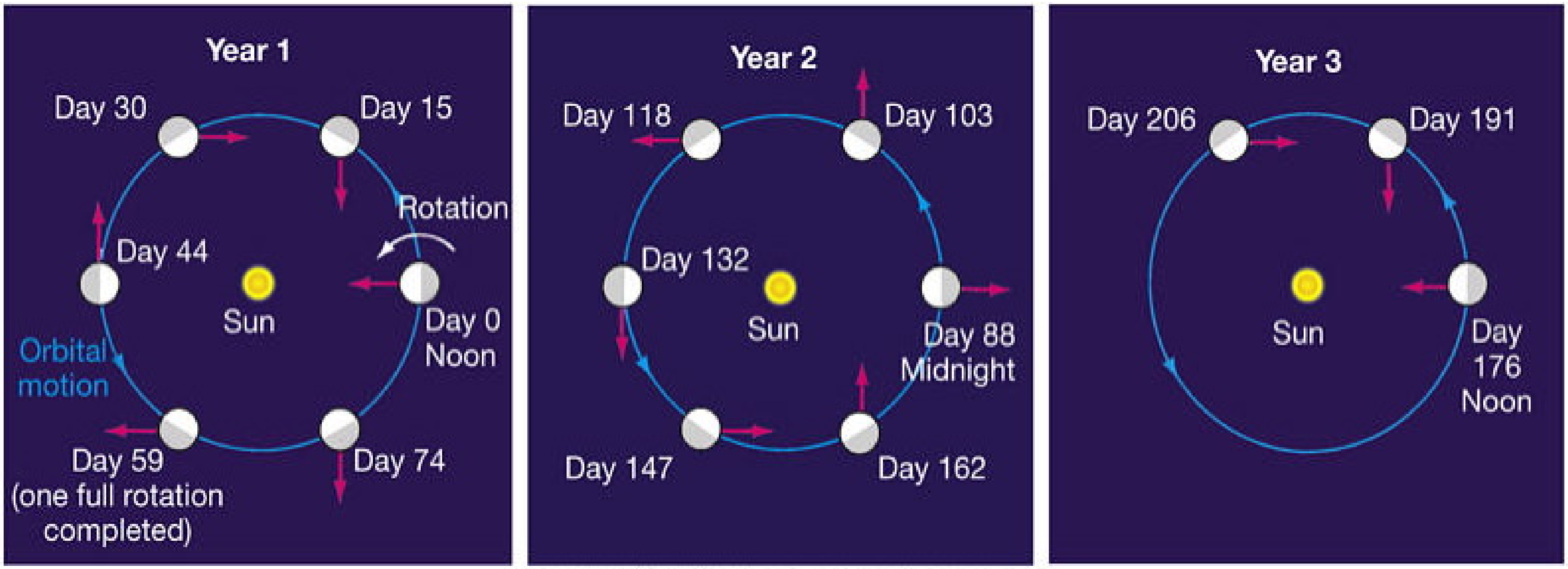

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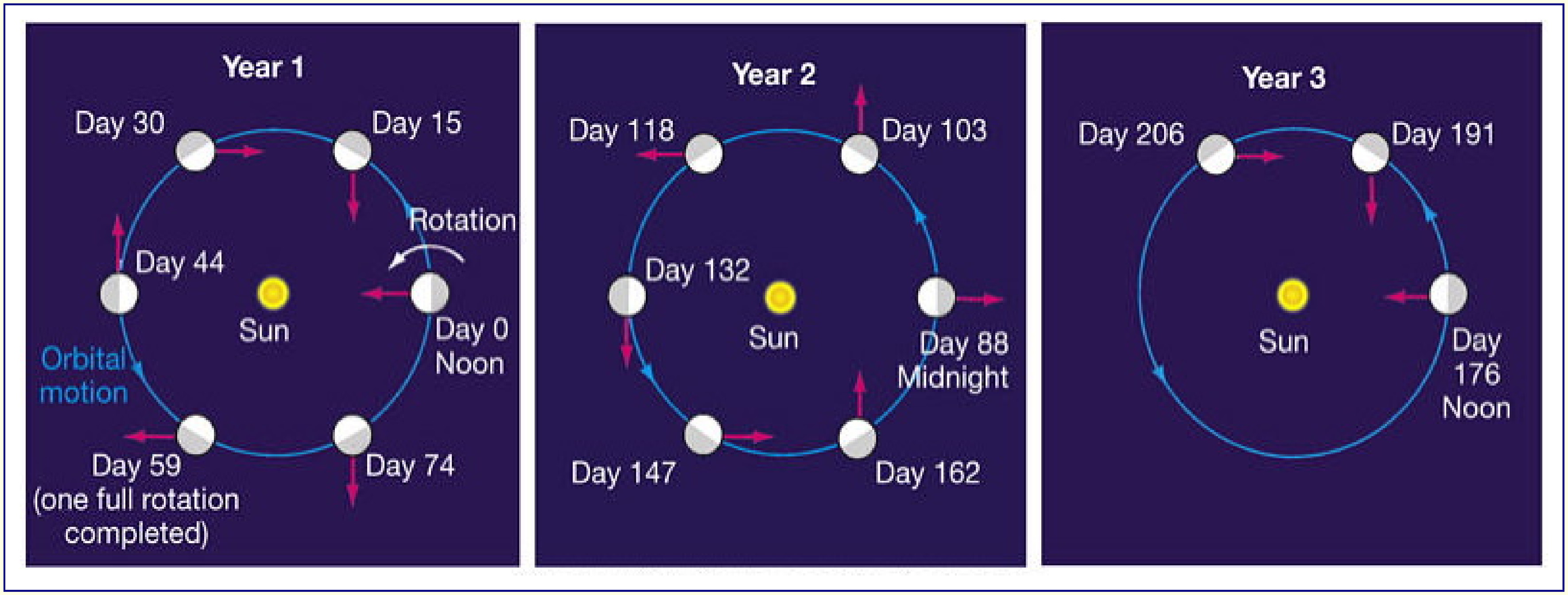

