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HOMOGENEOUS AND ISOTROPIC STATISTICAL SOLUTIONS OF THE NAVIER-STOKES EQUATIONS

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Abstract. Two constructions of homogeneous and isotropic statistical solutions of the 3D Navier-Stokes system are presented. First, homogeneous and isotropic probability measures supported by weak solutions of the Navier-Stokes system are produced by averaging over rotations the known homogeneous probability measures, supported by such solutions, of [VF1], [VF2]. It is then shown how to approximate (in the sense of convergence of characteristic functionals) any isotropic measure on a certain space of vector fields by isotropic measures supported by periodic vector fields and their rotations. This is achieved without loss of uniqueness for the Galerkin system, allowing for the Galerkin approximations of homogeneous statistical Navier-Stokes solutions to be adopted to isotropic approximations. The construction of homogeneous measures in [VF1], [VF2] then applies to produce homogeneous and isotropic probability measures, supported by weak solutions of the Navier-Stokes equations. In both constructions, the restriction of the measures at $t = 0$ is well defined and coincides with the initial measure.

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1. Introduction

The Kolmogorov theory of turbulence describes the behavior of homogeneous and isotropic fluid flows, i.e. flows with statistical properties independent of translations, reflections, and rotations in 3D-space. The mathematical equivalent of such flows are translation, reflection, and rotation invariant probability measures P supported by Navier-Stokes solutions. Given an appropriate initial measure μ on the space of initial conditions and any finite time interval $[0, T]$, the measure P should yield μ at $t = 0$ is some sense. It should also yield measures μ_t for each time t in $[0, T]$, each invariant under space translations, reflections and rotations, that represent the flow of μ under the Navier-Stokes equations as in [H].

The existence of such measures P that satisfy all the assumptions above for translations in space, called **homogeneous**, was proved by Vishik and Fursikov in [VF1] and is included in detail in [VF2]. See also [FT]. This note adopts the construction from [VF1], [VF2] to produce measures P that, in addition to being invariant under translations, are also invariant under rotations and reflections. Such measures will be called **homogeneous and isotropic**. (To emphasize the new elements of the construction, the convention here will be that "isotropic" does not imply "homogeneous.") One of the main results of this note is:

Theorem 1.1. *Given $\widehat{\mu}$ homogeneous and isotropic measure on a space $\mathcal{H}^0(r)$ of vector fields on $\mathbb{R}^3$ with the finite energy density, there exists measure $\widehat{P}$ on $L^2(0, T, \mathcal{H}^0(r))$, homogeneous and isotropic with respect to the space variables, supported by weak solutions of the Navier-Stokes equations, and with finite energy density that satisfies the standard energy inequality. On the support of $\widehat{P}$ right limits with respect to time are well-defined in an appropriate norm and the right limit at $t = 0$ yields $\widehat{\mu}$.*

(For the definitions of all notions used in the formulation of Theorem 1.1, see sections 2 and 3 below.)

To prove Theorem 1.1 the homogeneous statistical solution P constructed in [VF1], [VF2] with initial measure $\widehat{\mu}$ is averaged over all rotations and reflections. The resulting measure $\widehat{P}$ satisfies all conditions of Theorem 1.1 above and is therefore the desired homogeneous and isotropic statistical solution. This plan is realized in sections 2 and 3.

It has to be emphasized, however, that existence theorems obtained via convergent sequences of "simpler" approximations of the constructed solution are as a rule much more useful in mathematical physics than the so called "pure existence theorems." For this reason, sections 4, 5, and 6 below are devoted to constructing such approximations of isotropic statistical solutions.

The construction of homogeneous statistical solutions in [VF1], [VF2] is based on Galerkin approximations of measures that are supported by divergence free periodic vector fields with trigonometric polynomials as components. The main difficulty in extending this construction to isotropic measures is that the space of such vector fields, whereas invariant under translations, is not invariant under rotations.

The construction in sections 4, 5, and 6 is based on the observation that the space of such vector fields AND all their rotations and reflections should suffice for invariance under rotations and reflections.

It is then necessary to construct Galerkin approximations of isotropic measures on this class of vector fields. In this sense, the crux of the matter is Sections 4.2, 4.3, and 6.1. Note that the constructions of sections 4, 5, and 6 not only offer approximations of the isotropic statistical solution constructed in section 3 but they also allow for a construction of isotropic statistical solutions that is formally independent on the results of section 3.

This paper considers the case of 3D Navier-Stokes equations, although the arguments here are applicable in 2D case as well.

2. Isotropic measures

2.1. **Definitions.** Non-trivial measures invariant under translations exist on weighted Sobolev spaces of vector fields defined over $\mathbb{R}^3$, p. 208, [VF2]:

Definition 2.1. *For k non negative integer and $r < -3/2$, define $\mathcal{H}^k(r)$ to be the space of solenoidal vector fields*

$$\begin{gathered} u(x) = (u_1(x), u_2(x), u_3(x)), \quad x = (x_1, x_2, x_3) \in \mathbb{R}^3, \\ \operatorname{div} u = \sum_{j=1}^{3} \frac{\partial u_j}{\partial x_j} = 0, \end{gathered} \tag{2.1}$$

with finite (k, r)-norm:

$$\|u\|_{k,r}^2 = \int_{\mathbb{R}^3} \left(1 + |x|^2\right)^r \sum_{|\alpha| \le k} \left| \frac{\partial^{|\alpha|} u(x)}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial x_3^{\alpha_3}} \right|^2 \, dx, \quad r < -\frac{3}{2}, \tag{2.2}$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is multi index and $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$.

Here the equality div $u = 0$ is to be understood weakly, i.e.

$$\int u(x) \cdot \nabla \phi(x) \, dx = 0 \quad \forall \ \phi \in C_0^\infty(\mathbb{R}^3) \tag{2.3}$$

Observe that the restriction on r implies that constant vector fields are in $\mathcal{H}^k(r)$. This paper uses $\mathcal{H}^k(r)$ only for $k = 0$ and $k = 1$.

For u in $\mathcal{H}^0(r)$ let T_h be the translation operation defined, also weakly, by

$$T_h u(x) = u(x + h). \tag{2.4}$$

For M a metric space denote by $\mathcal{B}(M)$ the σ-algebra of Borel sets of M. Let M_1, M_2 be metric spaces, and $\Psi : M_1 \to M_2$ a measurable map, i.e.

$$\forall \ B \in \mathcal{B}(M_2) \quad \Psi^{-1} B := \{m \in M_1 : \Psi(m) \in B\} \in \mathcal{B}(M_1). \tag{2.5}$$

It is well known that Ψ generates a map on measures: For every measure $\nu(A), A \in \mathcal{B}(M_1)$

$$\Psi^*\nu(B) = \nu(\Psi^{-1}B) \quad \forall\ B \in \mathcal{B}(M_2). \tag{2.6}$$

The measure $\Psi^*\nu$ is called the **push forward of the measure ν under the map Ψ**. Equality (2.6) is equivalent to

$$\int f(u)\ \Psi^*\nu(du) = \int f(\Psi(v))\ \nu(dv). \tag{2.7}$$

Definition 2.2. *A measure μ defined on $\mathcal{B}(\mathcal{H}^0(r))$ is called* **homogeneous** *if it is translation invariant:*

$$T_h^*\mu = \mu \Longleftrightarrow \int_{\mathcal{H}^0(r)} F(u)\ T_h^*\mu(du) = \int_{\mathcal{H}^0(r)} F(T_h u)\ \mu(du) = \int_{\mathcal{H}^0(r)} F(u)\ \mu(du), \tag{2.8}$$

for any μ-integrable F on $\mathcal{H}^0(r)$ and for all h in $\mathbb{R}^3$.

