DYNAMICS ON THE BOUNDARY OF FATOU COMPONENTS

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INTRODUCTION TO HOLOMORPHIC ITERATION

$f\colon S \to S$ holomorphic, $S = \mathbb{C}$ or $S = \widehat{\mathbb{C}}$.

$$f^n = f \circ \overset{n}{\cdots} \circ f$$

Totally invariant partition of S:

Fatou set: Set of stability (normality). Open. $\mathcal{F}(f)$.

Julia set: Chaotic set. Closed. $\mathcal{J}(f) = S \setminus \mathcal{F}(f)$.

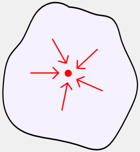
Escaping set: points which escape to ∞. $\mathcal{I}(f)$.

Fatou components: connected components of the Fatou set.

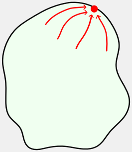
FATOU COMPONENTS

THEOREM (Fatou-1919)

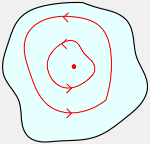
U simply-connected invariant Fatou component. Possibilities:



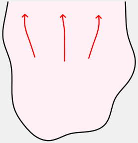
1. $f^n_{|U} \to z_0 \in U$
Attracting basin
$|f'(z_0)| < 1$



2. $f^n_{|U} \to z_0 \in \partial U$
Parabolic basin
$f'(z_0) = 1$

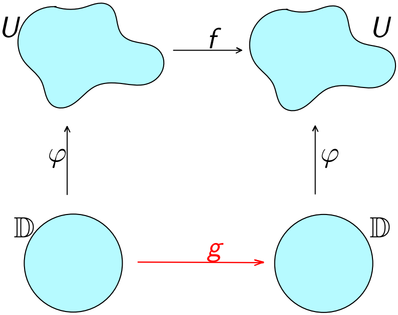


3. $f_{|U} \sim e^{2\pi i\theta} z$, $\theta \notin \mathbb{Q}$
Siegel disk



4. f transcendental,
$f^n_{|U} \to \infty$
Baker domain

DYNAMICS INSIDE A FATOU COMPONENT



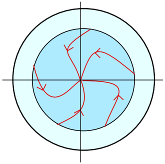
$\varphi\colon \mathbb{D} \to U$ (Riemann map) and $f_{|U} \sim g$, where $g\colon \mathbb{D} \to \mathbb{D}$

Tools to study the dynamics of $g\colon \mathbb{D} \to \mathbb{D}$:

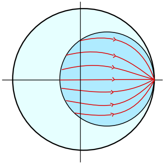
- Denjoy-Wolff Theorem
  If g is not a rotation, all orbits converge to the same point $p \in \overline{\mathbb{D}}$.
- Cowen's classification

DYNAMICS OF $g\colon \mathbb{D} \to \mathbb{D}$. Cowen's classification

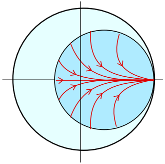
Existence of an **absorbing domain** where g is conjugate to $\phi\colon \Omega \to \Omega$ (Möbius).



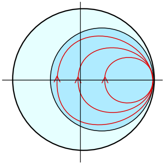
1. $\Omega = \mathbb{C}$
$\phi(z) = \lambda z$, $|\lambda| < 1$.
(elliptic)



3. $\Omega = \mathbb{H}$
$\phi(z) = \lambda z$, $\lambda > 1$.
(hyperbolic)

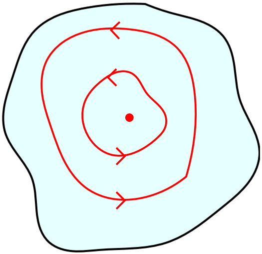


2. $\Omega = \mathbb{C}$
$\phi(z) = z + 1$.
(doubly-parabolic)

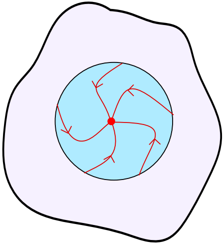


4. $\Omega = \mathbb{H}$
$\phi(z) = z + 1$.
(simply-parabolic)

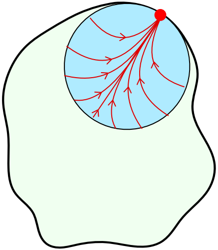
DYNAMICS INSIDE A FATOU COMPONENT



(a) Siegel disk
(irrational rotation)



