A topological method for the detection of normally hyperbolic invariant manifolds

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Joint work with

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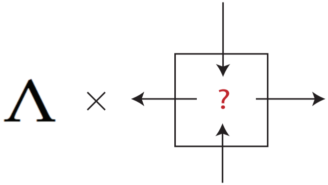
Plan of the presentation

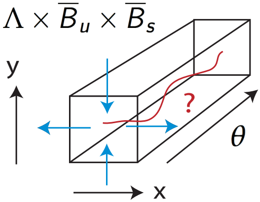
- Statement of the problem
- Normally hyperbolic invariant manifold theorem
- Covering relations and cone conditions
- Existence of the normally hyperbolic invariant manifold
- Foliation of W^u and W^s
- Verification of conditions
- Example

Statement of the problem

$$f : \Lambda \times \overline{B}_u \times \overline{B}_s \to \Lambda \times \mathbb{R}^u \times \mathbb{R}^s$$

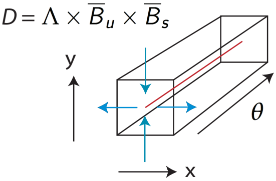
Λ is compact manifold without a boundary $\qquad (\Lambda = \mathbb{S}^1)$



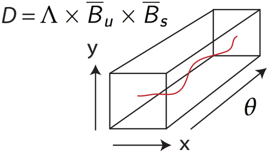


Do we have an invariant manifold in $\Lambda \times \overline{B}_u \times \overline{B}_s$?

Normally hyperbolic invariant manifold theorem



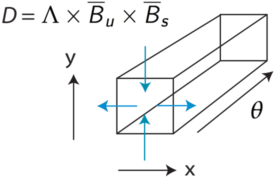
$$f : D \to \Lambda \times \mathbb{R}^2$$



$$f_\varepsilon = f + \varepsilon g$$

- we start with the region D and devise conditions which ensure the existence of the manifold
- the conditions are verifiable with rigorous numerics

Normally hyperbolic invariant manifold theorem



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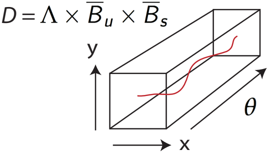

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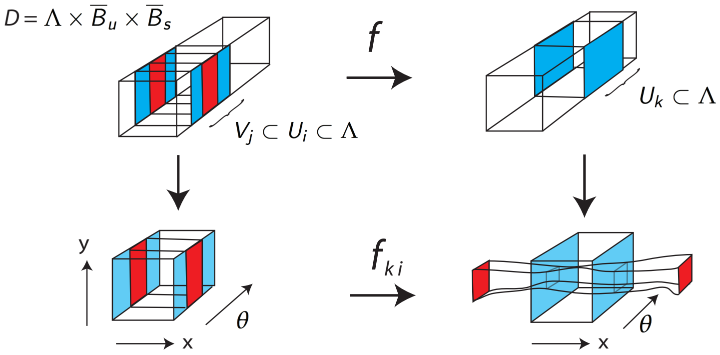
Local maps

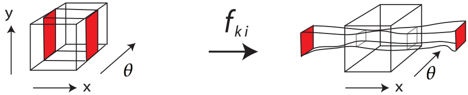
Topological conditions (covering relations)



$\{V_j\}$ and $\{U_i\}$ are coverings of Λ

$$f_{k\,i}\left(V_j \times \overline{B}_u \times \overline{B}_s\right) \subset U_k \times \mathbb{R}^u \times \mathbb{R}^s$$

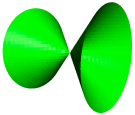
Cones



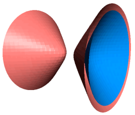
In local coordinates we define

$$Q(\theta, x, y) = \|x\|^2 - \|y\|^2 - \|\theta\|^2$$

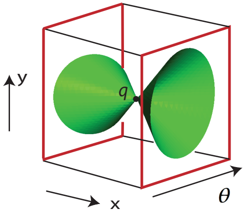
Horizontal cone $Q \geq 0$:



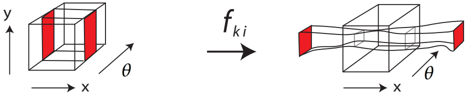
$Q = a$ and $Q = b$ for $0 < a < b$:



For each point $q \in D$ we have local coordinates which contain cones starting from q.



