Universitat de Barcelona
Facultat de Matemàtiques i Informàtica

Projecting Newton maps of Entire functions via the Exponential map

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OVERVIEW

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- Motivations: Complexification of real systems
  Logarithmic change of coordinates
  Projection of transcendental meromorphic maps

- Goal: Get familiar with projections & essential singularities

- Contents:
  - A♣ *Newton maps*
  - 2♣ *Projectable maps*
  - 3♣ *Class K (meromorphic outside a small set)*
  - 4♣ Logarithmic lift of Fatou components
  - 5♠ Exploration of a simple Newton map & its Projection
  - 6♠ Searching possible Wandering domains

1. NEWTON maps

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☞ *Barański, Fagella, Jarque, Karpińska* [2015]

- Entire function $f : \mathbb{C} \to \mathbb{C}$ | Root of f | Critical point of f

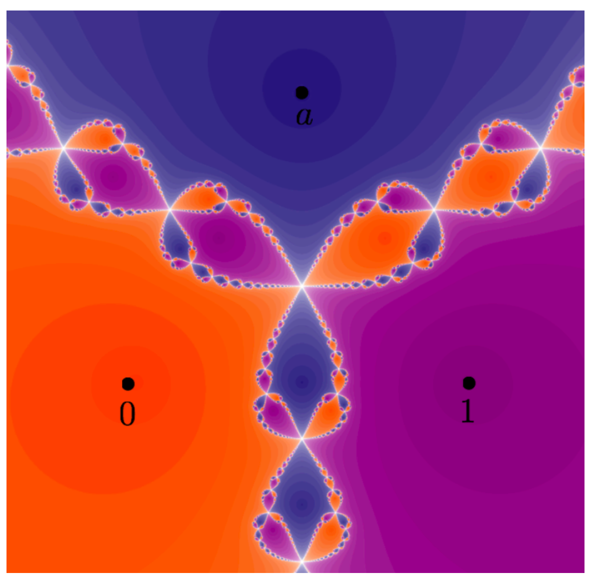
↓ ↓ ↓

- Newton map $N_f : \mathbb{C} \to \hat{\mathbb{C}}$ | Attracting fixed point of f | Pole of f

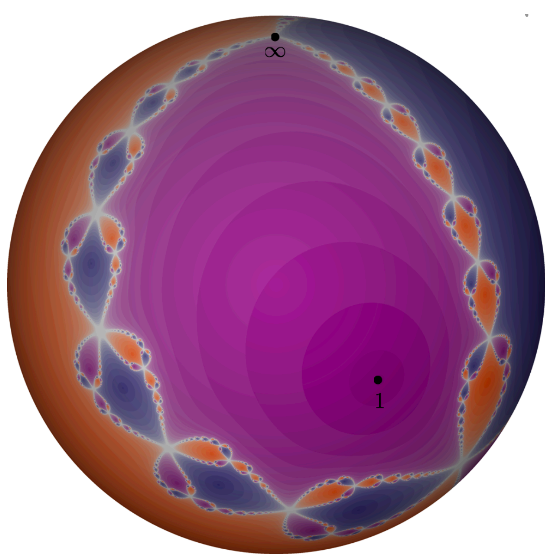
$$N_f(z) = z - \frac{f(z)}{f'(z)}$$

$f(z) = z(z-1)(z-a)$

N_f rational map



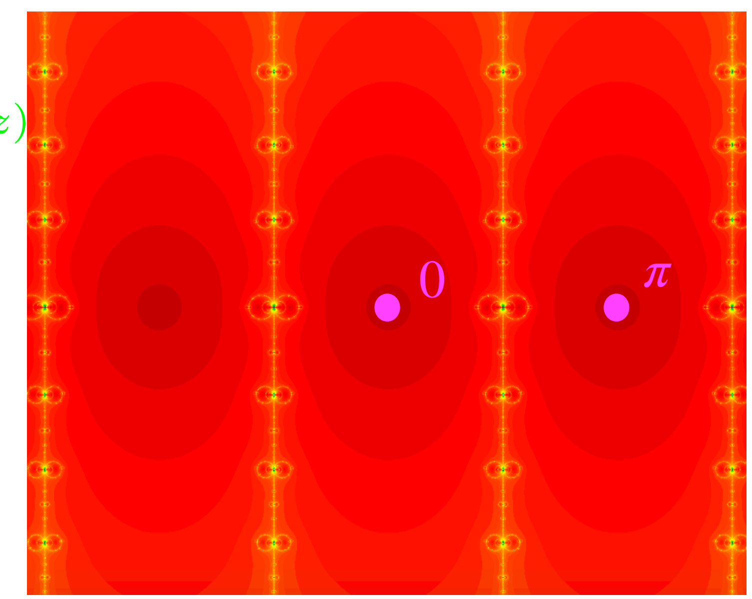
Stereographic projection



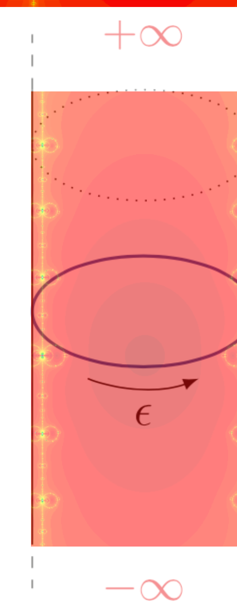
Riemann sphere:
$\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$

$f(z) = sin(z) \neq p(z)e^{q(z)}$

$N_f(z) = z - tan(z)$
transcendental mero.