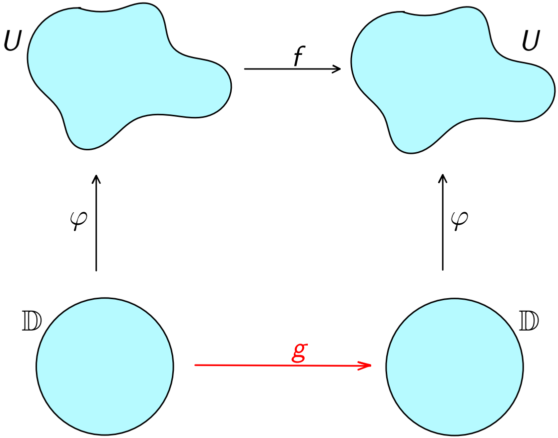
(b) Attracting basin
(elliptic)



(c) Parabolic basin
(doubly-parabolic)

For Baker domains, doubly-parabolic, hyperbolic and simply-parabolic types are possible $\rightsquigarrow$ classification of Baker domains

QUESTION: Dynamics on ∂U?



Intuitive idea: study $g_{|\partial\mathbb{D}}$.

But g and φ may not be defined on $\partial\mathbb{D}$...

INNER FUNCTIONS

DEF: Radial limit

Let $g\colon \mathbb{D} \to \mathbb{D}$ holomorphic, $e^{i\theta} \in \partial\mathbb{D}$. The **radial limit** of g at $e^{i\theta}$ is:

$$g^*(e^{i\theta}) = \lim_{r \to 1^-} g(re^{i\theta}).$$

THEOREM (Fatou, Riez and Riez)

For Lebesgue-almost every θ, $g^*(e^{i\theta})$ exists.

DEF: Inner function

A holomorphic function $g\colon \mathbb{D} \to \mathbb{D}$ is an **inner function** if $\left|g^*(e^{i\theta})\right| = 1$, for Lebesgue-almost all θ.

g^* induces a dynamical system almost everywhere on $\partial\mathbb{D}$.

ERGODICITY AND RECURRENCE

Ergodic properties of measurable maps

Let $(X, \mathcal{A}, \mu)$ be a measure space and $T\colon X \to X$ measurable. Then we say that T is:

- **ergodic**, if for every $A \in \mathcal{A}$ such that $T^{-1}(A) = A$, there holds $\mu(A) = 0$ or $\mu(X \setminus A) = 0$.
- **recurrent**, if for every $A \in \mathcal{A}$ and μ-almost every $x \in A$, $T^n(x) \in A$ for infinitely many n's.

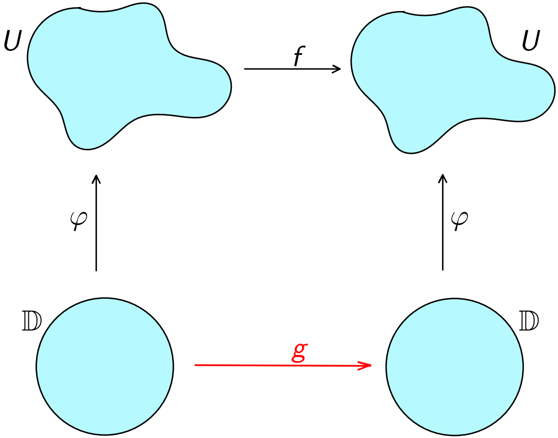
Ergodicity and recurrence are independent notions.

THEOREM[1]

If T is ergodic and recurrent with respect to the Lebesgue measure, then Lebesgue-almost every point has a dense orbit.

[1] General result in ergodic theory. A proof can be found in Aaronson. *Introduction to Infinite Ergodic Theory.*

QUESTION: Dynamics on ∂U?



Intuitive idea: study $g_{|\partial\mathbb{D}}$ (defined almost everywhere).

But φ may not be defined on $\partial\mathbb{D}$...

MEASURE ON ∂U. THE HARMONIC MEASURE

DEF: Harmonic measure

Let $U \subset \widehat{\mathbb{C}}$ be simply-connected and let $\varphi \colon \mathbb{D} \to U$ be a Riemann map, such that $\varphi(0) = z \in U$. The **harmonic measure** ω of ∂U with base point z is the image under φ of the normalized Lebesgue measure of $\partial\mathbb{D}$.

