TRANSCENDENTAL DYNAMICS

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HOLOMORPHIC DAYS

12-13 juliol de 2021

Transcendental functions are those with (at least) one **ESSENTIAL SINGULARITY** which we assume to be at infinity.

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$$\lambda z e^{z + \frac{1}{z}},$$

- **Meromorphic** if $\exists$ non-omitted preimages of the ess. sing.

$$\lambda \tan z, \quad \frac{e^z + 1}{e^{-z} + 2}, \quad N(z) = z - \frac{f(z)}{f'(z)} \text{ with } f \text{ entire}, \ldots$$

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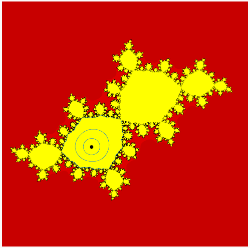
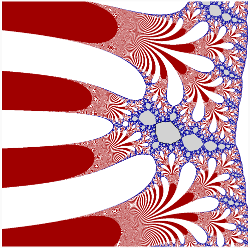


polynomial



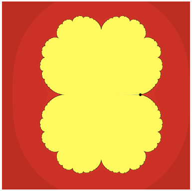
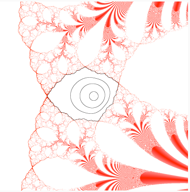
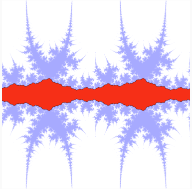
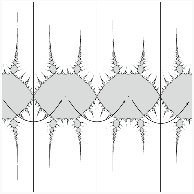
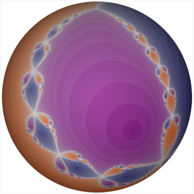
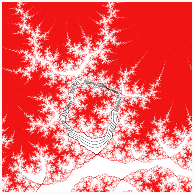
transcendental

FATOU COMPONENTS

Fatou's Classification Theorem (1920-26)

Connected components of $F(f)$ can be:

- **Periodic**
  - basins of attraction
  - rotation disks (Siegel)
  - rotation rings (Herman) (need poles)
  - **Baker domains**
- **Preperiodic**
- **Wandering**

SPECIAL FATOU COMPONENTS

- Baker domains and **wandering domains** (= **wandering components**) can only exist for transcendental maps.
  - U is a **Baker domain** of period 1 if $f^n|_U \to \infty$ loc. unif.

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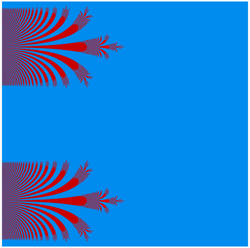
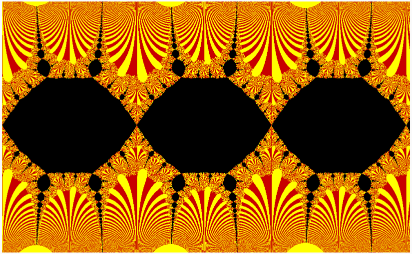
  - U is a **Baker domain** of period 1 if $f^n|_U \to \infty$ loc. unif.
  - U is a **wandering domain** if $f^n(U) \cap f^m(U) = \emptyset$ for all $n \neq m$.



$z + 1 + e^{-z}$



$z + 2\pi + \sin(z)$

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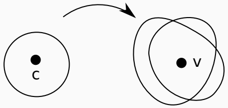
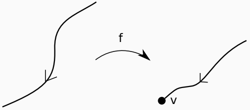


critical value



asymptotic value

$f : \mathbb{C} \setminus f^{-1}(S(f)) \to \mathbb{C} \setminus S(f)$ is a **covering map**.

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Examples: $z \mapsto \lambda \exp(z)$ or $z \mapsto \lambda \frac{z}{\sin z}$.

DINÀMICA TRANSCENDENT: TOPICS

LÍNIES PRINCIPALS

(1) Dinàmica global i topologia del conjunt de Julia

- Conectivitat y conectivitat local
- Cantor bouquets, continus indecomposables, "dynamic rays",...

(2) Components de Fatou: dinàmica interna y relació amb els valors singulars

- Dominis de rotació
- Dominis de Baker: classificació, dinàmica a la frontera, ...
- Dominis errants: existència, dinàmica interna, dinàmica a la frontera, ...

(3) Espais de paràmetres

- Bifurcacions en famílies transcendents enteres
- Bifurcacions en famílies meromorfes

DINÀMICA TRANSCENDENT: TOPICS

(1) Dinàmica global i topologia del conjunt de Julia

- **Funcions enteres transcendents**
  - Conjunts de Julia no acotats, "hairs", Cantor Bouquets, :.
- **Funcions meromorfes MANSES a l'infinit**
  - Condicions per a tenir connectivitat local?
  - Condicions per a no tenir "hairs"? (Funcions calves)
  - Exemples de famílies en concret.

U. WARSAW: **Krzysztof Baranski, Boguslawa Karpinska**

U. LIVERPOOL: Lasse Rempe, James Waterman, David Martí-Pete

OPEN U. (UK): Vasiliki Evdoridou, Phil Rippon, Gwyneth Stallard.

U. PARMA: Anna Benini

CUNY, New York: Linda Keen, Tao Chen, Yunping Yiang,...

(2) Components de Fatou: dinàmica interna y relació amb els valors singulars

- **Dominis de Baker**
  - Classificació, funcions internes, ...
  - Relació amb (1): dinàmica (ergòdica) a la frontera. Exemples.
- **Dominis errants**
  - Classificació, dinàmica a l'interior, sistemes dinàmics no autònoms, ...
  - Relació amb les singularitats.

U. WARSAW: Krzysztof Baranski, Boguslawa Karpinska

OPEN U. (UK): Phil Rippon, Gwyneth Stallard.

U. KIEL (Germany): Walter Bergweiler

(3) Espais de paràmetres

- Per funcions enteres de tipus finit: resolt històricament
- Per funcions meromorfes: desenvolupaments recents. Moltes novetats.
- Diversos estudis de famílies model.

U. PARMA: Anna Benini

U. ORLEANS: Matthieu Astorg

CUNY, New York: Linda Keen, Tao Chen, Yunping Yiang

Històricament: Alexander Eremenko, Misha Lyubich

U. LIVERPOOL: Lasse Rempe (exponential+)

U.MARSEILLE: Dierk Schleicher (exponential)

DINÀMICA TRANSCENDENT: ALTRES TOPICS

(4) Thurston theory for transcendental entire maps

- Determinar si la combinatòria dels valors singulars és suficient per determinar les funcions.

U.MARSEILLE: Dierk Schleicher+ estudiants

(5) Holomorphic self-maps de $\mathbb{C}^*$

- Extendre la teoria de funcions enteres, i trobar comportaments novedosos.

U. LIVERPOOL: David Marí-Pete

(6) Funcions de tipus finit (en sentit més general) i classe K

- Estudiar funcions meromorfes en subconjunts del pla. E.g. funcions amb un conjunt numerable de singularitats essencials.

U PUEBLA: Patricia Domínguez, Marcos Montes de la Oca. U. WARWICK: Adam Epstein

Gràcies per l'atenció!!!