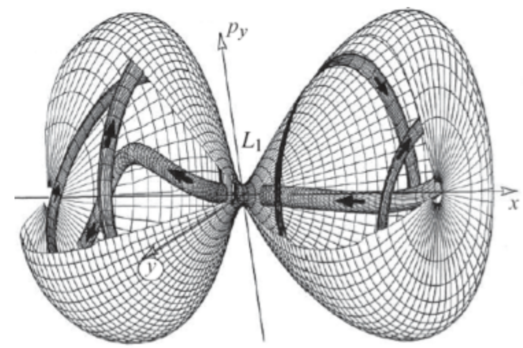
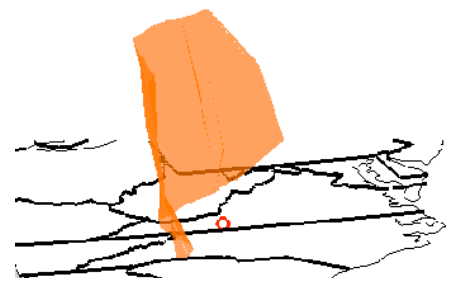
VIRGINIA TECH · WAKE FOREST UNIVERSITY
School of Biomedical Engineering and Sciences

VirginiaTech

Dynamical boundaries in a variety of mechanical systems





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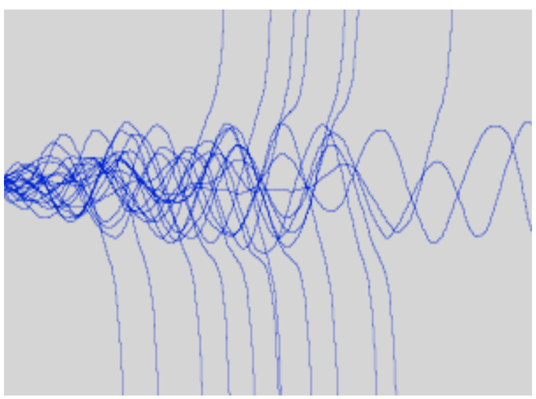
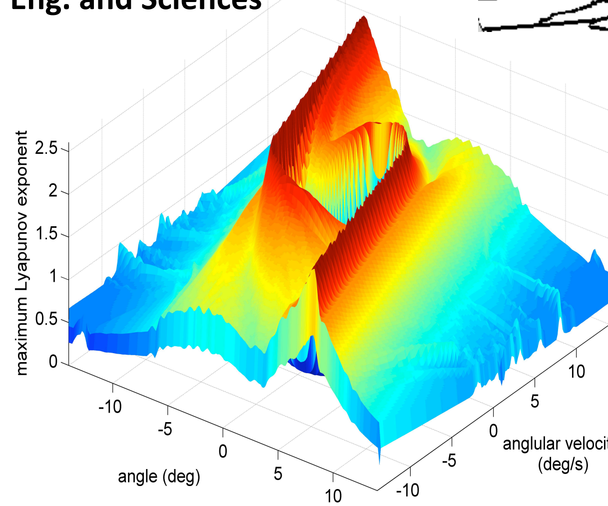
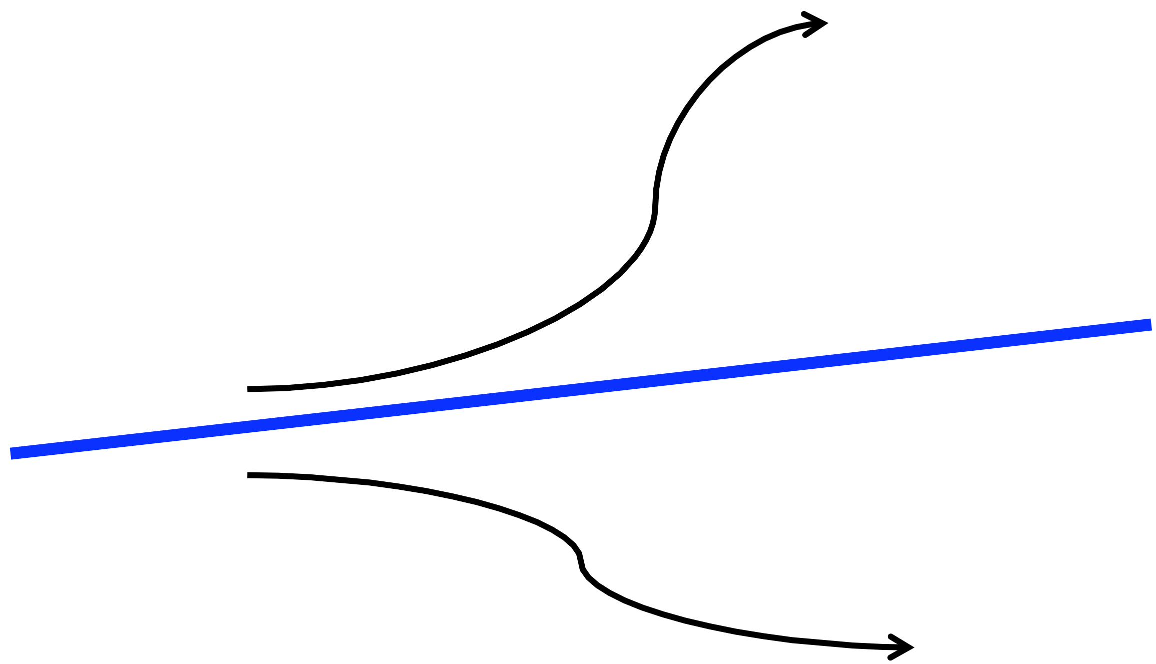


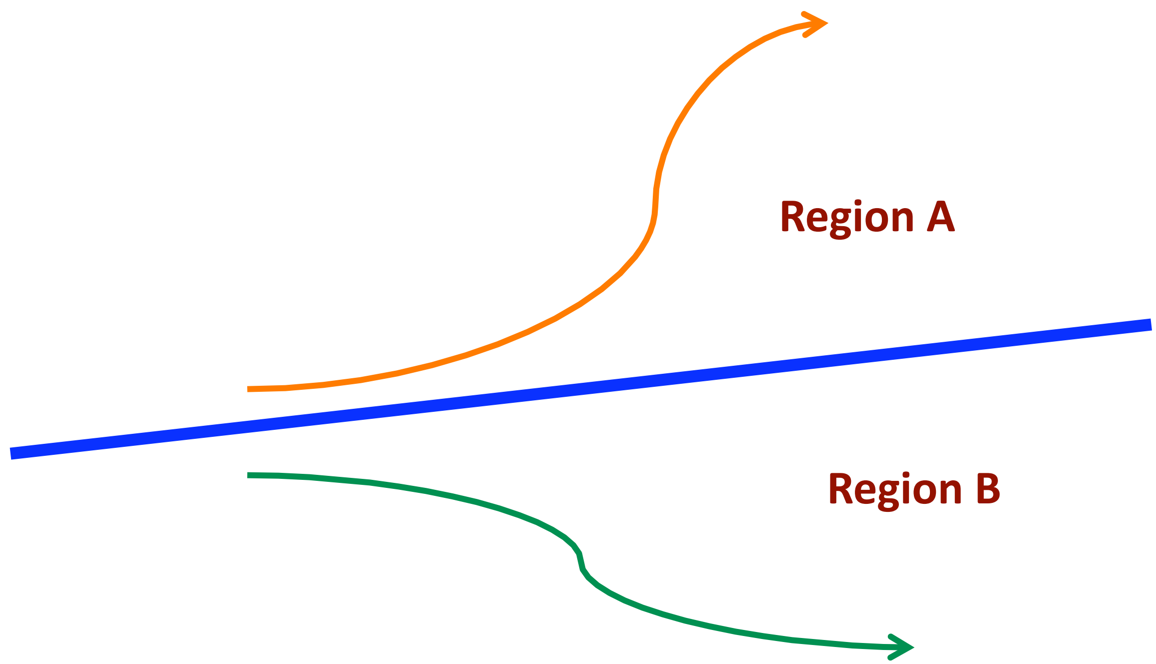


Separatrices: dynamical boundaries



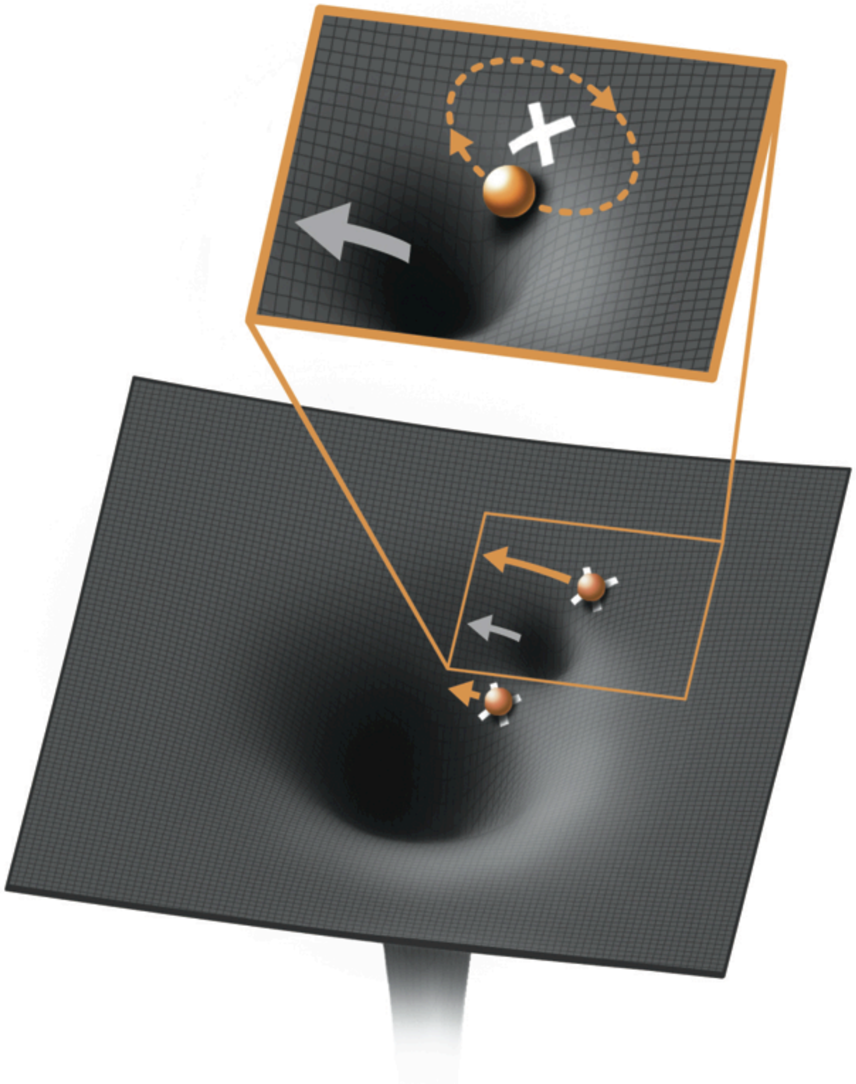
Transport barriers in state space separating qualitatively different kinds of behavior

Separatrices: dynamical boundaries



Transport barriers in state space separating qualitatively different kinds of behavior

Separatrices: high dimensions



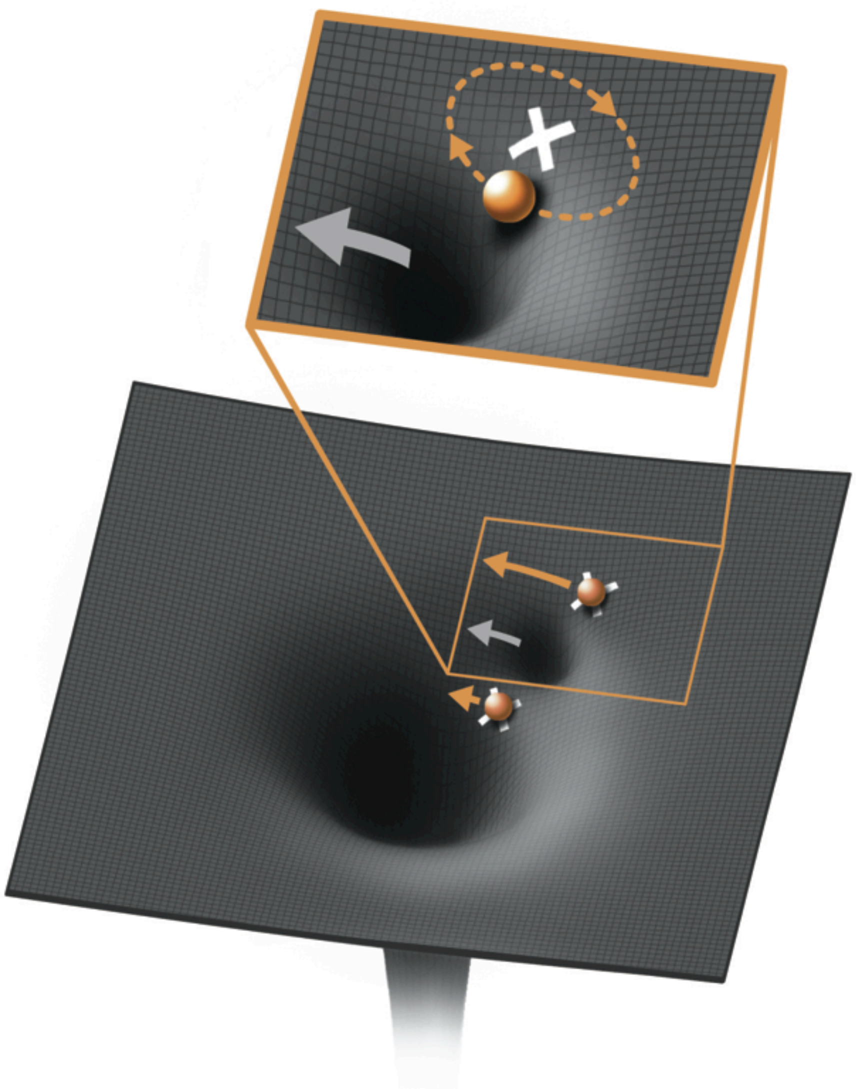
System with many basins, not necessarily attracting sets or attractors

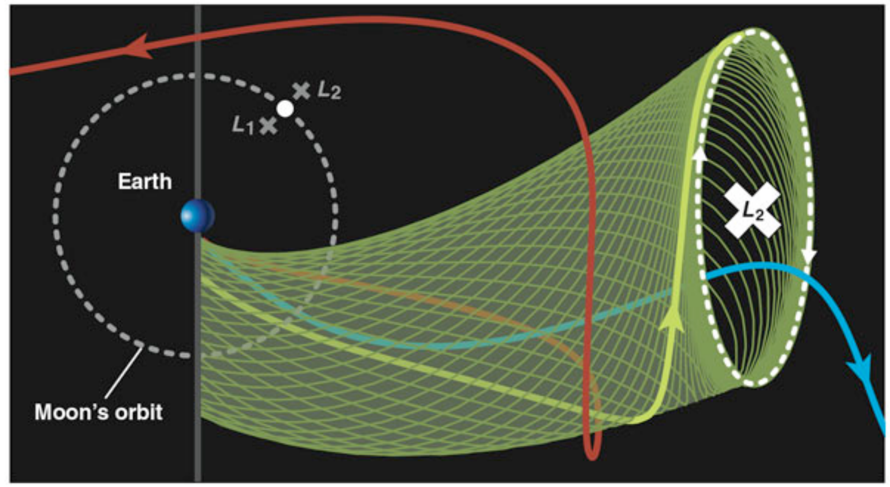
Potential surface with several minima ("bowls") separated by ridges

Basins are "almost-invariant structures"

System known analytically: vector field or map

Separatrices: high dimensions





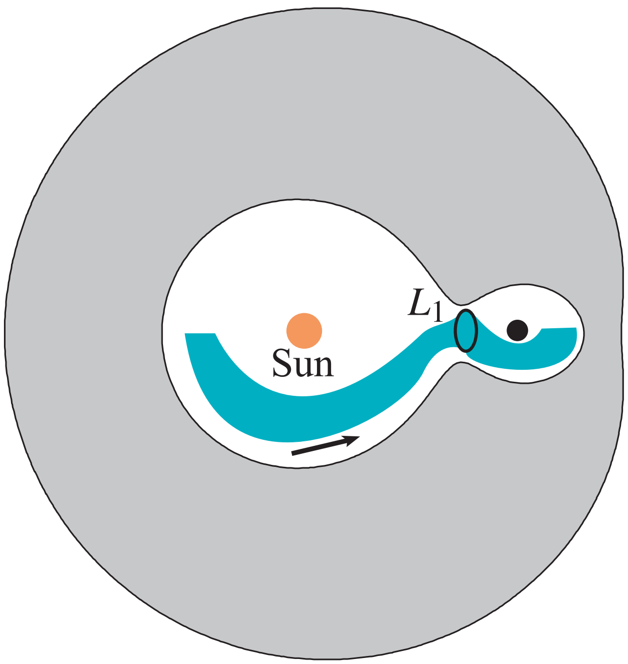
***Small body in solar system*: Transport from one basin to another controlled by high dimensional separatrix surfaces**

Geometrically tubes in this case

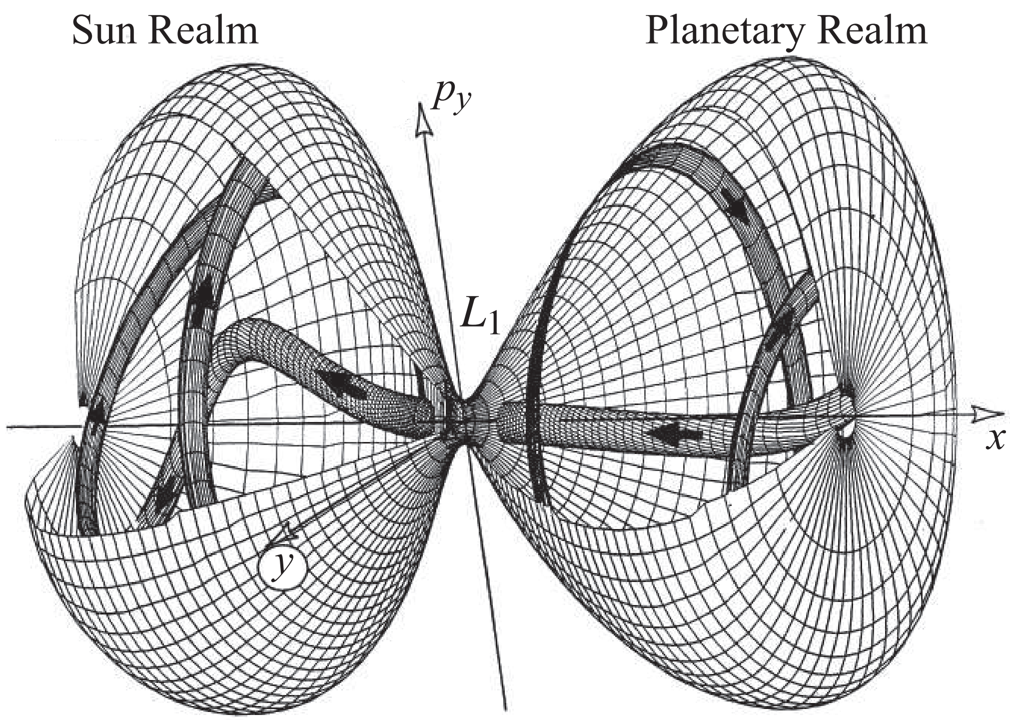
Tubes are attached to practically unobservable periodic orbits or other bound orbits

Realms and tubes

- Planetary and sun realms connected by tubes[1]



Position Space

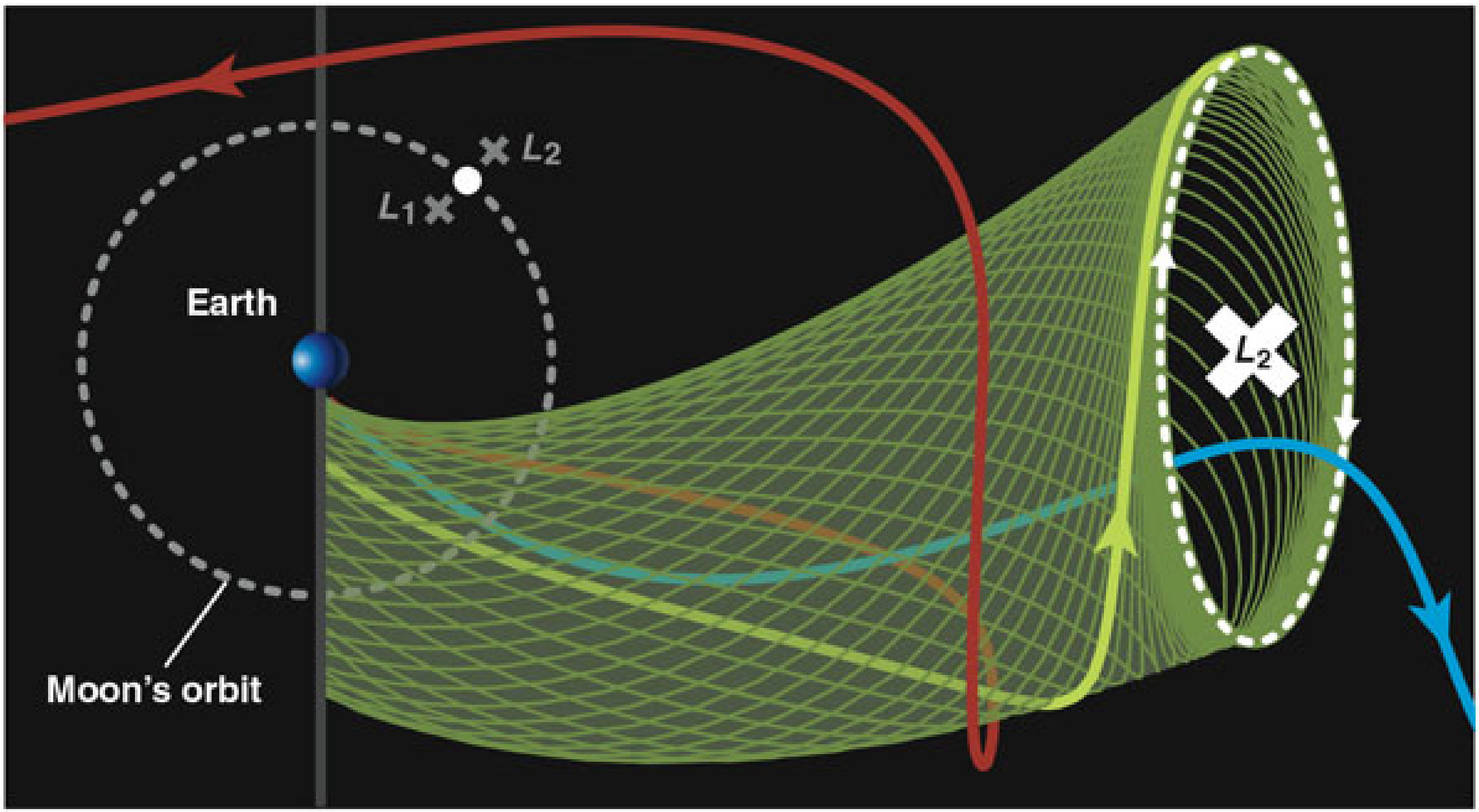


Phase Space (Position + Velocity)

[1]Conley & McGehee, 1960s, found these locally, speculated use for "low energy transfers"

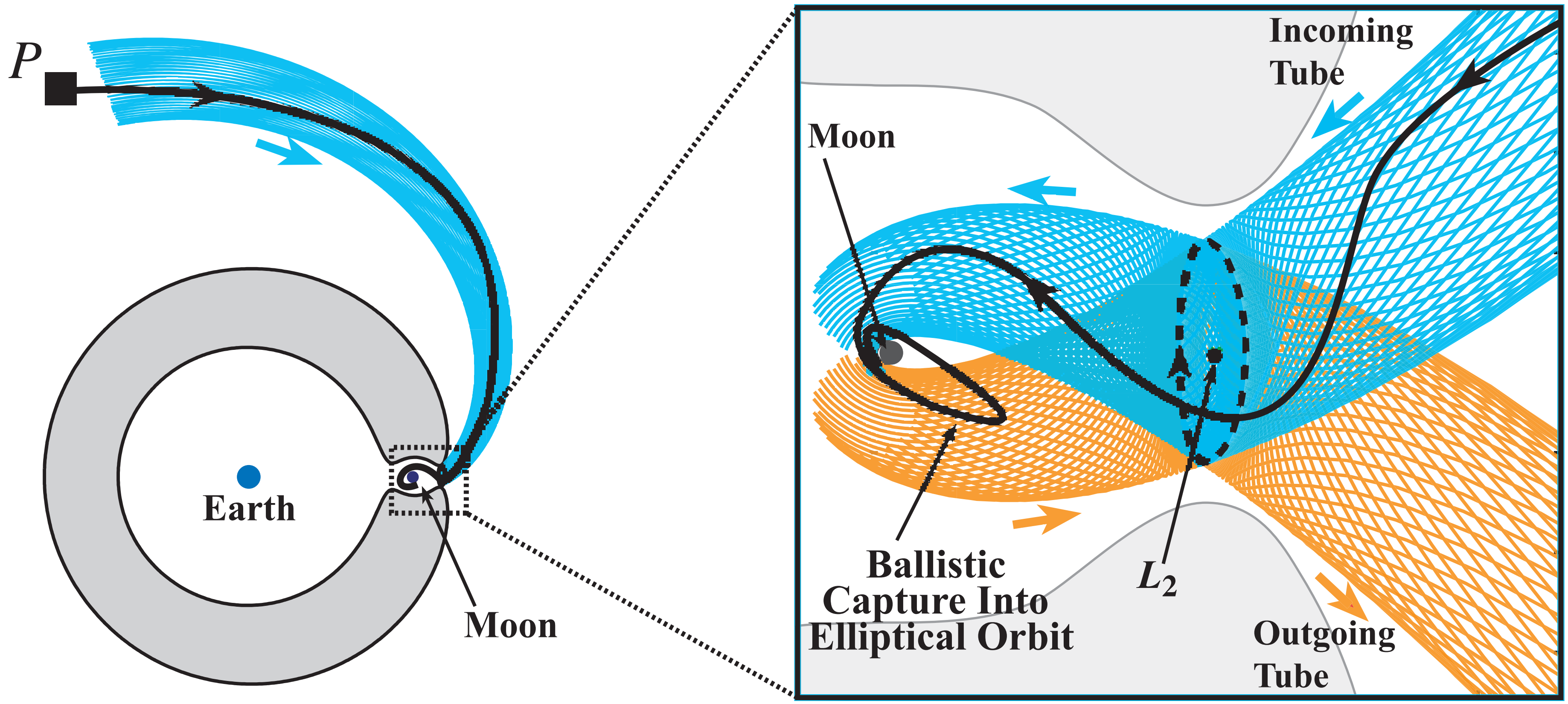
Transport between realms

- Asymptotic orbits form **4D invariant manifold tubes** ($S^3 \times \mathbb{R}$), separatrices in 5D energy surface[2]