Modulo π



Cylinder:
$\mathbb{C}/(\pi\mathbb{Z})$

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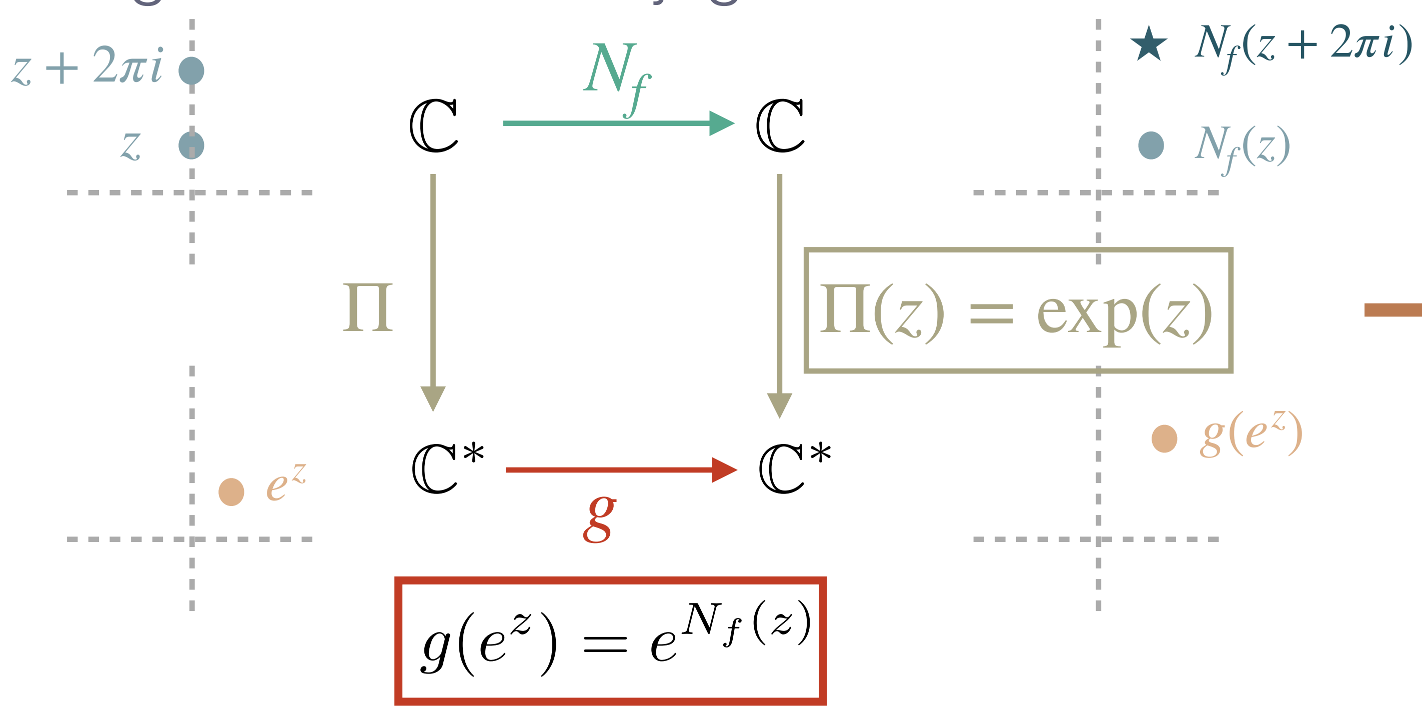
☞ *Keen* [1988]

- Diagram of semi-conjugation:



$\longrightarrow$ $\Pi(z) = \exp\left(\frac{2\pi i}{P} z\right)$

- Projectable maps under the exponential:

$$N_f(z+2\pi i) = N_f(z) + 2\pi i m, \quad m \in \mathbb{Z}$$

$\longrightarrow$ $N_f(z+P) = N_f(z) + mP$

(Semi-period $P \in \mathbb{C}^*$)

- Interesting class of Newton maps:

$$N_f(z) = z - M(e^z), \quad M(w) = \frac{aw+b}{cw+d} \text{ Möbius map}$$

  - If $a \neq 0$ & $b \neq 0$: $f(z) = ke^{\frac{d}{b}z}(ae^z+b)^{\frac{bc-ad}{ab}}$
  - If $a = 0$: $f(z) = ke^{\frac{d}{b}z}e^{\frac{c}{b}e^z}$
  - If $b = 0$: $f(z) = ke^{\frac{d}{b}z}e^{-\frac{d}{a}e^{-z}}$