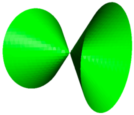
Cone conditions



$m > 1$. If $Q(x_1 - x_2) \geq 0$ then

$$Q(f_{ki}(x_1) - f_{ki}(x_2)) > mQ(x_1 - x_2)$$

Horizontal cone $Q \geq 0$:



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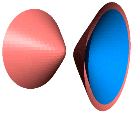

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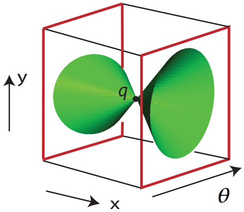
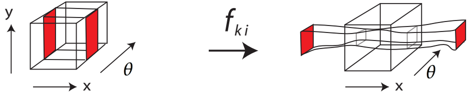
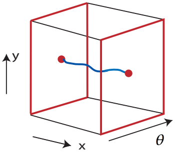


Horizontal discs

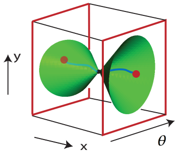


If $Q(x_1 - x_2) \geq 0$ then $Q(f_{ki}(x_1) - f_{ki}(x_2)) > mQ(x_1 - x_2)$

A horizontal disc:
$b : \overline{B}_u \to V_j \times \overline{B}_u \times \overline{B}_s$



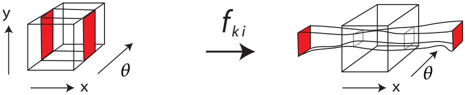
A horizontal disc which satisfies cone conditions:



Lemma

An image of a horizontal disc which satisfies cone conditions is a horizontal disc which satisfies cone conditions.

Horizontal discs



If $Q(x_1 - x_2) \geq 0$ then $Q(f_{ki}(x_1) - f_{ki}(x_2)) > mQ(x_1 - x_2)$

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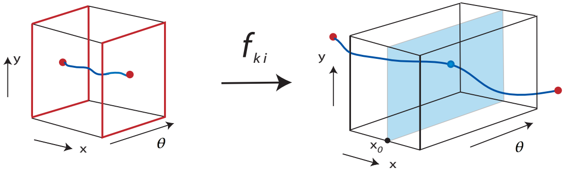
Proof.



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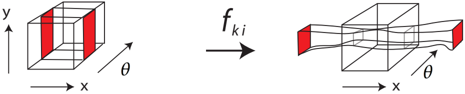

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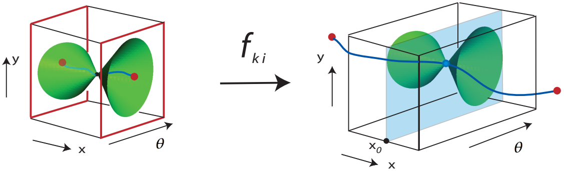

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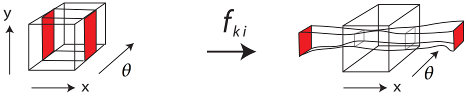

$$\text{If } Q(x_1 - x_2) \geq 0 \quad \text{then} \quad Q(f_{ki}(x_1) - f_{ki}(x_2)) > mQ(x_1 - x_2)$$

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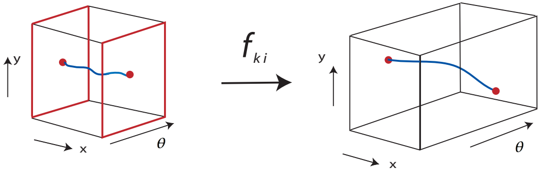

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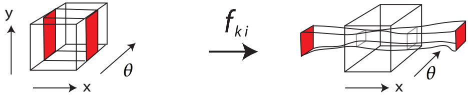
Forward iterations

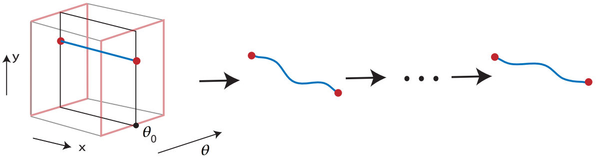


If $Q(x_1 - x_2) \geq 0$ then $Q(f_{ki}(x_1) - f_{ki}(x_2)) > mQ(x_1 - x_2)$

Lemma

For any $\theta_0 \in \Lambda$ we have a vertical disc of points in $\{\theta_0\} \times \overline{B}_u \times \overline{B}_s$ which stay inside of $\Lambda \times \overline{B}_u \times \overline{B}_s$.