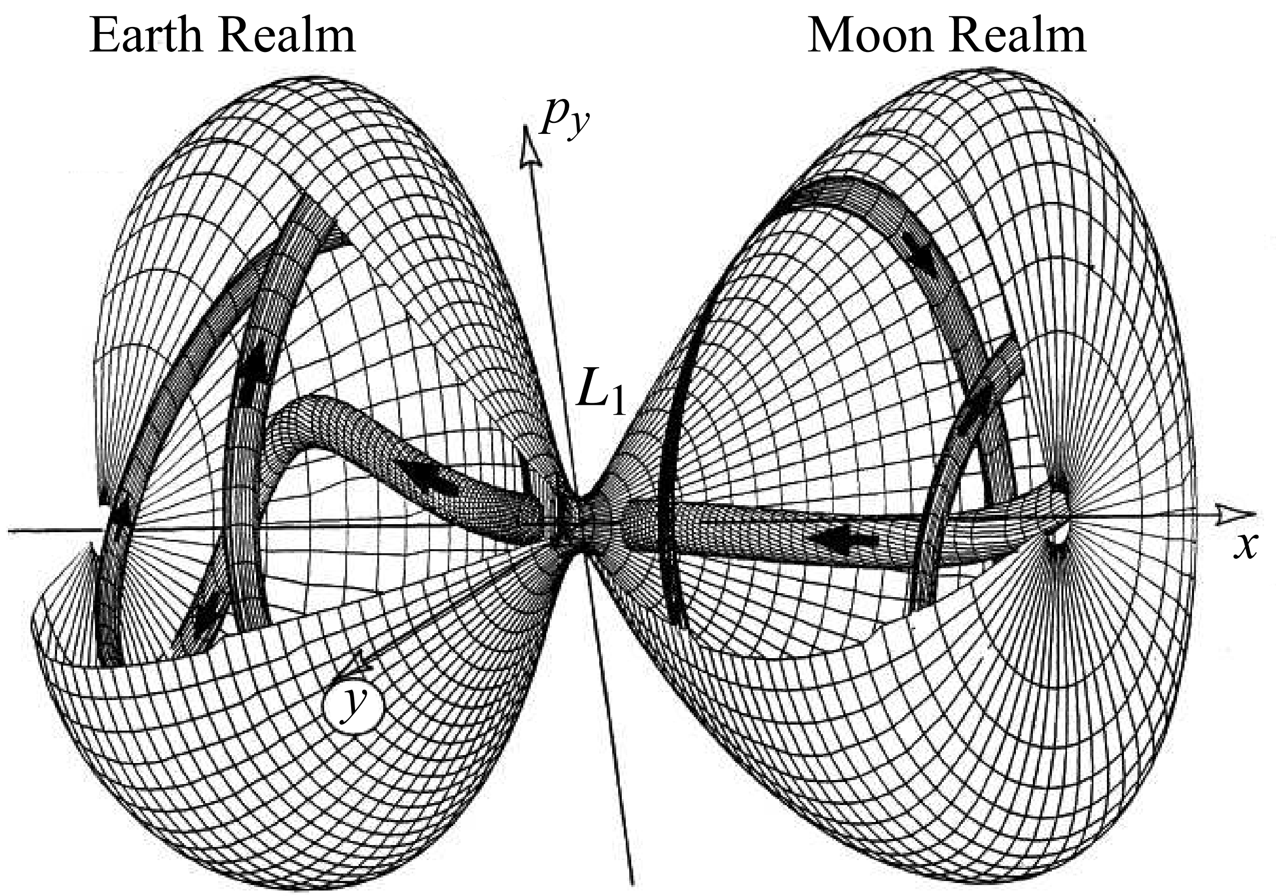
[2] Ross [2006] The interplanetary transport network, *American Scientist*

Transport between realms



- **Tubes in phase space**
  - Objects mediating transport through bottlenecks

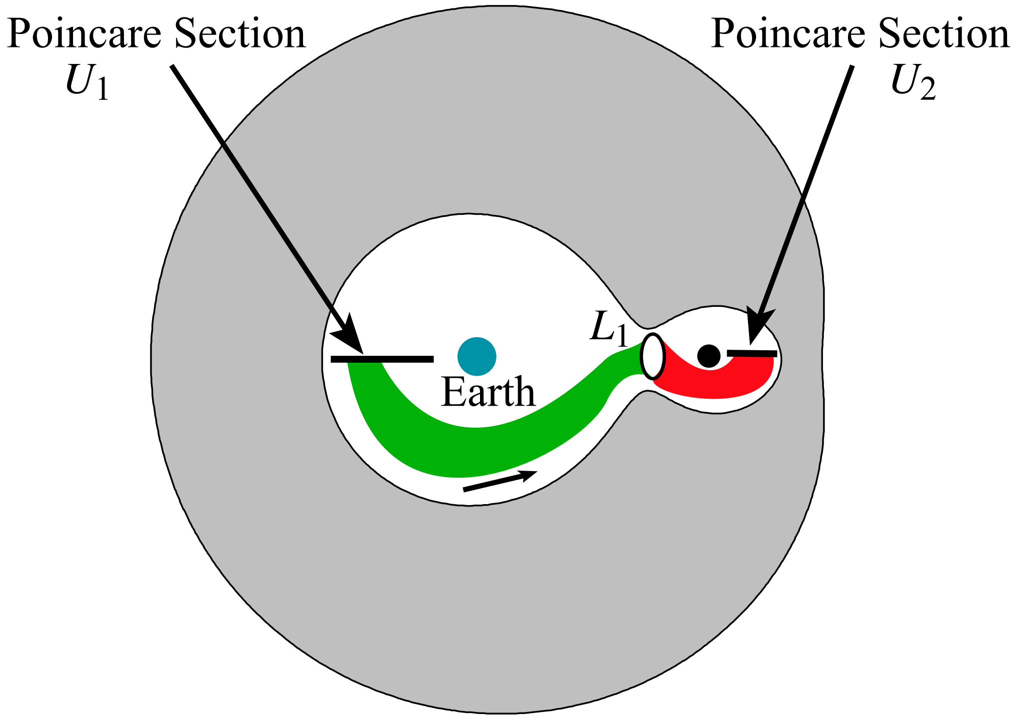
Tube dynamics

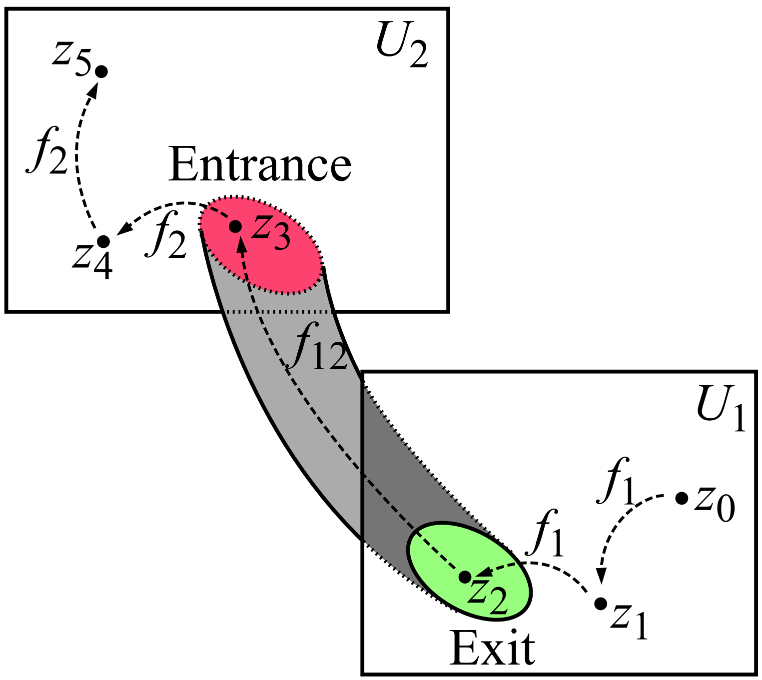


- **Tube dynamics:** All motion between realms connected by bottlenecks must occur through the interior of tubes

Multi-scale dynamics

- Slices of energy surface: Poincaré sections U_i
- Tube dynamics: evolution **between** U_i
- What about evolution on **on** U_i?

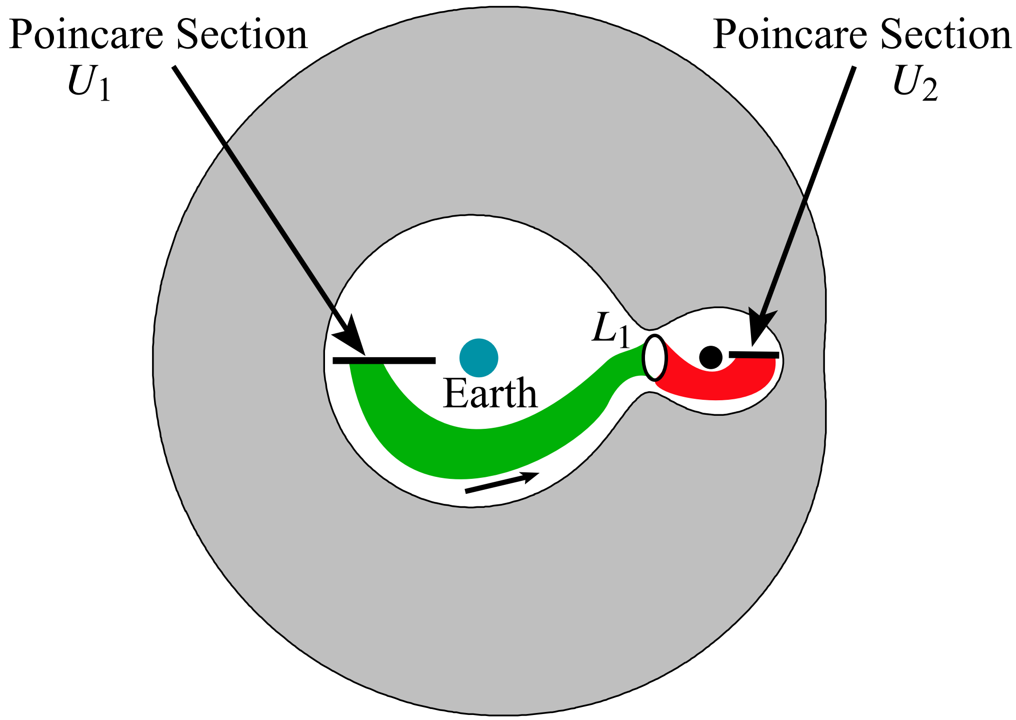


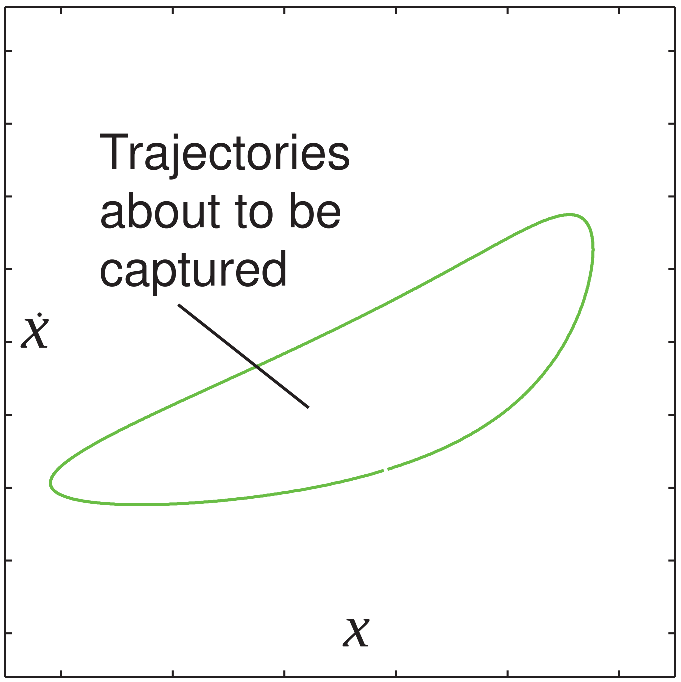


Some remarks on tube dynamics

- Tubes are general; consequence of rank 1 saddle – saddle $\times$ center $\times \cdots \times$ center – e.g., ubiquitous in chemistry
- Tubes persist
  – in presence of additional massive body
  – when primary bodies' orbit is eccentric

Tubes in elliptic restricted 3-body problem





Consider first cut of stable manifold of L_1 NHIM

Tubes in elliptic restricted 3-body problem

Gawlik, Marsden, Du Toit, Campagnola [2008] "Lagrangian coherent structures in the planar elliptic restricted three-body problem," submitted to Celestial Mechanics and Dynamical Astronomy.

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  - saddle $\times$ center $\times \cdots \times$ center
  - e.g., ubiquitous in chemistry
- Tubes persist
  - in presence of additional massive body
  - when primary bodies' orbit is eccentric
- Observed in the solar system (e.g., Oterma)
- Even on galactic and atomic scales!

Koon, Lo, Marsden, & Ross [2000], Gómez, Koon, Lo, Marsden, Masdemont, & Ross [2004], Yamato & Spencer [2003], Wilczak & Zgliczyński [2005], Ross & Marsden [2006], Gawlik, Marsden, Du Toit, Campagnola [2008], Combes, Leon, Meylan [1999], Heggie [2000], Romero-Gómez, et al. [2006,2007,2008]

Multi-scale dynamics

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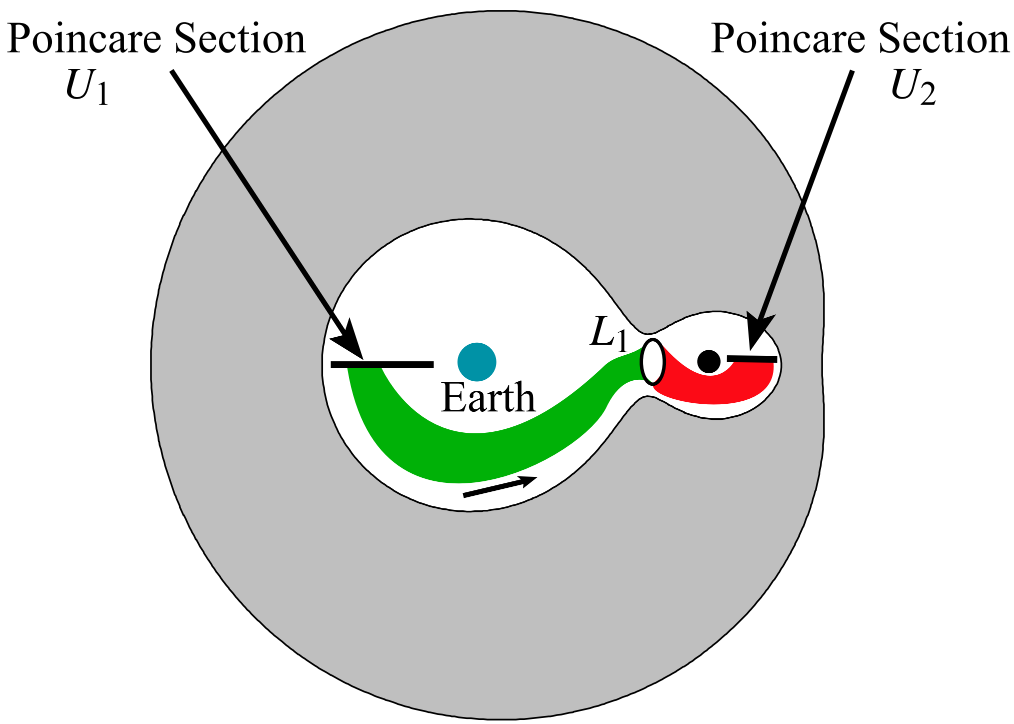
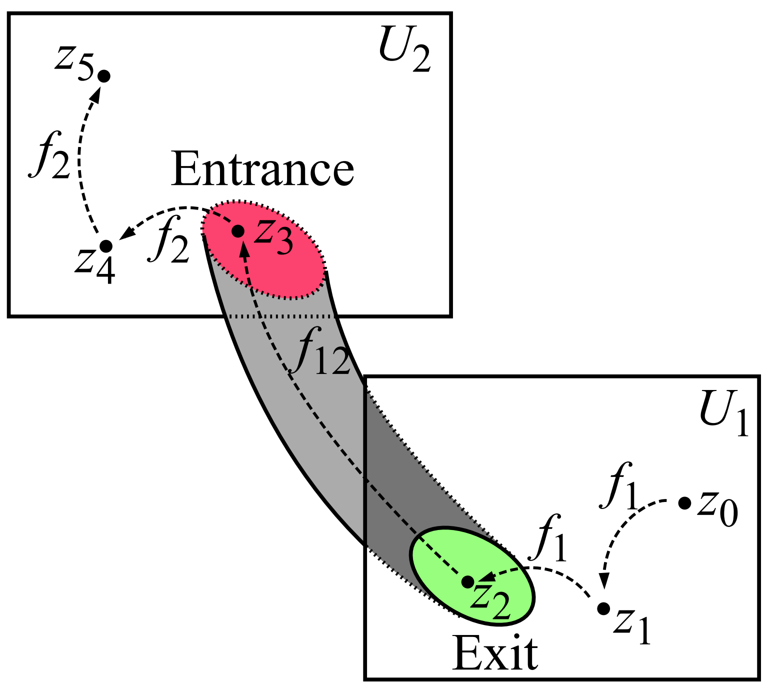
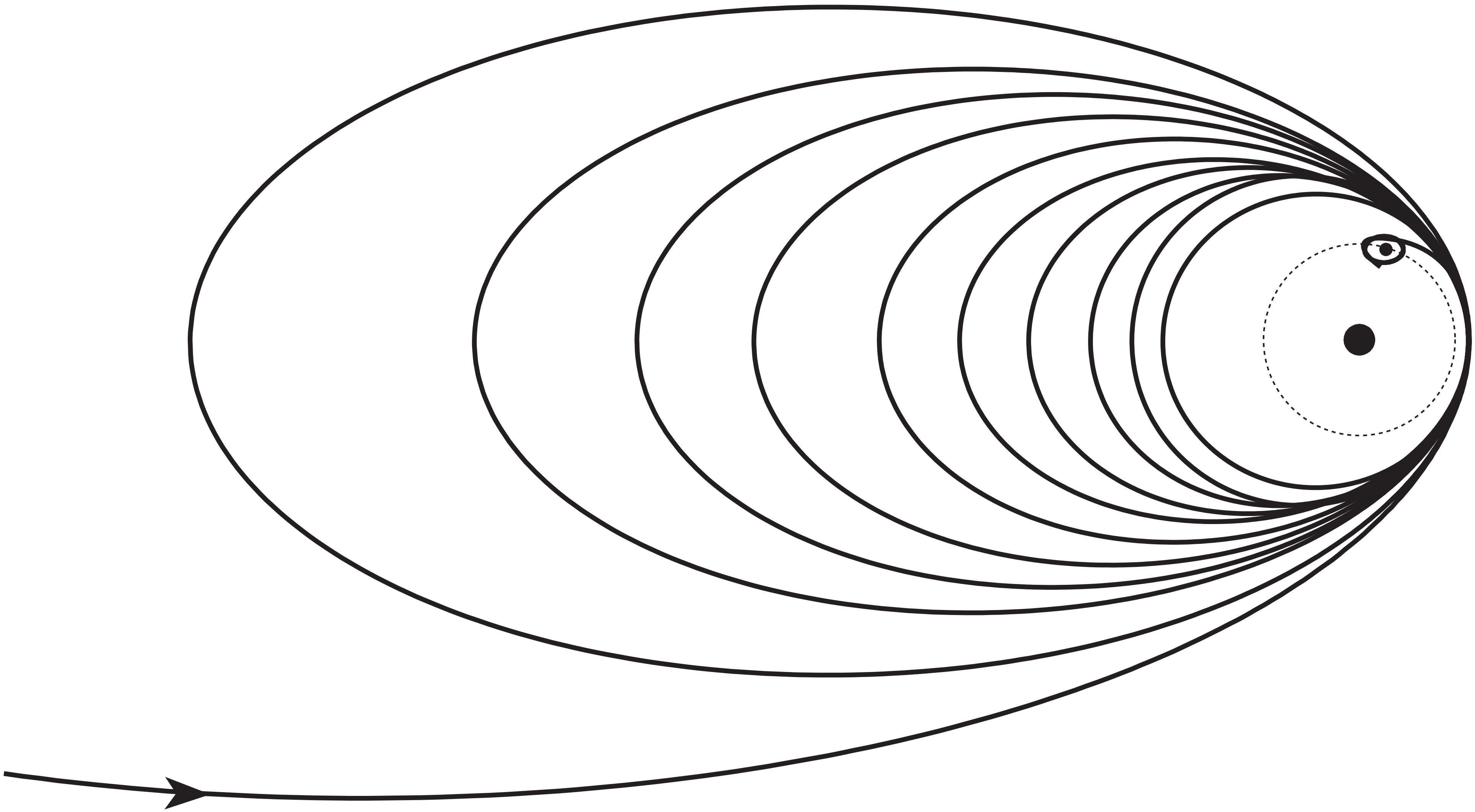




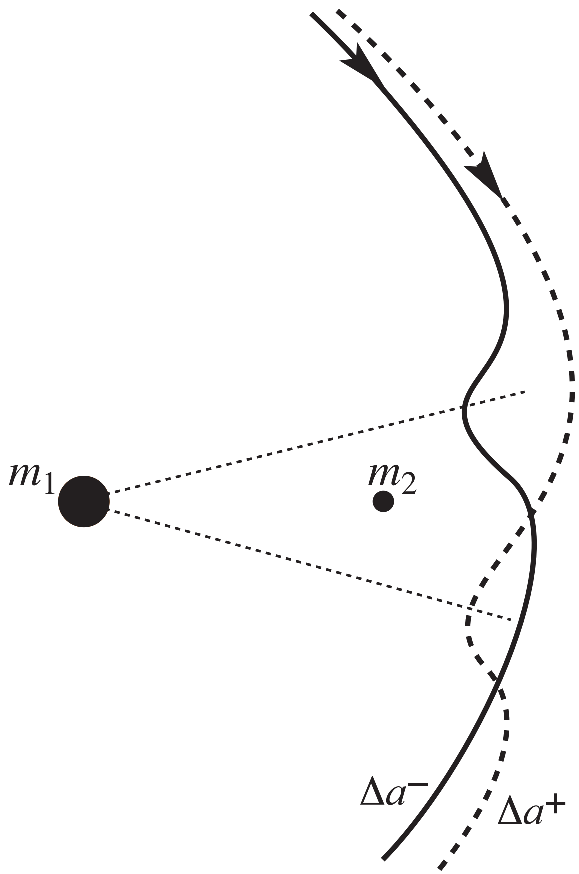
Infinity to capture about small companion in binary pair?

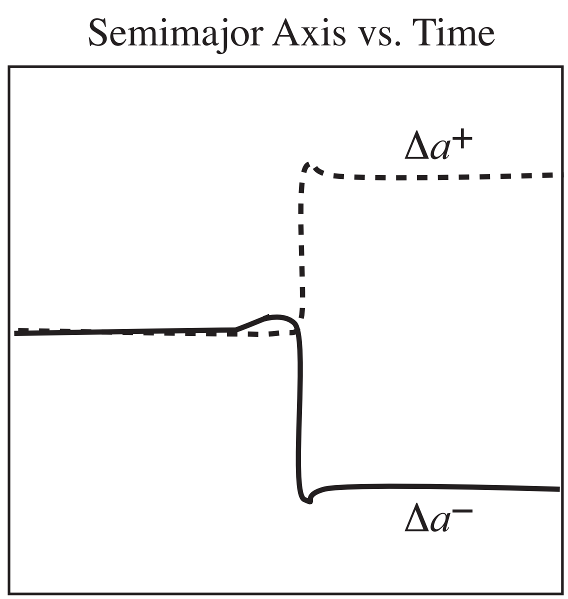


- After **consecutive gravity assists**, large orbit changes

Kicks at periapsis

- Key idea: model particle motion as "**kicks**" at periapsis

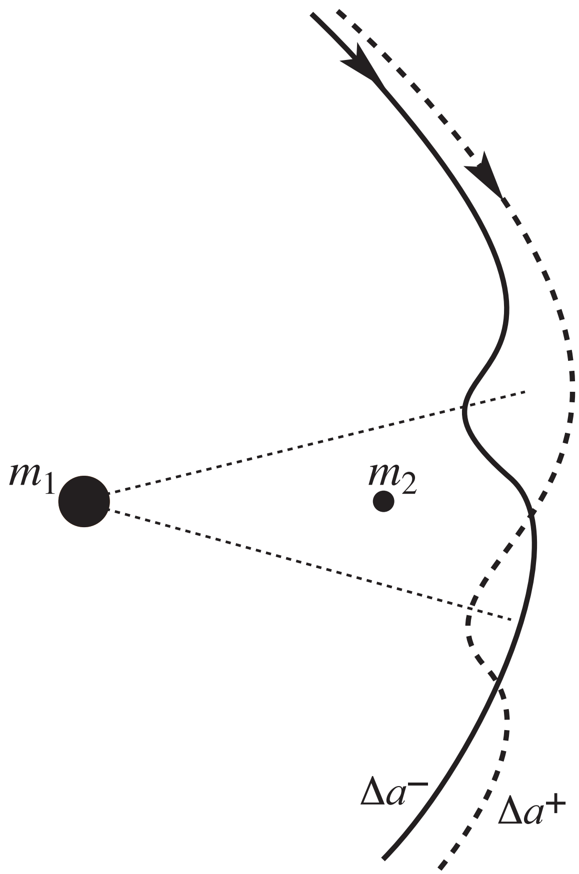


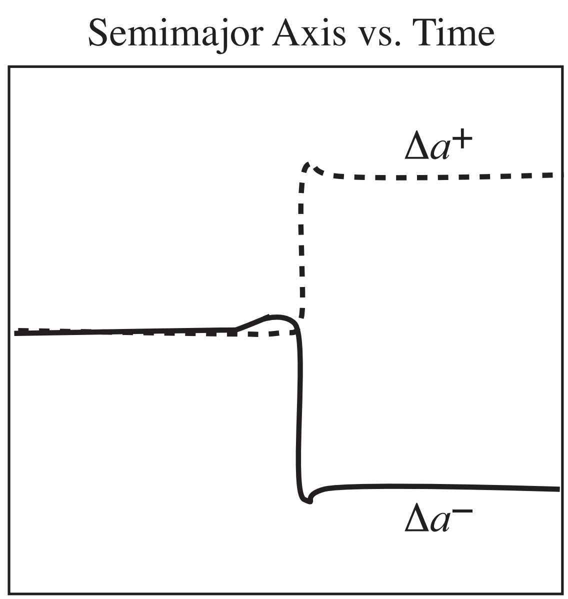


In rotating frame where m_1, m_2 are fixed

Kicks at periapsis

- Sensitive dependence on **argument of periapse** ω

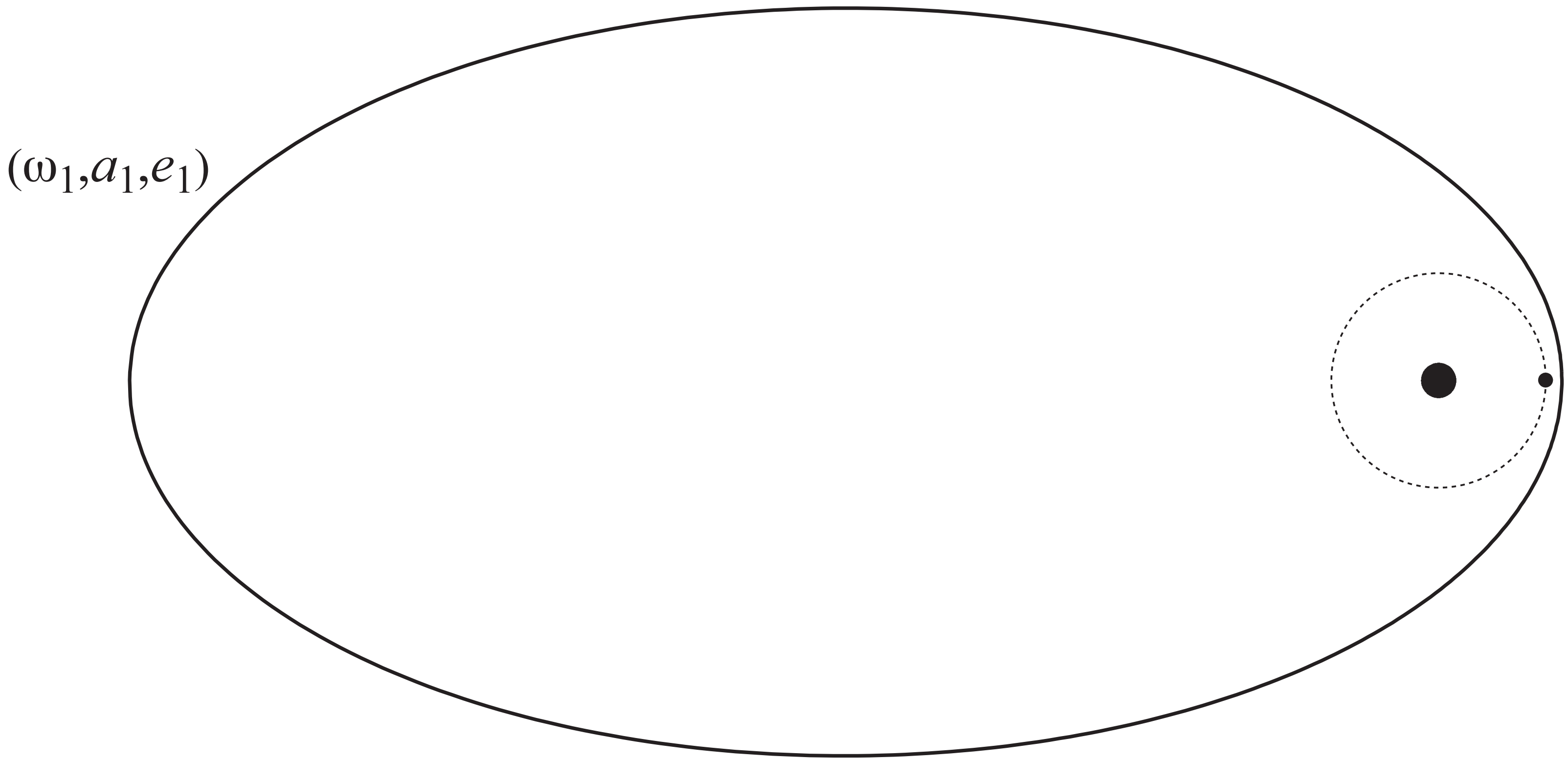




In rotating frame where m_1, m_2 are fixed

Kicks at periapsis

- Construct **update map** $(\omega_1, a_1, e_1) \mapsto (\omega_2, a_2, e_2)$ using average perturbation per orbit by smaller mass