As for isotropic flows, they should have statistical properties invariant under rotations of the coordinate system, [MY], [T]. To find how a vector field $u(x) = (u_1(x), u_2(x), u_3(x))$ is transformed under rotation of the coordinate system it is convenient to write it in the usual manifold notation, cf. [DFN], p. 15:

$$u(x) = u_k(x)\frac{\partial}{\partial x_k} \tag{2.9}$$

(using summation on repeated indices). Let $v(y) = v_j(y)\frac{\partial}{\partial y_j}$ be the description of the vector field (2.9) after the transformation $y = \omega x$ where $\omega = \{\omega_{ij}\}$ is a rotation matrix (i.e. $\omega^{-1} = \omega^*$). Since $\dfrac{\partial}{\partial x_k} = \omega_{jk}\dfrac{\partial}{\partial y_j}$, then

$$v_j(y)\frac{\partial}{\partial y_j} = u_k(\omega^{-1}y)\omega_{jk}\frac{\partial}{\partial y_j}.$$

In other words, returning to the standard notation for vector fields on $\mathbb{R}^3$ where $u(x) = (u_1(x), u_2(x), u_3(x)), v(y) = (v_1(y), v_2(y), v_3(y))$,

$$v(y) = \omega u(\omega^{-1}y). \tag{2.10}$$

Observe here that since ω is othogonal the differential form $\sum u_i\ dx_i$ transforms under ω in the same way, cf. [DFN], page 156.

Then for ω belonging to the group $O(3)$ of all orthogonal matrices (with $\det\omega = \pm 1$), define its action on vector fields as

$$(R_\omega u)(x) = \omega u(\omega^{-1}x) \tag{2.11}$$

Observe that the standard action identity holds

$$R_{\omega_1}(R_{\omega_2}u)(x) = \omega_1(R_{\omega_2}u)(\omega_1^{-1}x) = \omega_1\omega_2 u(\omega_2^{-1}\omega_1^{-1}x) = R_{\omega_1\omega_2}u(x), \tag{2.12}$$

that

$$R_\omega u = v \Leftrightarrow u = (R_{\omega^{-1}})v, \tag{2.13}$$

and that

$$T_h R_\omega u = R_\omega T_{\omega^{-1}h} u. \tag{2.14}$$

Lemma 2.3. *For every* $\omega \in O(3)$ *the operator* $R_\omega : \mathcal{H}^0(r) \to \mathcal{H}^0(r)$ *is an isometry, i.e. if* $\mathrm{div} u = 0$ *then* $\mathrm{div} R_\omega u = 0$ *and*

$$\|R_\omega u\|_{\mathcal{H}^0(r)} = \|u\|_{\mathcal{H}^0(r)}. \tag{2.15}$$

Proof. The transformation formula for multiple integrals, [A], page 421, gives for the change of variables $y = \omega x$:

$$\int_{\mathbb{R}^3} f(\omega x)\ dx = |(\det\,\omega)^{-1}| \int_{\mathbb{R}^3} f(y)\ dy = \int_{\mathbb{R}^3} f(y)\ dy,\ \forall\ \omega \in O(3), f \in L_1(\mathbb{R}^3), \tag{2.16}$$

since $|\det\,\omega| = 1$ for any ω in $O(3)$.

Now (2.2) and (2.16) yield

$$\begin{aligned} \|R_\omega u\|^2_{0,r} &= \int_{\mathbb{R}^3} \left(1 + |x|^2\right)^r |\omega u(\omega^{-1}x)|^2\ dx \\ &= \int_{\mathbb{R}^3} \left(1 + |\omega^{-1}x|^2\right)^r |u(\omega^{-1}x)|^2\ dx \\ &= \int_{\mathbb{R}^3} \left(1 + |x|^2\right)^r |u(x)|^2\ dx \\ &= \|u\|^2_{0,r}, \end{aligned} \tag{2.17}$$

which proves (2.15).

If $\omega = (\omega_{ij}) \in O(3)$ then $x = \omega y$ is equivalent to $x_i = \omega_{ij} y_j$ and $y = \omega^{-1}x = \omega^* x$ is equivalent to $y_l = \omega_{kl} x_k$ (using summation on repeated indexes). Then

$$\frac{\partial}{\partial x_k} = \omega_{kl} \frac{\partial}{\partial y_l}, \quad \omega_{kj}\omega_{kl} = \delta_{jl}, \tag{2.18}$$

where δ_{kl} is Kroneker symbol. Using these equalities, (2.16), and assuming that u satisfies (2.3), obtain

$$\begin{aligned} \int R_\omega u(x) \cdot \nabla\phi(x)\ dx &= \int \omega u(\omega^{-1}x) \cdot \nabla\phi(x)\ dx \\ &= \int \omega_{kj} u_j(y) \omega_{kl} \frac{\partial\phi(\omega y)}{\partial y_l}\ d\omega y \\ &= \int u_j(y) \frac{\partial\phi(\omega y)}{\partial y_j}\ dy = 0, \end{aligned} \tag{2.19}$$

where the last equality holds because of (2.3) and the inclusion $\phi \circ \omega \in C_0^\infty(\mathbb{R}^3)$. Therefore $\mathrm{div} R_\omega u = 0$ if $\mathrm{div} u = 0$. □

Definition 2.4. *A measure* μ *on* $\mathcal{H}^0(r)$ *is called* **isotropic** *if it is invariant under rotations: For all* ω *in* $O(3)$,

$$R^*_\omega \mu = \mu \quad \Longleftrightarrow \quad \int f(u)\ \mu(du) = \int f(R_\omega u)\ \mu(du) = \int f(u)\ R^*_\omega \mu(du), \tag{2.20}$$

for any μ*-integrable* f *on* $\mathcal{H}^0(r)$.

Remark 2.5. *The choice of $O(3)$ as space of rotations captures the usual conventions of isotropic flows as flows invariant under "proper" rotations and reflections with respect to coordinate planes, see* [Rob], *p 212. I.e. the measure is invariant under the transformations*

$$(u_1, u_2, u_3)(x) \mapsto (-u_1, u_2, u_3)(\overline{x}) \tag{2.21}$$

for $\overline{x} = (-x_1, x_2, x_3)$, and similarly for the indices 2 and 3.

It also follows from the definition that the correlation function and all statistical expressions of such a μ have the usual form for isotropic flows, see for example [MY], *p. 39.*

2.2. **Examples of isotropic measures.** Homogeneous and isotropic measures can easily be constructed from homogeneous measures by standard averaging:

Definition 2.6. *If μ is homogeneous, define $\widehat{\mu}$ on $\mathcal{B}(\mathcal{H}^0(r))$ as*

$$\widehat{\mu}(A) = \int_{O(3)} R_\omega^* \mu(A)\ d\omega \tag{2.22}$$

for any $A \in \mathcal{B}(\mathcal{H}^0(r))$ and for $d\omega = H$ the standard Haar measure on $O(3)$ normalized.

By definition (2.7) of the push forward measure and by Fubini's Theorem, equality (2.22) is equivalent to

$$\int f(u)\widehat{\mu}(du) = \int \int_{O(3)} f(R_\omega u)\ d\omega \mu(du) \tag{2.23}$$

for each μ-integrable function $f(u)$. This definition of $\widehat{\mu}$ is sometimes more convenient than (2.22), as will become clear below.

Proposition 2.7. *Let μ be a homogeneous probability measure on $\mathcal{H}^0(r)$. Then $\widehat{\mu}$ is isotropic and still homogeneous.*

Proof. Invariance under rotations follows from

$$\begin{aligned} \int f(u)\ R_{\omega_0}^* \widehat{\mu}(du) &= \int \int_{O(3)} f(R_{\omega\omega_0} u)\ d\omega\ \mu(du) \\ &= \int \int_{O(3)} f(R_\omega u)\ d\omega\ \mu(du) \\ &= \int f(u)\ \widehat{\mu}(du), \end{aligned} \tag{2.24}$$

with the second equality following from the fact that $O(3)$ is compact, therefore unimodular, therefore the "change of variables" $\omega \to \omega\omega_0$ has "Jacobian" 1, see [R], p. 498.

