Singular Perturbations

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1. Introduction

What is a singular perturbation?

$$F_\lambda(z) = f(z) + \lambda \cdot \frac{1}{g(z)}$$

- $f(z)$ is a polynomial or a rational map with a well understood dynamics. Easy model.
- λ (usually) is a small parameter.
- $\frac{1}{g(z)}$ introduce a big change when $g(z) = 0$.

STEP ONE. An interesting example

First Example: (McMullen, 1988)

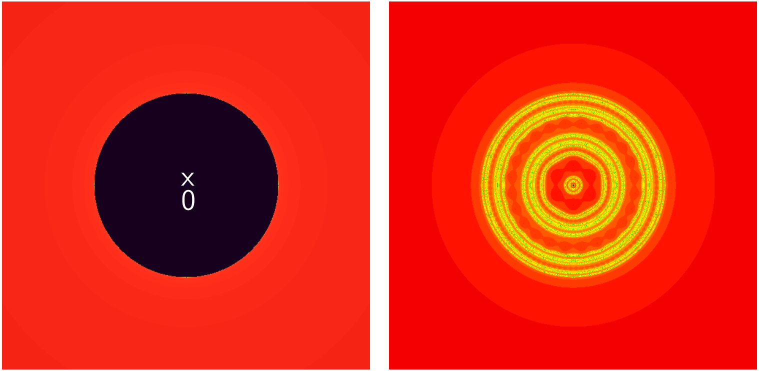
$$F_\lambda(z) = z^2 + 10^{-5} \cdot \frac{1}{z^3}$$

- Easy dynamics $f(z) = z^2$
- λ small parameter $\lambda = 10^{-5}$.
- Perturbing term $\frac{1}{z^3}$. Huge change near the origin.

$$F_\lambda(z) = z^2 + \lambda \cdot \frac{1}{z^3}$$

Theorem (McMullen'88)

The Julia set of F_λ for λ small enough is a cantor set of circles.



(a) The unperturbed map z^2.

(b) The perturbed map $f_\lambda(z) = z^2 + 10^{-5}/z^3$.

$$F_\lambda(z) = z^2 + \lambda \cdot \frac{1}{z^3}$$

PROVA

- All points in $U = \{z \in \mathbb{C}\,;\, |z| > 1\}$ escape to ∞ for λ sufficiently small.
- A neighborhood T of the origin is mapped onto U.

$$\lambda \cdot \frac{1}{z^3} \approx 1 \Rightarrow T = \{|z| < |\lambda|^{1/3}\}$$

- Compute the critical points c_λ and critical values v_λ of F_λ

$$c_\lambda = \left(\frac{3}{2}\lambda\right)^{1/5} \qquad v_\lambda = \frac{5}{3}\left(\frac{3}{2}\lambda\right)^{2/5}$$

- $v_\lambda \in T$ since $|\lambda|^{\frac{2}{5}} < |\lambda|^{\frac{1}{3}} \Leftrightarrow \frac{2}{5} > \frac{1}{3}$
- Apply the Riemann-Hurwitz formula to $F_\lambda : F_\lambda^{-1}(T) \mapsto T$ then $F_\lambda^{-1}(T)$ is an annulus containing the five critical points.
- Recursively we obtain that the Julia set of F_λ is a Cantor set of circles.

STEP TWO. An interesting family

Consider the family of maps (Devaney et al., 2004$-$)

$$F_\lambda(z) = z^n + \lambda \frac{1}{z^d}$$

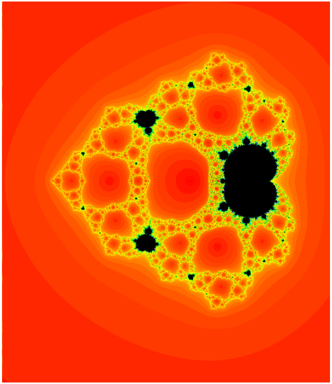
Where $n \geq 2$, $d \geq 1$ and λ now is a parameter (not small)

- ∞ superattracting fixed point. B_λ immediate basin of attraction of ∞.
- Family of rational maps with arbitrary large degree but with symmetries.
- The parameter space is 1-dimensional since we have "one critical orbit".