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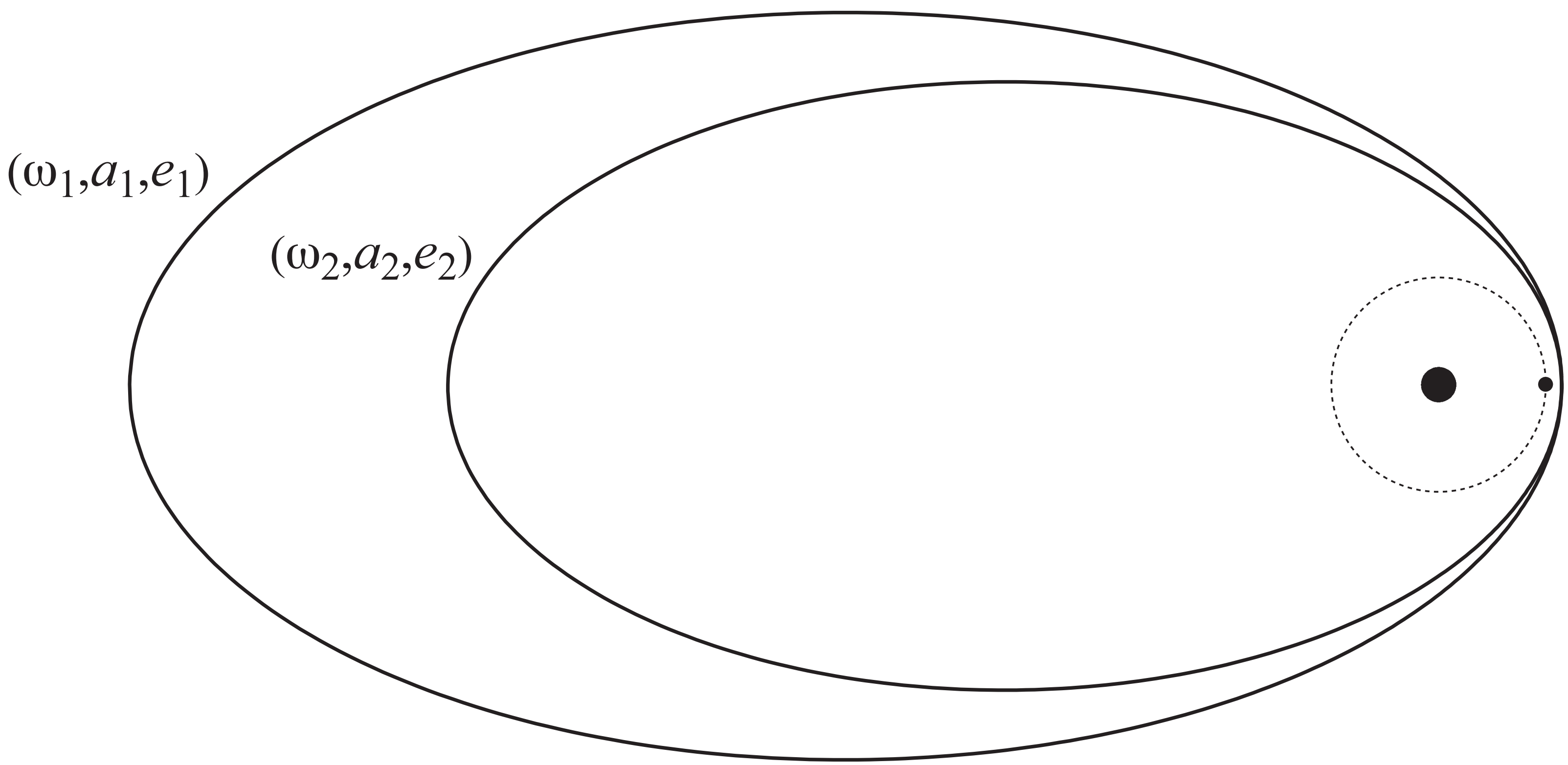
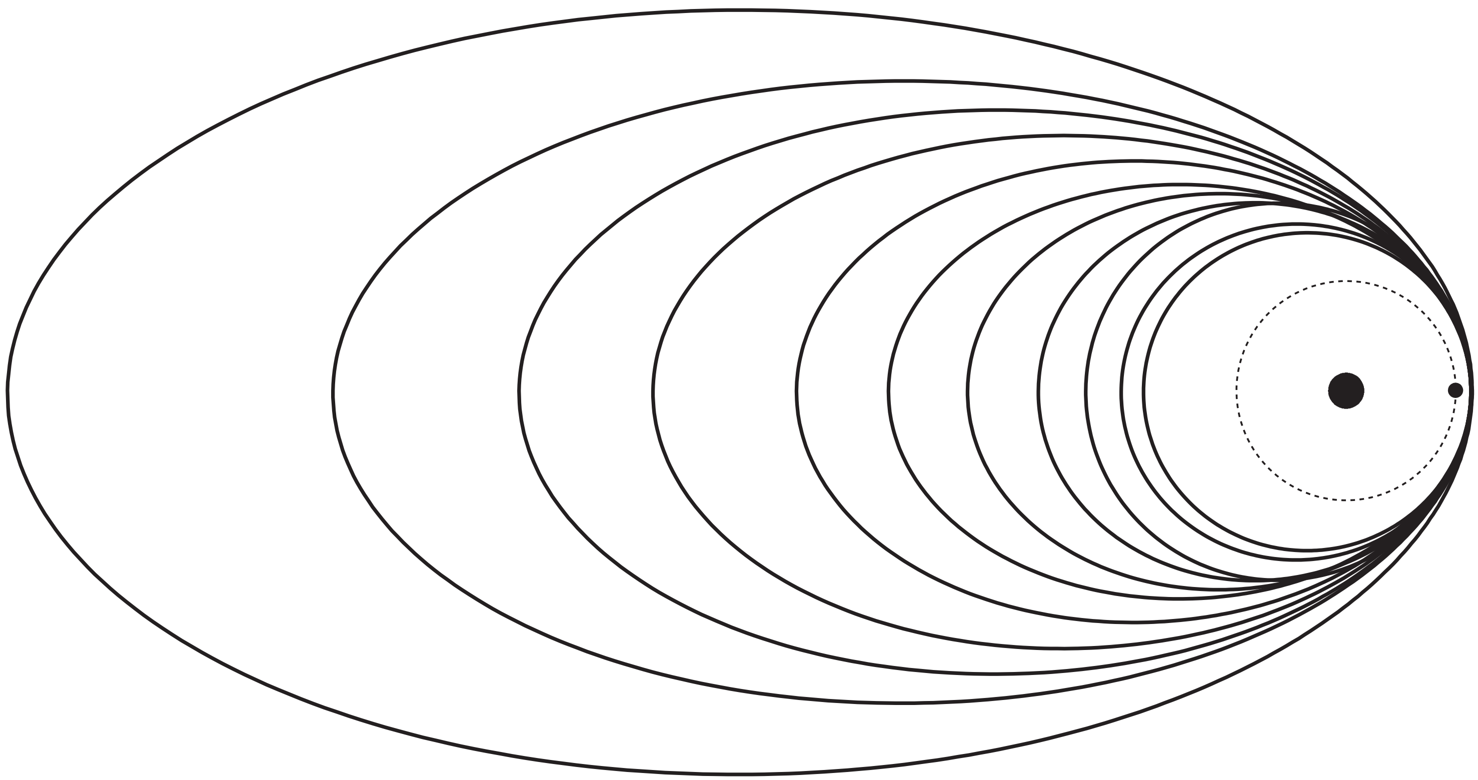


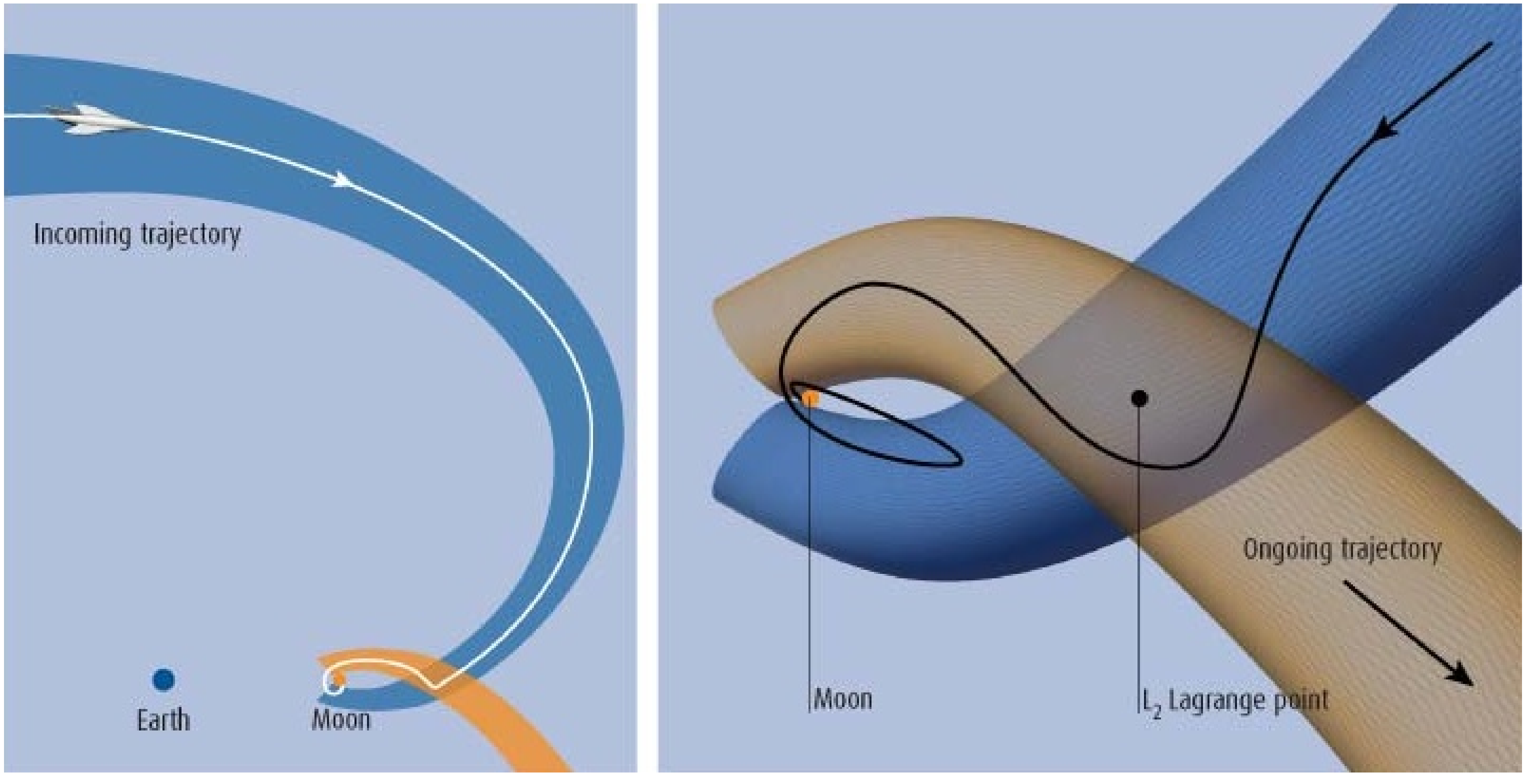
Not hyperbolic swing-by

- Occur **outside sphere of influence** (Hill radius) – not the close, hyperbolic swing-bys of Voyager



Capture by secondary

- Dynamically connected to capture thru tubes



Capture by secondary

- Particle assumed on **near-Keplerian orbit** around m_1
- In the frame co-rotating with m_2 and m_1,

$$H_{\mathrm{rot}}(l, \omega, L, G) = K(L) + \mu R(l, \omega, L, G) - G,$$

in Delaunay variables

- Evolution is Hamitlon's equations:

$$\frac{d}{dt}(l, \omega, L, G) = f(l, \omega, L, G)$$

- Jacobi constant, $C_J = -2H_{\mathrm{rot}}$ conserved along trajectories

Change in orbital elements over one particle orbit

- Evolution of G (angular momentum)

$$\frac{dG}{dt} = -\mu \frac{\partial R}{\partial \omega},$$

- Picard's approximation:

$$\begin{aligned}\Delta G &= -\mu \int_{-T/2}^{T/2} \frac{\partial R}{\partial \omega}\, dt \\ &= -\frac{\mu}{G}\left[\left(\int_{-\pi}^{\pi}\left(\frac{r}{r_2}\right)^3 \sin(\omega+\nu-t(\nu))\; d\nu\right) - \sin\omega\left(2\int_0^{\pi}\cos(\nu-t(\nu))\; d\nu\right)\right]\end{aligned}$$

- $\Delta K =$ **Keplerian energy change** over an orbit

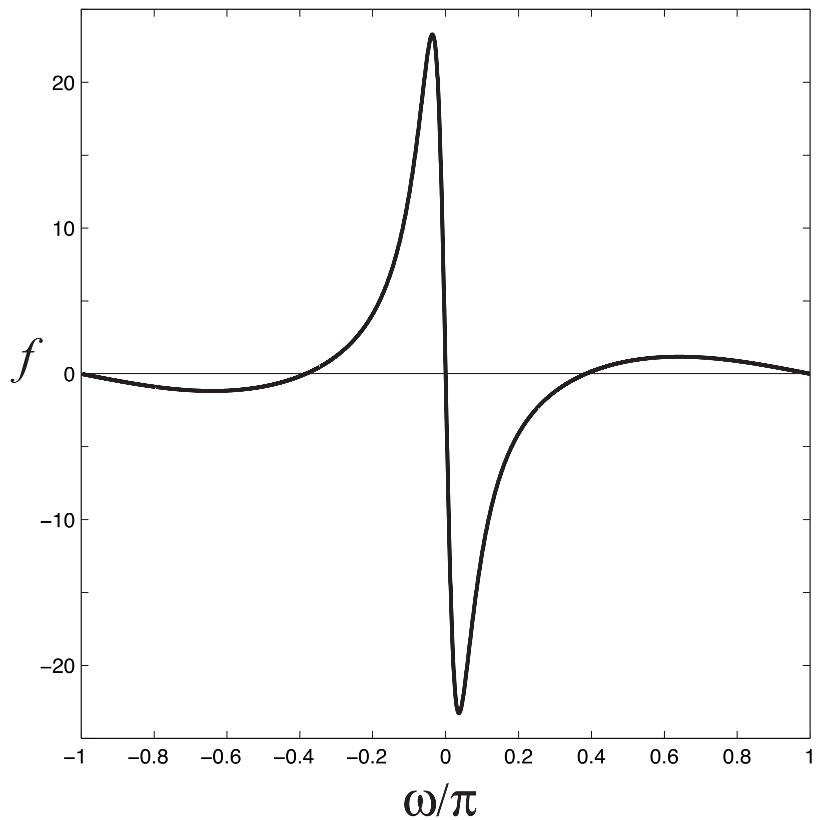
$$\Delta K = \Delta G - \mu \Delta R$$

Energy kick function

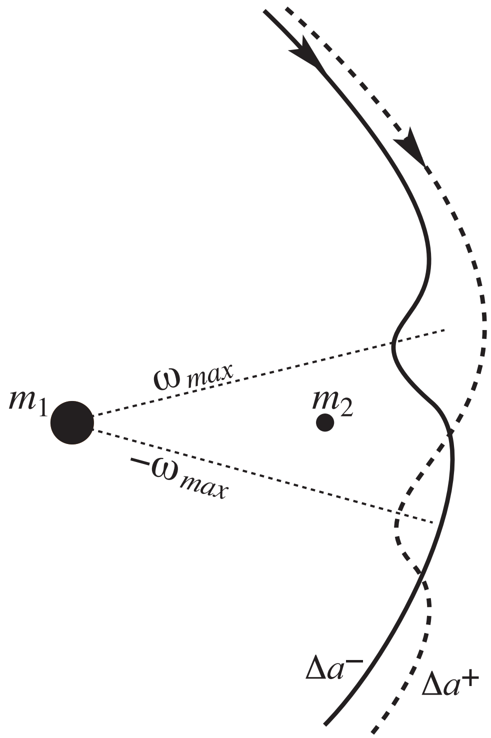
- Changes have form

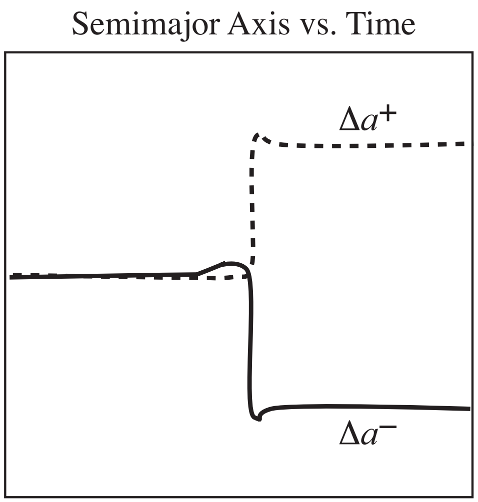
$$\Delta K = \mu f(\omega),$$

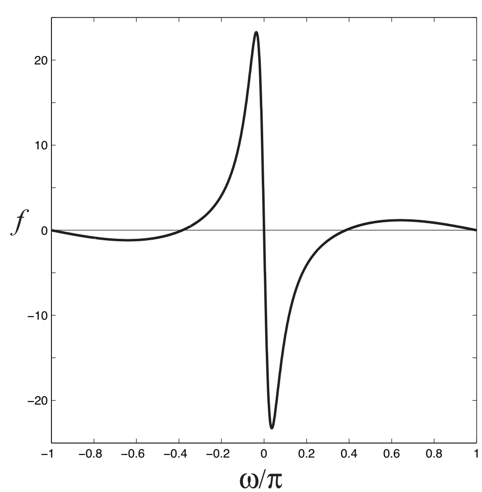
f is the **energy kick function** with parameters K, C_J



Maximum changes on either side of perturber







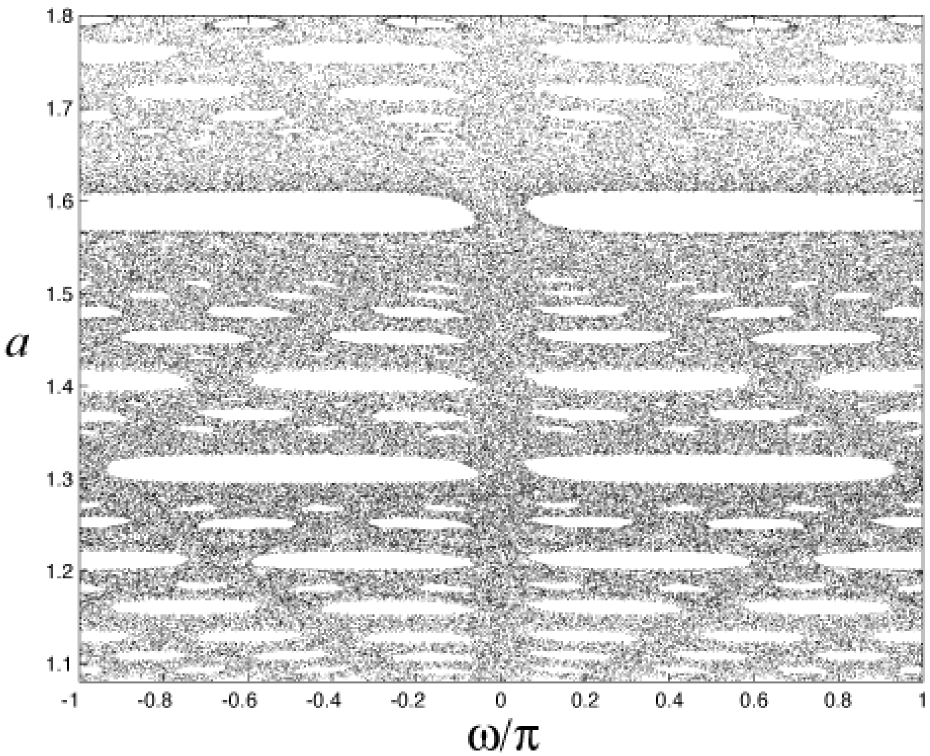
The periapsis kick map (Keplerian Map)

- Cumulative effect of **consecutive passes** by perturber
- Can construct an **update map** $(\omega_{n+1}, K_{n+1}) = F(\omega_n, K_n)$ on the cylinder $\Sigma = S^1 \times \mathbb{R}$, i.e., $F : \Sigma \to \Sigma$ where

$$\begin{pmatrix} \omega_{n+1} \\ K_{n+1} \end{pmatrix} = \begin{pmatrix} \omega_n - 2\pi(-2(K_n + \mu f(\omega_n)))^{-3/2} \\ K_n + \mu f(\omega_n) \end{pmatrix}$$

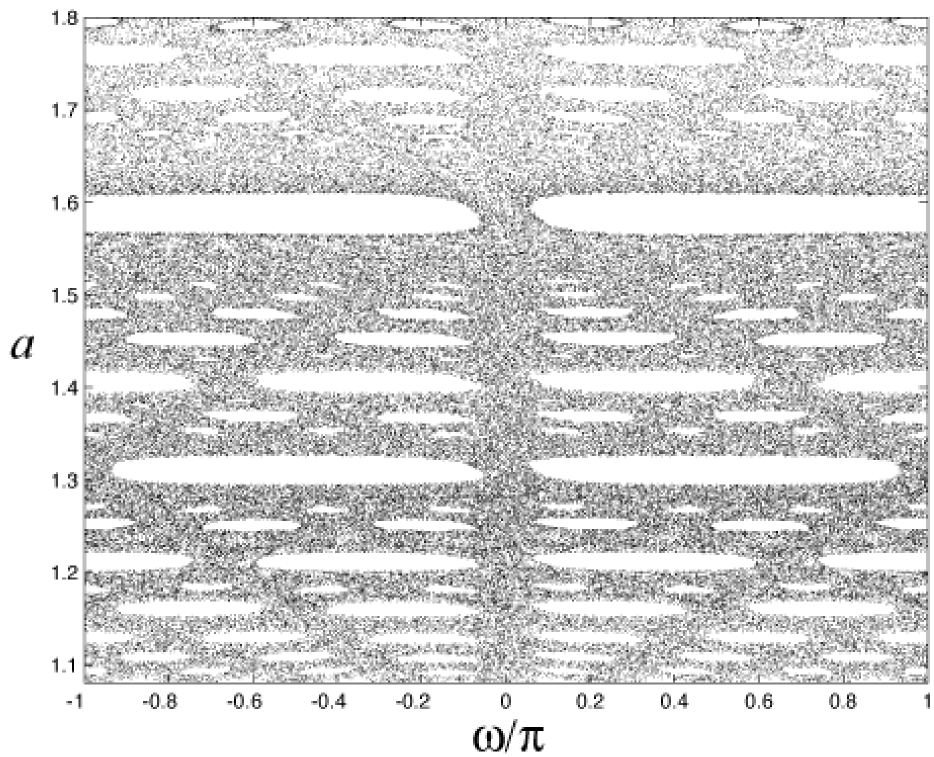
- **Area-preserving (symplectic twist) map**
- Example: particle in Jupiter-Callisto system $\mu = 5 \times 10^{-5}$

Verification of Keplerian map: phase portrait

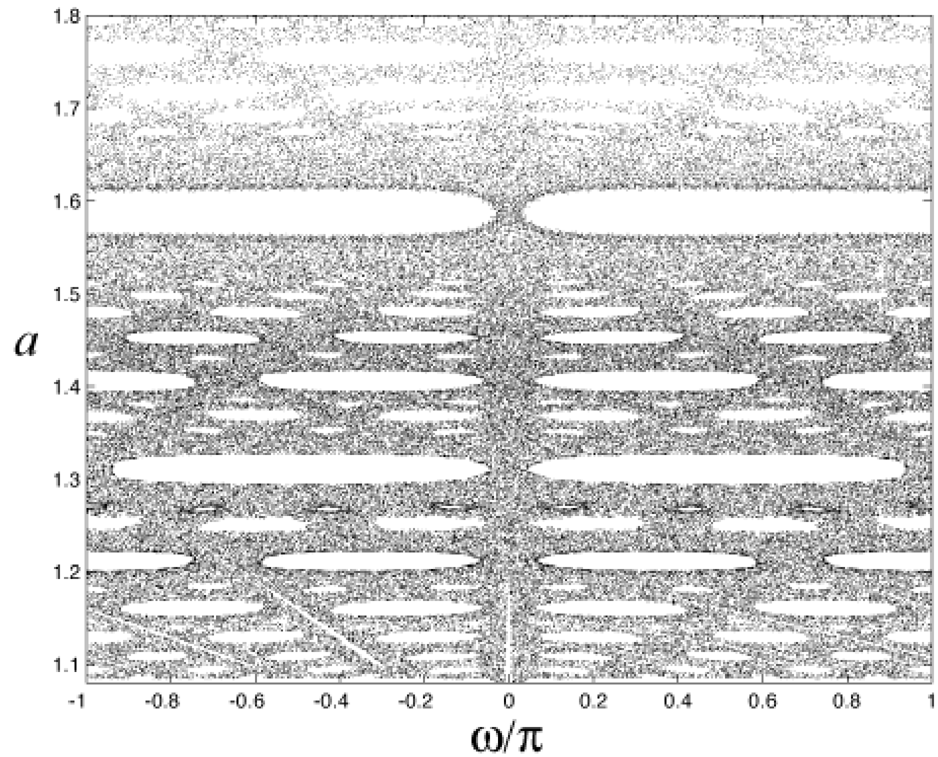


Keplerian map

Verification of Keplerian map: phase portrait



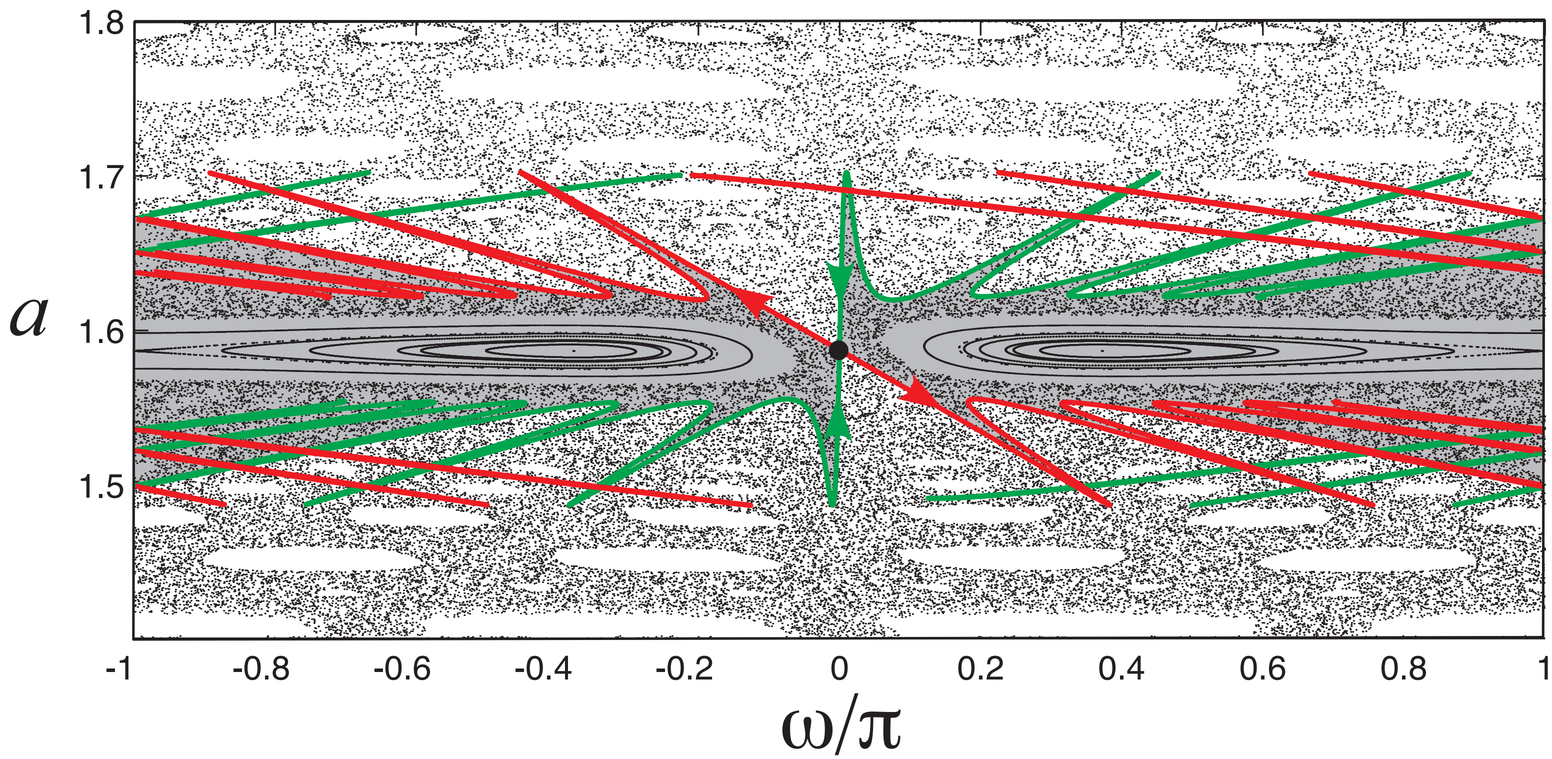
Keplerian map



numerical integration of ODEs

- Keplerian map = fast orbit propagator
- preserves phase space features
  — but breaks left-right symmetry present in original system
  — can be removed using another method (Hamilton-Jacobi)