☞ *Eccentricities for the 1:1 resonances:* 0.0001 (Thetis),

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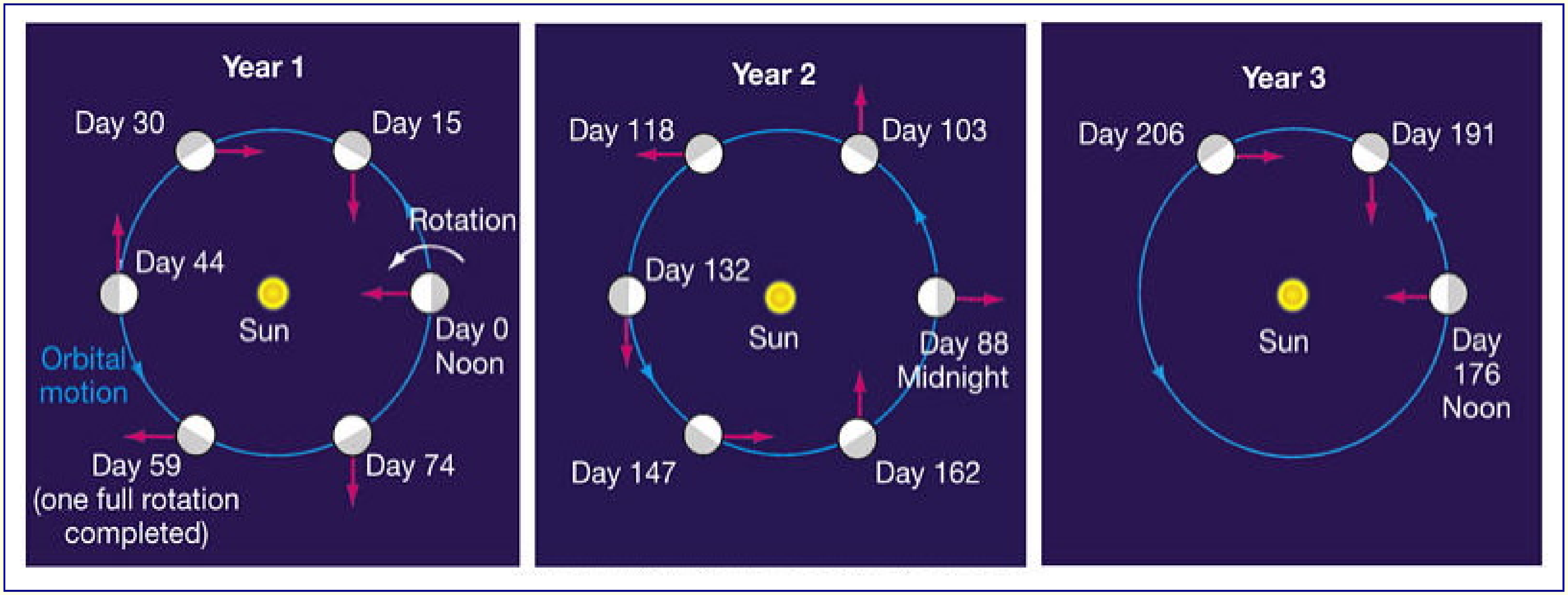

☞ *Eccentricities for the 1:1 resonances:* 0.0001 (Thetis), 0.0002 (Deimos),

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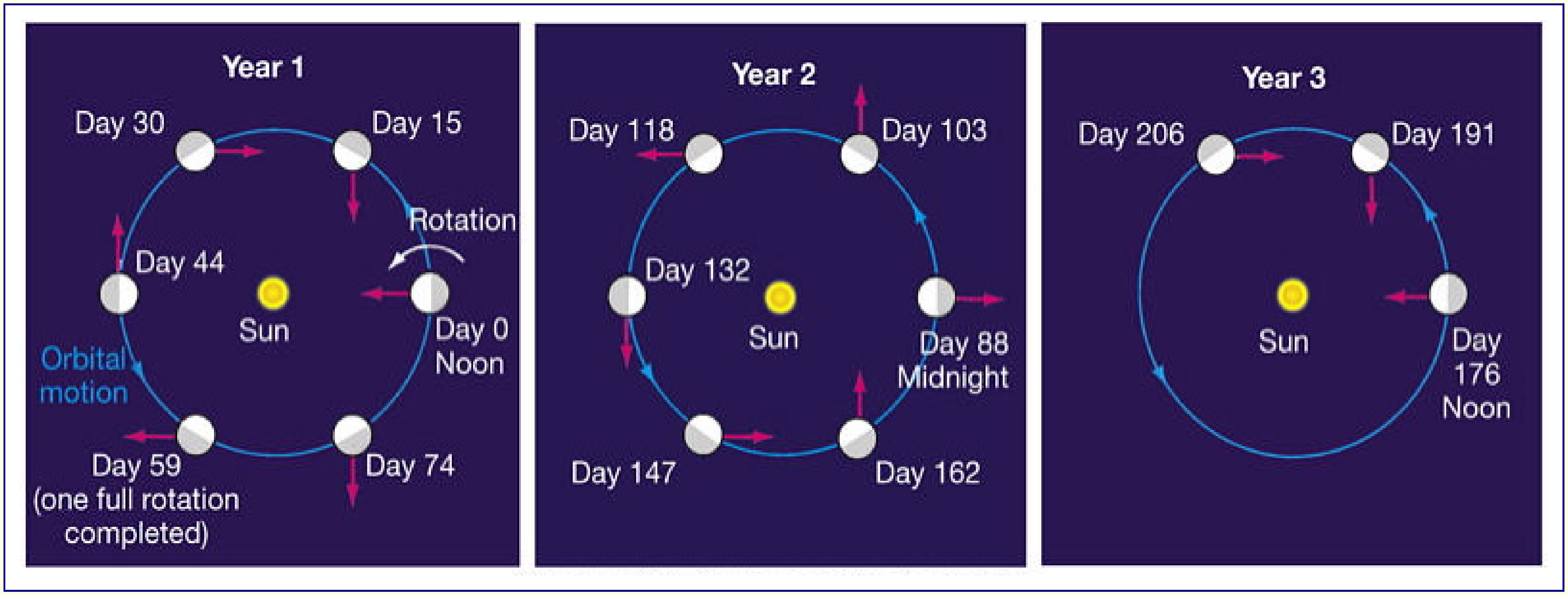

☞ *Eccentricities for the 1:1 resonances:* 0.0001 (Thetis), 0.0002 (Deimos), ... 0.0288 (Titan),