Proof.



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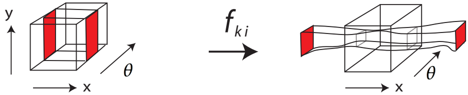

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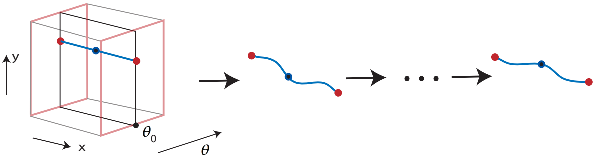

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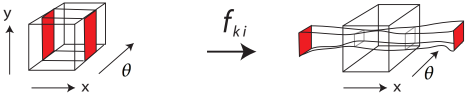

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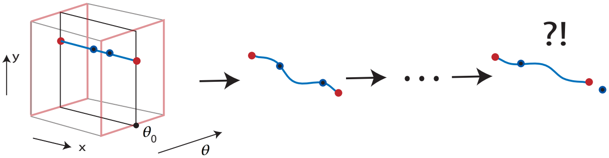

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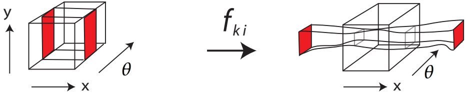

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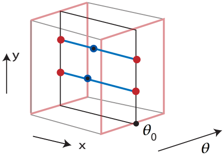

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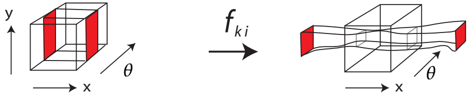

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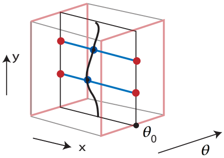

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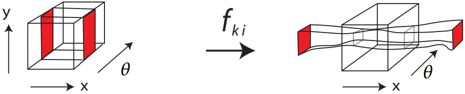

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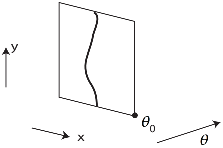

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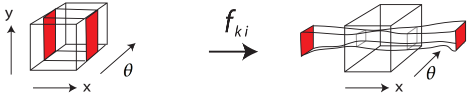
Main Result



If $Q(x_1 - x_2) \geq 0$ then $Q(f_{ki}(x_1) - f_{ki}(x_2)) > mQ(x_1 - x_2)$

Theorem

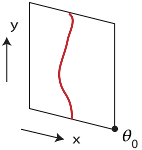
If f and f^{-1} satisfy the the topological and cone conditions then there exists a C^0 map $\chi : \Lambda \to \Lambda \times \overline{B}_u \times \overline{B}_s$ such that

$$\chi(\Lambda) = \mathrm{inv}(f, \Lambda \times \overline{B}_u \times \overline{B}_s)$$

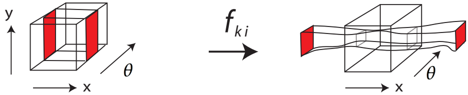
and C^0 stable and unstable manifolds W^s, W^u.

Proof

a vertical disc of forward invariant points:



Main Result



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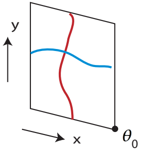
Proof

a horizontal disc of backward invariant points:



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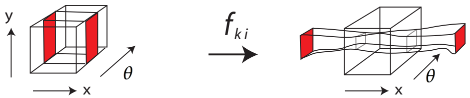

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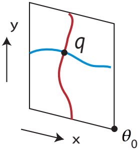
Proof

gives $\chi(\theta_0) := q$



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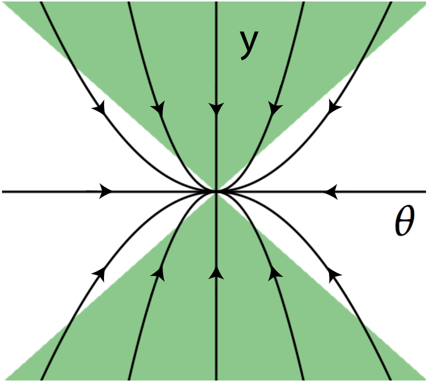
Foliation of W^s