☞ *Baker, Domínguez, Herring* [2001]
☞ *Bolsch* [1996]

- Meromorphic functions **outside** a compact and countable set $E(f)$, which is the closure of isolated **essential singularities**.
  - Class K is a closed system under iteration! $E(f\circ g) = E(g) \cup g^{-1}(E(f))$
  - (Essential) **Poles** & Prepoles (order $m \in \mathbb{N}^*$): $q \in \hat{\mathbb{C}} - E(f)$ s.t $f^m(q) = b \in E(f)$
  - Singular values: $S(f) = \overline{sing(f^{-1})}$
    - **Critical** values: $f(c) \in \hat{\mathbb{C}}$ s.t $f'(c) = 0$ with $c \in \hat{\mathbb{C}} - E(f)$
    - **Asymptotic** values: $a \in \hat{\mathbb{C}} - E(f)$ s.t $f(\gamma(t)) \underset{t\to\infty}{\longrightarrow} a$ for some path $\gamma(t) \underset{t\to\infty}{\longrightarrow} b \in E(f)$
- **Fatou** set: $F(f) = \{z \in \hat{\mathbb{C}} : \{f^n\} \text{ is defined \& normal in some nbd. of z}\}$
  - **Periodic** Fatou component ($f^p(U) \subseteq U$ for some $p \in \mathbb{N}$): Attracting basin (attracting p.), Parabolic basin (parabolic p.), Siegel disk (irrational neutral p.), Herman ring, Baker domain.
  - **Wandering** domain ($f^n(U) \cap f^m(U) = \emptyset$ for all $m > n \geq 0$)
- **Julia** set: $J(f) = \hat{\mathbb{C}} - F(f)$
  - If $J_\infty(f) = \bigcup_{n=0}^{\infty} f^{-n}(E(f))$ has at least 3 points $\Longrightarrow$ $J(f) = \overline{J_\infty(f)}$

4. LOG. LIFTS of FATOU components

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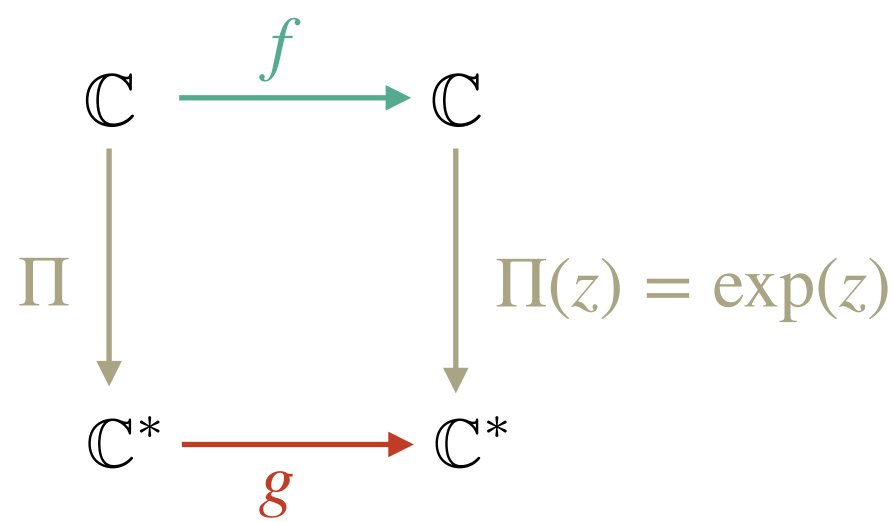
☞ *Bergweiler* [1996]
☞ *Zheng* [2005]

Theorem [Bergweiler, 1995]:

Let f be entire, g an analytic self-map of $\mathbb{C}^*$ and suppose $\exp(f(z)) = g(e^z)$.
If f is not linear or constant, then: $\exp^{-1}(F(g)) = F(f)$
$\exp^{-1}(J(g)) = J(f)$



$\exp^{-1}(V) = \{z : \ e^z \in V\}$

$g^n(e^z) = e^{f^n(z)}$

Theorem [Zheng, 2005]:

Let $f(z)$ and $g(z)$ both be in class K with either $\infty \in E(f)$ or $f(\infty) \neq \infty$.
If $\exp(f(z)) = g(e^z)$, then: $\exp^{-1}(F(g)) = F(f)$
$\exp^{-1}(J(g)) = J(f)$