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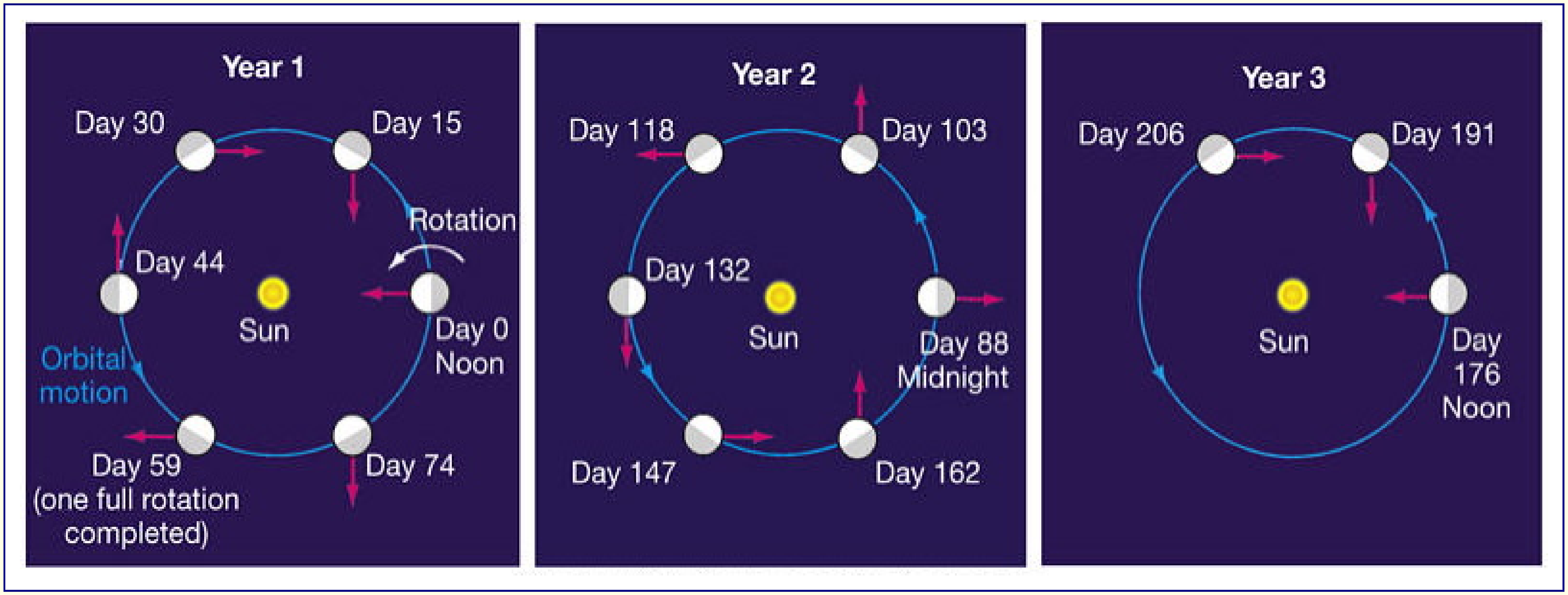

☞ *Eccentricities for the 1:1 resonances:* 0.0001 (Thetis), 0.0002 (Deimos), ... 0.0288 (Titan), 0.0554 (Moon);

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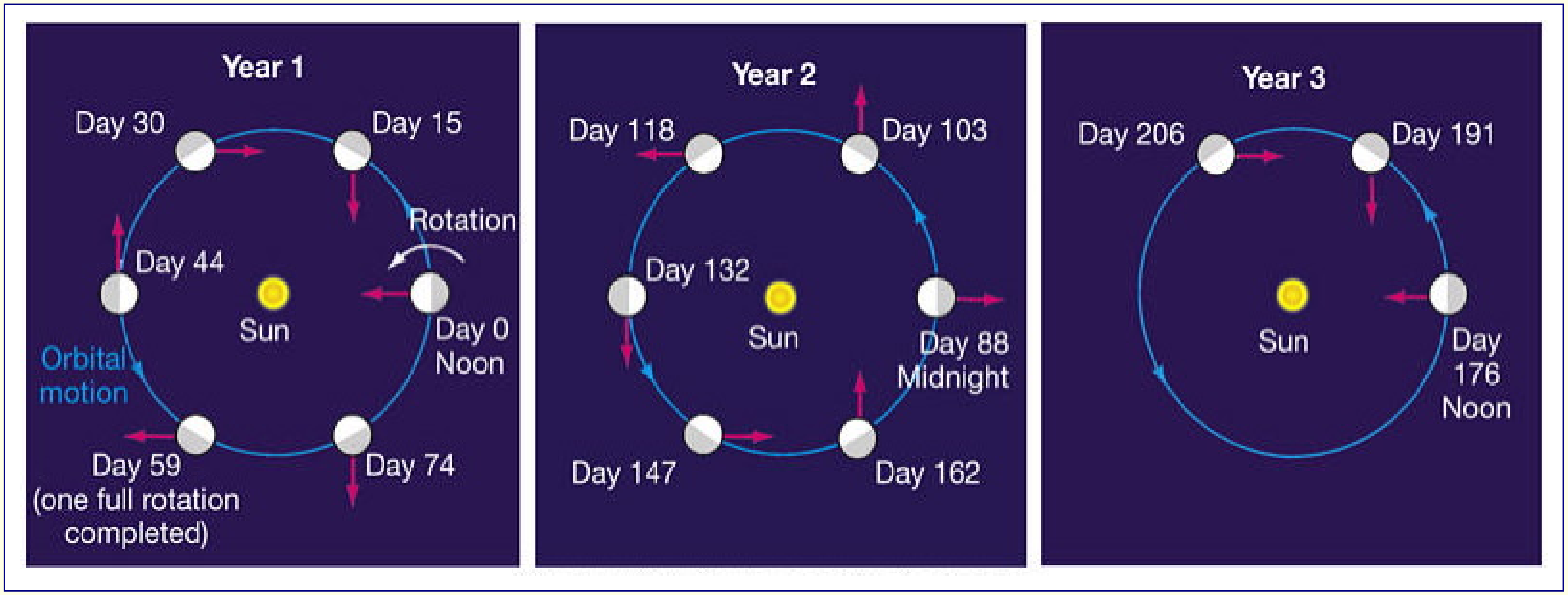