A very simple example

$$y' = -2y$$
$$\theta' = -\theta$$

$$y(t) = y_0 e^{-2t}$$
$$\theta(t) = \theta_0 e^{-t}$$



Vertical cone: $V \geq 0$

$$V(\theta, x, y) = -\|\theta\|^2 - \|x\|^2 + \|y\|^2$$

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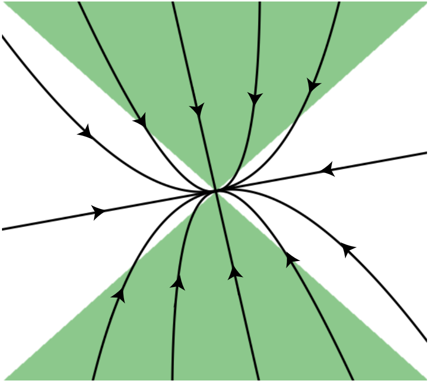

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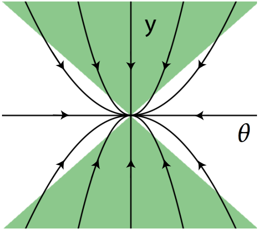
Foliation of W^s

Foliation conditions



$0 < \beta < \lambda$, $q_1 \neq q_2$

1. If $V(q_1 - q_2) \geq 0$ then $\|\pi_y(f(q_1) - f(q_2))\| < \beta \|\pi_y(q_1 - q_2)\|$
2. If $V(q_1 - q_2) < 0$ then $V(f(q_1) - f(q_2)) < \lambda^2 V(q_1 - q_2)$

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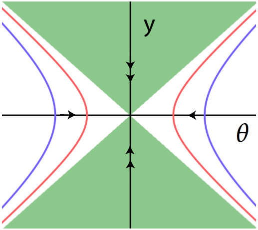
Foliation of W^s

Foliation conditions



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Foliation of W^s

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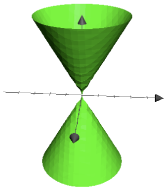

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Foliation of W^s

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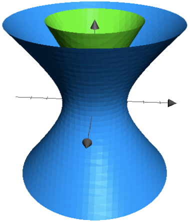

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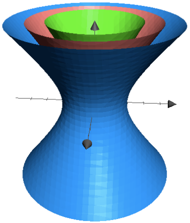
$$V(\theta, x, y) = -\|\theta\|^2 - \|x\|^2 + \|y\|^2$$

Foliation of W^s

Foliation conditions



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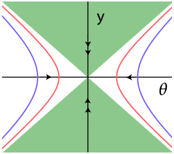

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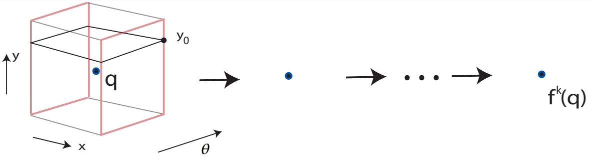

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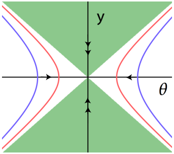

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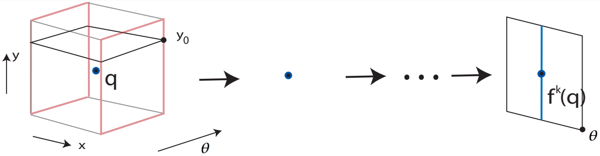

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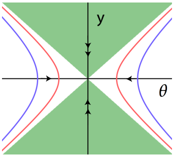

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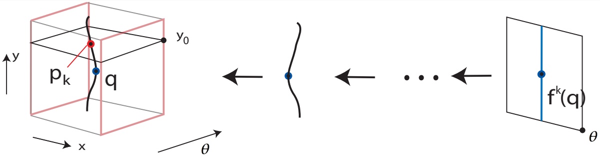

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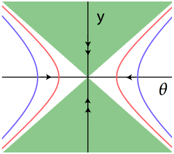