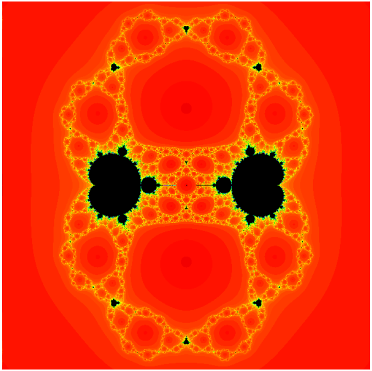
$$c_\lambda = \left(\lambda \frac{d}{n}\right)^{1/(n+d)} \qquad v_\lambda = \frac{n+d}{d}\left(\lambda \frac{d}{n}\right)^{n/(n+d)}$$

$$F_\lambda(z) = z^n + \lambda \cdot \frac{1}{z^d}$$

- RED — critical orbit tends to infinity
- BLACK — critical orbit tends to a bounded orbit
- YELLOW — Bifurcation parameters



(c) Parameter plane $n = d = 2$.



(d) Parameter plane $n = d = 3$.

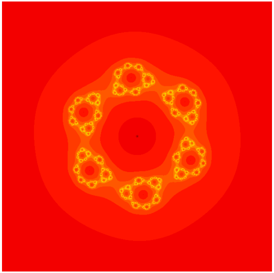
STEP TWO. An interesting family

- B_λ Immediate basin of attraction of ∞ and
- T_λ trap door, defined as $T_\lambda = F_\lambda^{-1}(B_\lambda) \setminus B_\lambda$

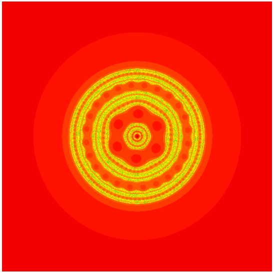
Theorem (Escape Trichotomy, Devaney-Look-Uminski'05)

Assume that the critical orbit escapes to infinity, then

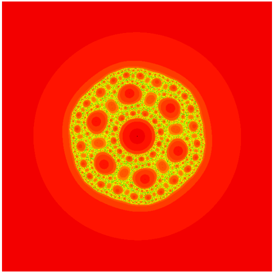
- *If the critical points belong to B_λ then the Julia set of F_λ is a Cantor set.*
- *If the critical values belong to T_λ then the Julia set of F_λ is a Cantor set of circles.*
- *In the rest of the cases then the Julia set of F_λ is a Sierpinski Carpet.*



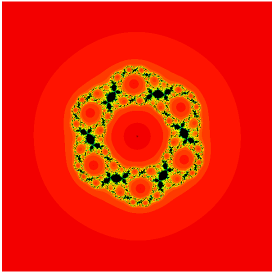
(e) Cantor set.



(f) Cantor set of Circles.



(g) Sierpinski Carpet.



(h) Not escaping.

STEP TWO. An interesting family

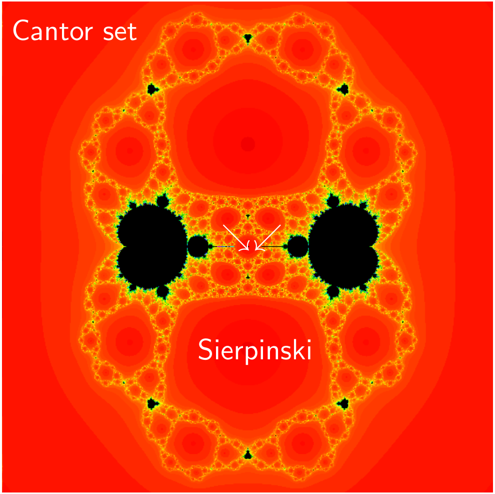


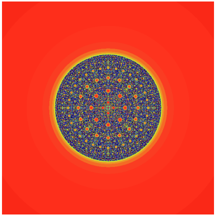
Figura: Parameter Plane of $z^3 + \lambda/z^3$.

STEP TWO. An interesting family

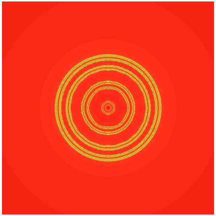
Consider the family $F_\lambda(z) = z^n + \lambda \cdot \frac{1}{z^n}$. The case $n = 2$ and $n > 2$ are different.

Theorem (Devaney, G., 2007)

- *Let $n = 2$. If λ_j is a sequence of parameters converging to 0, then the Julia sets of F_{λ_j} converge tot the closed unit disk.*
- *Let $n \geq 3$. This is not the case.*



(a) Julia set of $z^2 - 0.0001/z^2$.



(b) Julia set of $z^3 - 0.0001/z^3$.

More and more results about the topology of the Julia sets and the structure of the parameter planes. But .... are there other people interested in this family?