Dynamics of Keplerian map

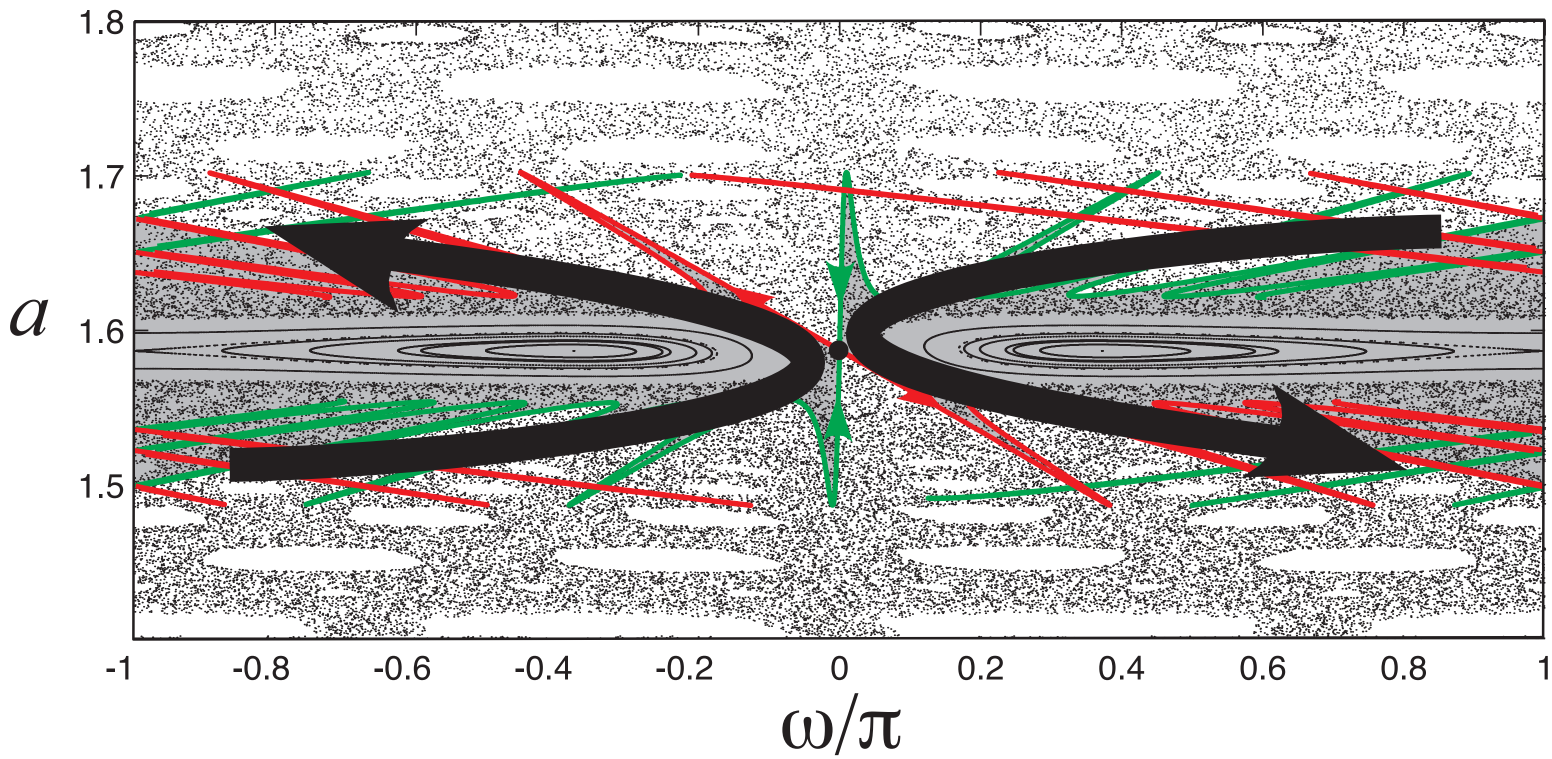


Resonance zone[3]

- Structured motion around resonance zones

[3]in the terminology of MacKay, Meiss, and Percival [1987]

Dynamics of Keplerian map



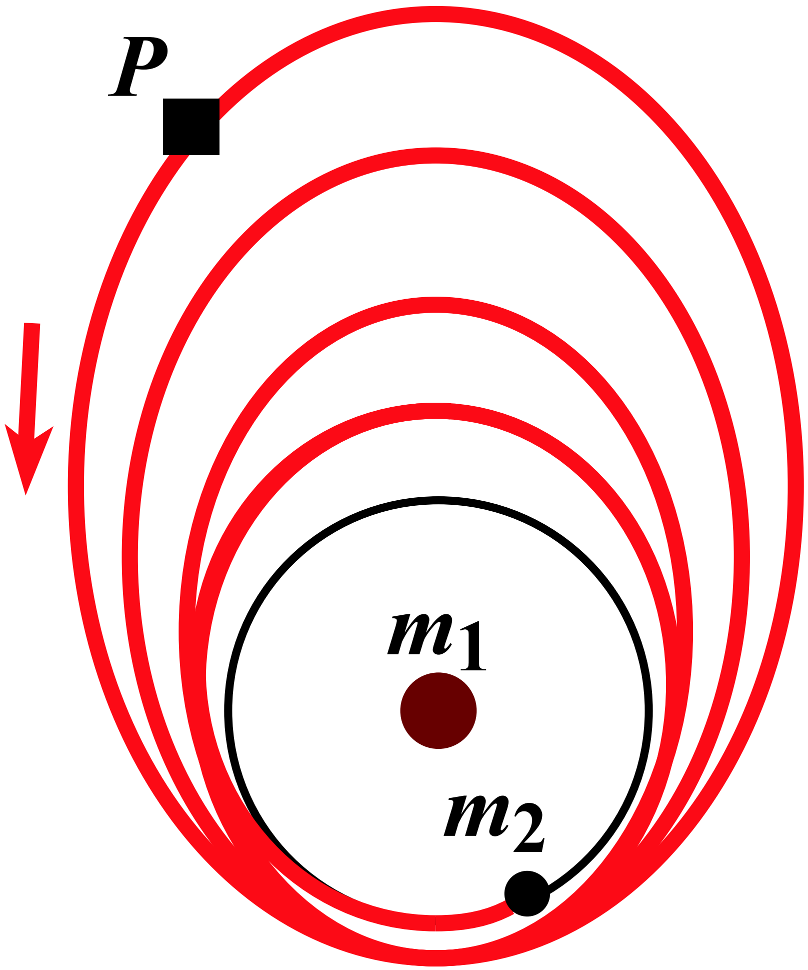
Resonance zone[4]

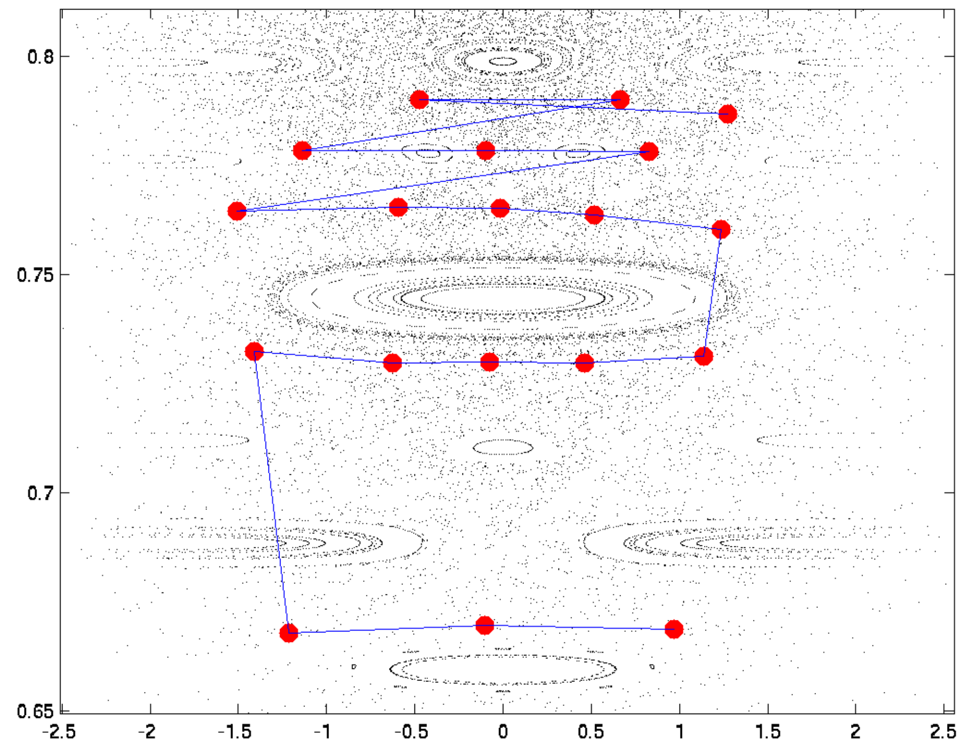
- Structured motion around resonance zones

[4]in the terminology of MacKay, Meiss, and Percival [1987]

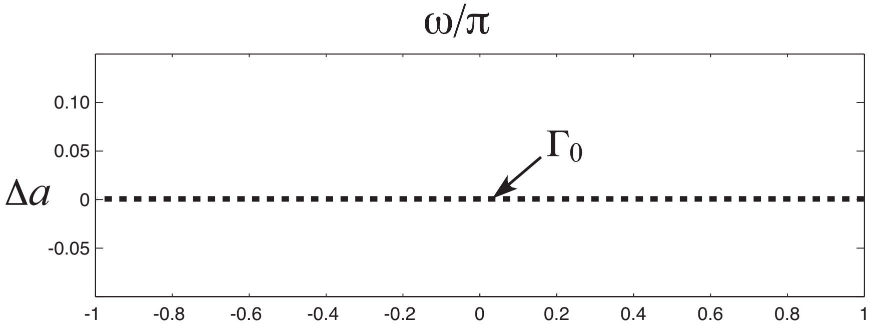
Large orbit changes via multiple resonance zones

- **multiple flybys** for **orbit reduction or expansion**





Large orbit changes, $\Gamma_n = F^n(\Gamma_0)$



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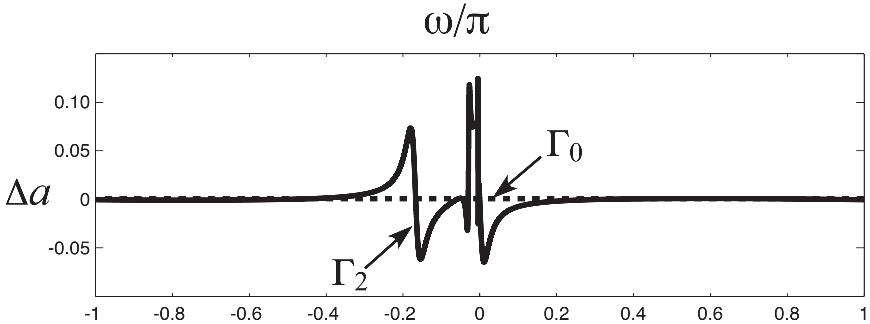

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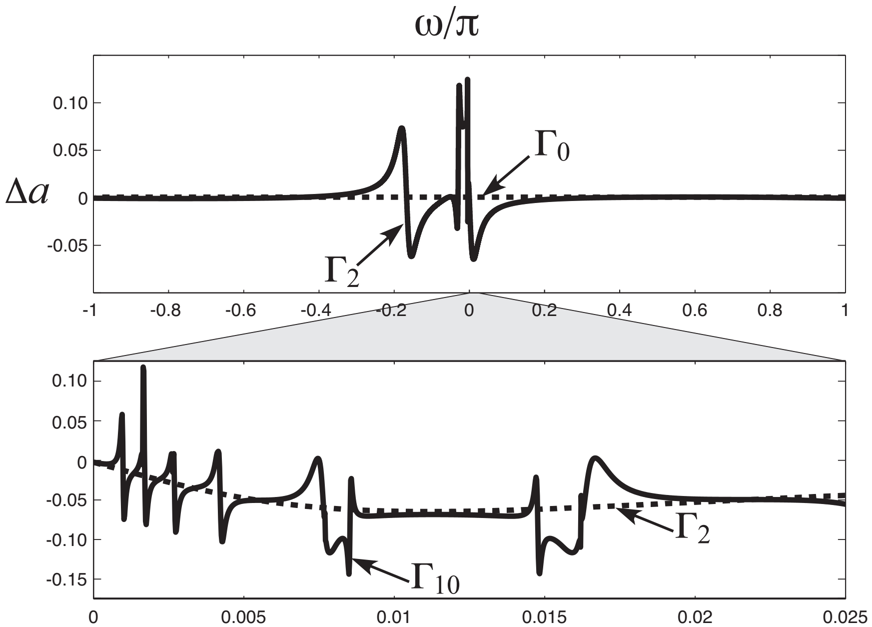

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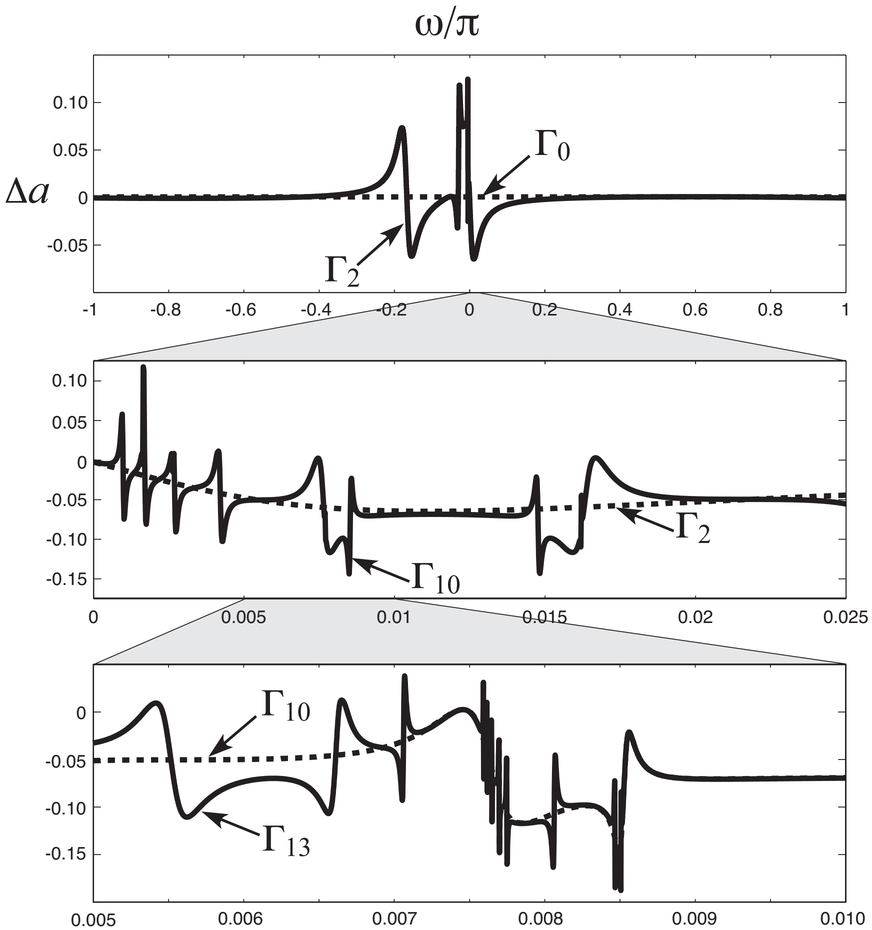

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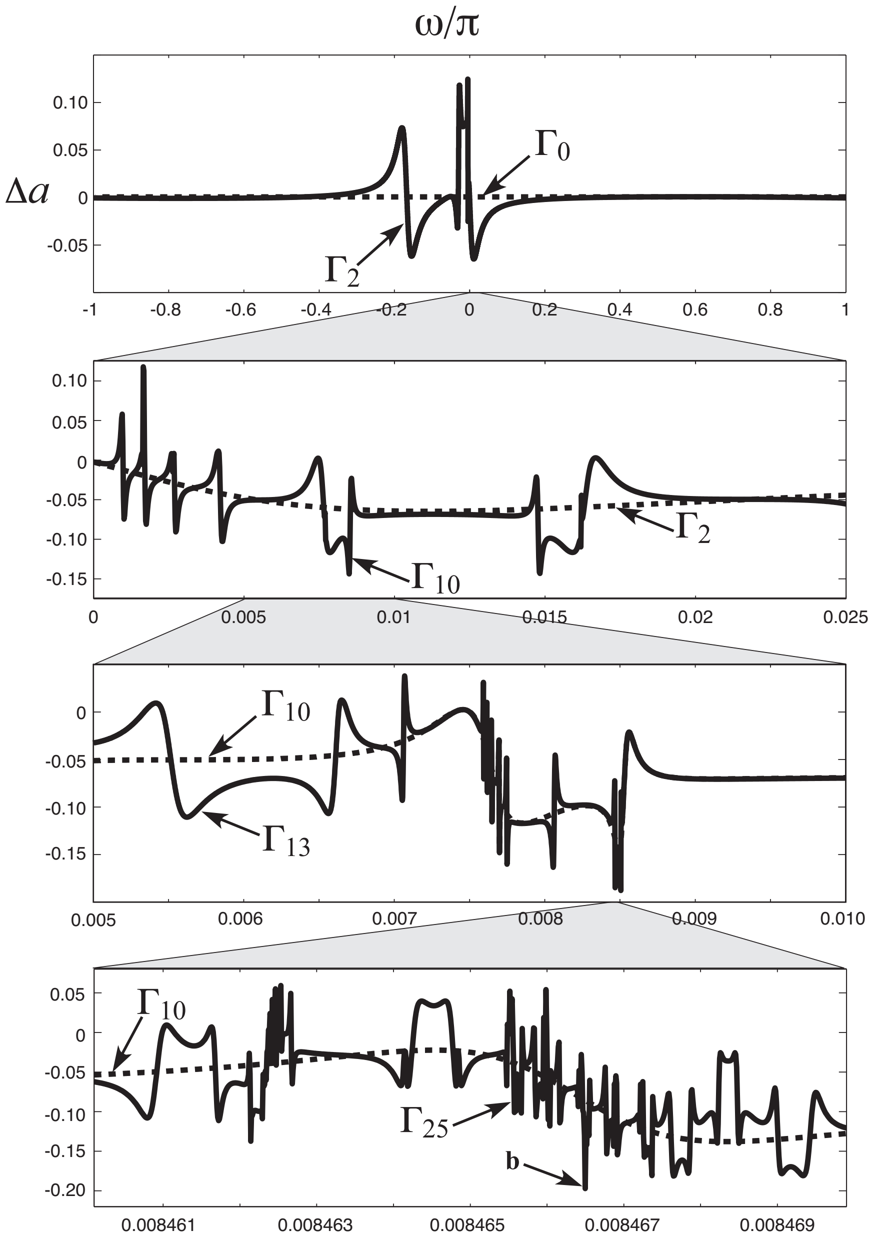

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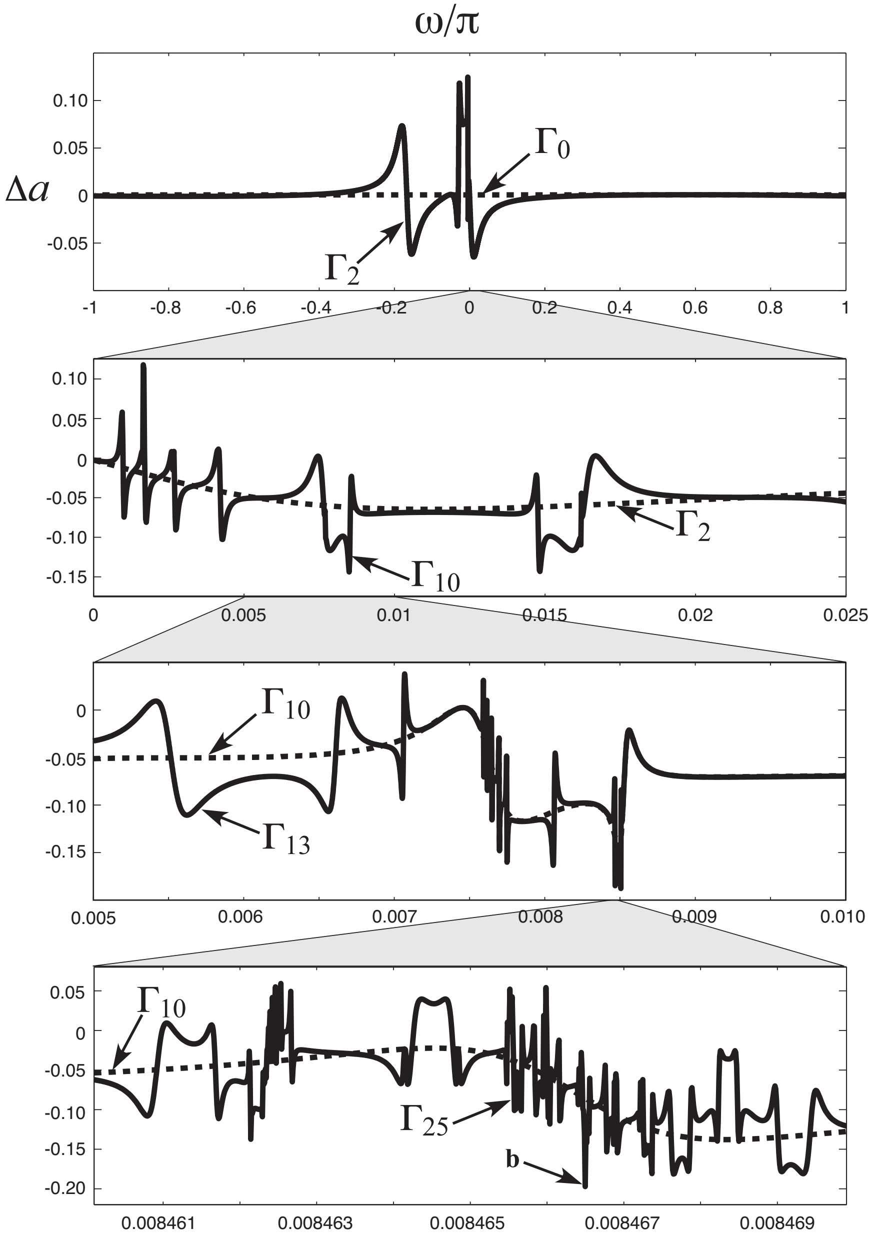
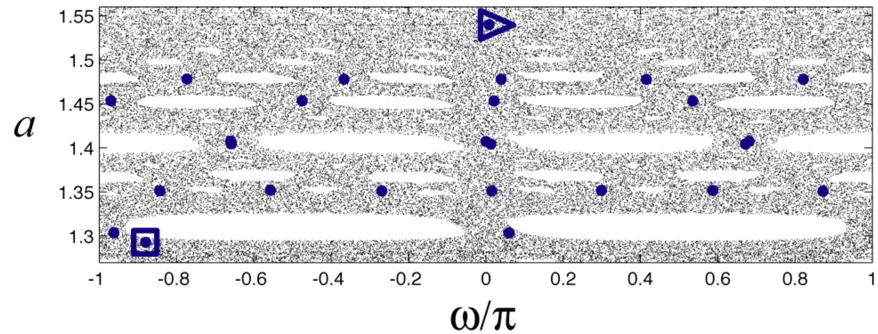
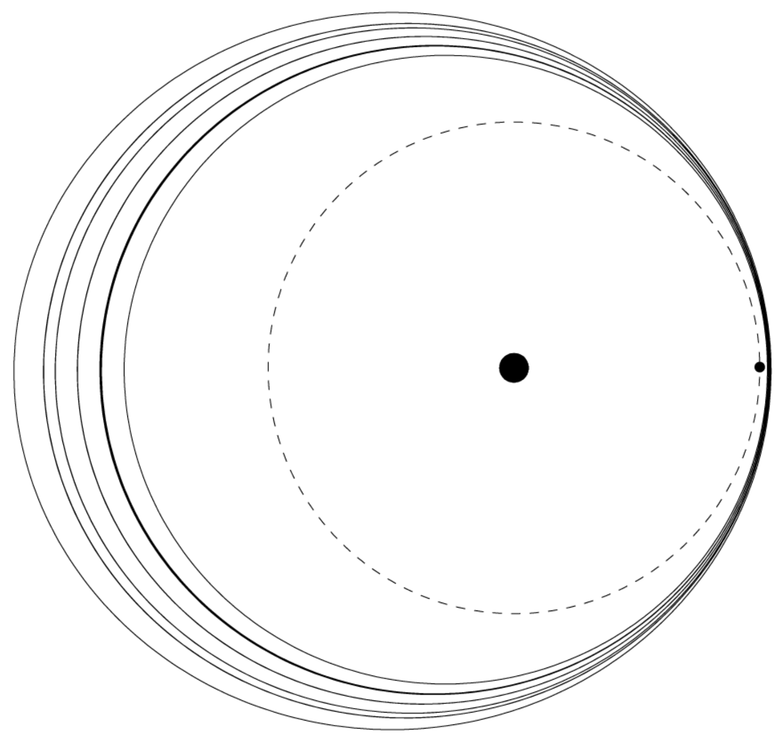
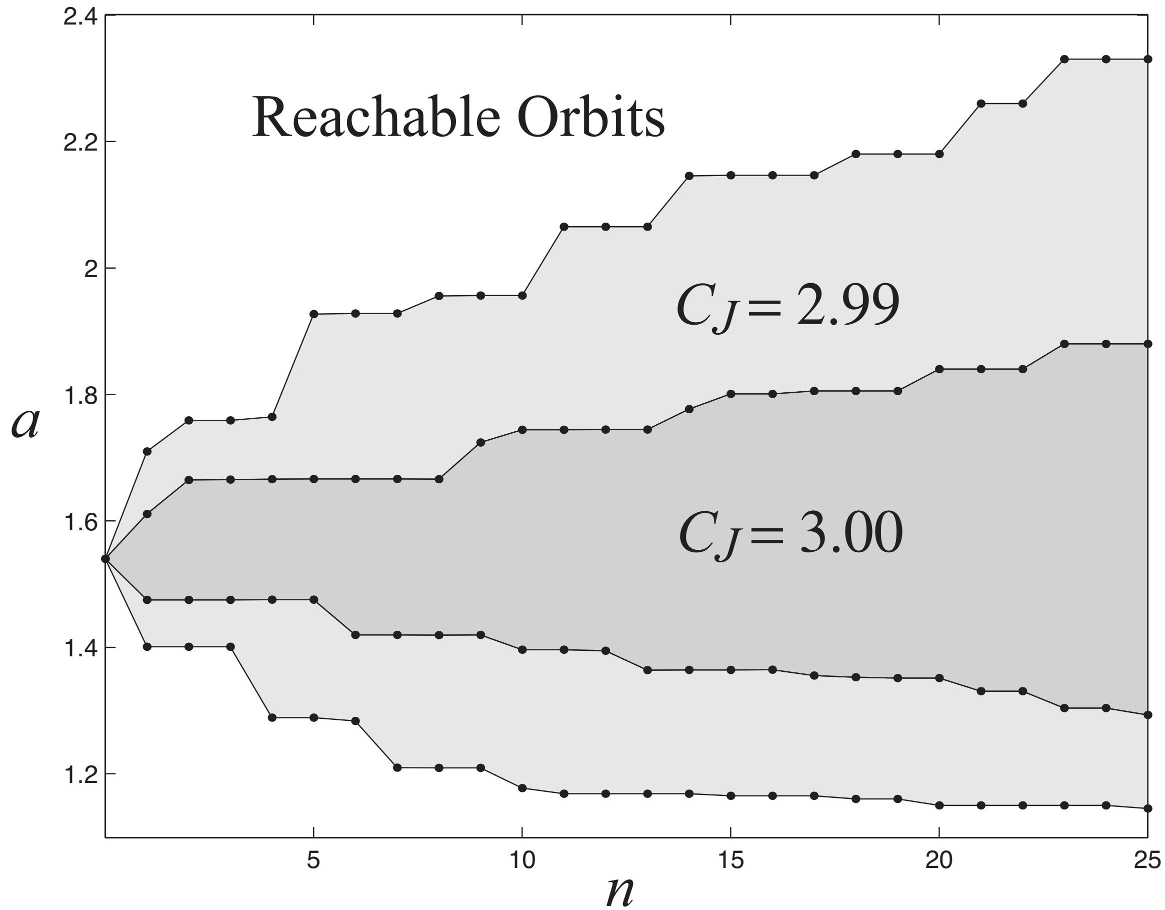