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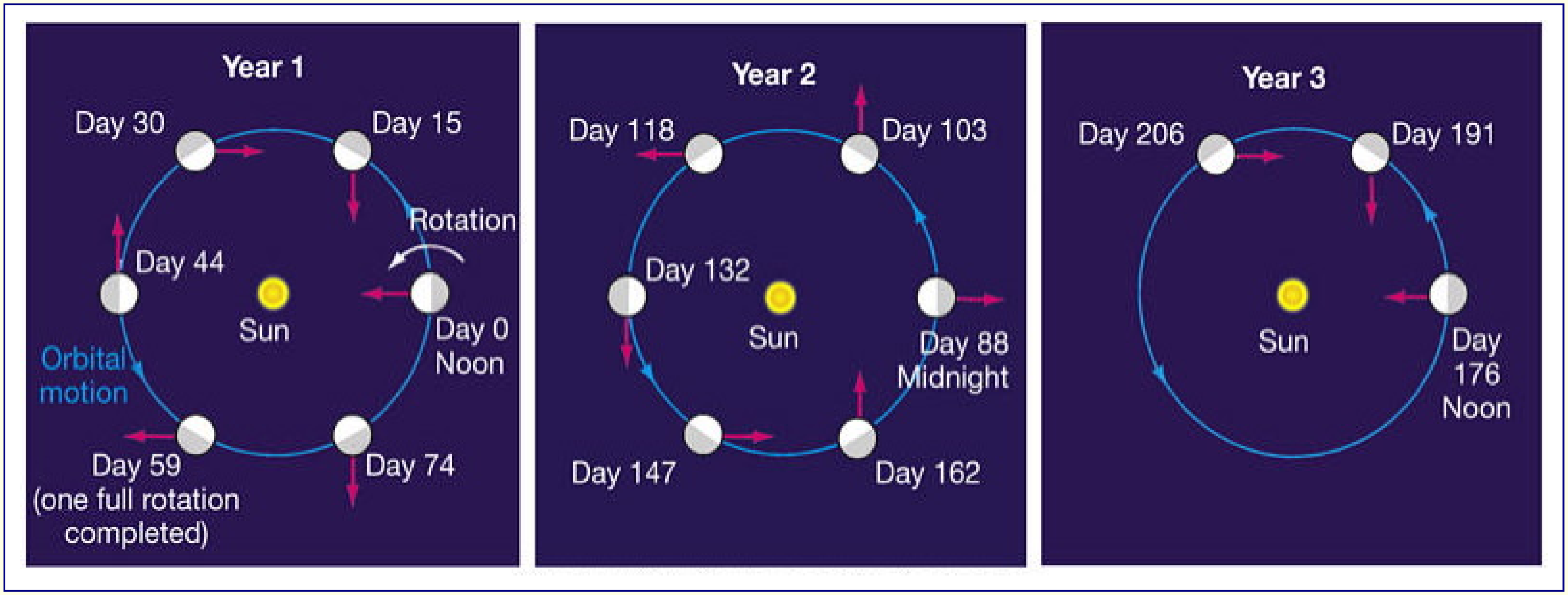

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☞ Provide a mathematical model for the spin–orbit problem and discuss a dynamical system approach

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(4) Basins of attraction of periodic attractors

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✍ (1)+(2) by A. Celletti and L. Chierchia, using Nash–Moser (KAM)

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✍ (3)+(4) by LB and L. Chierchia, using Lyapunov–Schmidt decomposition.

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Mathematical model

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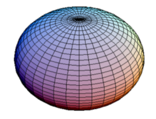
☞ The satellite/planet is a triaxial ellipsoid 

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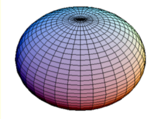
☞ The satellite/planet is a triaxial nearly–rigid ellipsoid 

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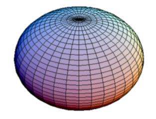
- ☞ The satellite/planet is a triaxial nearly–rigid ellipsoid 
- ☞ The satellite center of mass revolves on a given Keplerian ellipse

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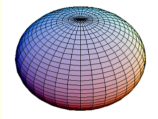
- The satellite/planet is a triaxial nearly–rigid ellipsoid 
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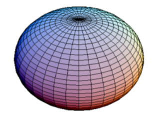
- ☞ The satellite/planet is a triaxial nearly–rigid ellipsoid 
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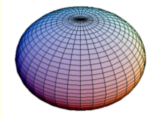
- The satellite/planet is a triaxial nearly–rigid ellipsoid 
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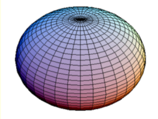
- ☞ The satellite/planet is a triaxial nearly–rigid ellipsoid 
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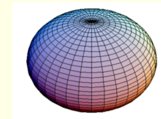
☞ The satellite/planet is a triaxial nearly–rigid ellipsoid 

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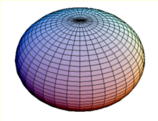
☞ The satellite/planet is a triaxial nearly–rigid ellipsoid 

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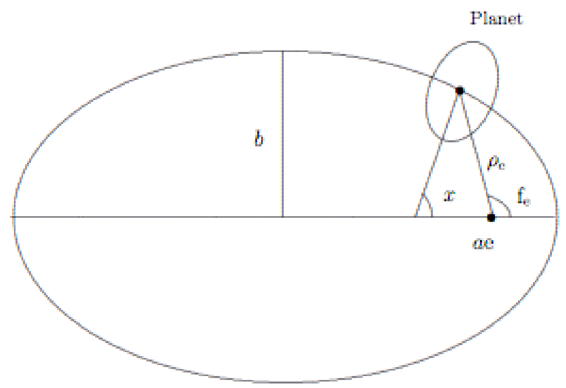

Equations of motion

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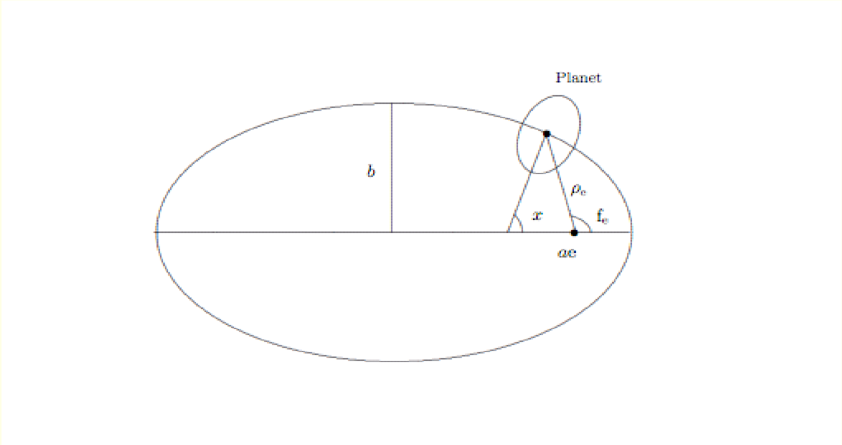
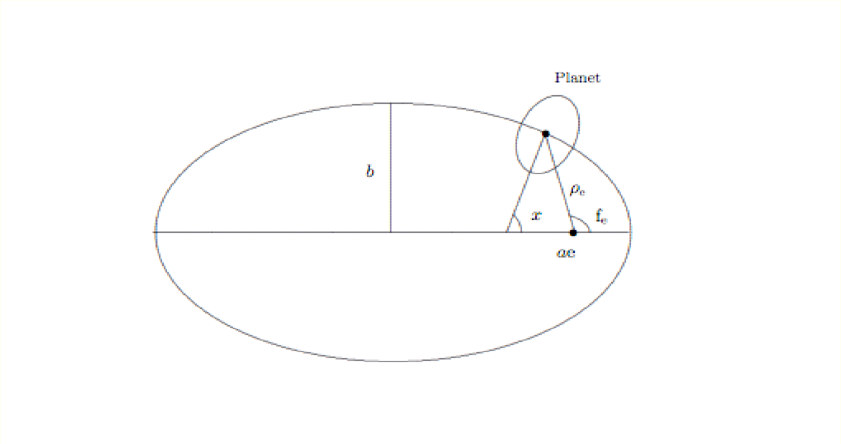