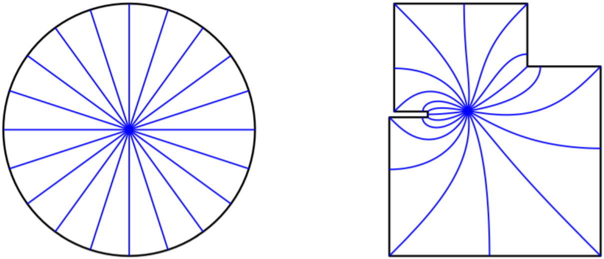


Figure: By Christopher Bishop.

With this measure, we only need to study $g^* \colon \partial\mathbb{D} \to \partial\mathbb{D}$.

ERGODIC PROPERTIES OF INNER FUNCTIONS

INNER FUNCTION	FATOU COMPONENT	Ergodicity	Recurrence
Rational rotation		✗	✓
Irrational rotation	Siegel disk	✓	✓
Elliptic *	Attracting basin	✓	✓
Doubly-parabolic *	Parabolic b./Baker d.	✓	?
Hyperbolic	Baker domain	✗	✗
Simply-parabolic	Baker domain	✗	✗

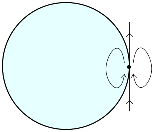
* In case of degree $d < \infty$, the boundary map is conjugate to $x \mapsto dx \mod 1$.

Summary of different results in:

Aaronson. *Ergodic theory for inner functions of the upper half plane.*
Aaronson. *A remark on the exactness of inner functions.*
Barański, Fagella, Jarque, Karpińska. *Escaping points in the boundaries of Baker domains.*
Bourdon, Matache, Shapiro. *On the convergence to the Denjoy-Wolff point.*
Doering, Mañé. *The dynamics of inner functions.*
Hamilton. *Absolutely continuous conjugacies of Blaschke products.*
Shub, Sullivan. *Expanding endomorphisms of the circle revisited.*

SIMPLY-PARABOLIC INNER FUNCTIONS

With non-singular Denjoy-Wolff point



- Linearization around the D-W point p: $g \sim z+1$ (Fatou coordinates)
- $g(\partial\mathbb{D}) \subset \partial\mathbb{D}$ (at least in a nbh. of p)

Therefore we have $I \subset \partial\mathbb{D}$ with $p \in \overline{I}$ and $g^*_{|I} \sim x+1$

- On I, $(g^*)^n \to p$
  $\rightsquigarrow$ g^* is not recurrent
- On $\mathbb{R}$, $x+1$ is non-ergodic: $\bigcup_{n\in\mathbb{Z}} \left(n, n+\frac{1}{2}\right)$ is invariant
  $\rightsquigarrow$ g^* is not ergodic

$\rightsquigarrow$ The same works for hyperbolic inner functions with non-singular DW point using Koenigs' coordinates

AARONSON'S DICHOTOMY

Let $g \colon \mathbb{D} \to \mathbb{D}$ be an inner function with Denjoy-Wolff point p. Then:

1. If $\sum_{n=1}^{\infty}(1-|g^n(z)|) < \infty$ for some $z \in \mathbb{D}$, then $p \in \partial\mathbb{D}$ and $(g^*)^n(z)$ converges to p for almost every $z \in \partial\mathbb{D}$.
2. If $\sum_{n=1}^{\infty}(1-|g^n(z)|) = \infty$ for some $z \in \mathbb{D}$, then g^* is recurrent.

Key idea of the proof: Relate the dynamics on the boundary with the dynamics on the disk (where g is holomorphic).

HARMONIC MEASURE

Let $A \subset \partial\mathbb{D}$. The **harmonic measure** (with base point $z \in \mathbb{D}$) of A is:

$$\omega_z(A) = \omega(z, A, \mathbb{D}) := \frac{1}{2\pi}\int_A \frac{1-|z|^2}{|w-z|^2}dw.$$

- $\omega_z((g^*)^{-1}(A)) = \omega_{g(z)}(A)$.

- (Bourdon-Matache-Shapiro)
  If $g : \mathbb{D} \to \mathbb{D}$ is a hyperbolic or simply-parabolic inner function, then $\sum_{n=1}^{\infty}(1 - |g^n(z)|) < \infty$ for all $z \in \mathbb{D}$ and g^* is not recurrent.
  Moreover, $(g^*)^n(z) \to p$ for almost every $z \in \partial\mathbb{D}$.

