A new construction of wandering domains: Wandering Lakes of Wada

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— joint work with Lasse Rempe and James Waterman —

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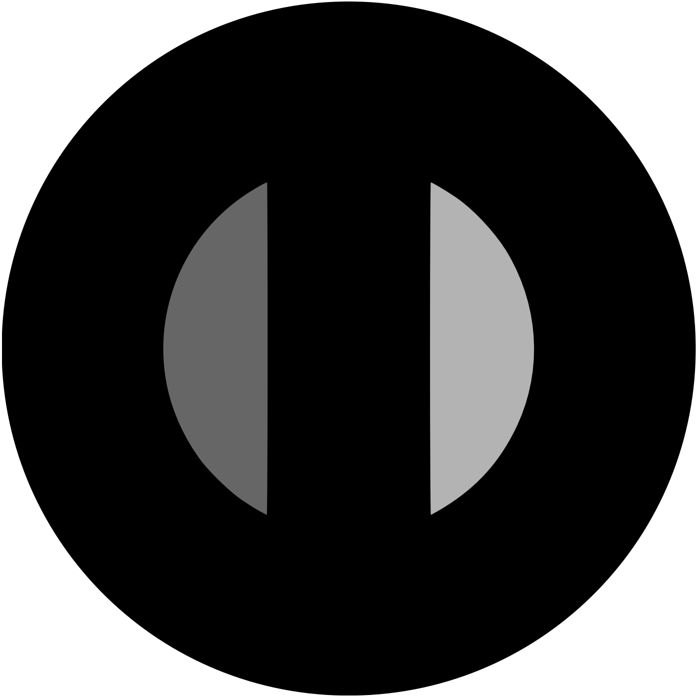
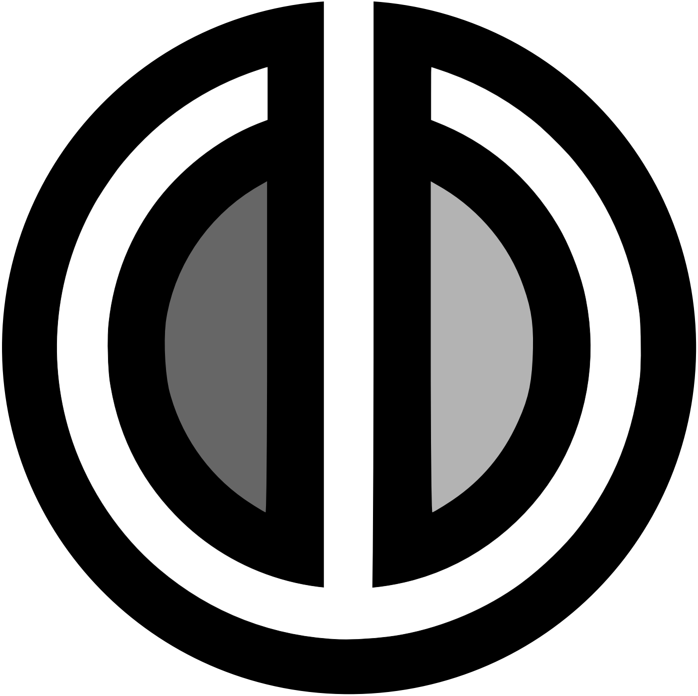
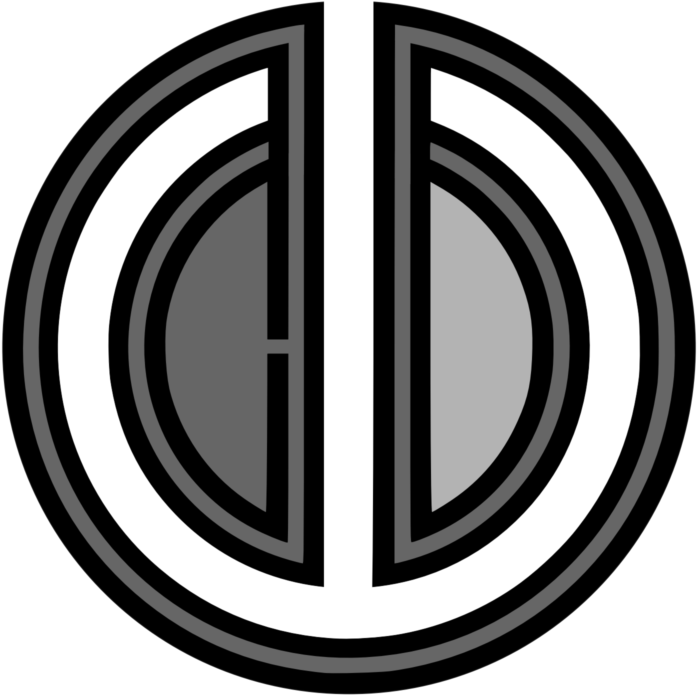
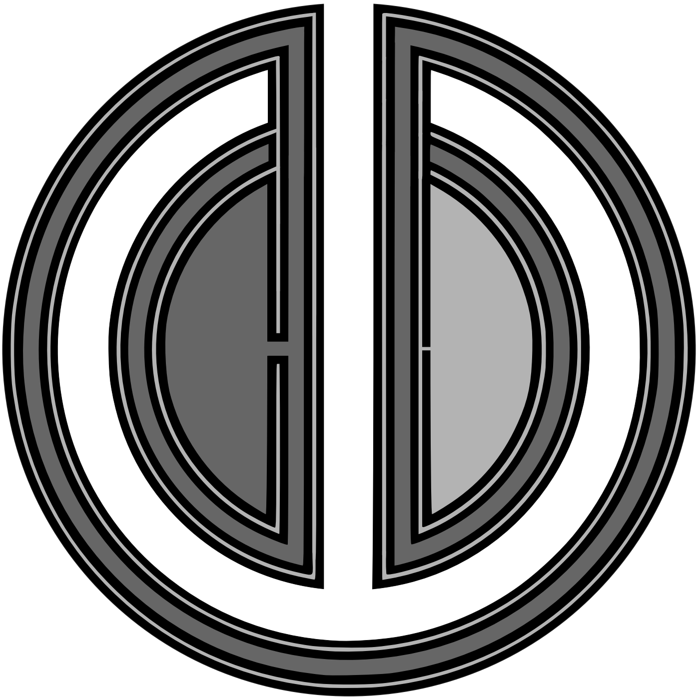