Invariance under translations follows from the fact that

$$T_h R_\omega u = R_\omega T_{\omega^{-1}h} u \Leftrightarrow T_{\omega h} R_\omega u = R_\omega T_h u \tag{2.25}$$

and

$$
\begin{aligned}
\int f(u)\ T_h^* \widehat{\mu}(du) &= \int \int_{O(3)} f(T_h R_\omega u)\ d\omega\ \mu(du) \\
&= \int_{O(3)} \int f(R_\omega T_{\omega^{-1}h} u)\ \mu(du)\ d\omega \\
&= \int_{O(3)} \int f(R_\omega u)\ \mu(du)\ d\omega \\
&= \int \int_{O(3)} f(R_\omega u)\ d\omega\ \mu(du) \\
&= \int f(u)\ \widehat{\mu}(du).
\end{aligned}
\tag{2.26}
$$

□

3. Existence of homogeneous and isotropic statistical solutions.

3.1. Definition of Statistical Solutions. The following preliminary definitions are required to state the properties of statistical solutions of the Navier-Stokes equations:

Definition 3.1. *Define $\mathcal{G}_{NS}$ to be the set of all generalized solutions of the Navier-Stokes system, i.e.*

$$
\mathcal{G}_{NS} = \Bigg\{ u \in L^2(0,T;\mathcal{H}^0(r)) :
$$

$$
L(u,\phi) \equiv \int_0^T \left(< u, \frac{\partial \phi}{\partial t} >_2 + < u, \Delta\phi >_2 + \sum_{j=1}^3 < u_j u, \frac{\partial \phi}{\partial x_j} >_2 \right) dt = 0, \tag{3.1}
$$

$$
\textit{for all}\ \ \phi \in C_0^\infty\left((0,T)\times\mathbb{R}^3\right) \cap C((0,T);\mathcal{H}^0(r)) \Bigg\},
$$

where $< u, v >_2 = \int_{\mathbb{R}^3} u(x)\cdot v(x)\ dx$.

To define restrictions at any time $t \in [0,T]$ one works with the following norms: For $B_N = \{|x| < N\}$ the ball of radius N in $\mathbb{R}^3$, and for $\|.\|_s$ the standard Sobolev norm in $W^{s,2}(\mathbb{R}^3) = L^2_s(\mathbb{R}^3)$, define the dual norm

$$
\|v|_{B_N}\|_{-s} = \sup_{w\in C_0^\infty(B_N)} \frac{< v, w >_2}{\|w\|_s}. \tag{3.2}
$$

Using this and following [VF2], p.245, define

$$
\|u\|_{BV^{-s}} = \|u\|_{L^2(0,T;\mathcal{H}^0(r))} + \sum_{N=1}^\infty \frac{1}{2^{2N}C(N)}|u|_N. \tag{3.3}
$$

Here $C(N)$ are constants from (5.3) and (5.4) (see below) and $|u|_N$ is defined as follows:

$$
|u|_N = \text{vrai} \sup_{t\in[0,T]} \|u(t,\cdot)|_{B_N}\|_{-s} + \sup_{\{t_j\}} \sum_{j=1}^{l} \text{vrai} \sup_{t,\tau\in[t_{j-1},t_j)} \|(u(t,\cdot) - u(\tau,\cdot))|_{B_N}\|_{-s}, \tag{3.4}
$$

where $\sup_{\{t_j\}}$ is the supremum over all partitions $t_0 < \cdots < t_l,\ l \in \mathbb{N}$ of the segment $[0,T]$.

Define

$$BV^{-s} = \{u \in L^2(0,T;\mathcal{H}^0(r)) : \|u\|_{BV^{-s}} < \infty\}. \tag{3.5}$$

The merit of the BV^{-s} norm is that for u in BV^{-s} the limits

$$\gamma_{t_0}(u) := \lim_{t \to t_0^+} u(t,.) \tag{3.6}$$

exist for all $t_0 \in [0,T]$, if taken with respect to the norm

$$\|u(t,.)\|_{\Phi^{-s}} = \left(\sum_{N=1}^{\infty} \frac{1}{2^{2N} C(N)} \|u(t,.)|_{B_N}\|_{-s}^2 \right)^{1/2}, \tag{3.7}$$

see [VF2], Chapter VII, Lemma 8.2.

Note also that for a homogeneous measure μ the pointwise averages

$$\int |u|^2(x)\ \mu(du), \quad \int |\nabla u|^2(x)\ \mu(du) \tag{3.8}$$

can be defined by the equalities

$$\int\int |u(x)|^2 \phi(x)\ dx\ \mu(du) = \int |u(x)|^2\ \mu(du) \int \phi(x)\ dx \tag{3.9}$$

$$\int\int |\nabla u(x)|^2 \phi(x)\ dx\ \mu(du) = \int |\nabla u(x)|^2\ \mu(du) \int \phi(x)\ dx \quad \forall\ \phi \in L_1(\mathbb{R}^3) \tag{3.10}$$

and they are independent of $x \in \mathbb{R}^3$, see Chapter VII, section 1 of [VF2]. The expressions in (3.8) are well defined since the left hand sides in (3.9), (3.10) are finite for each $\phi \in C_0^\infty(\mathbb{R}^3)$.

The first expression in (3.8) is **the energy density** and the second one is **the density of the energy dissipation**.

Since the translation operator T_h (along x) is well defined on the space $L^2(0,T;\mathcal{H}^0(r))$ of vector fields $u(t,x)$ dependent not only on x but on t as well, one can introduce the notion of homogeneity in x:

Definition 3.2. *A measure $P(A)$, $A \in \mathcal{B}(L^2(0,T;\mathcal{H}^0(r)))$ is called* **homogeneous in** x *if for each $h \in \mathbb{R}^3$:*

$$T_h^* P = P \Longleftrightarrow \int_{L^2(0,T;\mathcal{H}^0(r))} f(u)\ T_h^* P(du) = \int_{L^2(0,T;\mathcal{H}^0(r))} f(u)\ P(du), \tag{3.11}$$

for any P-integrable f on $\mathcal{H}^0(r)$.

The following definition summarizes the properties of homogeneous statistical solutions of the Navier-Stokes equations as they were produced in Chapter VII of [VF2]:

Definition 3.3. *Given homogeneous probability measure μ on $\mathcal{B}(\mathcal{H}^0(r))$ possessing finite energy density, a* **homogeneous statistical solution of the Navier-Stokes equations with initial condition** μ *is a probability measure P on $\mathcal{B}(L^2(0,T;\mathcal{H}^0(r)))$ such that:*

(1) *P is homogeneous in x.*

(2) *$P(\widehat{W}) = 1$, where $\widehat{W} = L^2(0,T;\mathcal{H}^1(r)) \cap BV^{-s} \cap \mathcal{G}_{NS}$, $s > \frac{11}{2}$.*

(3) *For all $A \in \mathcal{B}(\mathcal{H}^0(r))$,*

$$P(\gamma_0^{-1}A) = \mu(A), \quad \text{where} \quad \gamma_0^{-1}A = \{u \in \widehat{W} : \gamma_0 u \in A\}. \tag{3.12}$$

(4) *For each t in $[0,T]$,*

$$\int \left(|u(t,x)|^2 + \int_0^t |\nabla u|^2(\tau, x)\ d\tau \right)\ P(du) \le C \int |u(x)|^2\ \mu(du), \tag{3.13}$$

where the expression in the left hand side of (3.13) is defined similarly to (3.8).