- Careful!
  This does not imply that for a hyperbolic or simply-parabolic Baker domain the escaping set has full harmonic measure. $\rightsquigarrow$ The Riemann map can be highly discontinuous.

- (Barański-Fagella-Jarque-Karpińska)
  For a hyperbolic or simply-parabolic Baker domain of finite degree the escaping set has full harmonic measure.

- If $g : \mathbb{D} \to \mathbb{D}$ is elliptic, g^* is recurrent.
- Careful!
  There are examples of doubly-parabolic inner functions which are recurrent and others which are not.
- (Doering-Mañé)
  If $g : \mathbb{D} \to \mathbb{D}$ is doubly-parabolic inner function, either with non-singular Denjoy-Wolff point or associated to a parabolic basin, then $\sum_{n=1}^{\infty}(1 - |g^n(z)|) = \infty$ for all $z \in \mathbb{D}$ and g^* is recurrent.

THE EXAMPLE: $f(z) = z + e^{-z}$

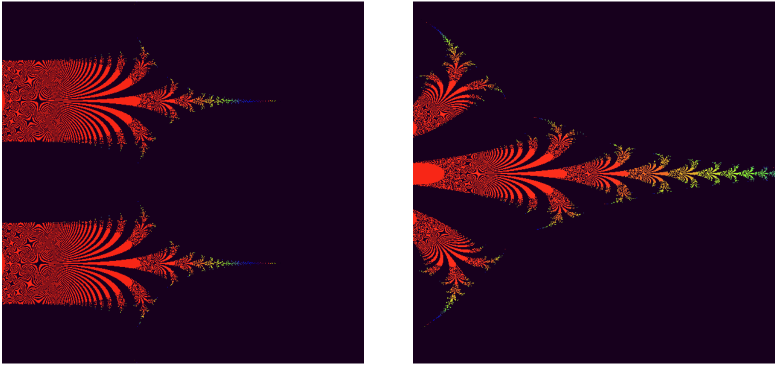


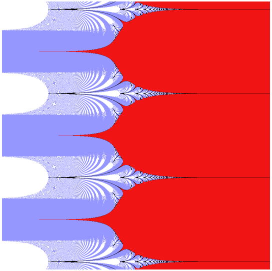
Figure: On the left, the dynamical plane of $f(z) = z + e^{-z}$. On the right, a zoom of it.

Previously studied in:

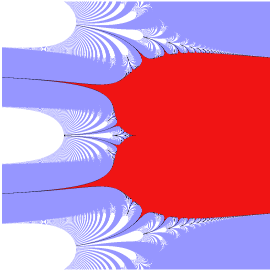
Baker, Domínguez. *Boundaries of unbounded Fatou components of entire functions.*
Fagella, Henriksen. *Deformation of entire functions with Baker domains.*
Barański, Fagella, Jarque, Karpińska. *Escaping points in the boundaries of Baker domains.*

THE EXAMPLE: $f(z) = z + e^{-z}$

Semiconjugacy to $h(w) = we^{-w}$



$\xrightarrow{w=e^{-z}}$



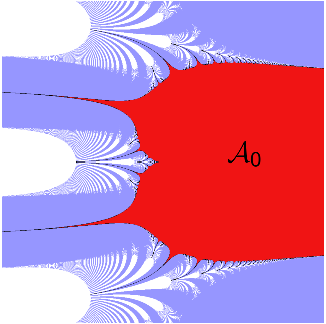
$z \mapsto f(z) = z + e^{-z}$

$w \mapsto h(w) = we^{-w}$

$$\begin{tikzcd}
z \arrow[r, "f"] \arrow[d, "w=e^{-z}"] & f(z) = z + e^{-z} \arrow[d, "w=e^{-z}"] \\
w \arrow[r, "h"] & h(w) = we^{-w}
\end{tikzcd}$$