Lakes of Wada

Let $D \subseteq \mathbb{C}$ be a bounded simply connected domain, and suppose that $U, V \subseteq D$ are simply connected domains such that

$$\overline{U} \subseteq D, \quad \overline{V} \subseteq D \quad \text{and} \quad \overline{U} \cap \overline{V} = \emptyset,$$

We obtain sequences $(D_n), (U_n), (V_n)$ of simply connected domains such that for $n \geqslant 0$,

$$\overline{U_n} \subseteq D_n, \quad \overline{V_n} \subseteq D_n \quad \text{and} \quad \overline{U_n} \cap \overline{V_n} = \emptyset,$$

and, moreover,

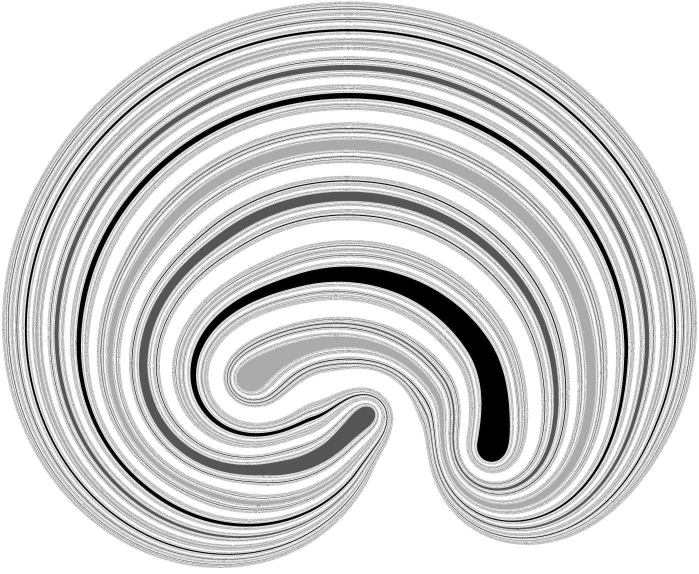
$$D_{n+1} \subseteq D_n, \quad U_{n+1} \supseteq U_n \quad \text{and} \quad V_{n+1} \supseteq V_n.$$

The canals are chosen such that the maximal distance from any point of $D_n \setminus (\overline{U_n} \cup \overline{V_n})$ to the boundary of any of the three domains tends to 0 as $n \to \infty$. In the limit, the three domains that started as the sea and the two lakes share the same boundary,

$$X = \bigcap_{n=0}^{\infty} \left(\overline{D_n} \setminus (U_n \cup V_n)\right).$$

Definition (Yoneyama, 1917)

We say that a compact and connected set $X \subseteq \mathbb{C}$ is a **Lakes of Wada continuum** if X is the common boundary of three or more disjoint domains.



A Lakes of Wada continuum.

Main result

Until now it was not known whether the Lakes of Wada occur in complex dynamics.

Question

Can the Julia set of a rational map be a Lakes of Wada continuum?

More generally, one may ask:

Question

Can the Lakes of Wada continuum occur as the boundary of a Fatou component of a holomorphic function (not necessarily as the whole Julia set)?