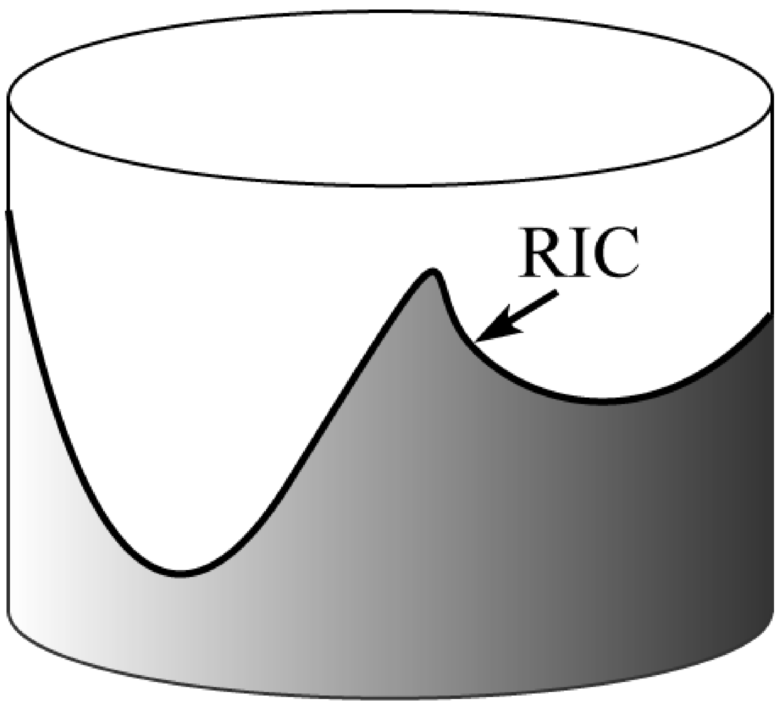
example trajectory

Reachable orbits and diffusion

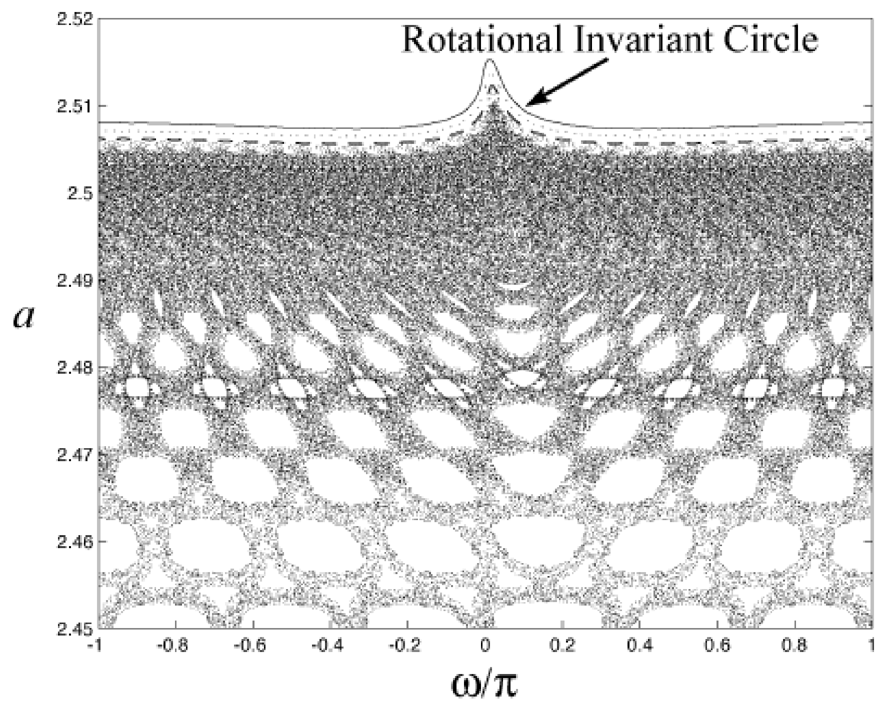


- Diffusion in semimajor axis
- ... increases as C_J decreases (larger kicks)

Reachable orbits: upper boundary for small μ

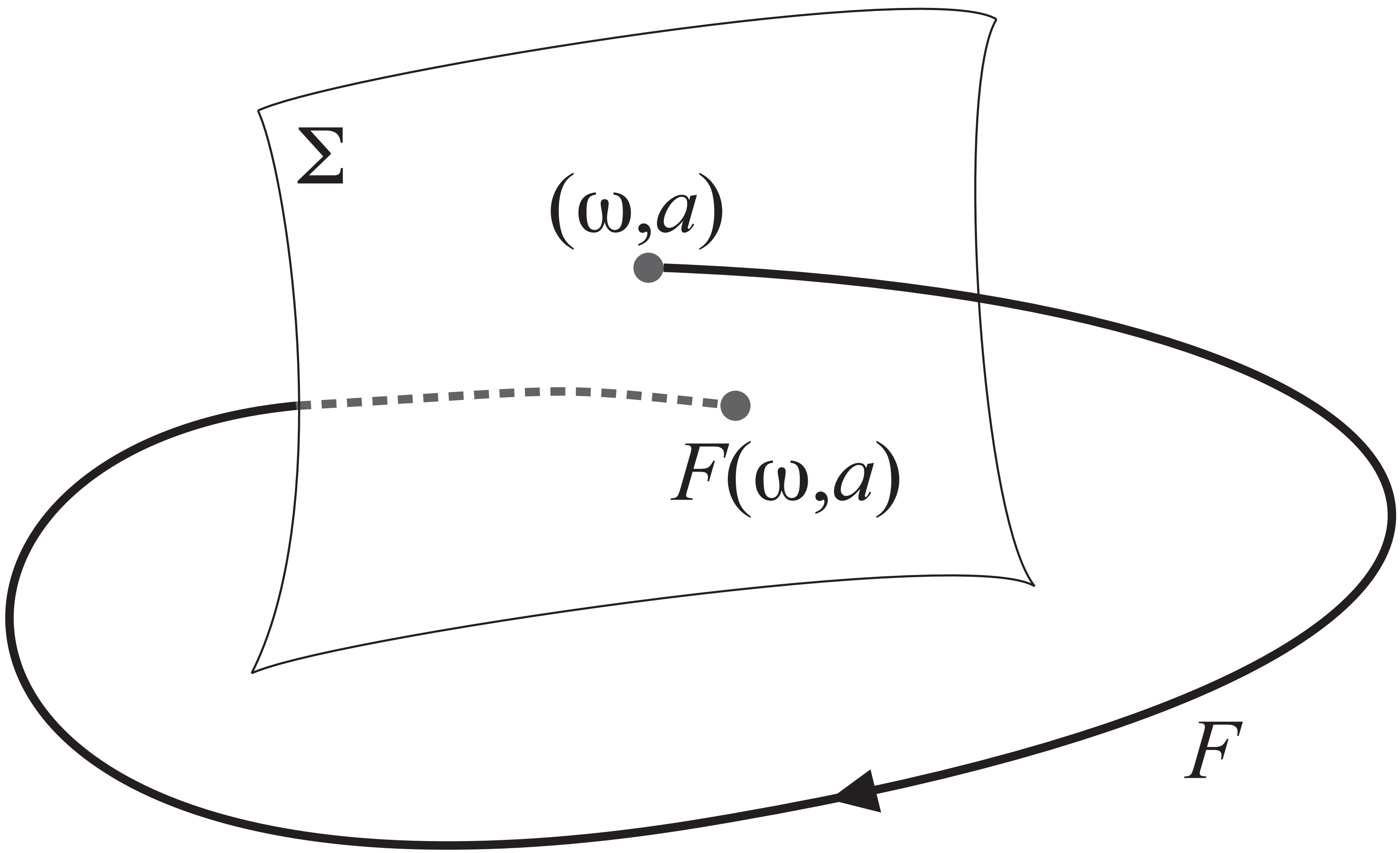


A rotational invariant circle (RIC)



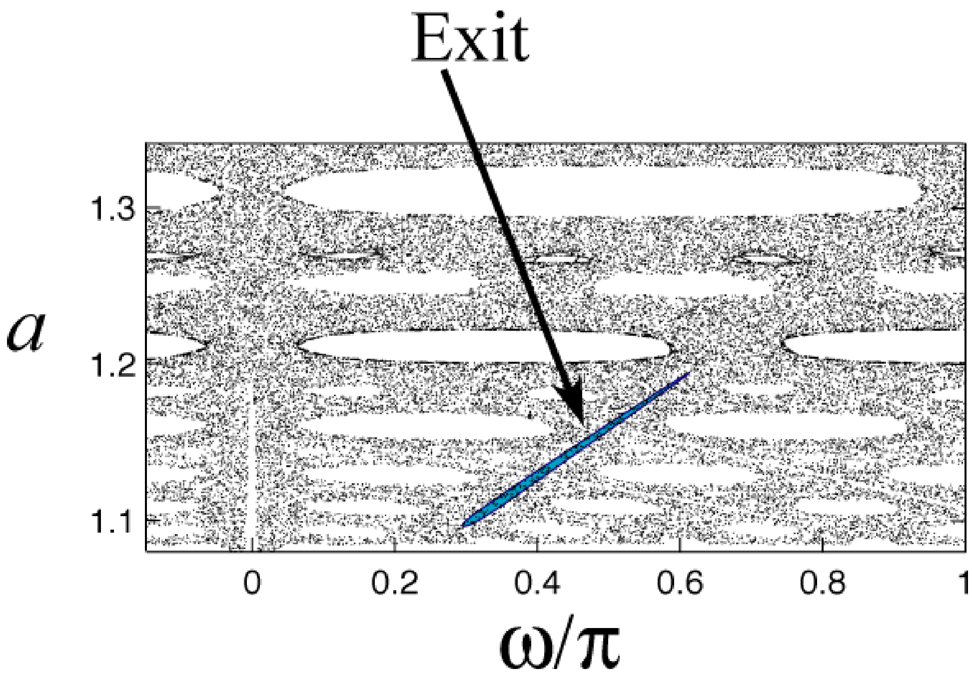
RIC found in Keplerian map for $\mu = 5 \times 10^{-6}$

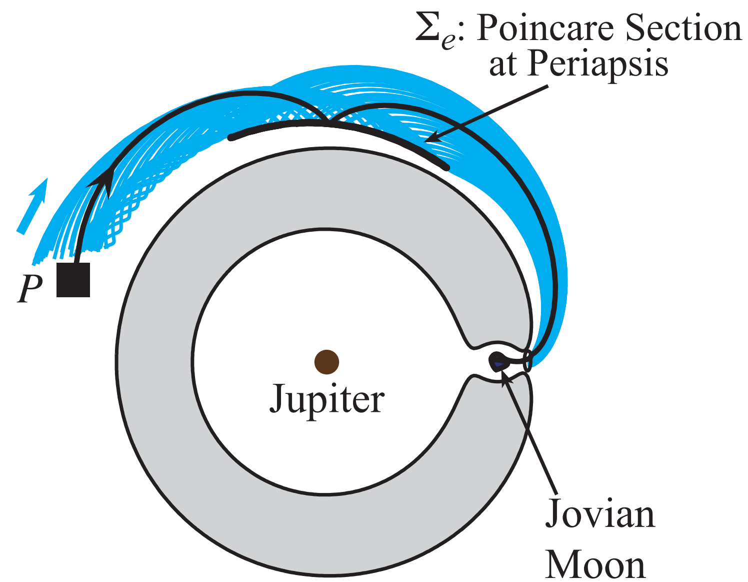
Identify Keplerian map as Poincaré return map



- **Poincaré map at periapsis** in orbital element space
- $F : \Sigma \rightarrow \Sigma$ where $\Sigma = \{l = 0 \mid C_J = \text{constant}\}$

Relationship to capture around perturber

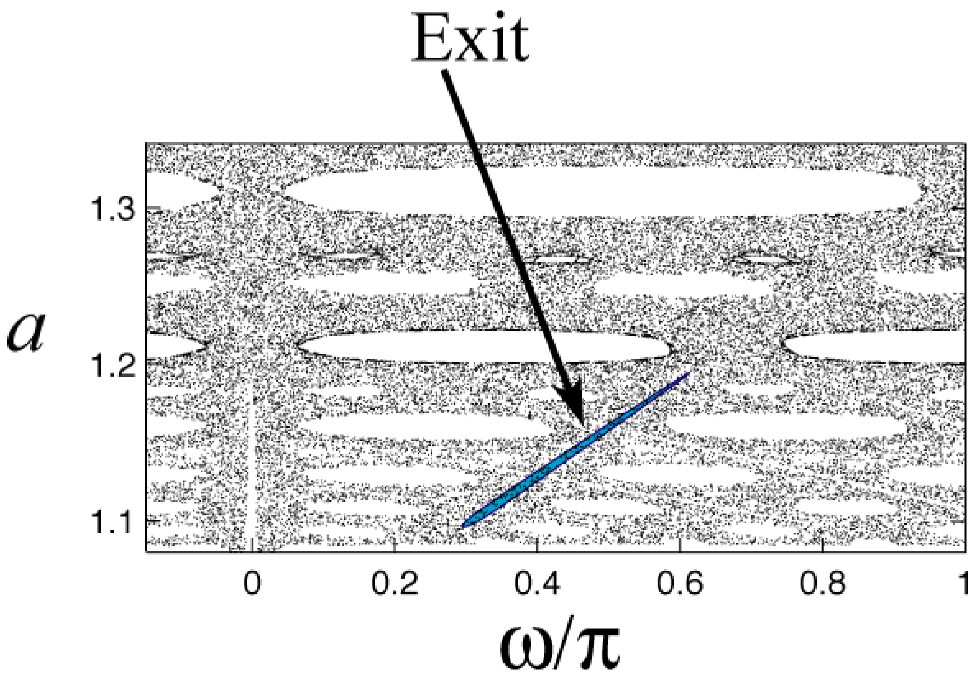




exit from jovicentric to moon region

- **Exit**: where tube of capture orbits intersects Σ

Relationship to capture around perturber



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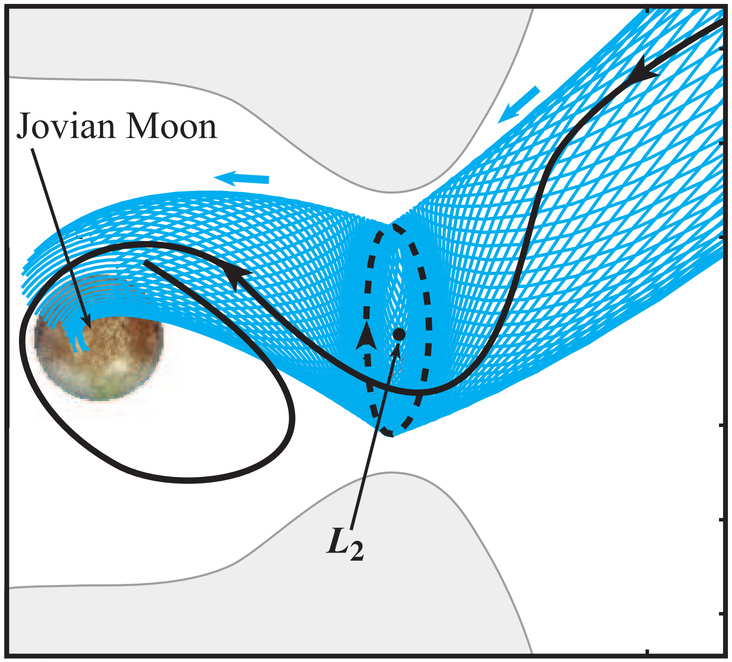
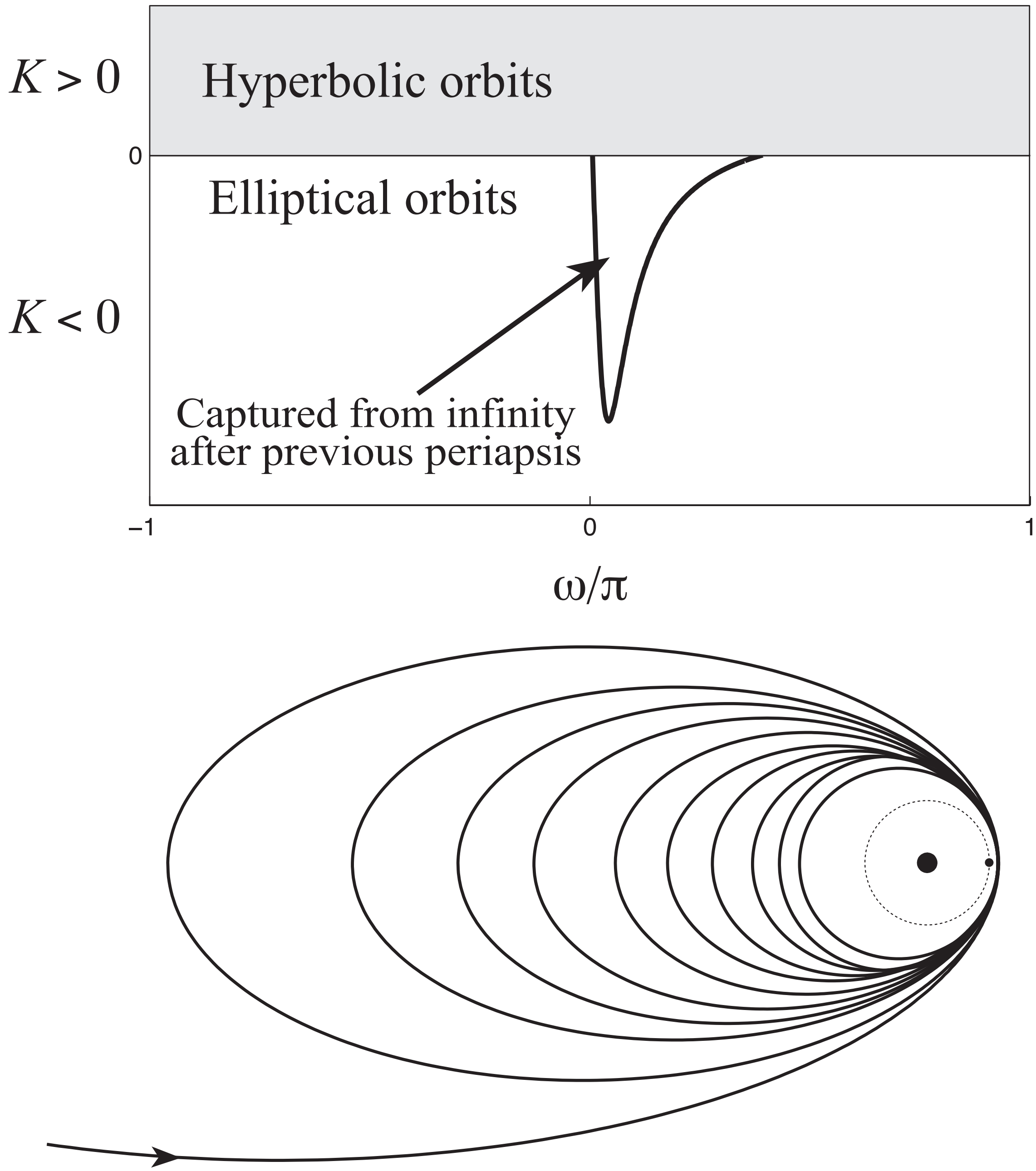


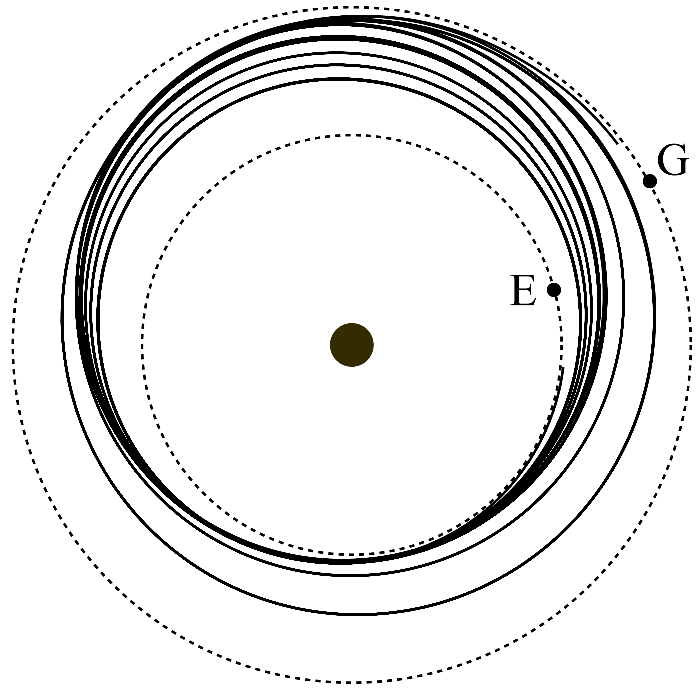
- **Exit**: where tube of capture orbits intersects Σ
- Orbits reaching exit are **ballistically captured**, passing by L_2

Relationship to capture from infinity



Final word about Keplerian map

- Extensions:
  - out of plane motion (**4D map**)
  - control in the presence of uncertainty
  - eccentric orbits for the perturbers
  - multiple perturbers
    transfer from one body to another

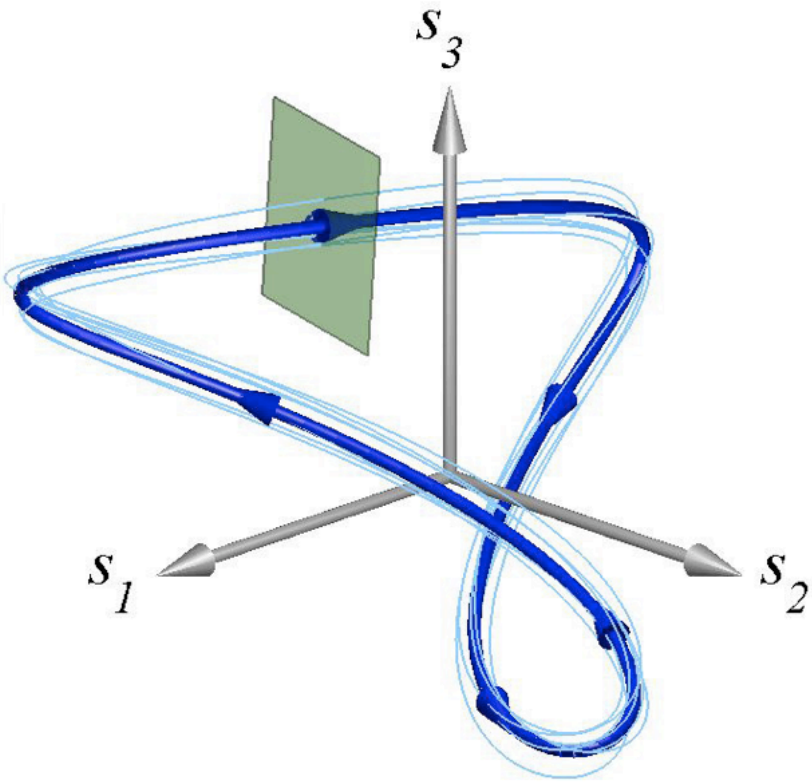


- Consider other problems with localized perturbations?
  – chemistry, vortex dynamics, ...

Reference:
Ross & Scheeres, *SIAM J. Applied Dynamical Systems*, 2007.