- W. Qiu, P. Roesch, X. Wang and Y. Yin (2012, 2014). Using Puzzle and Parapuzzle techniques they proven

Theorem

$F_\lambda(z) = z^n + \lambda/z^n$ with $n \geq 3$. Then,

- *∂B_λ is a Cantor set or a Jordan curve.*
- *The boundary of every hyperbolic component is a Jordan domain.*

STEP THREE. Other people interested ...

- K. Barański and M. Wardal (2015). Computing Hausdorff dimension

Theorem

For every $n \geq 2$, $\lambda \neq 0$, if $F_\lambda(z) = z^n + \lambda/z^n$ is hyperbolic then

$$dim_H(J(F_\lambda)) \geq \frac{\ln n}{\ln n + \ln 2 + \frac{n-1}{2n} \ln^+(|\lambda|/2^{\frac{n+1}{n-1}})}$$

....

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....

- H. Lu, W. Qiu and F. Yang (Preprint). More about Hausdorff dimension

Theorem

$F_{\lambda,n}(z) = z^n + \lambda/z^n$ *with* $n \geq 3$ *then*

$$dim_H(F_\lambda) = 1 + \frac{\ln 2}{\ln n} + \mathcal{O}(|\lambda|^{2-\frac{4}{n}}) \quad as \quad \lambda \to 0.$$

STEP THREE. Other people interested ...

- M. Bonk, M. Lyubich and S. Merenkov (2015).

Theorem

All Sierpinski carpet Julia sets of critically finite rational maps are homeomorphic to the same model but quasi-symmetrically rigid.

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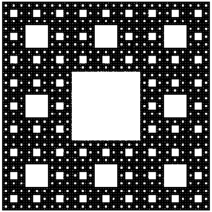
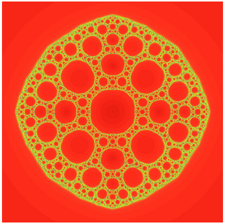
What? What? What?

- Sierpiński carpet Julia set?
- Critically finite rational map?
- Quasi-symmetrically ?
- Rigid?

Sierpiński carpet Julia set?



(c) The Sierpinski Carpet.



(d) Sierpinski Carpet Julia set.

Theorem (Whyburn' 1958)

Any planar set that is compact, connected, locally connected and nowhere dense and has de property of two complementary domains are bounded by disjoint simple closed curves is homeomorphic to the Sierpiński carpet

- A rational map f is critically finite if and only if all critical points of f has finite orbit.

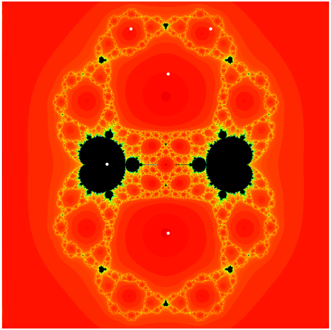
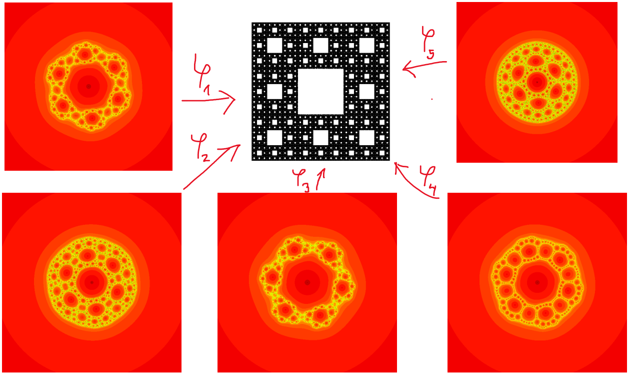


Figura: Parameter Plane of $z^3 + \lambda/z^3$.

- A map f is quasisymmetric in K if it is a restriction of a quasiconformal map, or equivalently

$$\frac{d(f(x), f(y))}{d(f(x), f(z))} \leq \eta\left(\frac{d(x,y)}{d(x,z)}\right) \quad \forall x, y, z \in K$$



(a) Rigid or no rigid?.

All the Sierpinski carpet Julia set are homeomorphic to the Sierpinski Carpet.

STEP THREE. Other people interested ...

Theorem (M. Bonk, M. Lyubich and S. Merenkov. 2015)

Let f and g be postcritically finite rational maps such that $J(f)$ and $J(g)$ are Sierpiński carpets. If φ is a quasisymmetric homeomorphism of $J(f)$ onto $J(g)$, then φ is the restriction to $J(f)$ of a Möbius transformation.