5. EXPLORATION of a simple Newton map (I)

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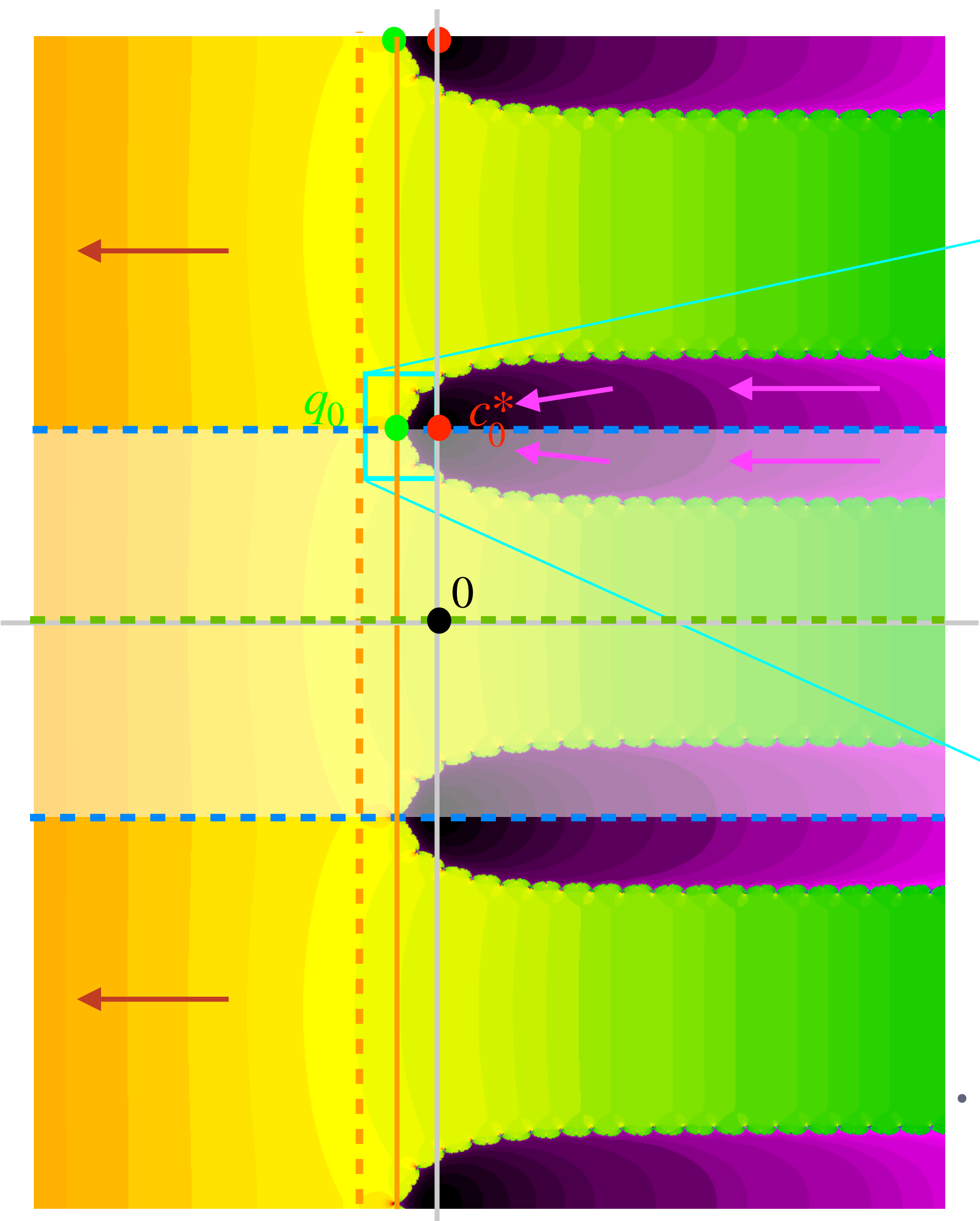
- Newton's map of $f(z) = e^z(1 + e^z)$: $N_f(z) = z - \dfrac{1 + e^z}{1 + 2e^z}$
- Fixed points: $c_k^* = \pi i + 2\pi i k, \quad k \in \mathbb{Z}$ (superattracting, deg. 2) ⟶ Basins of attraction: U_k
- Poles: $q_k = (-\ln 2 + \pi i) + 2\pi i k, \quad e^{-q_k} = -\dfrac{1}{2}$