Separatrices: biomechanics

- **Boundaries between qualitatively (functionally) different kinds of behavior**
- **For example, walking or standing versus falling**
- **Based on analytical models or experimentally observed data**



Planar 2-dof model of biped walking

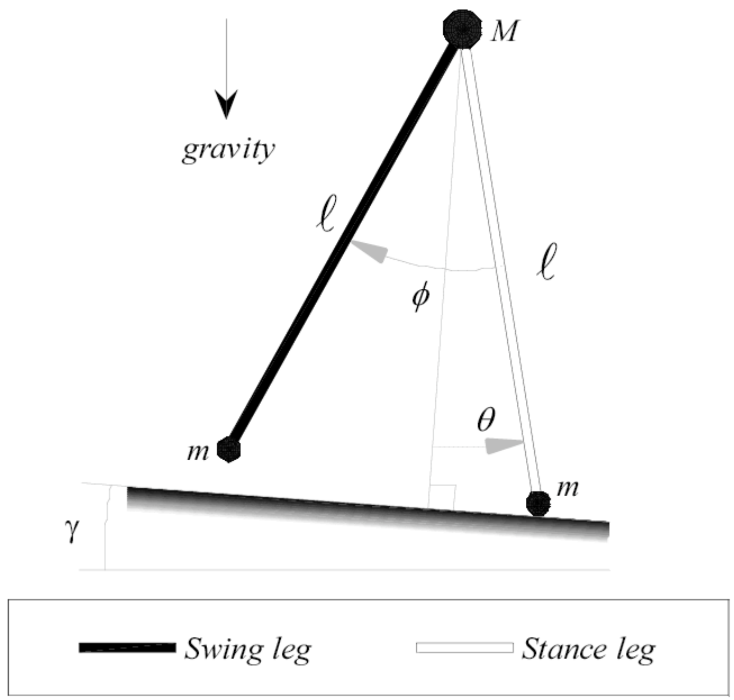
Two segment "compass biped" walker[1]

- **Simplest model of walking**
- **Double pendulum w/pin-joint at stance foot**
- **Point masses at hip and feet**
- **Massless legs of equal length**
- **Walks down slope**

- **Piecewise holonomic system**

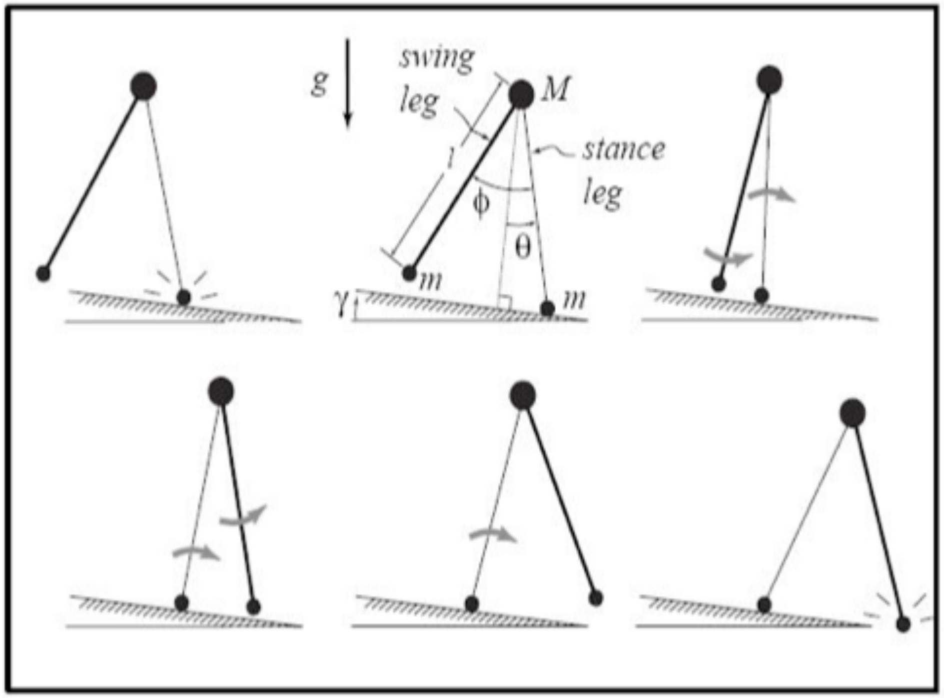
- **Swing phase: Hamiltonian dynamics**

- **Foot-strike event: discrete, dissipative**



[1]McGeer 1990. Garcia et al. 1998. Norris, Marsh, Granata, Ross, 2008, Physica D.

Planar 2-dof model of biped walking

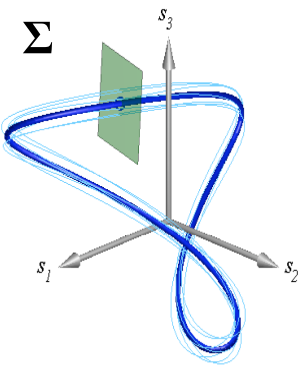


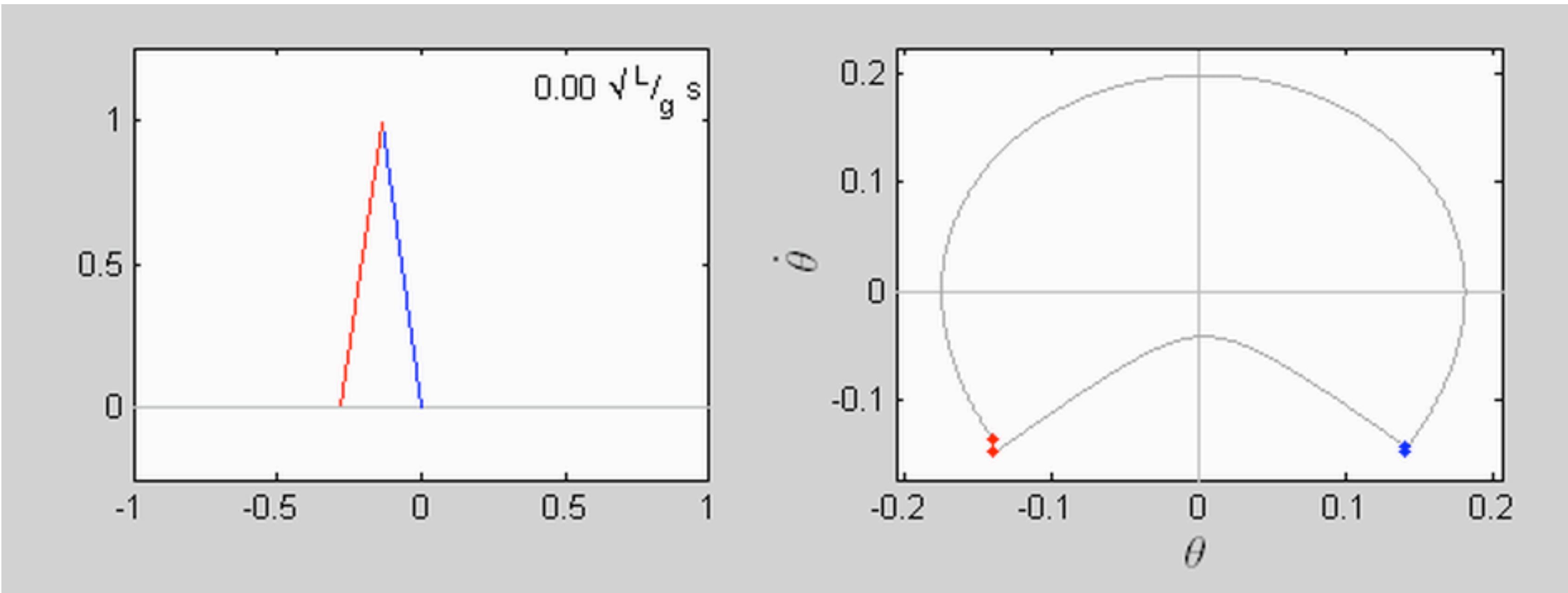
[1]McGeer 1990. Garcia et al. 1998. Norris, Marsh, Granata, Ross, 2008, Physica D.

Walking solutions

Poincare section at foot-strike

- **Search for period n solutions**
- **Period one solution shown**

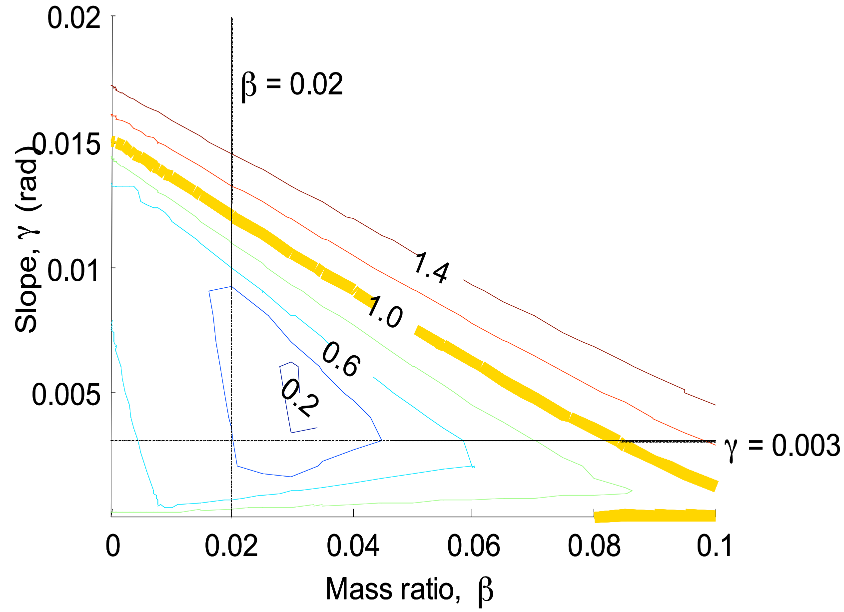




Walking solution stability

Orbital stability

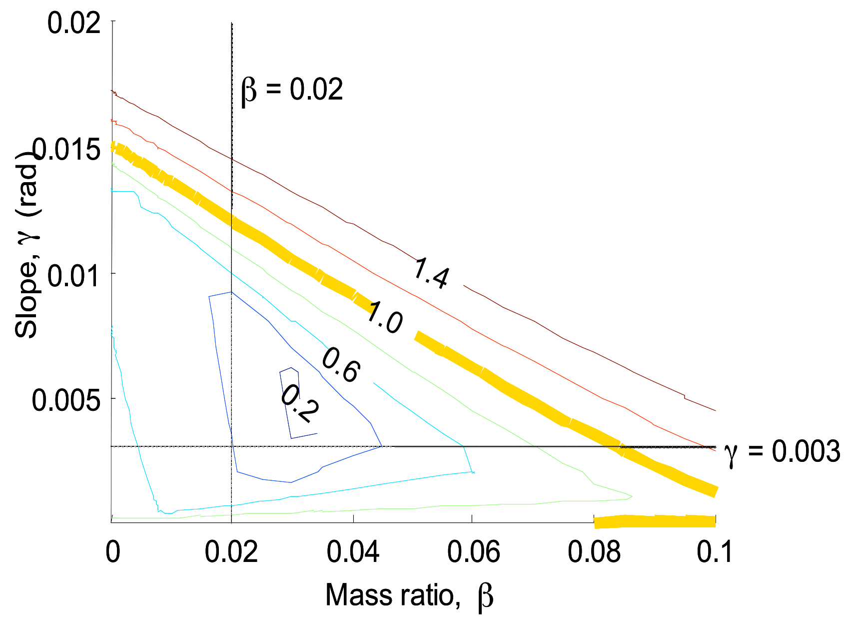
- **Maximum Floquet multipliers shown**

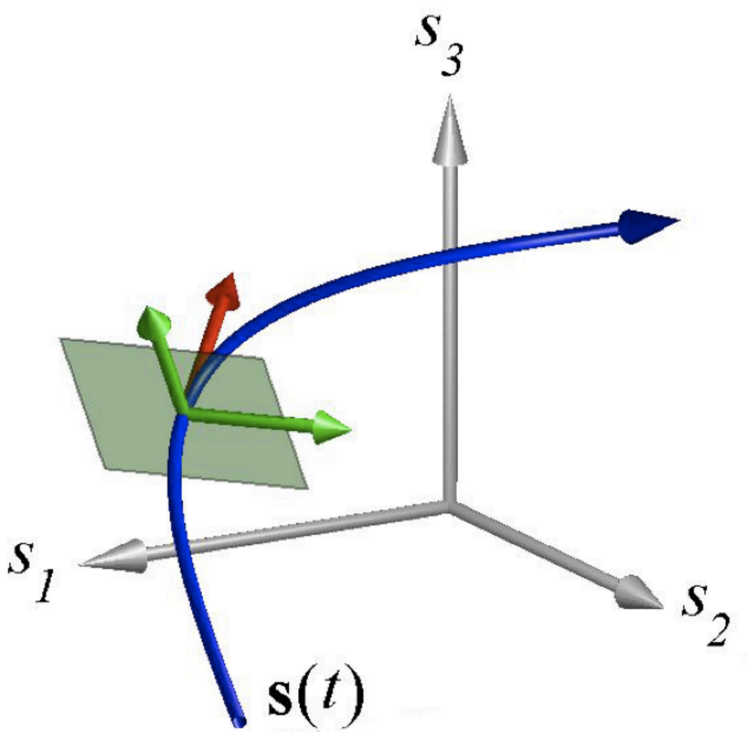


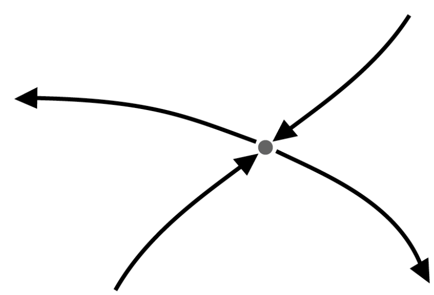
Walking solution stability

Orbital stability

- Maximum Floquet multipliers shown
- Swing phase has one dimensional unstable direction







Walking solution stability

Orbital stability

- Maximum Floquet multipliers shown
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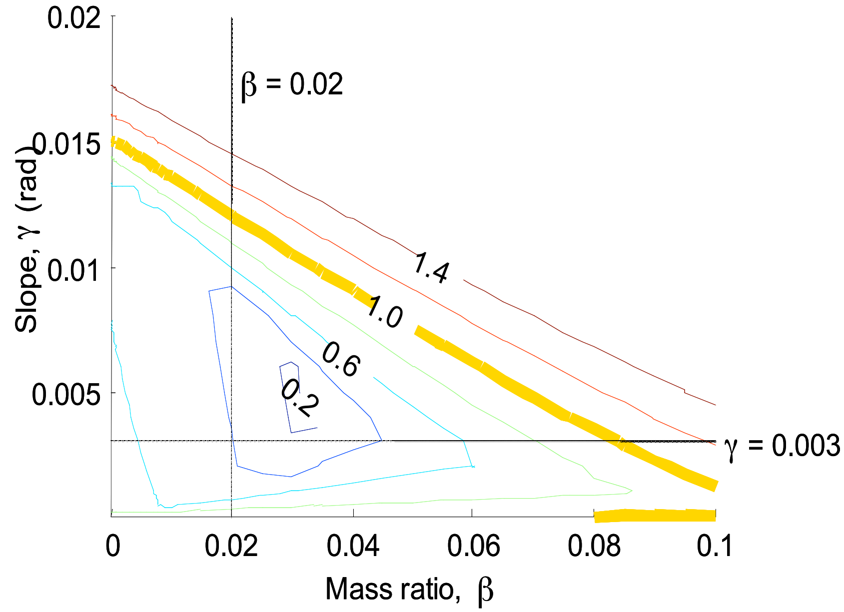
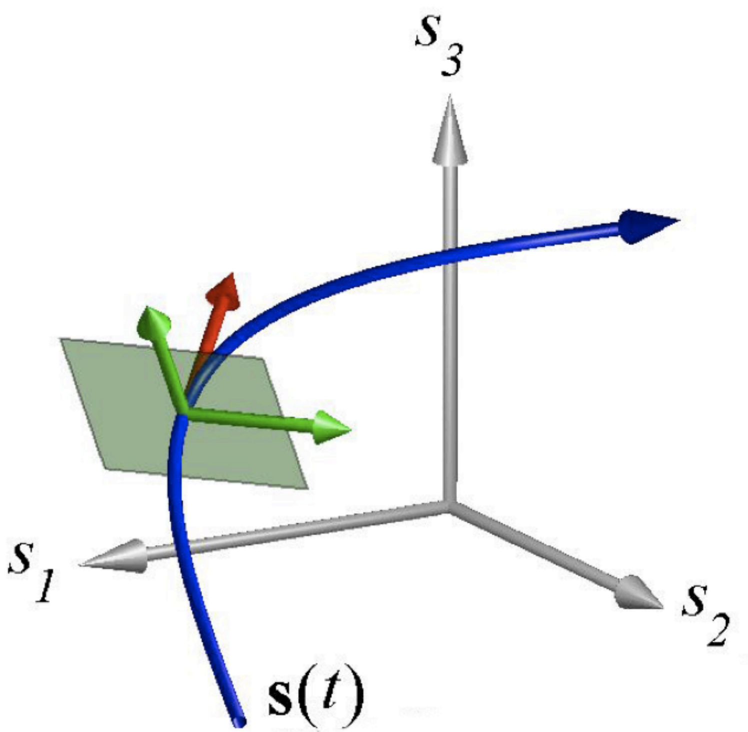
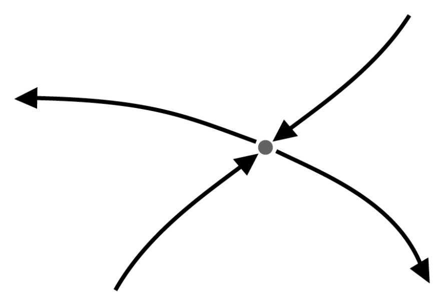
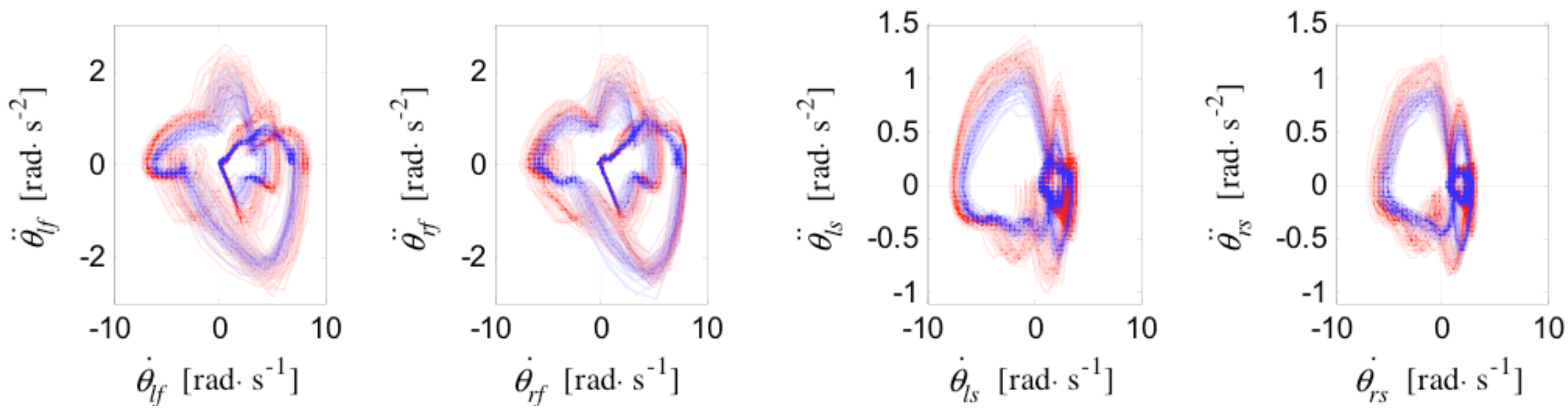






- Dissipation at foot-strike can lead to overall stability

Analysis of bipedal walking

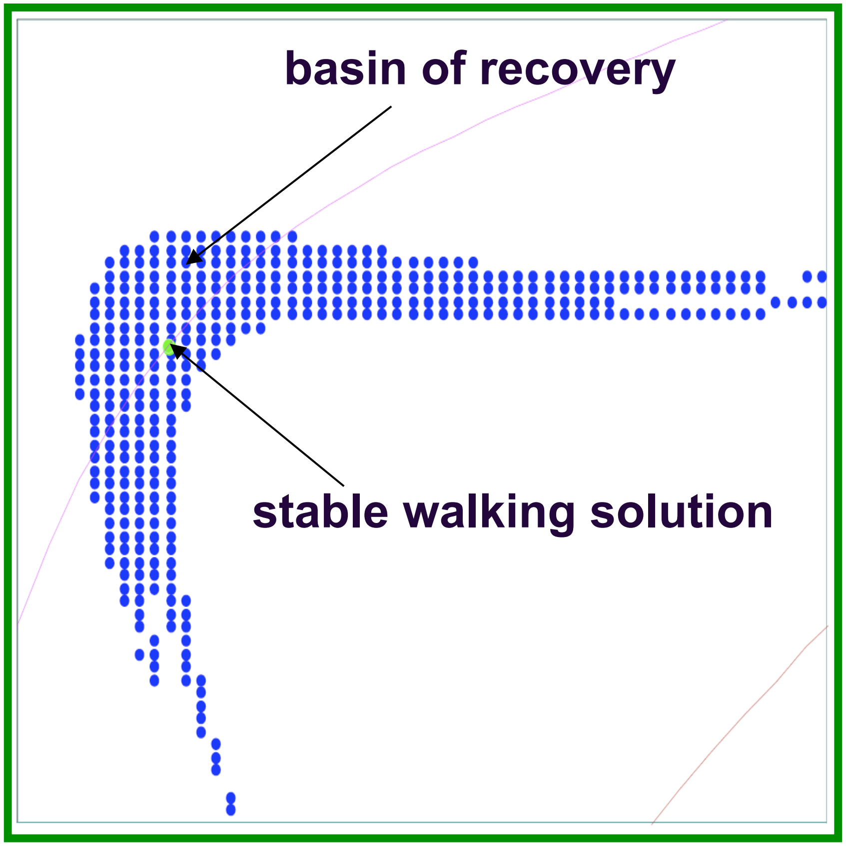


- **Walking gait data -- nearly cyclic in**
- **high dimensional phase space**

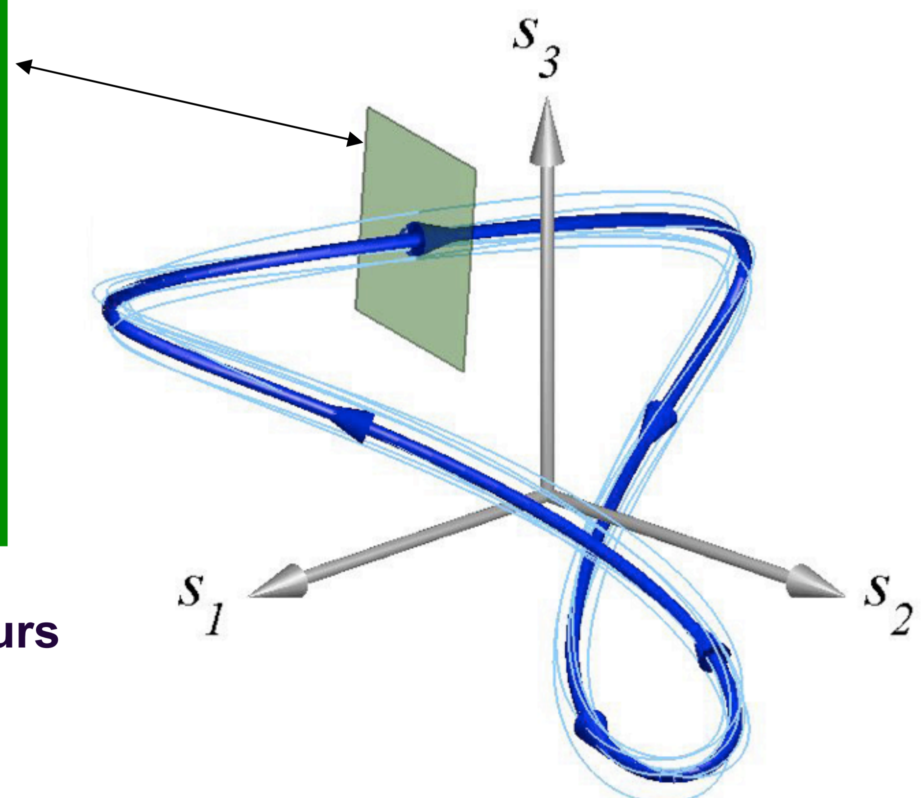
- **Orbital stability (maximum Floquet**
- **multiplier) typically around 0.7**

Norris, Marsh, Granata, Ross, 2007.

Walking solution stability



- From 4D model of bipedal walking (Norris, Marsh, Granata, Ross 2008)
- Separatrix bounds region of recovery around walking solution

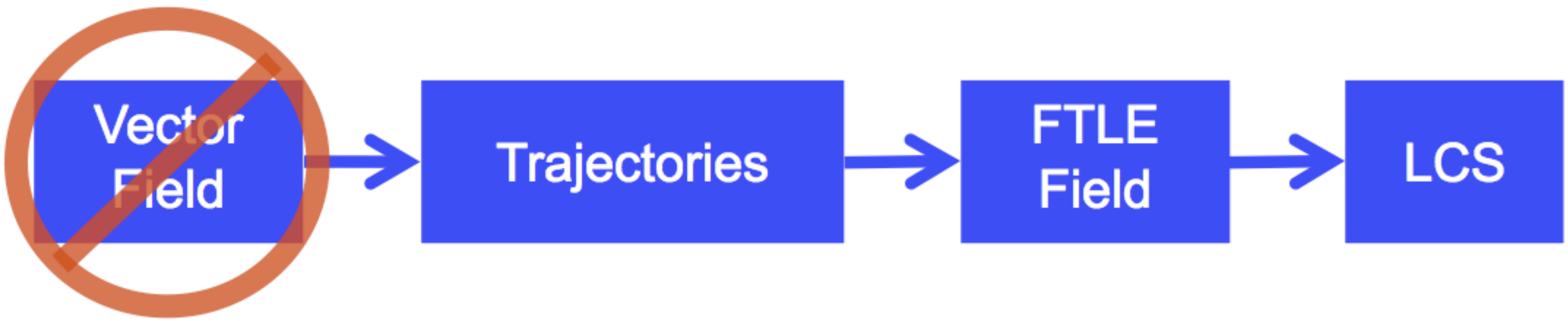


- Outside of this "tube", falling occurs

Norris et al, *Physica D* (2008)

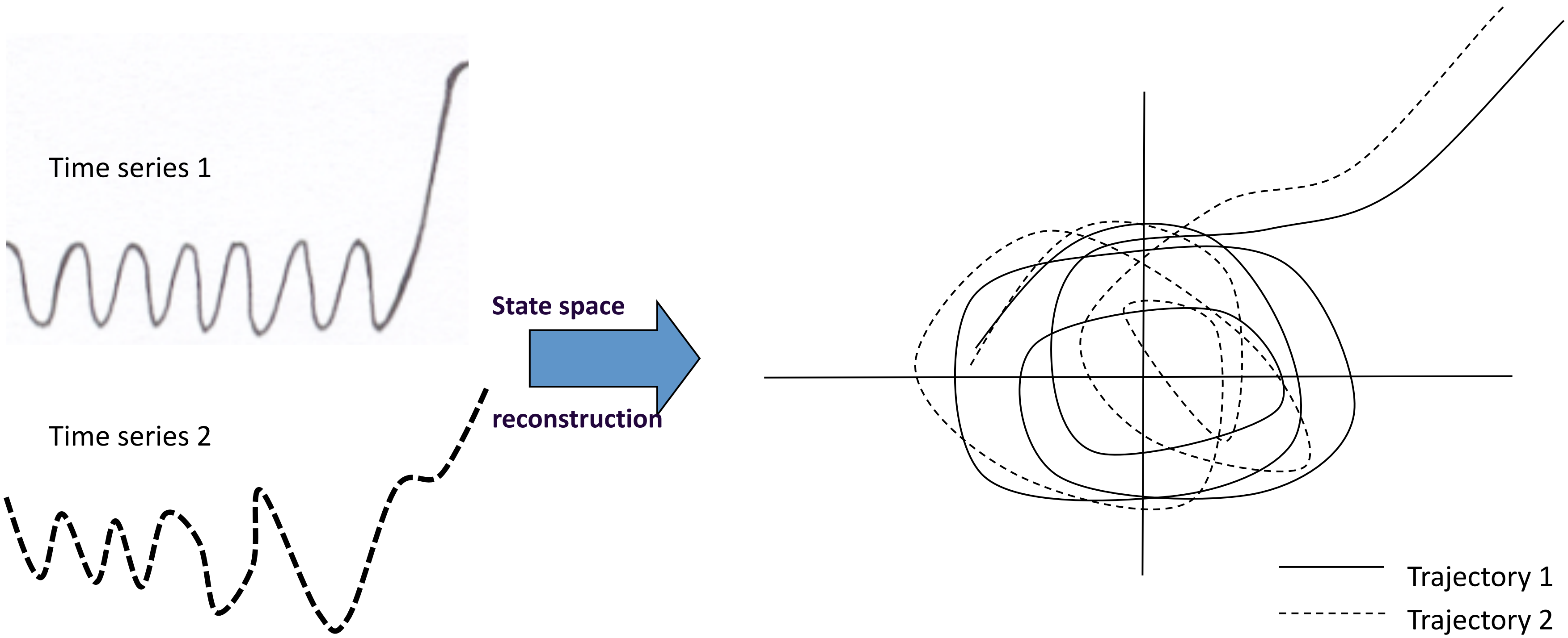
Separatrices: biological phenomena

- **Structure *even when no vector field known***
- **For example, experimental time series data**