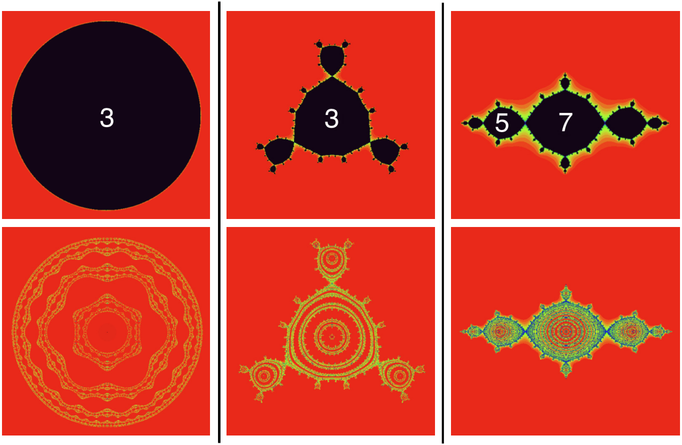
STEP FOUR. Other singular perturbations

- $z^n + \lambda \cdot \frac{1}{(z-a)^d}$. Changing the position of the pole. (Marotta, G.)
- $z^n + c + \lambda \cdot \frac{1}{z^d}$. Changing the polynomial. (Blanchard, Devaney, Marotta, Russell, G.)
- $z^n + c + \lambda \cdot \frac{1}{\prod_{i=1}^{k}(z-c_i)^{d_i}}$. Multiple poles. (Russell, Marotta, G.)
- McMullen like mappigns. (Godillon, G.)

$$F_\lambda(z) = p(z) + \lambda \cdot \frac{1}{\prod_{i=1}^{k}(z-c_i)^{d_i}}$$

where $p(z)$ is a postcritically finite polynomial.

STEP FOUR. Other singular perturbations



(b) McMullen-like mappings. z^3+10^{-2}/z^3 (left), z^3+i-10^{-7}/z^3 (center) and $z^2-1+10^{-22}/(z^7(z+1)^5)$ (right).

STEP FOUR. Other singular perturbations

... Examples of singular perturbations of Rational maps

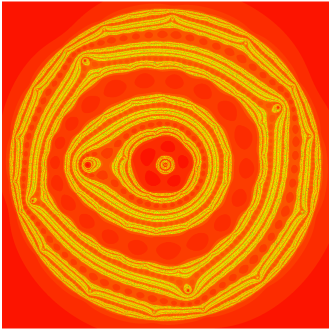
- $F_{\lambda,a}(z) = z^n \frac{z-a}{1-\bar{a}z} + \lambda \cdot \frac{1}{z^d}$ (J. Canela' 2016, 2017).
- $F_\lambda(z) = \frac{z^n(z-a)}{Q(z)} + \lambda \frac{1}{z^d}$ (J. Canela, X. Jarque, D. Paraschiv' 2020).

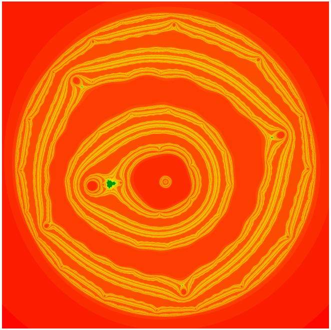
...obtaining Fatou components of large connectivity.

STEP FOUR. Other singular perturbations

Singular perturbations of Rational maps. J. Canela (2016,2017).

$$F_\lambda(z) = z^3 \frac{z - 0.5i}{1 + 0.5iz} + \lambda \cdot \frac{1}{z^2}$$





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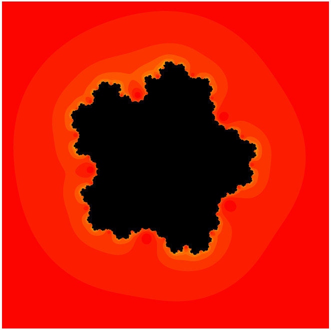
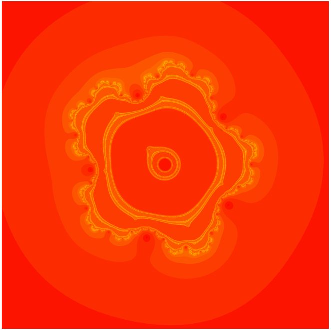


(e) $f(z) = \frac{z^6(z-(0.2+0.2i))}{z^5+(0.2-0.6i)}$.



(f) $f(z) + 3.610^{-21} \cdot \frac{1}{z^3}$