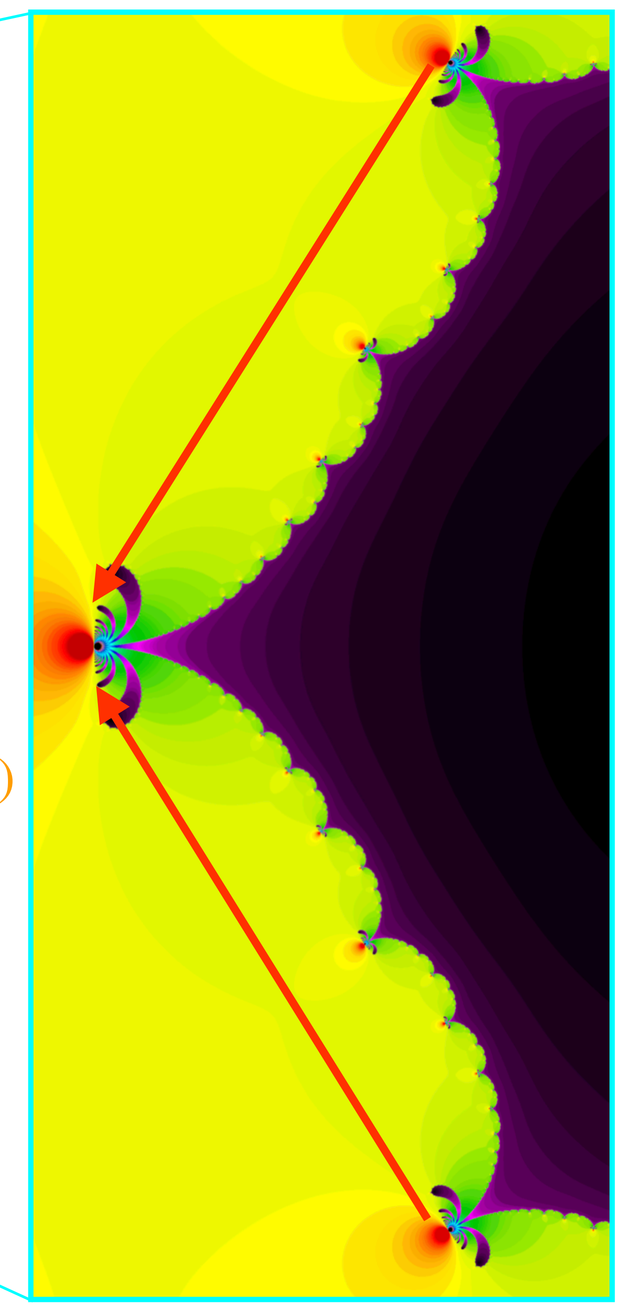
Invariant horizontal lines:

- $r_k(t) = t + i(2k + 1)\pi$
- $s_k(t) = t + i2k\pi$

Special vertical line:

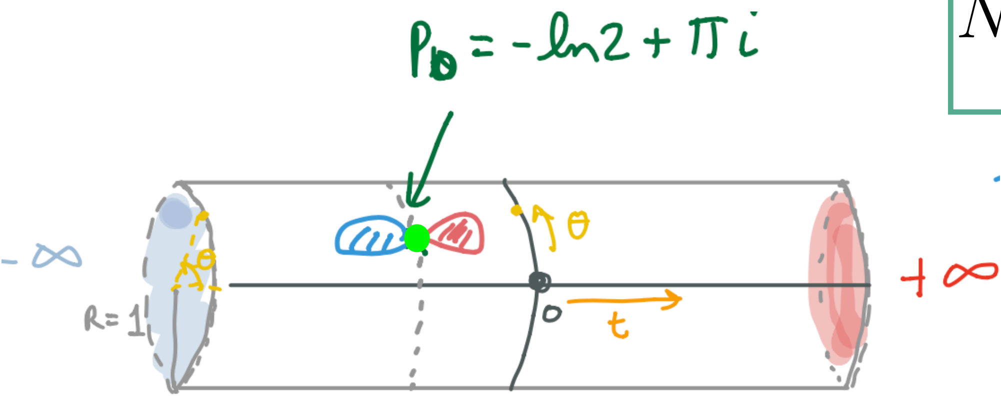
- $l(t) = -\ln 2 + it \quad ---> \quad N_f\big(l(t)\big) = -\dfrac{3}{4} + l(t)$

- Baker domain: U



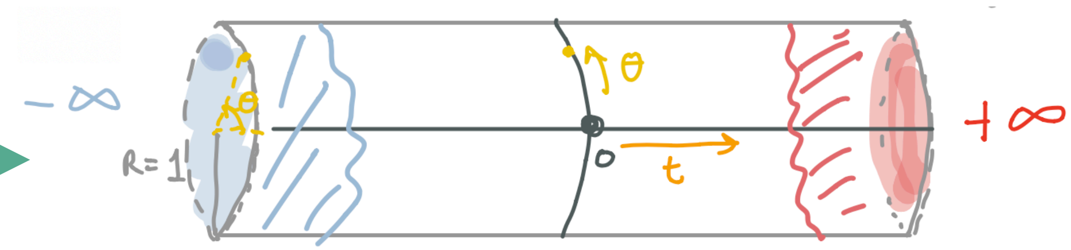
Cylinder: $\mathbb{C}/(2\pi i\mathbb{Z})$