THE EXAMPLE: $f(z) = z + e^{-z}$

The parabolic basin of $h(w) = we^{-w}$



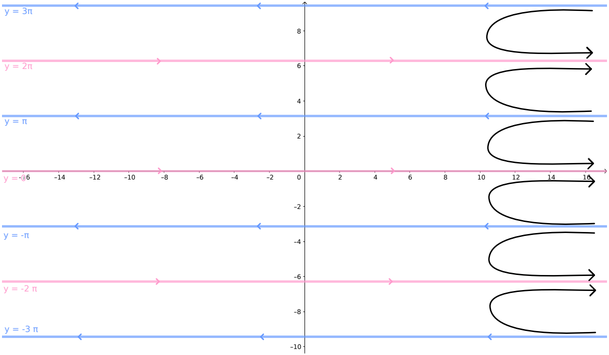
- 0 is a parabolic fixed point.
- Singular values: 0, $\frac{1}{e}$.
- $\frac{1}{e}$ converges to 0 under iteration.
- $\mathcal{F}(h) = \mathcal{A}$, parabolic basin of 0.
- $\mathcal{A}_0$, immediate parabolic basin.

THEOREM (Baker-Domínguez, Fagella-Henriksen)

- $\mathbb{R}_+ \subset \mathcal{A}_0$, so $\mathcal{A}_0$ is unbounded.
- $\mathbb{R}_- \subset \mathcal{J}(h)$.
- The map h has degree two on $\mathcal{A}_0$ and $h_{|\mathcal{A}_0} \sim \frac{3z^2+1}{z^2+3}$ (**doubly-parabolic**).

THE EXAMPLE: $f(z) = z + e^{-z}$

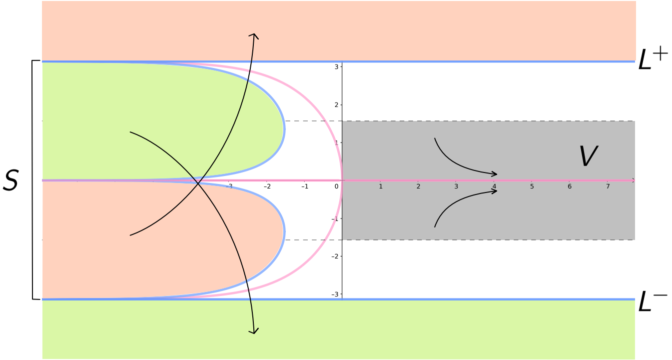
The dynamical plane of f



- $f(z + 2k\pi i) = f(z) + 2k\pi i$, for all $z \in \mathbb{C}$.
- The lines $\{\mathrm{Im}\ z = k\pi\}_{k\in\mathbb{Z}}$ are invariant.
- In each strip $\{(2k-1)\pi < \mathrm{Im}\ z < (2k+1)\pi\}_{k\in\mathbb{Z}}$, there is one preimage of $\mathcal{A}_0$, which is a **doubly-parabolic Baker domain** U_k.

THE EXAMPLE: $f(z) = z + e^{-z}$

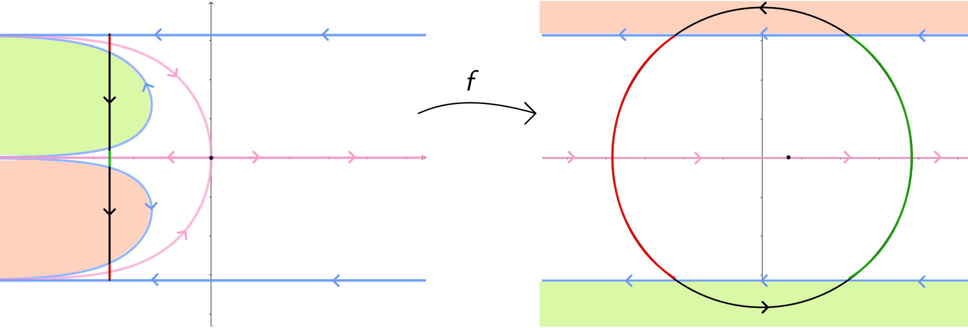
The dynamical plane of f



- $S := \{z\colon -\pi \leq \mathrm{Im}\ z \leq \pi\}$ and $U \subset S$, **invariant Baker domain**.
- $f\colon f^{-1}(S) \cap S \to S$ proper map of **degree 2**. Each point in $\mathbb{C} \setminus S$ has exactly one preimage in S.