The main result of [VF2], Chapter VII, then reads as follows:

Theorem 3.4. *Given μ homogeneous measure on $\mathcal{H}^0(r)$ with finite energy density,*

$$\int_{\mathcal{H}^0(r)} |u|^2(x)\ \mu(du) < \infty, \tag{3.14}$$

there exists homogeneous statistical solution of the Navier-Stokes equations P with initial condition μ.

Remark 3.5. *The definition above is a rephrasing of Definition 11.1 of* [VF2]*, with one minor change: It asks that P is supported by $L^2(0,T;\mathcal{H}^1(r)) \cap BV^{-s} \cap \mathcal{G}_{NS}$ rather than some subset of it. Since* [VF2] *produces some subset supporting a homogeneous statistical solution, it automatically produces a homogeneous statistical solution according to the definition here.*

Remark 3.6. *In addition, the family of homogeneous measures $\mu_t := P \circ \gamma_t^{-1}$ on $\mathcal{H}^0(r)$ satisfies the Hopf equation,* [VF2]*, Chapter VIII.*

Define now isotropic and homogeneous statistical solutions. First define **isotropic in x measures** P on $\mathcal{B}(L^2(0,T;\mathcal{H}^0(r)))$. (This can be done since for each $\omega \in O(3)$ operator $R_\omega u(t,x) \equiv \omega u(t, \omega^{-1}x)$ is well defined on $L^2(0,T;\mathcal{H}^0(r))$.)

Definition 3.7. *A measure $P(A)$, $A \in \mathcal{B}(L^2(0,T;\mathcal{H}^0(r)))$ is called* **isotropic in x** *if for each $\omega \in O(3)$:*

$$R_\omega^* P = P \Longleftrightarrow \int_{L^2(0,T;\mathcal{H}^0(r))} f(u)\ R_\omega^* P(du) = \int_{L^2(0,T;\mathcal{H}^0(r))} f(u)\ P(du), \tag{3.15}$$

for any P-integrable f on $L^2(0,T;\mathcal{H}^0(r))$.

For the definition of homogeneous and isotropic statistical solutions one has only to add in Definition 3.3 the property of rotation and reflection invariance:

Definition 3.8. *Given homogeneous and isotropic probability measure $\widehat{\mu}$ on $\mathcal{B}(\mathcal{H}^0(r))$, a* **homogeneous and isotropic statistical solution of the Navier-Stokes equations with initial condition** *$\widehat{\mu}$ is a probability measure $\widehat{P}$ on $\mathcal{B}(L^2(0,T;\mathcal{H}^0(r)))$ such that:*

(1) *$\widehat{P}$ is homogeneous and isotropic in x.*

(2) $\widehat{P}(\widehat{W})=1$, *where* $\widehat{W}=L^2(0,T;\mathcal{H}^1(r))\cap BV^{-s}\cap\mathcal{G}_{NS}$, $s>\frac{11}{2}$,.
(3) $\widehat{P}(\gamma_0^{-1}A)=\widehat{\mu}(A)$ *for every* $A\in\mathcal{B}(\mathcal{H}^0(r))$.
(4) *For each* t *in* $[0,T]$,

$$\int_{L^2(0,T;\mathcal{H}^0(r))}\left(|u(t,x)|^2+\int_0^t|\nabla u|^2(\tau,x)\;d\tau\right)\;\widehat{P}(du)\le C\int_{\mathcal{H}^0(r)}|u(x)|^2\;\widehat{\mu}(du). \tag{3.16}$$

3.2. **Construction of homogeneous and isotropic statistical solutions.** To construct homogeneous and isotropic statistical solutions several preliminary assertions need to be proved first. For these, use the definition of the norm $\|\cdot\|_s$ of Sobolev space $W^{s,2}(\mathbb{R}^3)$ through Fourier transform:

$$\|\phi\|_s^2=\int_{\mathbb{R}^3}(1+|\xi|^2)^s|\widehat{\phi}|^2(\xi)\;d\xi,\quad\text{where}\quad\widehat{\phi}(\xi)=\frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3}e^{-ix\cdot\xi}\phi(x)\;dx. \tag{3.17}$$

Lemma 3.9. *For any matrix* $\omega\in O(3)$ *the following equalities hold:*

$$\begin{aligned}\|R_\omega\phi\|_s&=\|\phi\|_s,\\ \|R_\omega\phi\|_{BV^{-s}}&=\|\phi\|_{BV^{-s}},\\ \|R_\omega\phi\|_{\Phi^{-s}}&=\|\phi\|_{\Phi^{-s}}.\end{aligned} \tag{3.18}$$

Proof. By the definition of Fourier transform and by virtue of (2.16)

$$\begin{aligned}\widehat{R_\omega\phi}(\xi)&=\frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3}e^{-i\omega^{-1}x\cdot\omega^{-1}\xi}\omega\phi(\omega^{-1}x)\;dx\\ &=\frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3}e^{-iy\cdot\omega^{-1}\xi}\omega\phi(y)\;dy=R_\omega\widehat{\phi}(\xi).\end{aligned} \tag{3.19}$$

By (3.17), (3.19), and (2.16)

$$\begin{aligned}\|R_\omega\phi\|_s^2&=\int_{\mathbb{R}^3}(1+|\omega^{-1}\xi|^2)^s|\omega\widehat{\phi}(\omega^{-1}\xi)|^2\;d\xi\\ &=\int_{\mathbb{R}^3}(1+|\eta|^2)^s|\omega\widehat{\phi}(\eta)|^2\;d\eta=\|\phi\|_s^2,\end{aligned} \tag{3.20}$$

which proves the first equality in (3.18).

Equalities (3.20), (2.16), and (3.2) give:

$$\begin{aligned}\|R_\omega v|_{B_N}\|_{-s}&=\sup_{\phi\in C_0^\infty(B_N)}\frac{<R_\omega v,\phi>_2}{\|\phi\|_s}\\ &=\sup_{\phi\in C_0^\infty(B_N)}\frac{<v,(R_\omega)^{-1}\phi>_2}{\|(R_\omega)^{-1}\phi\|_s}=\|v|_{B_N}\|_{-s},\end{aligned} \tag{3.21}$$

since $R_\omega:C_0^\infty(B_N)\to C_0^\infty(B_N)$ is an isomorphism.

Equality (3.21) and definition (3.4) of $|\cdot|_N$ imply the equality

$$|R_\omega\phi|_N=|\phi|_N. \tag{3.22}$$

This identity, (3.21), and the definitions (3.3), (3.7) of the norms $\|\cdot\|_{BV^{-s}}$, $\|\cdot\|_{\Phi^{-s}}$, imply directly the second and third equalities in (3.18). ☐

Lemma 3.10. *For every $\omega \in O(3)$ and for each homogeneous measure $\mu(A)$, $A \in \mathcal{B}(\mathcal{H}^0(r))$*

$$\begin{aligned} \int |u(x)|^2 \; R^*_\omega \mu(du) &= \int |u(x)|^2 \; \mu(du), \\ \int |\nabla u(x)|^2 \; R^*_\omega \mu(du) &= \int |\nabla u(x)|^2 \; \mu(du). \end{aligned} \tag{3.23}$$