Equations of motion

☞ $$\ddot{x} + \eta(\dot{x} - \upsilon) + \varepsilon V_x(x,t) = 0$$



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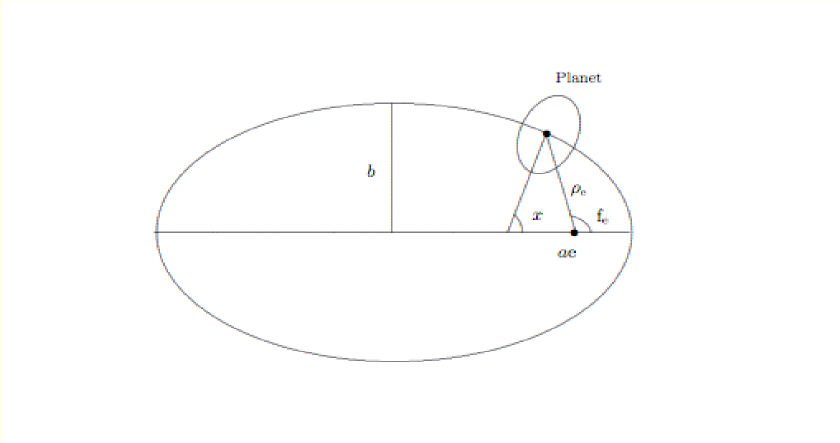

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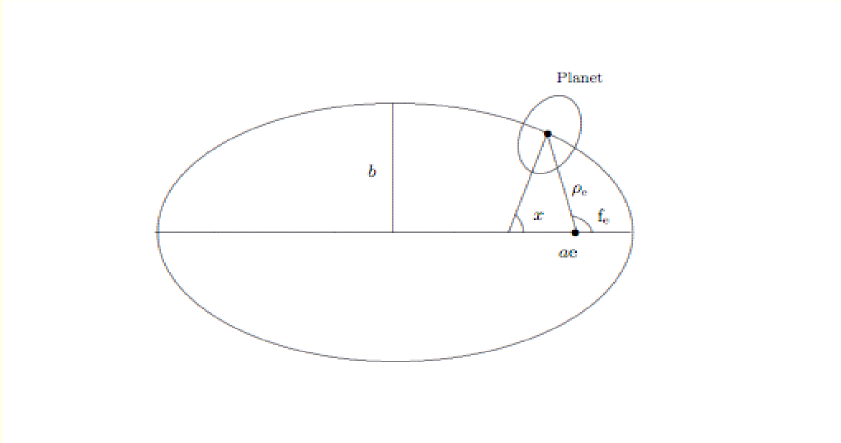
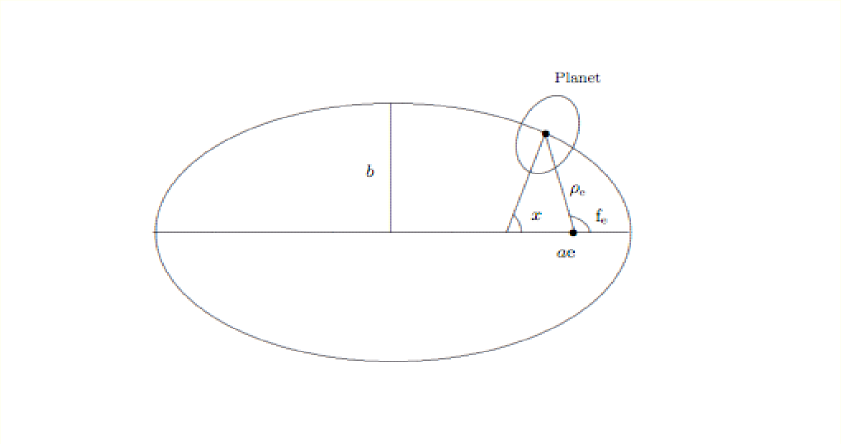


- V = “Keplerian potential”

Equations of motion



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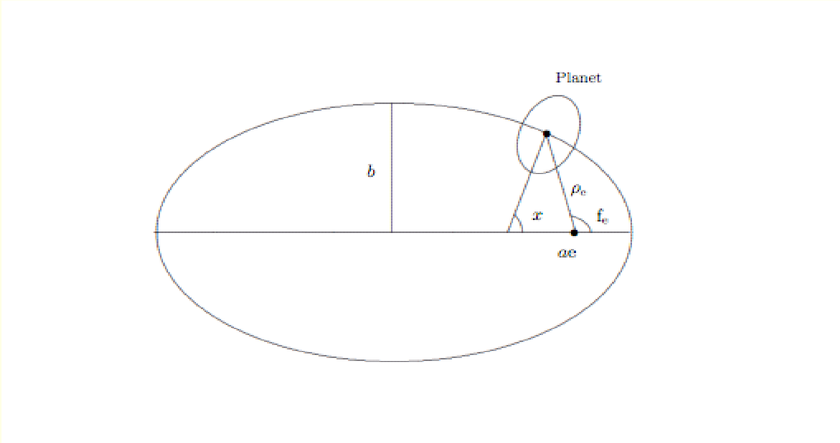

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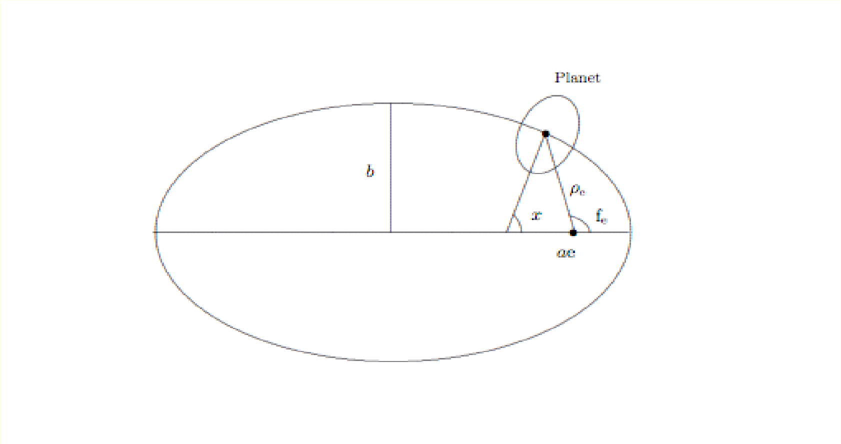
Equations of motion