THE EXAMPLE: $f(z) = z + e^{-z}$

The dynamical plane of f



THE EXAMPLE: $f(z) = z + e^{-z}$

Questions and goals

$$U \subset S, \text{ doubly-parabolic invariant Baker domain.}$$

$$f_{|\partial U} \text{ is ergodic and recurrent.}$$

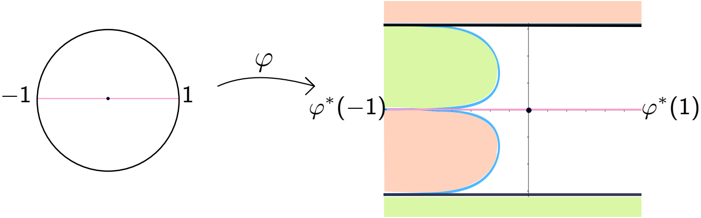
$$\omega\text{-almost every orbit is dense and } \mathcal{I}(f) \text{ has zero measure.}$$

Goal: Study the boundary of the Baker domain U and its dynamics.

- Accesses to infinity from U.
  $\rightsquigarrow$ Complete characterization by means of the inner function.
- Periodic points in ∂U.
  $\rightsquigarrow$ Complete characterization of periodic points in ∂U.
  $\rightsquigarrow$ All are accessible.
- Escaping points in the boundary?
  $\rightsquigarrow$ For a general Baker domain, it is an open question.
  $\rightsquigarrow$ We construct uncountably many curves of escaping points in ∂U.

THE EXAMPLE: $f(z) = z + e^{-z}$

Accesses to infinity



Fix φ Riemann map such that $\varphi(0) = 0$ and $\varphi(\mathbb{R} \cap \mathbb{D}) = \mathbb{R}$.

$$\Theta := \left\{ e^{i\theta} \in \partial\mathbb{D} \colon \varphi^*(e^{i\theta}) = \infty \right\}$$

THEOREM (Baker-Domínguez)

The set Θ consists precisely of points $e^{i\theta} \in \partial\mathbb{D}$ such that $g^n(e^{i\theta}) = 1$.
Equivalently, accesses from U to ∞ are defined by the preimages of $\mathbb{R}_+$ under f.

THEOREM

Let $z_0 \in \partial U$ be periodic under f, i.e. $f^p(z_0) = z_0$, for some p. Then z_0 is accessible.

THEOREM

Let $e^{i\theta} \in \partial \mathbb{D}$ be periodic under g, i.e. $g^p(e^{i\theta}) = e^{i\theta}$ for some $p > 1$. Then, $\varphi^*(e^{i\theta})$ exists and it is a periodic point of period p.

Consequence: Characterization of periodic points in ∂U.
A point $z \in \partial U$ satisfies $f^p(z) = z$ for some $p \geq 1$ if, and only if, $z = \varphi^*(e^{i\theta})$ for some $e^{i\theta} \in \partial \mathbb{D}$ satisfying $g^p(e^{i\theta}) = e^{i\theta}$.

Goal: Describe the escaping set constructing the Cantor Bouquet of f.

$$\widehat{S} := \{z \in S \colon f^n(z) \in S, \text{ for all } n\}$$

$$S_0 := S \cap \mathbb{H}^+ \qquad S_1 := S \cap \mathbb{H}^-$$

$$\Sigma_2 = \left\{ k = \{k_j\}_j : k_j = 0 \text{ or } k_j = 1, \text{ for all } j \geq 0 \right\}$$

To $z \in \widehat{S}$, we associate a sequence $k = \{k_n\}_n \in \Sigma_2$ (its **itinerary**) such that $f^n(z) \in S_j$ if and only if $k_n = j$, with $j = 0$ or 1.

THEOREM

For every sequence $k = \{k_j\}_j \in \Sigma_2$ there exists a curve $\gamma_k \subset S$ whose points belong to $\mathcal{I}(f) \cap \widehat{S}$, with itinerary prescribed by k and $\gamma_k \subset \partial U$.

- The studied points have zero measure (periodic points, escaping set)
  $\rightsquigarrow$ find oscillating points (typical points w.r.t. harmonic measure)
- Periodic points in ∂U
  $\rightsquigarrow$ are accessible periodic points dense in ∂U?[2]
- Curves of non-accessible escaping points
  $\rightsquigarrow$ are all escaping points non-accessible?[2]
  $\rightsquigarrow$ construction of the Cantor Bouquet

[2] Stated as a conjecture in Barański, Fagella, Jarque, Karpińska. *Escaping points in the boundaries of Baker domains.*

Thank you for your attention!!!