Refs: Tanaka & Ross (2008), *Nonlinear Dynamics*
Tanaka, Nussbaum, Ross (2008), *Journal of Biomechanics*

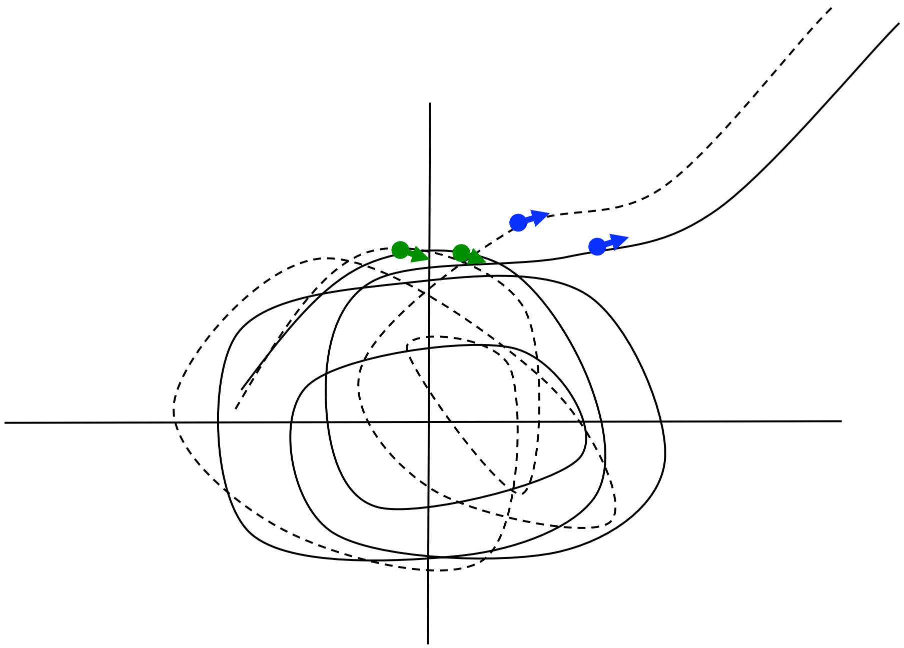
Time series to state space structure



- **From time-series data to state space trajectories**

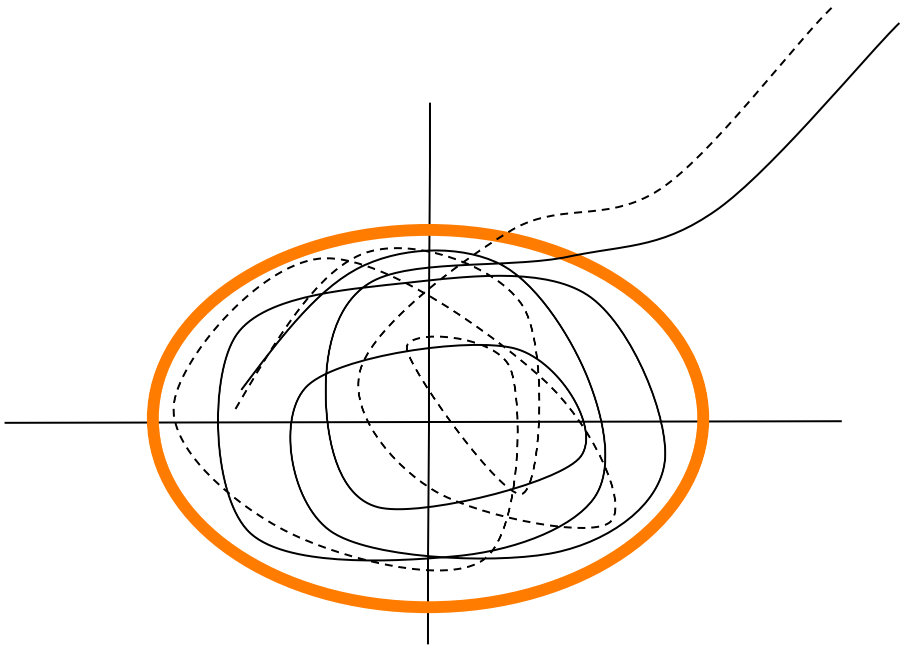
Time series to state space structure

- **Two green points have small divergence**
- **Two blue points have small divergence**
- **Green and blue points diverge rapidly**



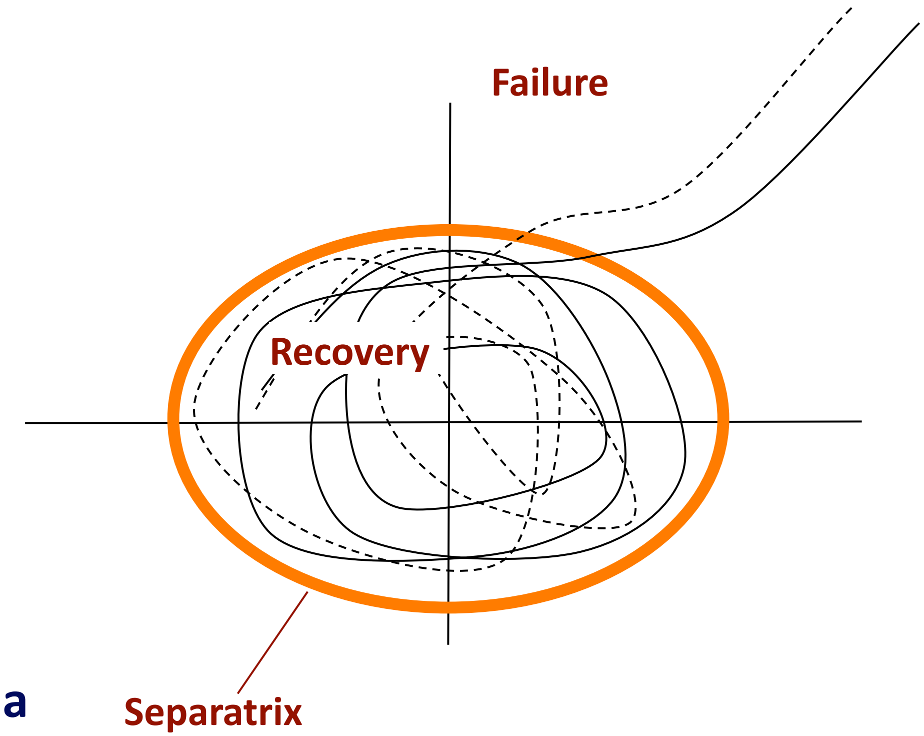
Time series to state space structure

- **Two green points have small divergence**
- **Two blue points have small divergence**
- **Green and blue points diverge rapidly**
- **A thin region of high divergence**



Time series to state space structure

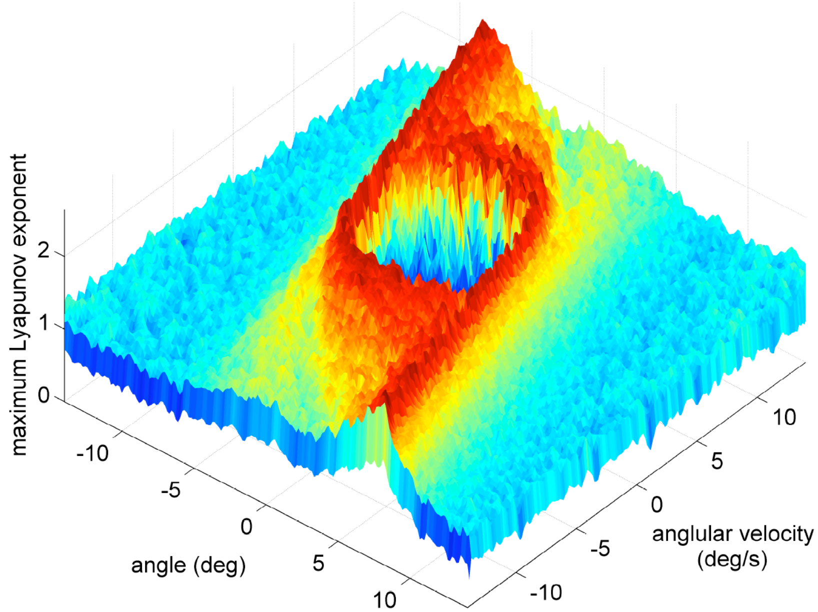
- **Two green points have small divergence**
- **Two blue points have small divergence**
- **Green and blue points diverge rapidly**
- **A thin region of high divergence**
- **This dynamical boundary separates a region of recovery from one of failure**



Comparison with other methods

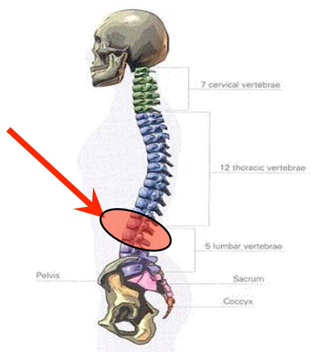
- **Other methods obtaining divergence measures from time series:**
  - **Yield only *one number (a state-space average)***
  - **Assume an attracting set exists *and there may not be one***
  - **E.g., Wolf et al 1985, Sano & Sawada 1985, Kantz & Schreiber 2004**

- **This method yields a *sensitivity field*, yielding important state space structure**
  - **Even for noisy, messy experimental data**
  - **Or mechanical systems with no attractors**

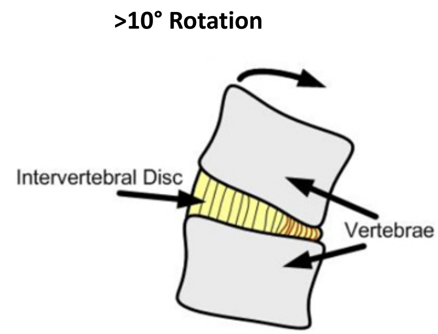


Torso instability is often associated with LBP

Local Instability

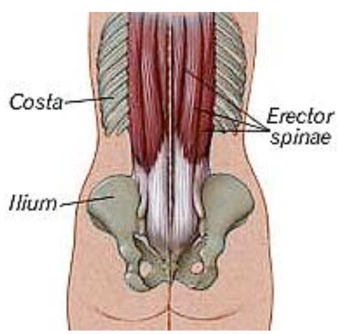


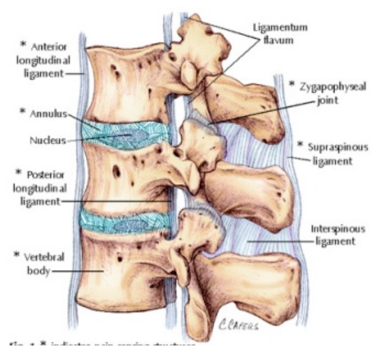
Torso Instability



Excessive Strain

Injury





Low Back Pain

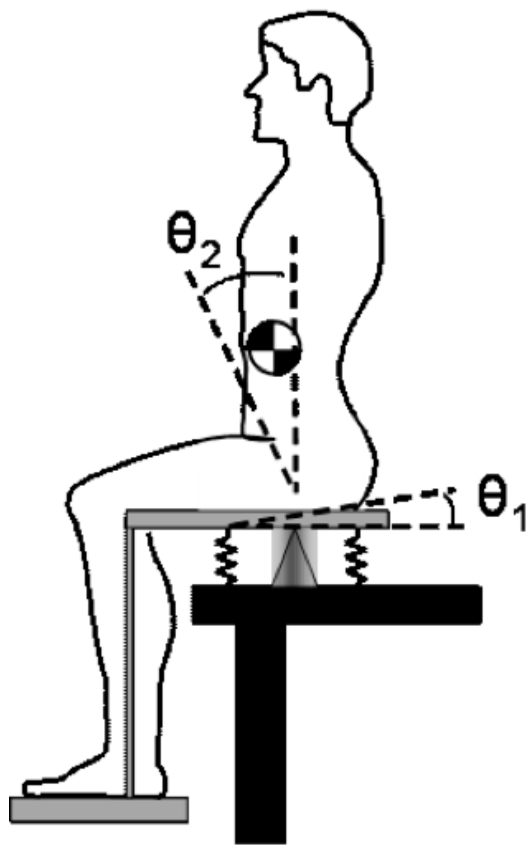
Basins of recovery in balance control

- **Wobble chair (measure torso stability, linked to low back pain)**

- **Movement of lumbar spine maintains stability**

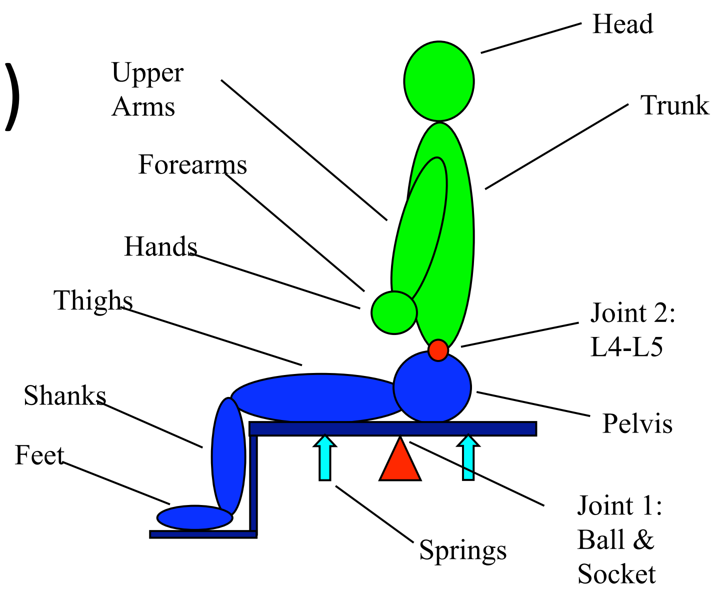
Wobble chair

- **Wobble chair schematic (slice through fore-aft plane)**



Wobble Chair Model (2 dof)

- Anthropometric data was used to calibrate the model
- System reduced to a double inverted pendulum (2D CS 4D SS)
- Solved using Lagrange's equations



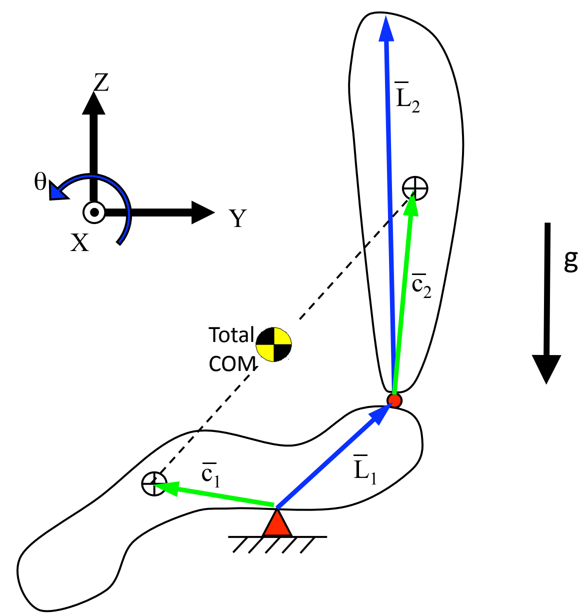
$$M\ddot{\theta} + C\dot{\theta}^2 + G(\theta) = \tau$$

$$M = \begin{bmatrix} m_1\|\vec{c}_1\|^2 + m_2\|\vec{L}_1\|^2 + I_1 & m_2\left(R'_{\theta 1}\vec{L}_1\right)\cdot\left(R'_{\theta 2}\vec{c}_2\right) \\ m_2\left(R'_{\theta 1}\vec{L}_1\right)\cdot\left(R'_{\theta 2}\vec{c}_2\right) & m_2\|\vec{c}_2\|^2 + I_2 \end{bmatrix}$$

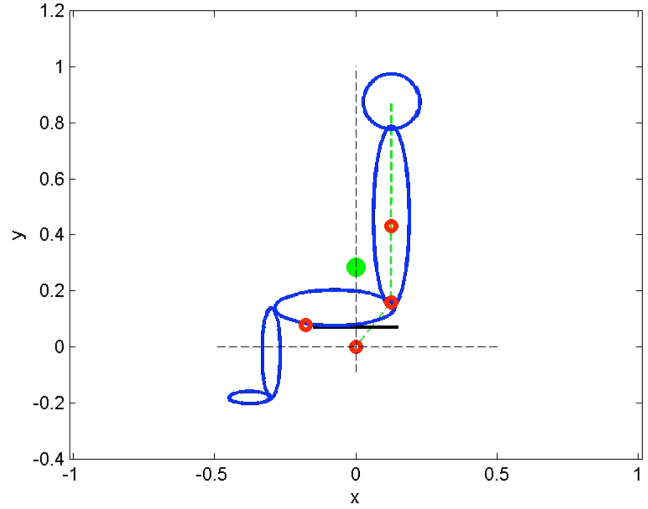
$$C = \begin{bmatrix} 0 & m_2\left(R'_{\theta 1}\vec{L}_1\right)\cdot\left(R''_{\theta 2}\vec{c}_2\right) \\ m_2\left(R''_{\theta 1}\vec{L}_1\right)\cdot\left(R'_{\theta 2}\vec{c}_2\right) & 0 \end{bmatrix}$$

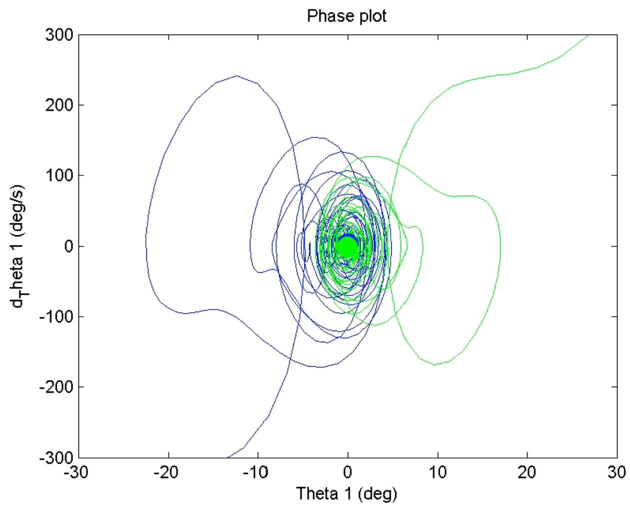
$$G = \begin{bmatrix} m_1\vec{g}\cdot\left(R'_{\theta 1}\vec{c}_1\right) + m_2\vec{g}\cdot\left(R'_{\theta 1}\vec{L}_1\right) \\ m_2\vec{g}\cdot\left(R'_{\theta 2}\vec{c}_2\right) \end{bmatrix}$$

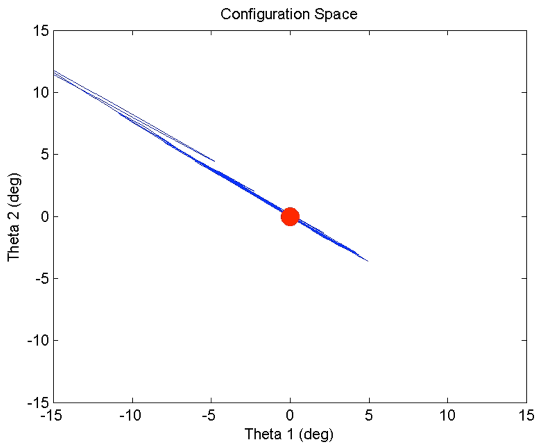
$$\tau = \begin{bmatrix} T_{Spr} - \left(T_{sk} + T_{sd} + C_p + C_d + Noise\right) \\ T_{sk} + T_{sd} + C_p + C_d + Noise \end{bmatrix} \qquad \theta = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$$



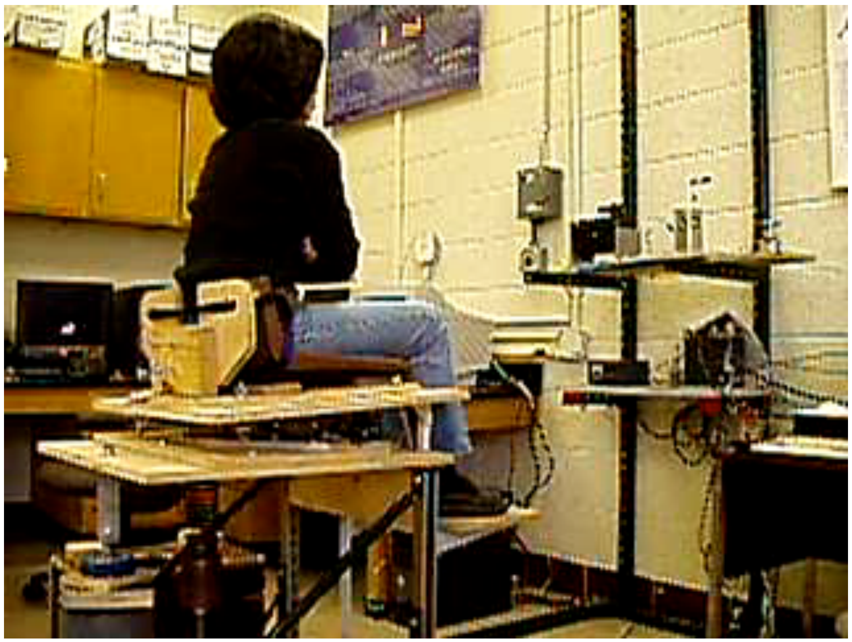
Wobble Chair Simulation





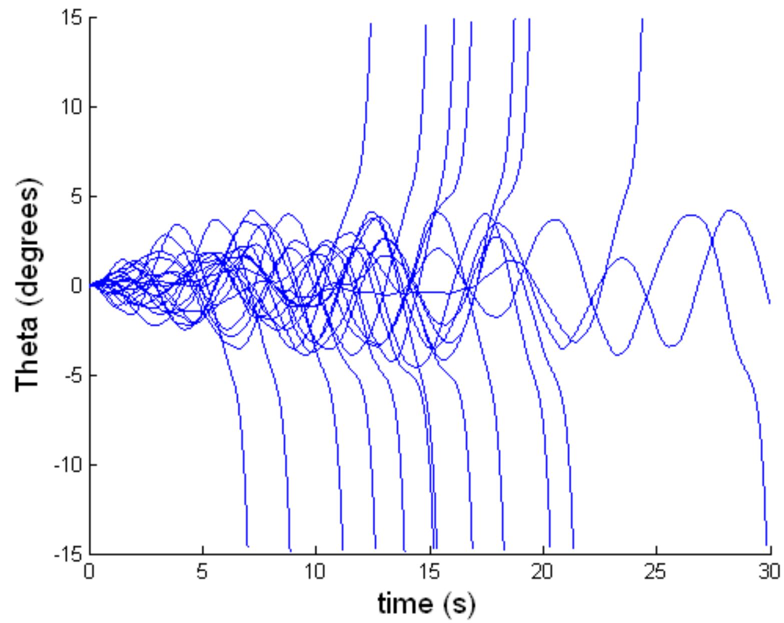


Experimental trials: sampling state space



Previous wobble chair experiments
[Tanaka and Granata 2007, Lee and Granata 2008]

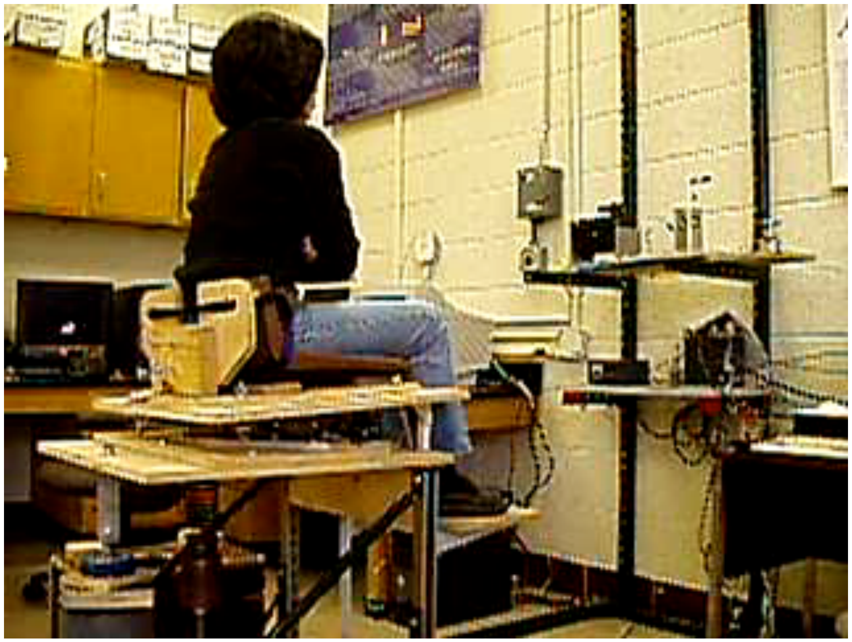
- Averaged over the entire time series
- Result was a single scalar value λ_{max}



Measure orientation time history for multiple trials

Attempt to identify boundaries

Experimental trials: sampling state space



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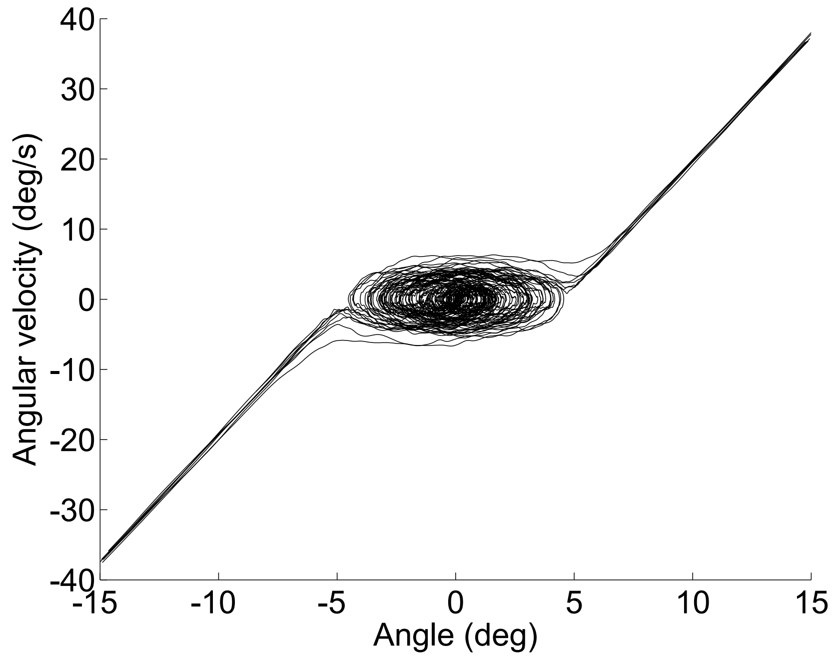
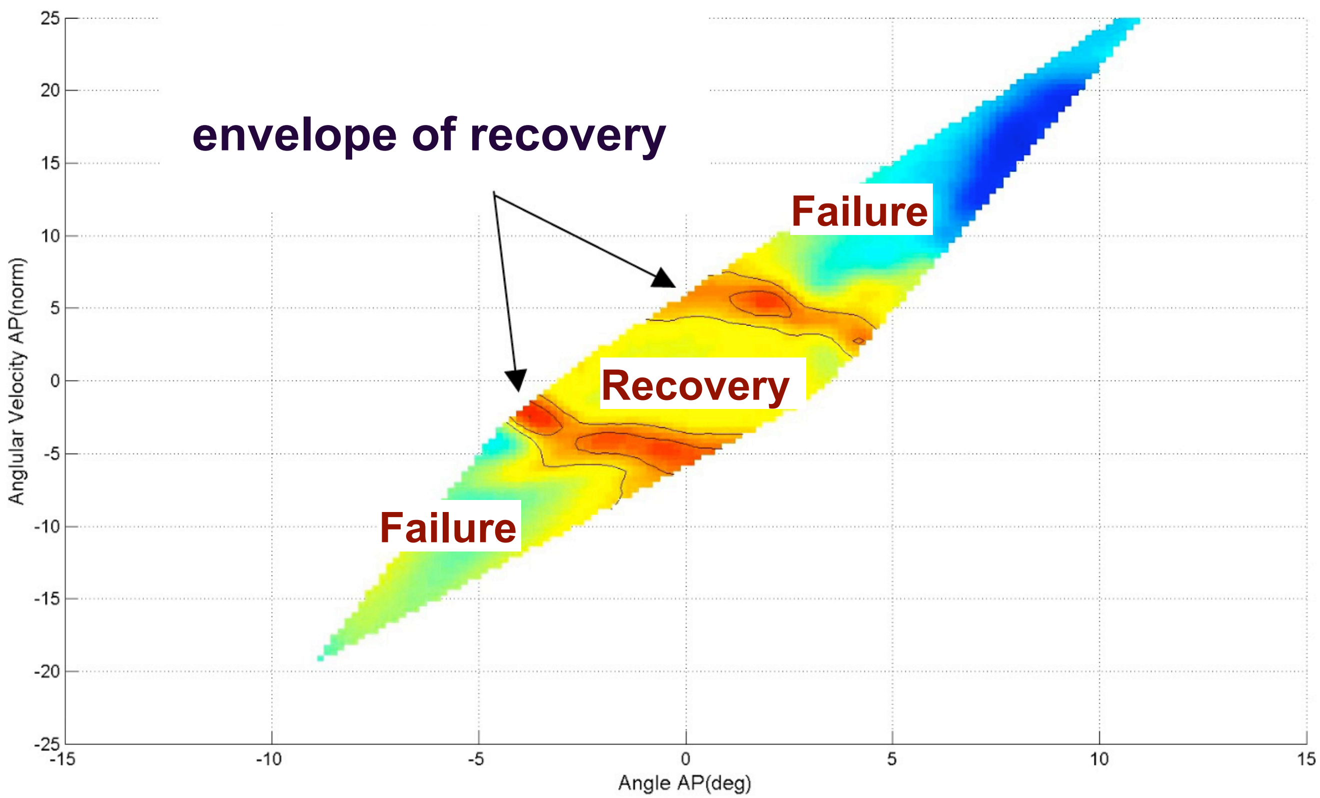