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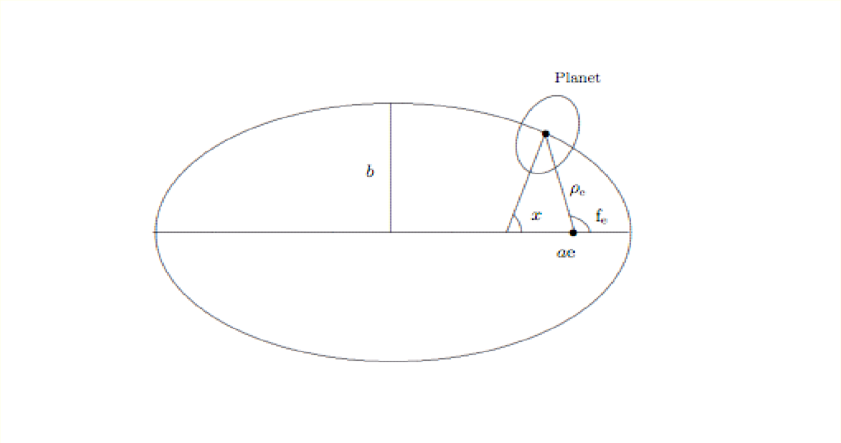

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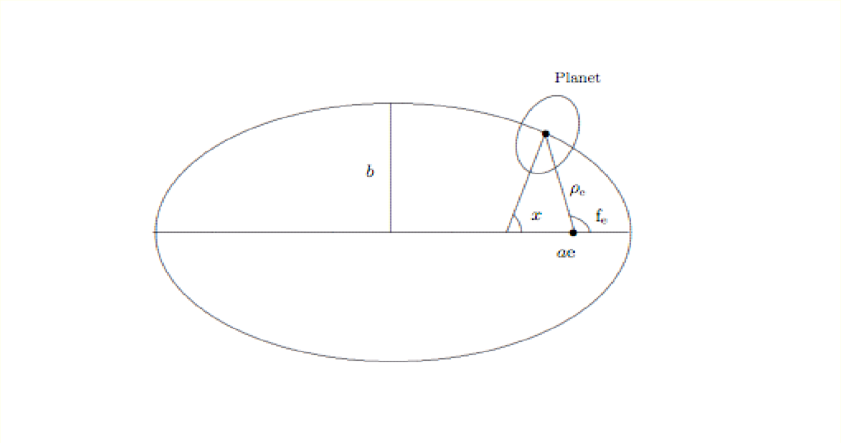

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  ➪ $\varepsilon = \frac{3}{2}\,\frac{B-A}{C}$

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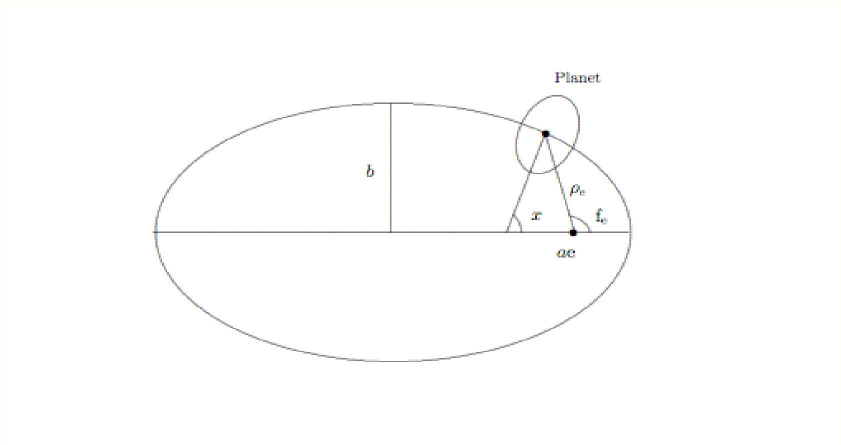

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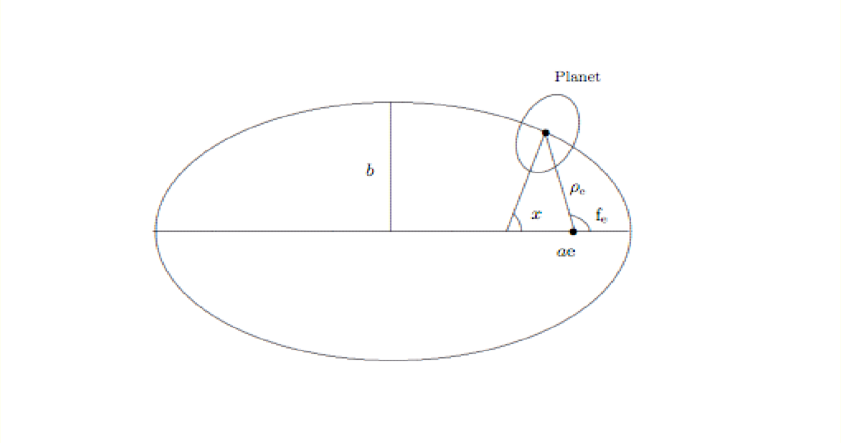

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  - ➪ $\eta = K\Omega_e$

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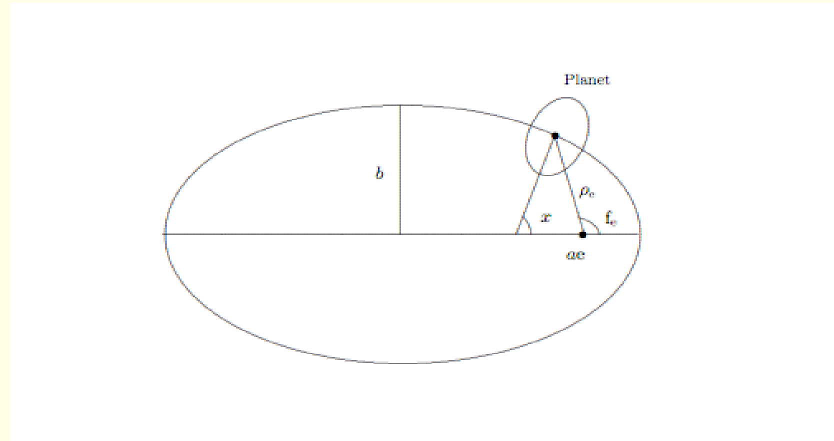