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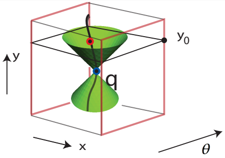

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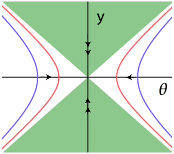

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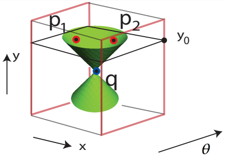

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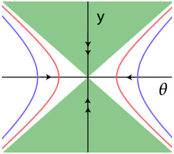

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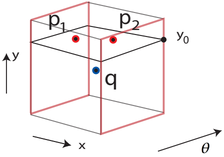

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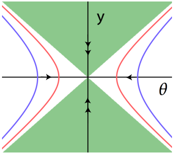

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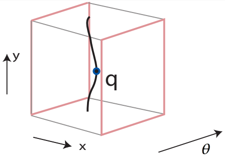

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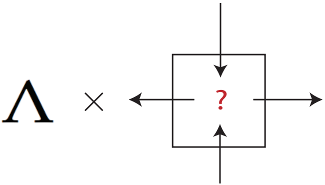
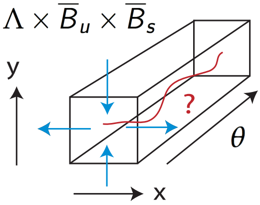
What can we do so far?



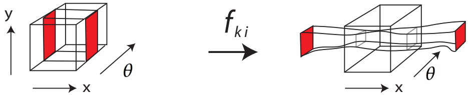


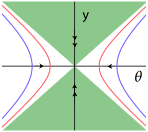
- We have a normally hyperbolic invariant manifold in $\Lambda \times B_u \times B_s$
- We have it's stable and unstable manifolds W^s and W^u
- We have foliations of W^s and W^u

Question:

How can we verify our assumptions in practice?

Verification of conditions





$$\text{If } Q(x_1 - x_2) \geq 0 \quad \text{then} \quad Q(f_{ki}(x_1) - f_{ki}(x_2)) > mQ(x_1 - x_2)$$

We need

$$[Df(V_j)] \quad \longleftrightarrow \quad \begin{bmatrix} \left\| \frac{\partial f_1}{\partial \theta} \right\| \leq C & \varepsilon & \varepsilon \\ \varepsilon & \left\| \frac{\partial f_2}{\partial x} \right\| \geq \alpha & \varepsilon \\ \varepsilon & \varepsilon & \left\| \frac{\partial f_3}{\partial y} \right\| \leq \beta \end{bmatrix}$$

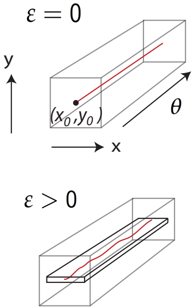
where

$$\beta < C < \alpha$$

with $\beta < 1 < \alpha$ and ε appropriately small.

Example of applications

Rotating Hénon map



$$\bar{\theta} = \theta + \omega \qquad (\text{mod } 1),$$
$$\bar{x} = 1 + y - ax^2 + \varepsilon\cos(2\pi\theta),$$
$$\bar{y} = bx$$

For $a = 0.68$, $b = 0.1$ and $\varepsilon \leq \frac{1}{2}$

$$\Lambda \subset U_\varepsilon = \mathbb{T}^1 \times [x_0 - 1.1\varepsilon, x_0 + 1.1\varepsilon] \times [y_0 - 0.12\varepsilon, y_0 + 0.12\varepsilon],$$

where

$$x_0 = \frac{-(1-b) - \sqrt{(1-b)^2 + 4a}}{2a} \approx -2.043\,3,$$
$$y_0 = bx_0 \approx -0.204\,33.$$

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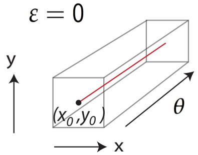

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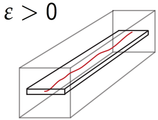

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Closing remarks

- The method works in a more general setting.
- We do not need an invariant manifold to start with.
- If we start with an invariant manifold then we can estimate the size of the perturbation for which it survives.
- We only know that the invariant manifold, W^u, W^s, and foliations are C^0.
- The method still waits to be tested on a more challenging example.