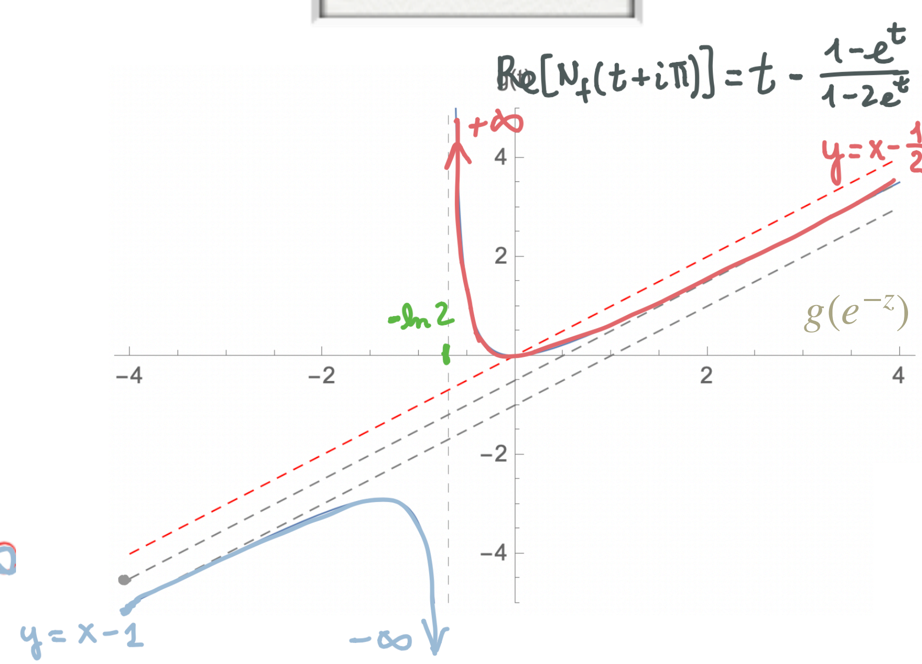
$$N_f(z) = z - \frac{1+e^z}{1+2e^z}$$



N_f

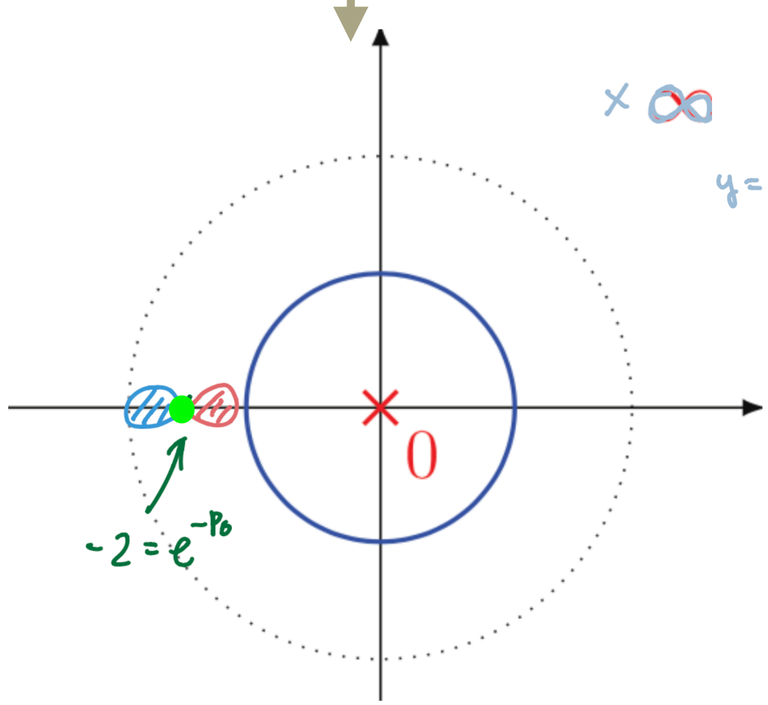
$P = 2\pi i$



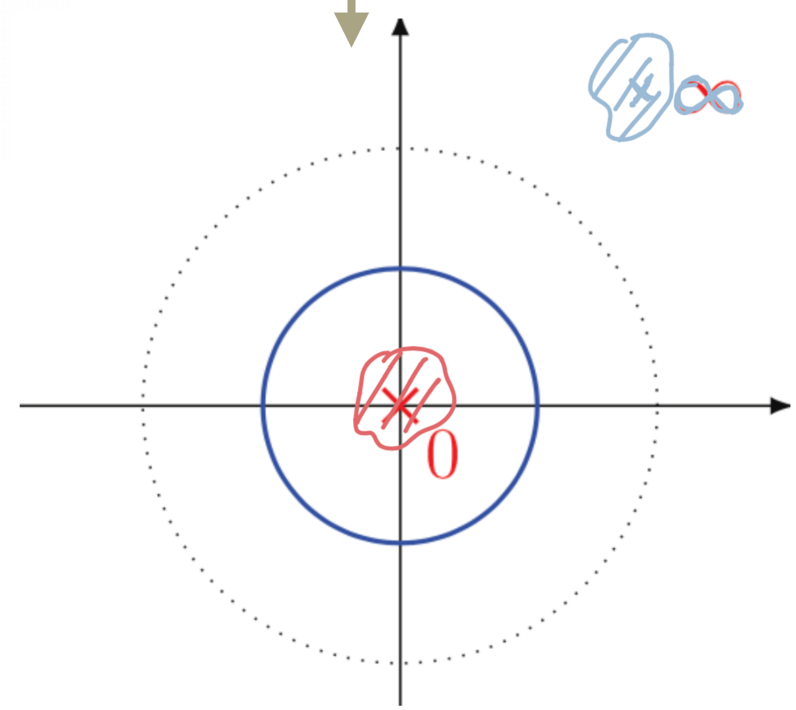


$w = \Pi(z) = \exp(-z)$

$$g(e^{-z}) = \exp\left(-N_f(z)\right) = \exp\left(z - \frac{1+e^z}{1+2e^z}\right) = e^{-z}\exp\left(\frac{e^{-z}+1}{e^{-z}+2}\right)$$