Proof. By (2.6), (2.7), (3.9), and (2.16), for each $\omega \in O(3)$

$$\begin{aligned} \int |u(x)|^2 \; R^*_\omega \mu(du) \int \phi(x)\; dx &= \int \int_{\mathbb{R}^3} |\omega u(\omega^{-1}x)|^2 \phi(\omega\omega^{-1}x)\; dx\; d\mu(u) \\ &= \int \int_{\mathbb{R}^3} |u(y)|^2 \phi(\omega y)\; dy\; d\mu(u) \\ &= \int |u(y)|^2 \; d\mu(u) \int_{\mathbb{R}^3} \phi(\omega y)\; dy \\ &= \int |u(x)|^2 \; d\mu(u) \int_{\mathbb{R}^3} \phi(x)\; dx, \end{aligned} \tag{3.24}$$

which proves the first equality of (3.23). If $y = \omega^{-1}x$, i.e. $y_l = \omega_{kl}x_k$, then by (2.16), (2.18), obtain

$$\begin{aligned} \int \sum_j |\omega \frac{\partial u(\omega^{-1}x)}{\partial x_j}|^2 \phi(x)\; dx &= \int \sum_j |\frac{\partial u(\omega^{-1}x)}{\partial x_j}|^2 \phi(\omega\omega^{-1}x)\; dx \\ &= \int \sum_{j,p} \omega_{jl} \frac{\partial u_p(y)}{\partial y_l} \omega_{jm} \frac{\partial u_p(y)}{\partial y_m} \phi(\omega y)\; dy \\ &= \int |\nabla_y u(y)|^2 \phi(\omega y)\; dy. \end{aligned} \tag{3.25}$$

Using these identities, the second equality of (3.23) can be proved similarly to (3.24). □

Recall now that the set $\mathcal{G}_{NS}$ has been introduced in Definition 3.1.

Lemma 3.11. *For each $\omega \in O(3)$ the equality $R_\omega \mathcal{G}_{NS} = \mathcal{G}_{NS}$ holds.*

Proof. Prove first that if u satisfies (3.1) then $R_\omega u$ satisfies (3.1) as well, for every $\omega \in O(3)$. For this note that, by Lemma 2.3,

$$\begin{gathered} u \in L^2(0,T;\mathcal{H}^0(r)) \Rightarrow R_\omega u \in L^2(0,T;\mathcal{H}^0(r)), \\ \phi \in C((0,T);\mathcal{H}^0(r)) \cap C_0^\infty((0,T)\times\mathbb{R}^3) \Rightarrow R_\omega \phi \in C((0,T);\mathcal{H}^0(r)) \cap C_0^\infty((0,T)\times\mathbb{R}^3). \end{gathered} \tag{3.26}$$

So let u satisfy (3.1). Then (2.16) and the well-known fact that Laplace operator is invariant under orthogonal change of variables yield:

$$< R_\omega u, \frac{\partial \phi}{\partial t} + \Delta\phi > = < u, \frac{\partial (R_{\omega^{-1}})\phi}{\partial t} + \Delta(R_{\omega^{-1}})\phi > . \tag{3.27}$$

If $y = \omega^{-1}x = \omega^* x$, i.e. $y_l = \omega_{kl}x_k$, then taking into account (2.18) and (2.16) calculate:

$$
\begin{aligned}
\int (R_\omega u)_j R_\omega u \cdot \frac{\partial \phi}{\partial x_j}\, dx &= \int \omega_{jk}u_k(\omega^{-1}x)\omega_{lm}u_m(\omega^{-1}x)\frac{\partial \phi_l(x)}{\partial x_j}\, dx \\
&= \int \omega_{jk}u_k(y)\omega_{lm}u_m(y)\omega_{jp}\frac{\partial \phi_l(\omega y)}{\partial y_p}\, dy \\
&= \int u_k(y)u_m(y)\frac{\partial \omega_{lm}\phi_l(\omega y)}{\partial y_k}\, dy.
\end{aligned} \tag{3.28}
$$

Then

$$
\sum_{j=1}^{3} < (R_\omega u)_j R_\omega u, \frac{\partial \phi}{\partial x_j} >_2 = \sum_{j=1}^{3} < u_j u, \frac{\partial (R_{\omega^{-1}})\phi}{\partial x_j} >_2 . \tag{3.29}
$$

Adding (3.27) and (3.29), integrating the resulting equality with respect to t over $[0,T]$, and taking into account that u satisfies (3.1), shows that $R_\omega u$ satisfies (3.1) as well. $\square$

Lemma 3.12. *γ_0 commutes with R_ω for any rotation ω.*

Proof. By Lemma 3.9, $\|u(t,.)\|_{\Phi^{-s}} = \|R_\omega u(t,.)\|_{\Phi^{-s}}$. Therefore if $\lim_{t\to 0^+} u(t,.) = \gamma_0(u)$, then $\lim_{t\to 0^+} R_\omega u(t,.) = R_\omega \gamma_0(u)$, and $\lim_{t\to 0^+} R_\omega u(t,.) = \gamma_0(R_\omega u)$, i.e. $\gamma_0(R_\omega u) = R_\omega \gamma_0(u)$. $\square$

Recall that for each $B \in \mathcal{B}(\mathcal{H}^0(r))$

$$
\gamma_0^{-1}B = \{u(t,x) \in \widehat{W} : \gamma_0 u \in B\}, \tag{3.30}
$$

where $\widehat{W}$ is the set of Definition 3.3 or (equivalently) of Definition 3.8.

Lemma 3.13. *For $B \in \mathcal{B}(\mathcal{H}^0(r))$,*

$$
R_\omega \gamma_0^{-1}(B) = \gamma_0^{-1}(R_\omega B), \quad \forall\, \omega \in O(3). \tag{3.31}
$$

Proof. Using (2.18), one can prove similarly to Lemma 2.3 that

$$
R_\omega \mathcal{H}^1(r) = \mathcal{H}^1(r) \quad \forall\, \omega \in O(3), \tag{3.32}
$$

for $\mathcal{H}^1(r)$ as in Definition 2.1. This, together with lemmas 3.9 and 3.11, imply that

$$
R_\omega \widehat{W} = \widehat{W} \quad \forall\, \omega \in O(3). \tag{3.33}
$$

Therefore

$$
\begin{aligned}
u \in R_\omega \gamma_0^{-1}(B) &\Rightarrow u = R_\omega v, v \in \gamma_0^{-1}(B) \\
&\Rightarrow \gamma_0 u = \gamma_0(R_\omega v), v \in \gamma_0^{-1}(B) \\
&\Rightarrow \gamma_0 u = R_\omega \gamma_0(v), v \in \gamma_0^{-1}(B), \text{ by Lemma 3.12,} \\
&\Rightarrow \gamma_0 u = R_\omega b, b \in B \\
&\Rightarrow u = \gamma_0^{-1} R_\omega b, b \in B \\
&\Rightarrow u \in \gamma_0^{-1}(R_\omega B).
\end{aligned} \tag{3.34}
$$

Conversely,

$$
\begin{aligned}
u \in \gamma_0^{-1}(R_\omega B) &\Rightarrow \gamma_0(u) = R_\omega b, b \in B \\
&\Rightarrow R_{\omega^{-1}}\gamma_0(u) = b, b \in B \\
&\Rightarrow \gamma_0(R_{\omega^{-1}}u) = b, b \in B, \text{ by Lemma 3.12}, \\
&\Rightarrow R_{\omega^{-1}}u = \gamma_0^{-1}(b), b \in B \\
&\Rightarrow R_{\omega^{-1}}u \in \gamma_0^{-1}(B), \\
&\Rightarrow u \in R_\omega \gamma_0^{-1}(B).
\end{aligned}
\tag{3.35}
$$

□

Theorem 3.14. *Given $\widehat{\mu}$ homogeneous and isotropic measure on $\mathcal{H}^0(r)$ with finite energy density,*

$$
\int_{\mathcal{H}^0(r)} |u|^2(x)\ \widehat{\mu}(du) < \infty,
\tag{3.36}
$$

there exists homogeneous and isotropic statistical solution $\widehat{P}$ of the Navier-Stokes equations with initial condition $\widehat{\mu}$.