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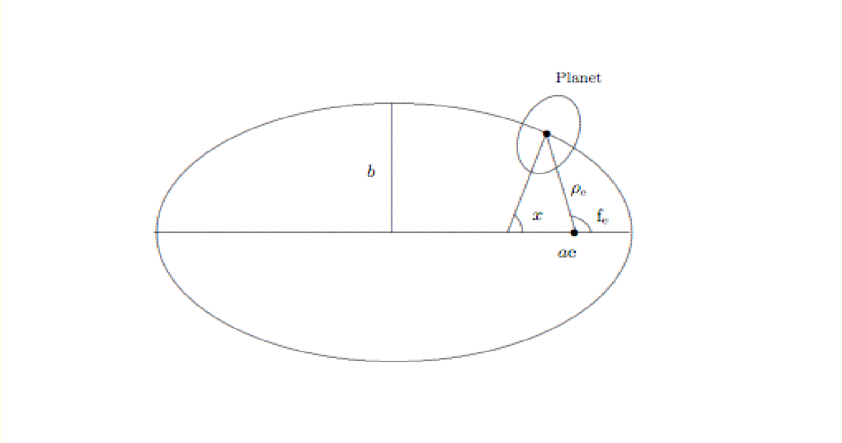

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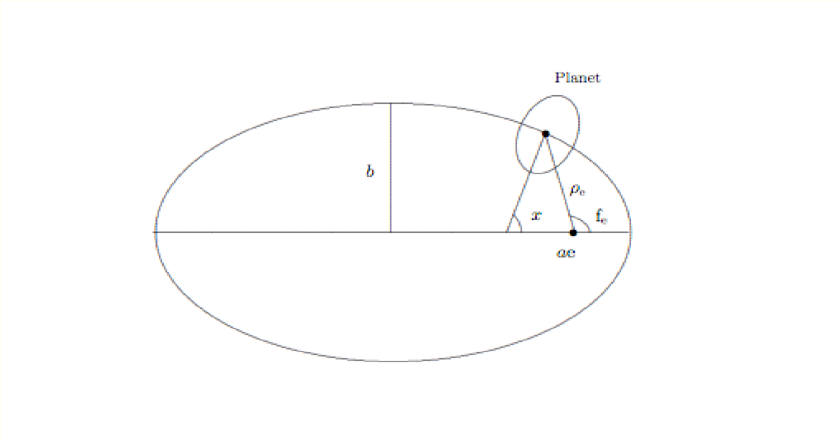
  - ➮ $\varepsilon = \frac{3}{2}\,\frac{B-A}{C}$, $(0 < A < B < C$ being the inertia moments of the planet)
  - ➮ $\eta = K\Omega_e$: $K \geq 0$ measures the non–rigidity of the planet ,
    $\Omega_e := 1 + \frac{15}{2}e^2 + O(e^4)$

Equations of motion



☞ $$\ddot{x} + \eta(\dot{x} - \upsilon) + \varepsilon V_x(x,t) = 0$$

- V = "Keplerian potential" $= -\frac{1}{2\,\rho_e(t)^3}\cos(2x - 2\,f_e(t)) = \sum_{\substack{j\neq 0\\ j\in\mathbb{Z}}} \alpha_j(e)\cos(2x - jt)$
- $\varepsilon, \eta, \upsilon$ are positive numbers
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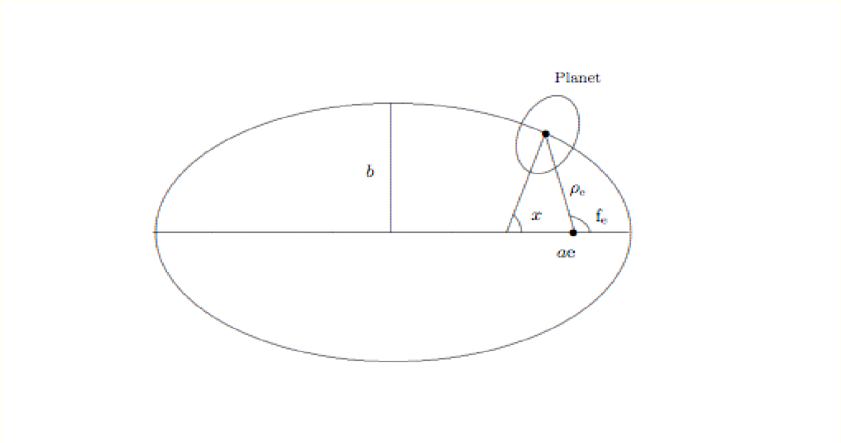

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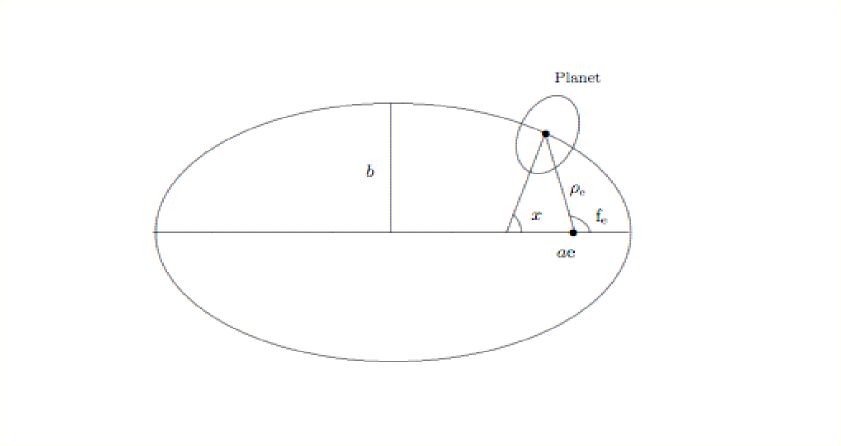
✎ The dissipative term $\eta(\dot{x} - \upsilon)$