Measure orientation time history for multiple trials

Attempt to identify boundaries

Boundary of basin of recovery

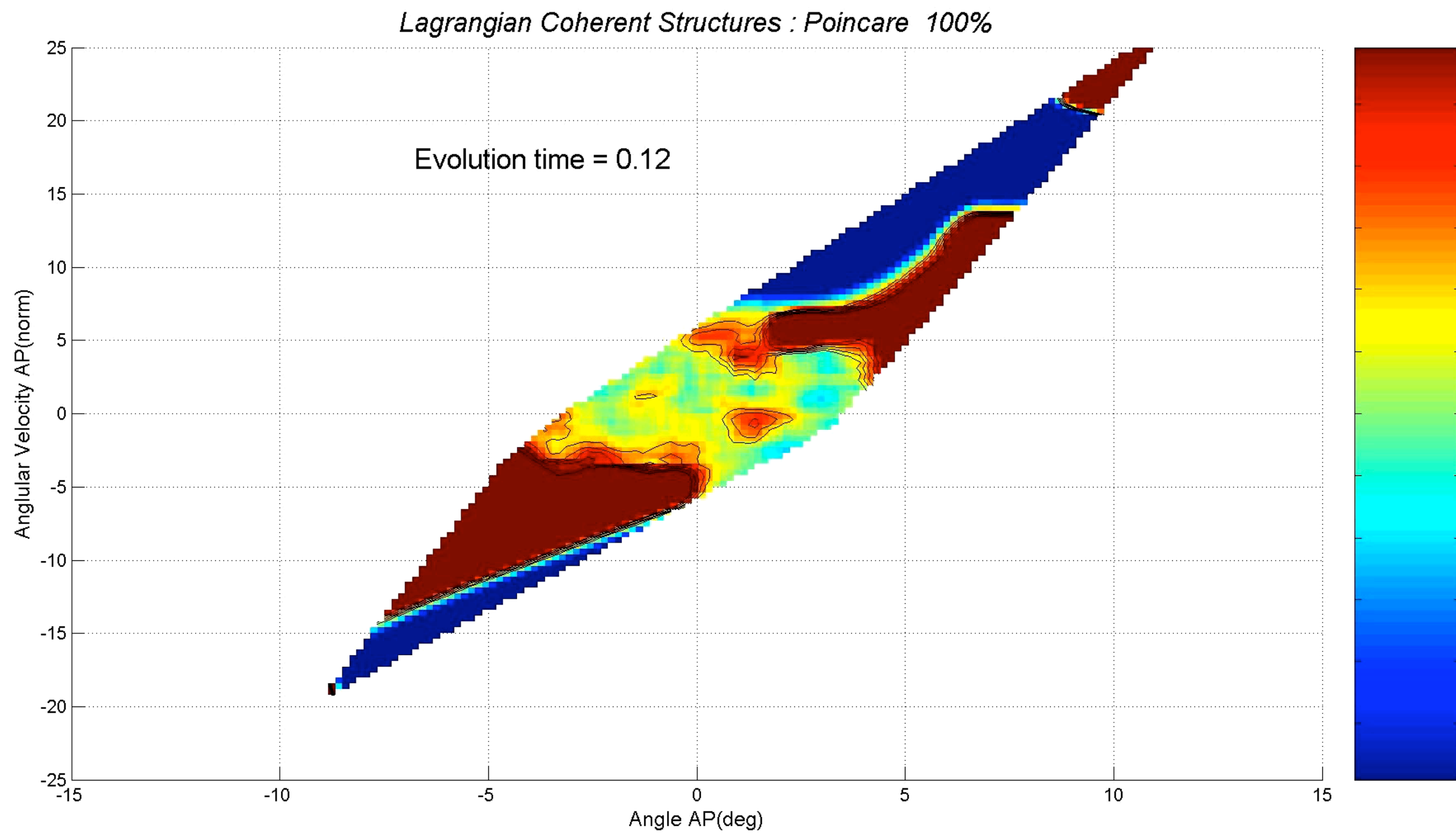
- **Measure motion**
- **Time-series data to state space trajectories, even sparse data**
- **Compute the recovery/failure boundary, the *<u>envelope of recovery</u>***



Basin of recovery bounded by ridges in sensitivity field

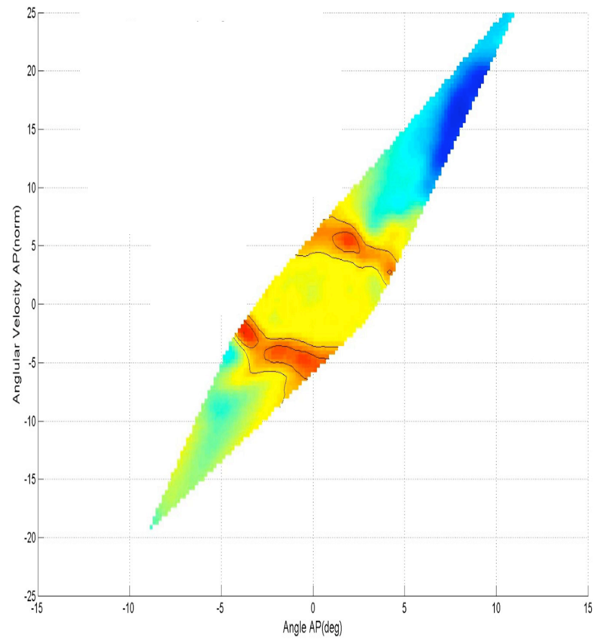
- <u>Sensitivity</u> field could be max finite-time Lyapunov exponent field
- Measures <u>divergence</u> between trajectories starting near a point
- Ridges are <u>boundaries</u> between qualitatively different trajectories; originally developed as "Lagrangian coherent structures" in fluid mechanics

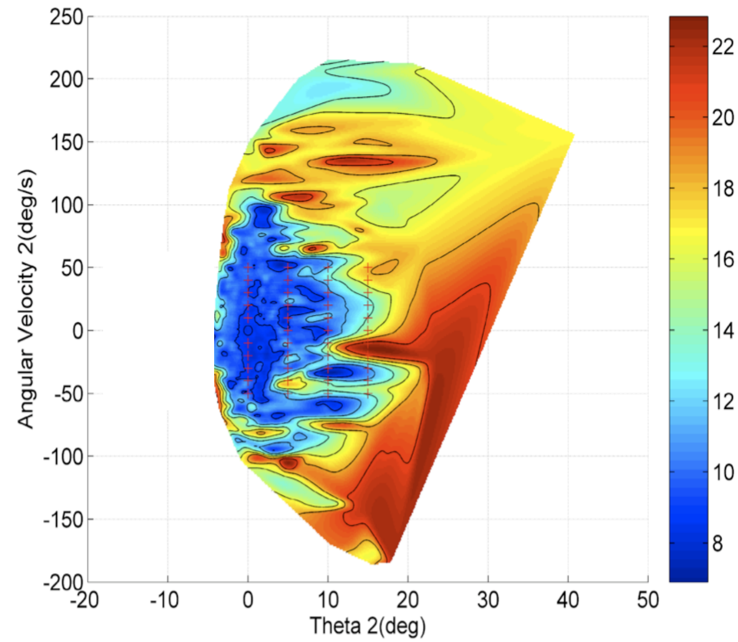
Sensitivity field reveals structure as <u>evolution time</u> increases



Boundaries in higher dimension

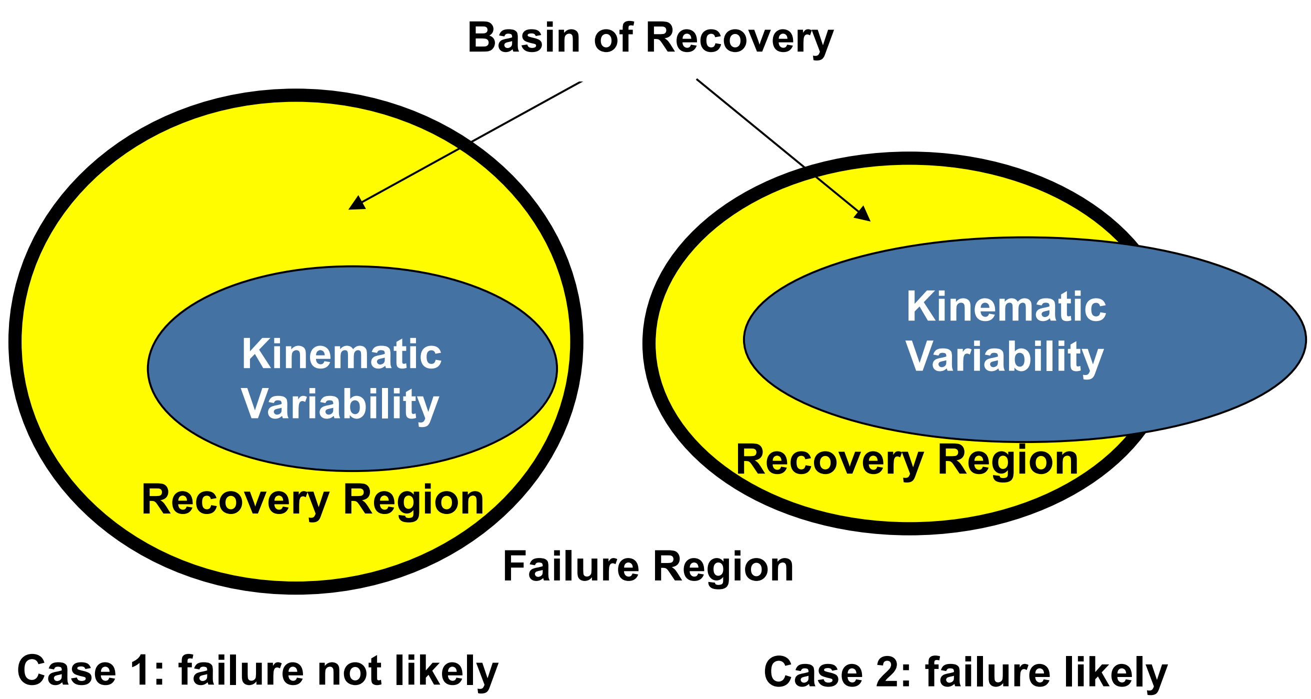
- **Basin of recovery in n-dim system bounded by (n-1)-dim surface**
- **2 angles and their rates (2D X 2D = 4D system)**
- **Hard to visualize, but can compute *4D basin bounded by 3D surface***
- **Basin size affected by environmental conditions**





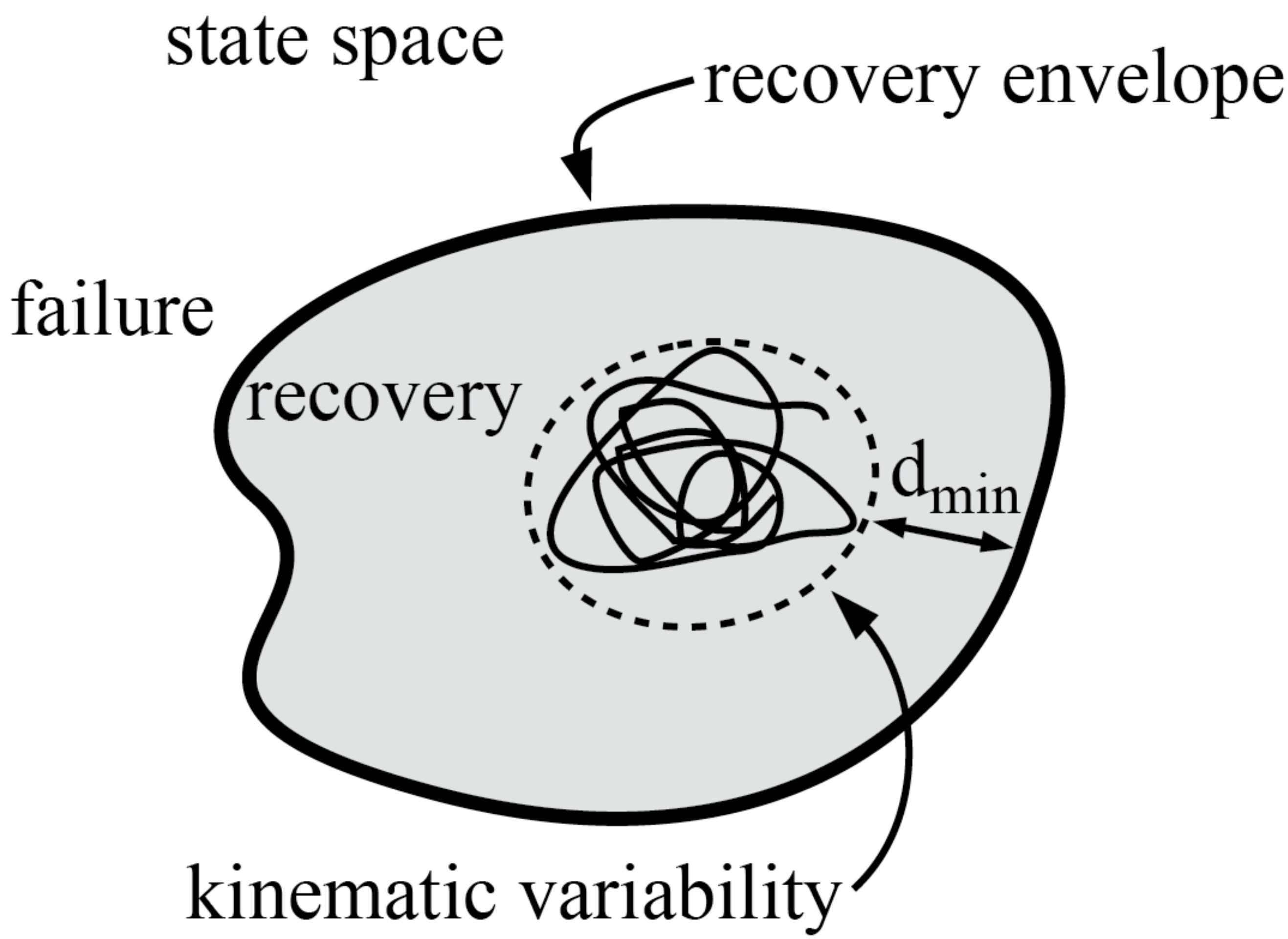
Recovery envelope: the basin boundary

- **Size and shape of basin is function of *possible* natural dynamics**
- **Kinematic variability: currently explored region of state space**
- **When kinematic variability exceeds basin of recovery => *failure***



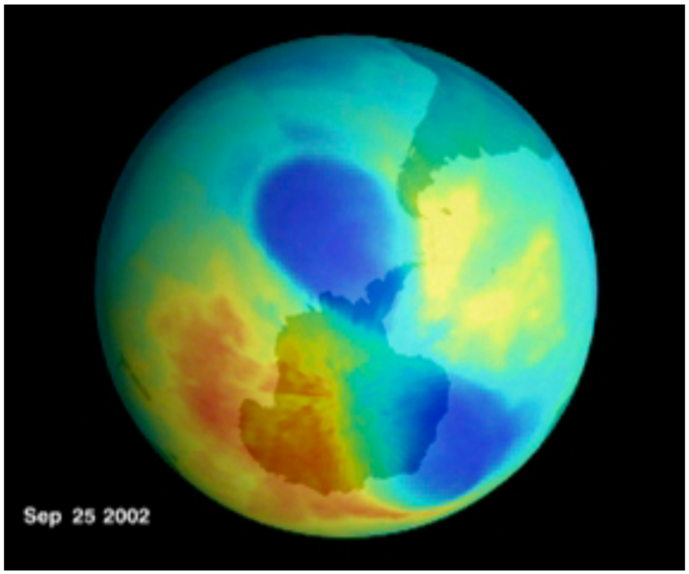
A new tool in the evaluation of risk of failure?

- **Minimum distance between kinematic variability and recovery envelope can measure risk of failure**

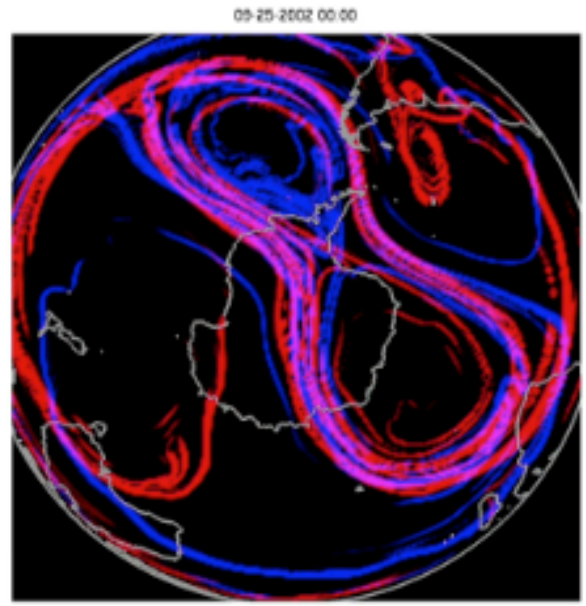


Separatrices (LCS)

- **Structure of trajectories and transport in chaotic, time-dependent vector field**



Ozone concentration

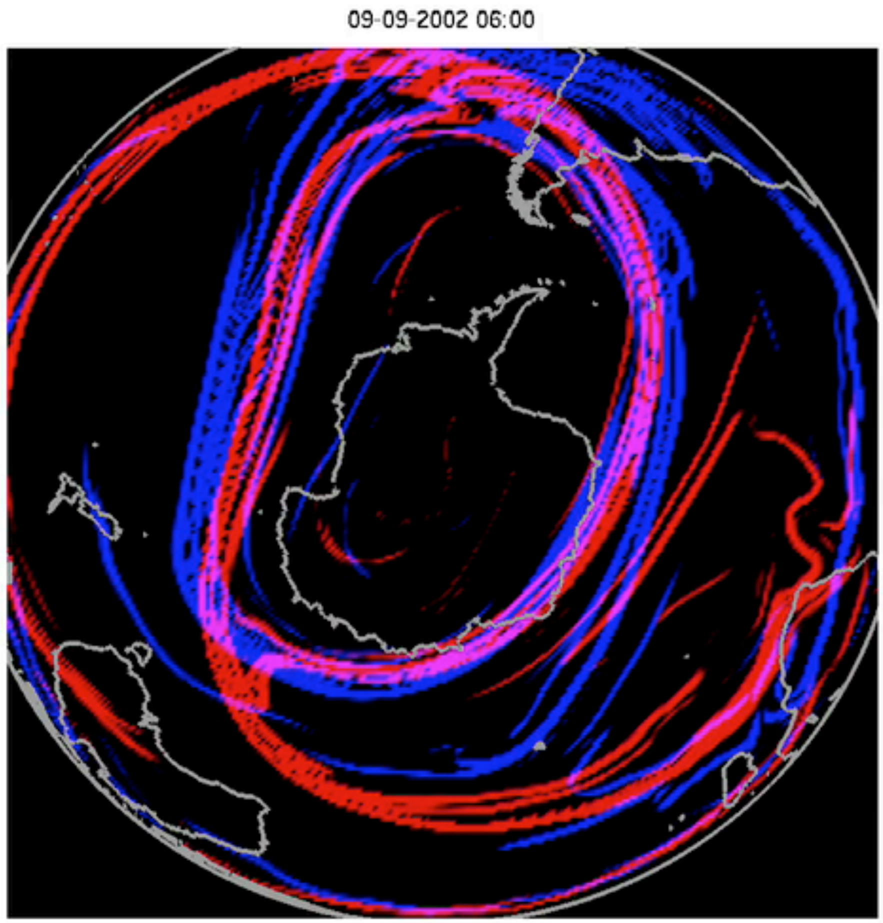


Separatrices from wind data

2002 Ozone Hole Splitting Event

Atmospheric transport barriers

Are there isolated “aero-ecosystems”?



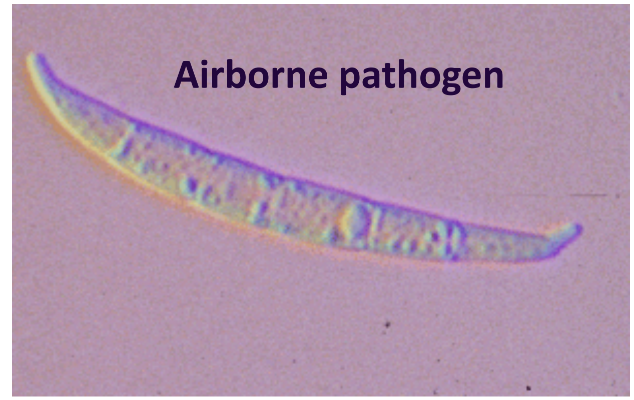
Atmospheric transport of pathogens

Determine atmospheric pathways

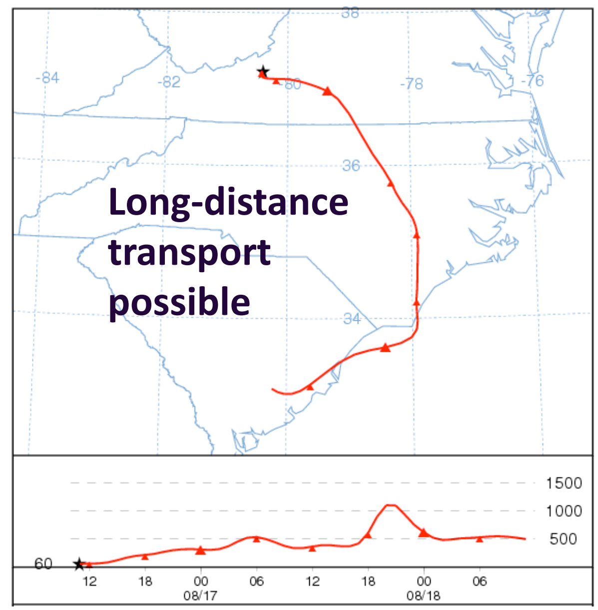
- For airborne biological pathogens which lead to spread of plant disease

Relevant scales

- Horizontal spread over large areas
- Pathogens may separate from fluid flow and change shape or clump

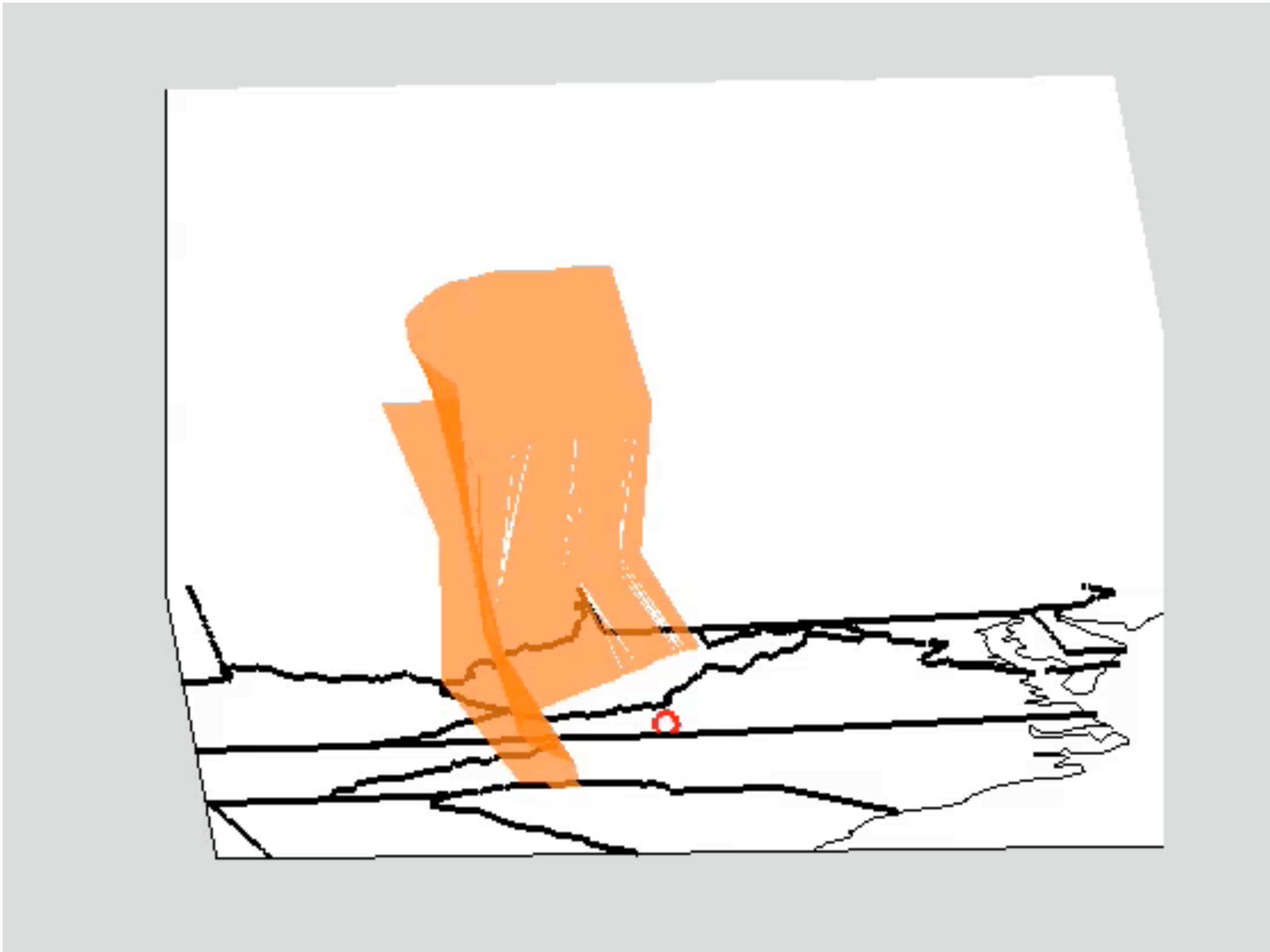


Crop diseases spread by airborne spores



[1]Schmale, et al [2006]

Atmospheric transport barriers (3D)



Outlook

- **Dynamical boundaries/separatrices reveal state space structure displayed by trajectories; even in noisy, experimental time series data**

- **Ridges in finite-time Lyapunov exponent field over sampled state space**
  - **Field average gives usual Lyapunov exponent quoted by others**

- **Applications: torso stability, walking stability, bipedal robotics, prosthetics, ship capsize**

- **Limitations**
  - **Applicable only to mechanical systems?**
  - **Data requirements? Quantity and quality of data to see structure?**
  - **High dimensions difficult; choose appropriate coarse variables**