Proof. Ignoring for the moment that $\widehat{\mu}$ is also isotropic, let P be the homogeneous statistical solution with initial condition the homogeneous $\widehat{\mu}$ guaranteed by Theorem 3.4. The set $\widehat{W} = L^2(0,T;\mathcal{H}^1(r)) \cap BV^{-s} \cap \mathcal{G}_{NS}$ is invariant under rotations by (3.33). Applying the analogue of operation (2.22) on the homogeneous measure P obtain:

$$
\widehat{P}(A) = \int_{O(3)} R_\omega^* P(A)\ d\omega = \int_{O(3)} P(R_{\omega^{-1}}A)\ d\omega,
\tag{3.37}
$$

for any $A \in (B)(\widehat{W})$. Repeating the proof of Proposition 2.7 for the measure $\widehat{P}$ shows that $\widehat{P}$ is homogeneous and isotropic in x. Since $P(\widehat{W}) = 1$ by Theorem 3.4, equality (3.33) implies that $R_\omega^* P(\widehat{W}) = P(R_\omega^{-1}\widehat{W}) = P(\widehat{W}) = 1$ for each $\omega \in O(3)$. Hence, $\widehat{P}(\widehat{W}) = 1$ by definition (3.37).

That $\widehat{P}$ has initial condition $\widehat{\mu}$ follows from

$$
\begin{aligned}
\widehat{P}(\gamma_0^{-1}B) &= \int_{O(3)} P(R_{\omega^{-1}}\gamma_0^{-1}(B))\ d\omega \\
&= \int_{O(3)} P(\gamma_0^{-1}(R_{\omega^{-1}}B))\ d\omega, \text{ by Lemma 3.13} \\
&= \int_{O(3)} \widehat{\mu}(R_{\omega^{-1}}B)\ d\omega, \text{ by (3.12)} \\
&= \widehat{\mu}(B), \text{ since } \widehat{\mu} \text{ is also isotropic},
\end{aligned}
\tag{3.38}
$$

for any $B \in \mathcal{B}(\mathcal{H}^0(r))$.

For the energy inequality, use (3.37), Lemma 3.10, and (3.13) to get:

$$
\begin{aligned}
&\int_{L^2(0,T;\mathcal{H}^0(r))} \left(|u(t,x)|^2 + \int_0^t |\nabla u|^2(\tau,x)\, d\tau\right) \widehat{P}(du) \\
&\qquad = \int_{O(3)} \int_{L^2(0,T;\mathcal{H}^0(r))} \left(|u(t,x)|^2 + \int_0^t |\nabla u|^2(\tau,x)\, d\tau\right) R_\omega{}^* P(du)\, d\omega \\
&\qquad = \int_{L^2(0,T;\mathcal{H}^0(r))} \left(|u(t,x)|^2 + \int_0^t |\nabla u|^2(\tau,x)\, d\tau\right) P(du) \\
&\qquad \leq C \int_{\mathcal{H}^0(r)} |u(x)|^2\, \widehat{\mu}(du),
\end{aligned} \tag{3.39}
$$

which proves (3.16). □

4. Galerkin approximation of isotropic statistical solutions

4.1. **Isotropic measures on periodic vector fields.** Let $\mathcal{M}_l$ be as in [VF2]:

$$
\mathcal{M}_l = \Bigg\{ \sum_{\substack{k \in \frac{\pi}{l}\mathbb{Z}^3, \\ |k| \leq l}} a_k e^{ik\cdot x} : a_k \cdot k = 0,\ a_k = \overline{a}_{-k}\ \forall\ k \Bigg\}, \tag{4.1}
$$

the finite-dimensional space of divergence-free, $3D$, real, vector valued trigonometric polynomials of degree l and period $2l$. Then the inclusion

$$
\mathcal{M}_l \subset \mathcal{H}^0(r) \tag{4.2}
$$

holds for all l. [VF2], Appendix II, shows explicitly how, starting from any homogeneous probability measure on $\mathcal{H}^0(r)$, one can construct homogeneous probability measures μ_l on $\mathcal{H}^0(r)$, supported solely by $\mathcal{M}_l$ for each l, and approximating μ in the sense of characteristic functionals. The trouble, of course, is that $\mathcal{M}_l$ is not invariant under rotations. The following definitions address this point.

Definition 4.1. *Let $\widehat{\mathcal{M}_l}$ be the union of all rotations of elements of $\mathcal{M}_l$:*

$$
\widehat{\mathcal{M}_l} = \bigcup_{\omega \in O(3)} R_\omega M(l) \tag{4.3}
$$

in $\mathcal{H}^0(r)$.

Consider on $\widehat{\mathcal{M}_l}$ the topology τ generated by sets of the form

$$
\{R_\omega m : \omega \in \rho, m \in \sigma,\ \text{where } \rho \subset O(3), \sigma \subset \mathcal{M}_l \text{ are open sets}\}. \tag{4.4}
$$

Since $O(3)$ and $\mathcal{M}_l$ are finite-dimensional sets, the topology τ coincides with the topology generated by the enveloping space $\mathcal{H}^0(r)$. Therefore, the Borel σ-algebra $\mathcal{B}(\widehat{\mathcal{M}_l})$ is generated by sets of the form (4.4). Moreover, it is clear that

$$
\mathcal{B}(\widehat{\mathcal{M}_l}) = \mathcal{B}(\mathcal{H}^0(r)) \cap \widehat{\mathcal{M}_l} \equiv \{A \cap \widehat{\mathcal{M}_l} : A \in \mathcal{B}(\mathcal{H}^0(r))\} \tag{4.5}
$$

Note that for each fixed ω the elements of $R_\omega \mathcal{M}_l$ are of the form

$$\sum_{\substack{k\in\frac{\pi}{l}\mathbb{Z}^3,\\ |k|\le l}} b_k e^{i\omega k\cdot x},\ b_k\cdot\omega k = 0 \text{ for all } k. \tag{4.6}$$

Using definition (2.6) of the push forward measure, proceed to:

Definition 4.2. *Let $\widehat{\mu_l}(A),\ A\in\mathcal{B}(\widehat{\mathcal{M}_l})$ be the push-forward of the product of the Haar measure on $O(3)$ and the measure μ_l on $\mathcal{M}_l$ via the map $(\omega,u)\mapsto R_\omega u$:*

$$\widehat{\mu_l}(A) = (H\times\mu_l)\{(\omega,u)\in O(3)\ \times\mathcal{M}_l : R_\omega u\in A\}. \tag{4.7}$$

As for any push-forward measure, by (2.7),

$$\int_{\widehat{\mathcal{M}_l}} f(v)\ \widehat{\mu_l}(dv) = \int_{\mathcal{M}_l}\int_{O(3)} f(R_\omega u)\ d\omega\mu_l(du), \tag{4.8}$$

for any $\widehat{\mu_l}$-integrable f.