Theorem (MP, Rempe and Waterman, 2021)

There exists a transcendental entire function with a bounded Fatou component whose boundary is a Lakes of Wada continuum.

The Fatou components from the function in the theorem are **wandering domains**. We may choose the boundary of the Fatou component in the theorem to be **any** Lakes of Wada continuum. In particular, this continuum can be the boundary of infinitely many bounded wandering domains.

More questions

If U is a **completely invariant** Fatou component, i.e. $f^{-1}(U) = U$, then $\partial U = J(f)$ and, in particular, $J(f)$ has no buried points. Makienko conjectured that the converse is also true.

Conjecture (Makienko, 1989)

If U is a Fatou component such that $\partial U = J(f)$, then U is completely invariant for an iterate f^n of f for some $n \in \mathbb{N}$.

Dudko and Lyubich have announced a proof that the boundary of a Siegel disc of a *quadratic* polynomial is also a Jordan curve, ruling out the existence of Lakes of Wada boundaries in this family. The question remains open for polynomials of degree at least 3.

One may expect that, for polynomials and rational maps, the boundary of a Fatou component can never be a Lakes of Wada continuum. This would imply a positive answer to Makienko's conjecture.

Question

Let f be an entire function and suppose that U is a bounded invariant Fatou component of f. Must ∂U be a simple closed curve?

Wandering domains

Let $f : \mathbb{C} \to \mathbb{C}$ be a transcendental entire functon.

A Fatou component U of f is a **wandering domain** if $f^m(U) \cap f^n(U) = \emptyset$ for all $m \neq n$.

Wandering domains can be classified into:

- **Multiply connected wandering domains**:
  - dynamics: they are always in the fast escaping set $A(f) \subset I(f)$
  - geometry: eventually they contain large annuli and connectivity becomes 2 or ∞
  - boundary: can have smooth boundaries but may also be very complicated

  See Baker, Kisaka-Shishikura, Zheng, Bergweiler-Rippon-Stallard...

- **Simply connected wandering domains**:
  - dynamics: they can be in $I(f)$, $BU(f)$ or even $K(f)$ (?)
  - geometry: they may be very small
  - boundary: **this talk is about their boundaries!**

  See Herman, Baker, Eremenko-Lyubich, Bishop, MP-Shishikura, Lazebnik, Fagella-Jarque-Lazebnik, Benini-Evdoridou-Fagella-Rippon-Stallard, Evdoridou-Rippon-Stallard, Boc Thaler...

Boc Thaler's result

Let $U \subseteq \mathbb{C}$ be a bounded domain. Recall that U is **regular** if $\text{int}(\overline{U}) = U$.

Theorem (Boc Thaler, 2021)

Let U be a bounded simply connected domain such that U is regular and $\mathbb{C} \setminus \overline{U}$ is connected. Then there exists a transcendental entire function f for which U is a wandering domain.

Boc Thaler proved that being regular is a necessary condition.

Question (Boc Thaler, 2021)

Is it true that the closure of any bounded simply connected Fatou component of an entire function has a connected complement?

Our Lakes of Wada wandering domains provide a negative answer to this question.

Our result on the wandering Lakes of Wada is a corollary of the following result.

Theorem (MP, Rempe and Waterman, 2021)

Let $K \subseteq \mathbb{C}$ be a full compact set. Then, there exists a transcendental entire function f such that $f^n|_K \to \infty$ uniformly as $n \to \infty$, $\partial K \subseteq J(f)$ and $f^n(K) \cap f^m(K) = \emptyset$ for $n \neq m$.

This is a generalization of the result by Boc Thaler, and poses the following question.

Question (MP, Rempe and Waterman, 2021)

Suppose that U is a bounded simply connected Fatou component of a transcendental entire function, and let $K = \text{fill}(\overline{U})$. Is it true that $\partial U = \partial K$?

A negative answer to this question would imply that a transcendental entire function has at most one completely invariant Fatou component.

I am working with Phil Rippon, Gwyneth Stallard and Dave Sixsmith to answer this question for the case that U is a wandering domain.

Polynomial Julia sets appear in transcendental dynamics

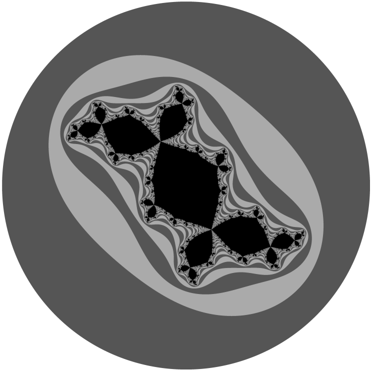
Corollary (MP, Rempe and Waterman, 2021)

Let P be a polynomial of degree at least 2. Then there exists a transcendental entire function f such that

(a) $J(P) \subseteq J(f)$;

(b) every bounded Fatou component of P is an escaping simply connected wandering domain of f;

(c) every connected component of $J(P)$ is a wandering continuum of f.