Equations of motion



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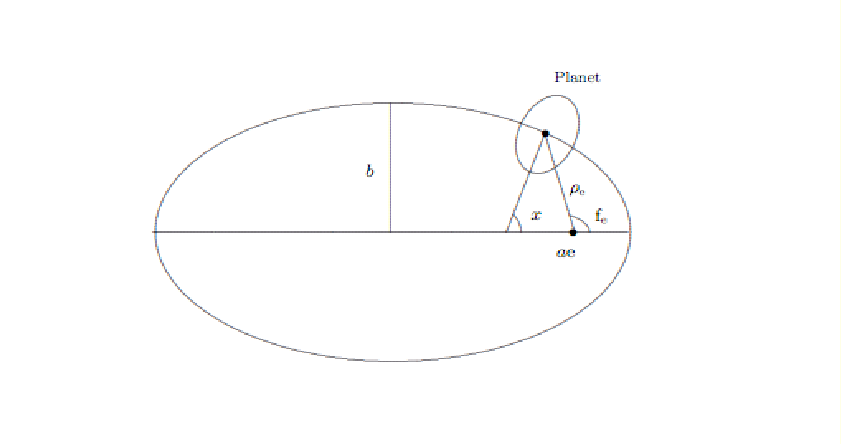

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☞ Some remarks on the Equation

$$\ddot{x} + \eta(\dot{x} - \upsilon) + \varepsilon V_x(x,t) = 0 \qquad (*)$$

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no periodic solutions

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Numerical results

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ω_{attract}	1/1	3/2	2/1
BAM	*96.7%*	*3%*	*0.3%*

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☞ "Mercury" parameters: $\text{e} = 0.206$, $\varepsilon = 10^{-3}$, $K = 5 \cdot 10^{-6}$, ($\upsilon = 1.256$)

ω_{attract}	1/1	5/4	1.256	3/2	2/1	5/2	3/1
BAM	*4.7%*	*6.8%*	*71.6%*	*13.3%*	*2.5%*	*0.6%*	*0.3%*

On the existence of quasi-periodic attractors

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$$(*) \qquad \boxed{\ddot{x} + \eta(\dot{x} - \upsilon) + \varepsilon V_x(x,t) = 0}$$

$$(\star) \begin{cases} \eta = K\Omega_{\mathrm{e}} = K(1 + O(\mathrm{e}^2)) \\ \upsilon = \upsilon_{\mathrm{e}} = 1 + O(\mathrm{e}^2) \end{cases}$$

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- Denote $\mathcal{D}_{\kappa,\tau} := \{\omega \in \mathbb{R} : |\omega n_1 + n_2| \geq \kappa |n_1|^{-\tau}, \forall n \in \mathbb{Z}^2, n_1 \neq 0\}$.

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Theorem.

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Theorem. *Fix* $0 < \kappa < 1 \leq \tau$ *and* $\eta_0 > 0$.

On the existence of quasi-periodic attractors

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$$(*) \qquad \boxed{\ddot{x} + \eta(\dot{x} - \upsilon) + \varepsilon V_x(x,t) = 0} \qquad\qquad (\star) \left\{ \begin{array}{l} \eta = K\Omega_{\rm e} = K(1 + O({\rm e}^2)) \\ \upsilon = \upsilon_{\rm e} = 1 + O({\rm e}^2) \end{array} \right.$$

- Let V be real–analytic on $\mathbb{T}^2$.
- Denote $\mathcal{D}_{\kappa,\tau} := \{\omega \in \mathbb{R} : |\omega n_1 + n_2| \geq \kappa |n_1|^{-\tau}, \forall n \in \mathbb{Z}^2, n_1 \neq 0\}$.

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➠ for $\omega > 1$, $\exists!$ function $\mathrm{e} = \mathrm{e}(\eta, \varepsilon, \omega)$ such that $\upsilon_{\mathrm{e}(\eta,\varepsilon,\omega)} = \omega(1 + \langle u^2_{\theta_1} \rangle) = \omega + O(\varepsilon^2)$.

Co–existence of spin–orbit resonances

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Co–existence of spin–orbit resonances

[LB, L. Chierchia: *On the basins of attraction of low-order resonances in weakly dissipative spin–orbit models*, Journal of Differential Equation, to appear]

$$\ddot{x} + \eta(\dot{x} - \upsilon) + \varepsilon V_x(x,t) = 0 \qquad \text{with} \qquad V = \sum_{\substack{j \neq 0 \\ j \in \mathbb{Z}}} \alpha_j(\mathrm{e}) \cos(2x - jt)$$

Theorem 1. *Let p and q be positive co–prime integers with $q = 1, 2, 4$. Then, for ε and η small and positive, there exist (elliptic) spin–orbit resonances of type (p,q)*

provided

$$\left|\upsilon(\mathrm{e}) - \frac{p}{q}\right| < \begin{cases} r_{pq} := |\beta_{pq}|(\varepsilon/\eta) & \text{if } q = 1, 2 \\ r_{pq} := |\beta_{pq}|\, 16\,(\varepsilon^2/\eta) & \text{if } q = 4 \end{cases}$$

where: $\beta_{p1} = -2\alpha_{2p}$, $\beta_{p2} = -2\alpha_p$ and $\beta_{p4} = \sum_{\substack{j \in \mathbb{Z} \\ j \neq 0, p}} \frac{\alpha_{p-j}\alpha_j}{(p-2j)^2}$

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K	(1,1)	(5,4)	(3,2)	(2,1)	(5,2)	(3,1)
10^{-3}	1.05	**0.0058**	0.7	−0.27	−1.0	−1.67
$5 \cdot 10^{-4}$	2.35	0.017	1.65	**0.19**	−0.84	−1.59
10^{-4}	12.81	0.11	9.24	3.92	**0.75**	−1.01
$5 \cdot 10^{-5}$	25.88	0.22	18.74	8.60	2.75	−0.28
10^{-5}	130.46	1.16	94.69	45.99	18.76	5.55
$5 \cdot 10^{-6}$	261.17	2.33	189.62	92.72	38.77	12.86

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In the functional equation $(\star)$, $\eta, \nu, \varepsilon, p, q$ are parameters, while the unknowns are the (2π–periodic with zero average) function u and the "phase" $\xi \in \mathbb{T}^1$.

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$$u(t) = u(t;\xi,\varepsilon) = \varepsilon u_1(t;\xi) + \varepsilon^2 u_2(t;\xi) + \cdots \qquad u_1 = L^{-1}[\hat{\Phi}(0)].$$

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Since $w(t) := e^{-t\eta/2}z(t)$, to prove that $w(t) \stackrel{t\to+\infty}{\longrightarrow} 0$ exponentially fast, we have to show that $|z(t)| \leq const.\ \forall\, t \geq 0$.

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In fact $\mathcal{L} \overset{\varepsilon\to 0}{\longrightarrow} \partial_t^2$, whose fundamental solutions are $c(t) \equiv 1$ and $s(t) \equiv t$ which is *not bounded!*

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