g



$$g(w) = w\exp\left(\frac{w+1}{w+2}\right)$$

5. EXPLORATION of a simple Newton map (III)

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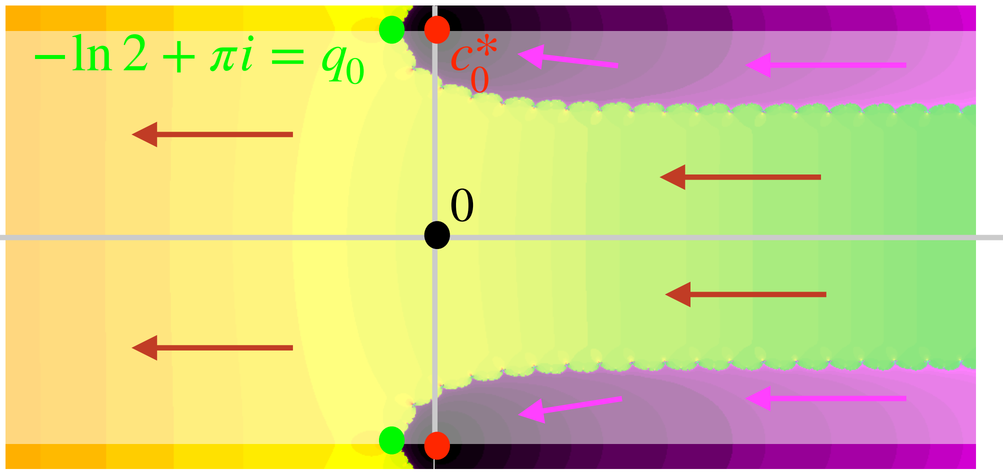