Since μ_l is supported on $\mathcal{M}_l\subset\mathcal{H}^0(r)$ and $\widehat{\mu_l}$ is supported on $\widehat{\mathcal{M}_l}\subset\mathcal{H}^0(r)$, the domains of integration $\widehat{\mathcal{M}_l},\mathcal{M}_l$ in (4.8) can change to $\mathcal{H}^0(r)$. Comparing then (4.7), (4.8) to the definitions of averaging (2.22), (2.23) it follows that the measure $\widehat{\mu_l}$ is the averaging of μ_l over $O(3)$:

$$\widehat{\mu_l}(A) = \int_{O(3)} R^*_\omega\mu_l(A)\ d\omega \quad \forall\ A\in\mathcal{B}(\mathcal{H}^0(r)) \tag{4.9}$$

Proposition 4.3. *$\widehat{\mu_l}$ is homogeneous and isotropic.*

Proof. Using the equality $R_{\omega^{-1}}\widehat{\mathcal{M}_l} = \widehat{\mathcal{M}_l}$ and the invariance of the Haar measure, obtain for each $\widehat{\mu_l}$-integrable f:

$$\begin{aligned}
\int_{\widehat{\mathcal{M}_l}} f(w)R^*_{\omega_0}\widehat{\mu_l}(dw) &= \int_{\widehat{\mathcal{M}_l}} f(R_{\omega_0}v)\ \widehat{\mu_l}(dv)\\
&= \int_{\mathcal{M}_l}\int_{O(3)} f(R_{\omega_0}R_\omega u)\ d\omega\ \mu_l(du)\\
&= \int_{\mathcal{M}_l}\int_{O(3)} f(R_\omega u)\ d\omega\ \mu_l(du)\\
&= \int_{\widehat{\mathcal{M}_l}} f(v)\ \widehat{\mu_l}(dv),
\end{aligned} \tag{4.10}$$

i.e. $\widehat{\mu}$ is invariant with respect of rotations: $R^*_{\omega_0}\widehat{\mu} = \widehat{\mu}$ for each $\omega_0\in O(3)$.

Similarly, the equalities $T_h^{-1}\widehat{\mathcal{M}_l} = \widehat{\mathcal{M}_l}$, $T_{\omega h}^{-1}\mathcal{M}_l = \mathcal{M}_l$ imply:

$$
\begin{aligned}
\int_{\widehat{\mathcal{M}_l}} f(w) T_h^* \widehat{\mu}_l(dw) &= \int_{\widehat{\mathcal{M}_l}} f(T_h v)\ \widehat{\mu}_l(dv) \\
&= \int_{\mathcal{M}_l} \int_{O(3)} f(T_h R_\omega u)\ d\omega\ \mu_l(du) \\
&= \int_{O(3)} \int_{\mathcal{M}_l} f(R_\omega T_{\omega^{-1}h} u)\ \mu_l(du)\ d\omega \\
&= \int_{O(3)} \int_{\mathcal{M}_l} f(R_\omega u)\ \mu_l(du)\ d\omega, \text{ by the homogeneity of } \mu_l, \\
&= \int_{\mathcal{M}_l} \int_{O(3)} f(R_\omega u)\ d\omega\ \mu_l(du) \\
&= \int_{\widehat{\mathcal{M}_l}} f(v)\ \widehat{\mu}_l(dv).
\end{aligned}
\tag{4.11}
$$

□

4.2. **Galerkin equations for Fourier coefficients on $R_\omega \mathcal{M}_l$.** Let $H^s(\Pi_l)$ be the space of periodic vector fields

(4.12)
$$
H^s(\Pi_l) = \Big\{ u(x) = \sum_{k \in \frac{\pi}{l}\mathbb{Z}^3} a_k e^{ik\cdot x},\ a_k = (a_{k_1}, a_{k_2}, a_{k_3}),\ \|u\|_s^2 = \sum_{k \in \frac{\pi}{l}\mathbb{Z}^3} (1+|k|^2)^s |a_k|^2 < \infty \Big\}.
$$

Here

$$
\Pi_l = \{x = (x_1, x_2, x_3) :\ |x_j| \le l,\ j = 1,2,3\} \tag{4.13}
$$

is the cube of periods for these vector fields.

On the space $C^1(0,T; H^2(\Pi_l))$ the Navier-Stokes system can be written in the form:

$$
\partial_t u - \Delta u + \pi(u,\nabla)u = 0,\ \text{div u=0}, \tag{4.14}
$$

where $\pi : L^2(\Pi_l) \to \{u \in L^2(\Pi_l) : \text{div}u = 0\}$ is the projection on solenoidal vector fields. It is standard that substitution of the Fourier series $u(x) = \sum_k a_k e^{ik\cdot x}$ into (4.14) yields the following system for the Fourier coefficients $a_k(t)$:

$$
\begin{gathered}
\partial_t a_k + |k|^2 a_k + \sum_{\substack{k'+k''=k,\\ k',k'' \in \frac{\pi}{l}\mathbb{Z}^3}} i((a_{k'}\cdot k'')a_{k''} - \frac{(a_{k'}\cdot k'')(a_{k''}\cdot k)}{|k|^2} k) = 0, \quad a_k \cdot k = 0, \\
k \in \frac{\pi}{l}\mathbb{Z}^3.
\end{gathered}
\tag{4.15}
$$

Let $p_l : H^2(\Pi_l) \to \mathcal{M}_l$ be projection on trigonometric polynomials:

$$
H^2(\Pi_l) \ni u(x) = \sum_{k \in \frac{\pi}{l}\mathbb{Z}^3} a_k e^{ikx} \mapsto p_l u(x) = \sum_{k \in \frac{\pi}{l}\mathbb{Z}^3, |k| \le l} a_k e^{ikx} \tag{4.16}
$$

As is well-known, to get Galerkin approximations of Navier-Stokes system one restricts (4.14) to $C^1(0,T;\mathcal{M}_l)$ and applies the operator p_l to (4.14) to obtain:

$$\partial_t u - \Delta u + p_l \pi(u,\nabla)u = 0, \text{ div u=0}, \tag{4.17}$$

where $u \in C^1(0,T;\mathcal{M}_l)$. In terms of the Fourier coefficients of the Galerkin approximations this will have the form:

$$\begin{gathered} \partial_t a_k + |k|^2 a_k + \sum_{\substack{k'+k''=k,\\ k',k''\in\frac{\pi}{l}\mathbb{Z}^3,\\ |k'|\le l,|k''|\le l}} i((a_{k'}\cdot k'')a_{k''} - \frac{(a_{k'}\cdot k'')(a_{k''}\cdot k)}{|k|^2}k) = 0, \quad a_k\cdot k = 0, \\ k\in\frac{\pi}{l}\mathbb{Z}^3,\ |k|\le l. \end{gathered} \tag{4.18}$$

Proposition 4.4. *For each $\omega \in O(3)$ the following holds:*

$$R_\omega \mathcal{M}_l = \left\{ v(x) = \sum_{\substack{m\in\frac{\pi}{l}\omega\mathbb{Z}^3,\\ |m|\le l}} b_m e^{imx} \right\}. \tag{4.19}$$

Moreover,

$$\left.\begin{aligned} u(x) &= \sum_{\substack{k\in\frac{\pi}{l}\mathbb{Z}^3\\ |k|\le l}} a_k e^{ikx} \\ R_\omega u(x) &= \sum_{\substack{m\in\frac{\pi}{l}\omega\mathbb{Z}^3\\ |m|\le l}} b_m e^{imx} \end{aligned}\right\} \Rightarrow b_m = \omega a_{\omega^{-1}m}. \tag{4.20}$$

Proof. Let $u(x) = \sum_{|k|\le l} a_k e^{ikx} \in \mathcal{M}_l$. Then using the definition $R_\omega u(x) = \omega u(\omega^{-1}x)$ and applying the change of variables $\omega k = m$ get:

$$R_\omega u(x) = \sum_{\substack{k\in\frac{\pi}{l}\mathbb{Z}^3,\\ |k|\le l}} \omega a_k e^{i\omega kx} = \sum_{\substack{m\in\frac{\pi}{l}\omega\mathbb{Z}^3,\\ |m|\le l}} \omega a_{\omega^{-1}m} e^{imx} \tag{4.21}$$

This proves (4.19) and (4.20). □

Proposition 4.5. *For each $\omega \in O(3)$ the Galerkin approximations for the Navier-Stokes equations on the space $C^1(0,T;R_\omega\mathcal{M}_l)$ are of the following form:*