The strong version of Eremenko's conjecture

For an entire function f, define the **escaping set** of f by

$$I(f) := \{z \in \mathbb{C} \colon f^n(z) \to \infty \text{ as } n \to \infty\}.$$

Eremenko proved that every component of $\overline{I(f)}$ is unbounded and asked whether the same holds for $I(f)$, which is still an open question nowadays. He also asked:

Conjecture (Eremenko, 1989)

Can every point of $I(f)$ be joined to ∞ by a curve of points in $I(f)$?

Theorem (Rottenfußer, Rückert, Rempe and Schleicher, 2011)

No, even in the class $\mathcal{B}$. But sometimes yes.

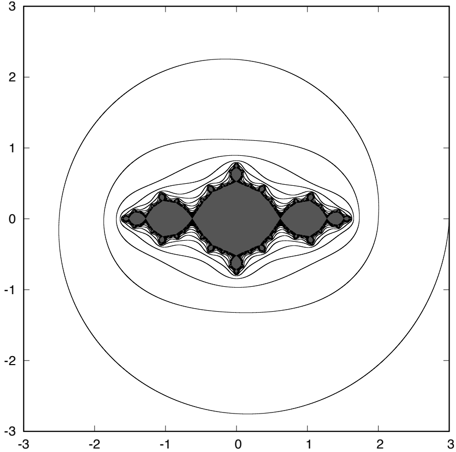
Further counterexamples to the strong Eremenko's conjecture were given in

- Bishop (2015): the strong Eremenko's conjecture fails in the class $\mathcal{S}$.
- Rempe (2016): examples from the arc-like continua paper.
- Benítez and Rempe (2021): a bouquet of pseudoarcs.

A new counterexample

Theorem (MP, Rempe and Waterman, 2021)

Let $X \subseteq \mathbb{C}$ be any full continuum. Then there exists a transcendental entire function f such that every path-connected component of X is a path-connected component of $I(f)$. In particular, no point of X can be connected to ∞ by a curve in $I(f)$.



Sketch of the proof

The proof uses the following classical result from approximation theory.

Theorem (Runge, 1885)

Let $K \subseteq \mathbb{C}$ be a compact set such that $\mathbb{C} \setminus K$ is connected. Suppose that $\varepsilon > 0$ and the function $\phi : K \to \mathbb{C}$ is analytic on a neighbourhood of K. Then, there exists a polynomial f such that

$$|f(z) - \phi(z)| \leqslant \varepsilon, \quad \text{for } z \in K.$$

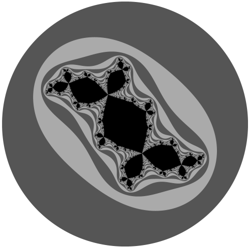
Let $K \subseteq \mathbb{C}$ be a full compact set. We choose sequences (K_j) and (L_j) of compact sets with the following properties:

(a) $L_j \subseteq \operatorname{int}(K_{j-1})$ and $K_j \subseteq \operatorname{int}(L_j)$ for all $j \geqslant 0$;

(b) $\bigcap K_j = \bigcap L_j = K$;

(c) Each of the sets L_j and K_j is a topological annulus.

We choose a sequence (p_j) with $p_j \in K_j \setminus L_{j+1}$ for all $j \geqslant 0$ so that every point of ∂K is an accumulation point of (p_j).



We define the sequence of discs

$$D_j := D(3j, 1), \quad \text{for } j \geqslant -1.$$

We may assume without loss of generality that $L_0 \subseteq D_0 = \mathbb{D}$.

The proof of the theorem follows from the following proposition:

Proposition

There is a transcendental entire function f with the following properties:

(a) $f(D_{-1}) \subseteq D_{-1}$;

(b) $f^{j+1}(p_j) \in D_{-1}$ *for all* $j \geqslant 0$;

(c) $f^j(L_j) \subseteq D_j$ *and* f^j *is injective on* K_j *for all* $j \geqslant 0$.

Since $K \subseteq L_j$, we have $f^j(K) \subseteq D_j$ for all $j \geqslant 0$. The fact that every point of ∂K is accumulated by the sequence (p_j), which lie in different preimages of a basin of attraction $B \subseteq D_{-1}$, this implies that $\partial K \subset J(f)$. So the components of $\text{int}(K)$ are wandering domains.

Thank you for your attention!

Everyone knows what a curve is, until they have studied enough mathematics to become confused through the countless number of possible exceptions.

Felix Klein (1849–1925)