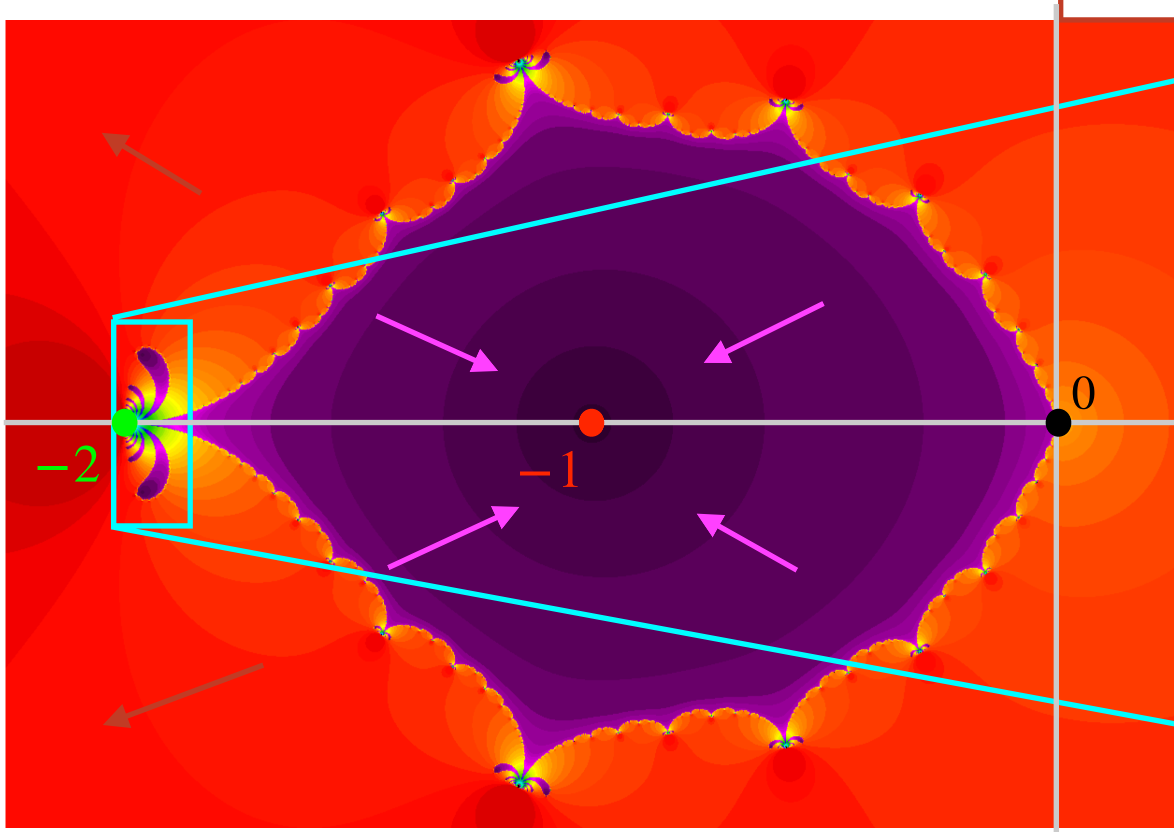
$$N_f(z) = z - \frac{1+e^z}{1+2e^z}$$

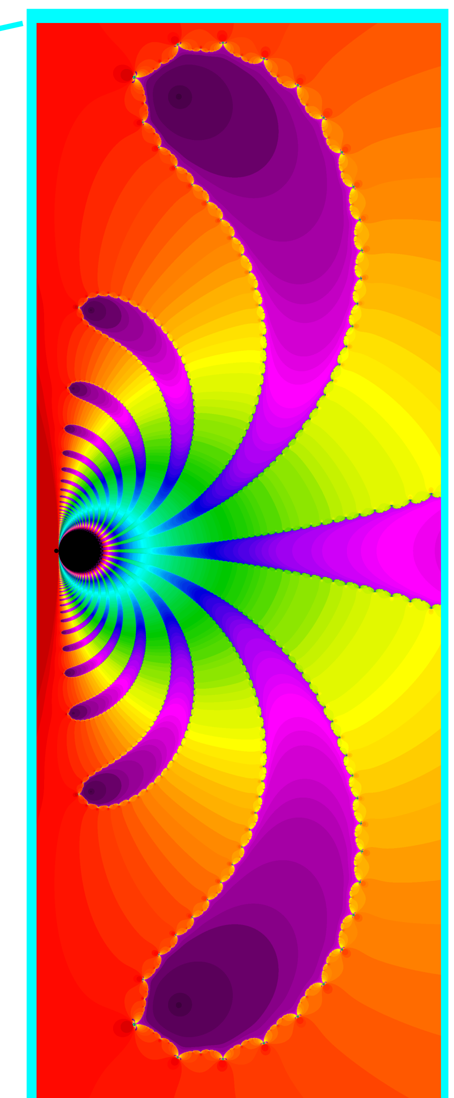
- Basins of attraction of c_k^* : U_k
- Baker domain: U

$w = \exp(-z)$

$$g(w) = w \exp\left(\frac{w+1}{w+2}\right)$$

- Basins of attraction of -1: V_k
- Basin of attraction of ∞: V





6. Searching possible WANDERING domains

Theorem [BFJK, 2020]:

Let f be a topologically hyperbolic meromorphic map, and U_n the Fatou component such that $f^n(U) \subset U_n$.
If $U_n \cap P(f) = \emptyset$, then for every compact $K \subset U$ and every $r > 0$,
there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ and every $z \in K$: $D(f^n(z), r) \subset U_n$
In particular, diam $U_n \xrightarrow[n\to\infty]{} \infty$ and $\operatorname{dist}(f^n(z), \partial U_n) \xrightarrow[n\to\infty]{} \infty$.

☞ *Barański, Fagella, Jarque, Karpińska* [2020]

☞ *Buff, Rückert* [2006]

$$P(f) = \overline{\bigcup_{n=0}^{\infty} f^n(S(f))}$$

☆ Candidate *Wandering domains* must contain ***postsingular** points.*

☆ *Finite-type functions* have **neither** Wandering domains **nor** Baker domains.

- **Buff-Rückert**'s family — Newton map of $f(z) = \exp\left(\frac{-1}{\alpha}\left(z + \frac{1}{2\pi i}e^{2\pi i z}\right)\right)$ for $\alpha \in (0,\infty)$:

$$N_f(z) = z + \frac{\alpha}{1 + e^{2\pi i z}}$$

$$N_f(z+1) = N_f(z) + 1$$

$$w = \Pi(z) = e^{2\pi i z}$$

$$g(e^{2\pi i z}) = e^{2\pi i N_f(z)}$$

$$g(w) = w \exp\left(\frac{2\pi i \alpha}{1+w}\right)$$

- $w = 0$: Fixed point, $g'(0) = e^{2\pi i \alpha}$
- $w = \infty$: Parabolic fixed point

☆ *"Note that if we choose $\alpha \in \mathbb{Q}$, N_α will have a wandering domain that projects to a parabolic basin".*

FUTURE RESEARCH

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- Non-trivial **Wandering** domains for Newton maps of entire functions with zeros?

- Further investigation on the **Newton class**:

$$N_f(z) = z - M(e^z), \qquad M(w) = \frac{aw+b}{cw+d} \text{ Möbius map}$$

$$f(z) = k e^{\frac{d}{b} z} \left(a e^z + b\right)^{\frac{bc-ad}{ab}}$$

- Other areas of exploration: **Quasiperiodically**-forced systems?

$$F : \mathbb{T} \times \mathbb{C} \longrightarrow \mathbb{T} \times \mathbb{C}$$

$$(\theta, z) \longrightarrow F(\theta, z) = (\theta + \omega,\ f_\theta(z)), \quad \omega \in \mathbb{R} - \mathbb{Q}$$

THANK YOU !

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PhD Thesis