(4.22)

$$\begin{gathered} \partial_t b_m(t) + |m|^2 b_m + \sum_{\substack{m'+m''=m,\\ m',m''\in\frac{\pi}{l}\omega\mathbb{Z}^3,\\ |m'|\le l,\ |m''|\le l}} i\left((b_{m'}\cdot m'')b_{m''} - \frac{(b_{m'}\cdot m'')(b_{m''}\cdot m)}{|m|^2}m \right) = 0, \quad b_m\cdot m = 0, \\ m\in\frac{\pi}{l}\omega\mathbb{Z}^3,\ |m|\le l. \end{gathered}$$

Proof. To obtain the Galerkin approximations on $C^1(0,T;R_\omega\mathcal{M}_l)$ for the Navier-Stokes equations, repeat the procedure above that leads to the Galerkin approximations (4.18): Re-write

(4.14) on the space of periodic fields $C^1(0,T;R_\omega H^2(\Pi_l))$ in terms of Fourier coefficients to get the following analog of (4.15):

(4.23)
$$\partial_t b_m + |m|^2 b_m + \sum_{\substack{m'+m''=m,\\ m',m''\in\frac{\pi}{l}\omega\mathbb{Z}^3}} i\left((b_{m'}\cdot m'')b_{m''} - \frac{(b_{m'}\cdot m'')(b_{m''}\cdot m)}{|m|^2}m\right) = 0, \quad b_m\cdot m = 0,$$
$$m \in \frac{\pi}{l}\omega\mathbb{Z}^3,$$

and then repeat the derivation of (4.17), (4.18) from (4.14), (4.15) to finally get (4.22) from (4.23). □

Now supplement (4.17) and (4.18) with the initial condition

$$u(t,x)|_{t=0} = \sum_{\substack{k\in\frac{\pi}{l}\mathbb{Z}^3,\\ |k|\le l}} a_k(t)e^{ikx}|_{t=0} = u_0(x) = \sum_{\substack{k\in\frac{\pi}{l}\mathbb{Z}^3,\\ |k|\le l}} a_{k0}e^{ikx} \tag{4.24}$$

and

$$a_k(t)|_{t=0} = a_{k0}, \quad k\in\frac{\pi}{l}\mathbb{Z}^3, |k|\le l \tag{4.25}$$

Moreover, supplement (4.22) with the initial condition

$$b_m(t)|_{t=0} = b_{m0}, \quad m\in\frac{\pi}{l}\omega\mathbb{Z}^3, |k|\le l \tag{4.26}$$

Then, as is well-known, the Cauchy problem (4.17), (4.24), (equivalently, the Cauchy problem for the ordinary differential equations (4.18), (4.25)) has a unique solution in $C^1(0,T;\mathcal{M}_l)$. Call this solution $S_l(u_0)$. Analogously, the Cauchy Problem (4.22),(4.26) possesses a unique solution. Write this solution as the Fourier polynomial

$$S_l(v_0) = \sum_{\substack{m\in\frac{\pi}{l}\omega\mathbb{Z}^3,\\ |k|\le l}} b_m(t)e^{imx}, \quad \text{where} \quad v_0 = \sum_{\substack{m\in\frac{\pi}{l}\omega\mathbb{Z}^3,\\ |k|\le l}} b_{m0}e^{imx} \tag{4.27}$$

Lemma 4.6. *Let $u_0\in\mathcal{M}_l$. Then $R_\omega S_l(u_0)$ solves (4.22) with initial condition $v_0 = R_\omega u_0$. Moreover, if $S_l(u_0)$ admits the Fourier decomposition*

$$S_l(u_0) = \sum_{\substack{k\in\frac{\pi}{l}\mathbb{Z}^3,\\ |k|\le l}} a_k(t)e^{ikx}, \tag{4.28}$$

then $\{\omega a_k\}$ satisfies

(4.29)
$$\partial_t\omega a_k + |k|^2\omega a_k + \sum_{\substack{k'+k''=k,\\ k',k''\in\frac{\pi}{l}\mathbb{Z}^3,\\ |k'|\le l,|k''|\le l}} i\left((\omega a_{k'}\cdot\omega k'')\omega a_{k''} - \frac{(\omega a_{k'}\cdot\omega k'')(\omega a_{k''}\cdot\omega k)}{|k|^2}\omega k\right) = 0,$$
$$\omega a_k\cdot\omega k = 0, \quad k\in\frac{\pi}{l}\mathbb{Z}^3,\ |k|\le l.$$

Proof. Let

$$R_\omega S_l(u_0) = \sum_{\substack{m\in\frac{\pi}{l}\omega\mathbb{Z}^3,\\ |m|\le l}} b_m(t)e^{imx} \tag{4.30}$$

where, by (4.20),

$$b_m = \ a_{\omega^{-1}m}. \tag{4.31}$$

The assertion of the Lemma will be proved once it shown that the $\{b_m(t)\}$ satisfy (4.22). Substitution of (4.31) into the left hand side of (4.22), the change of variables $m = \omega k$, and the fact that $\omega \in O(3)$ yield:

$$\begin{aligned}
&\forall\ k \in \frac{\pi}{l}\mathbb{Z}^3,\ |k| \le l;\\
&\partial_t\omega a_k + |\omega k|^2\omega a_k + \sum_{\substack{k'+k''=k,\\ k',k''\in\frac{\pi}{l}\mathbb{Z}^3,\\ |k'|\le l,|k''|\le l}} i\bigg((\omega a_{k'}\cdot\omega k'')\omega a_{k''} - \frac{(\omega a_{k'}\cdot\omega k'')(\omega a_{k''}\cdot\omega k)}{|\omega k|^2}\omega k\bigg)\\
&= \omega\bigg\{\partial_t a_k + |k|^2 a_k + \sum_{\substack{k'+k''=k,\\ k',k''\in\frac{\pi}{l}\mathbb{Z}^3,\\ |k'|\le l,|k''|\le l}} i((a_{k'}\cdot k'')a_{k''} - \frac{(a_{k'}\cdot k'')(a_{k''}\cdot k)}{|k|^2}k\bigg\} = 0,
\end{aligned} \tag{4.32}$$

since $\{a_k(t)\}$ satisfies (4.18). $\square$

Remark 4.7. *As will be shown, more is true: The Galerkin PDE has a unique solution for any initial condition v in $\widehat{\mathcal{M}_l}$, see (4.47) below.*

The task now is to show that inclusions $u_1 \in \mathcal{M}_l$ and $R_\omega u_1 = u_2 \in \mathcal{M}_l$, for some ω, implies that for the same ω, $R_\omega S_l u_1$ stays in $\mathcal{M}_l$ for all t, still solving the Galerkin system on $\mathcal{M}_l$. This will then show that $R_\omega S_l u_1 = S_l u_2$.

Definition 4.8. *Given an isometry ω of $\mathbb{R}^3$, let $\mathcal{K}_\omega$ be the set of all elements k in the lattice $\frac{\pi}{l}\mathbb{Z}^3$ such that ωk also belongs to the lattice.*

Lemma 4.9. *Let u_1, u_2 be vector fields of the form*

$$u_1(x) = \sum_{|k|\le l} a_k e^{ik\cdot x}, \quad u_2(x) = \sum_{|k|\le l} b_k e^{ik\cdot x}, \quad \textit{where all} \quad k \in \frac{\pi}{l}\mathbb{Z}^3, \tag{4.33}$$

not necessarily divergence free. Then for some ω isometry of $\mathbb{R}^3$, $R_\omega u_1 = u_2$ if and only if all k's in the representation of u_1 are in $\mathcal{K}_\omega$.

Proof. Since $R_\omega u_1 = u_2$, $R_\omega u_1$ is periodic of period $2l$. Therefore

$$\begin{aligned}\sum_{|k|\le l}\omega a_k e^{i\omega k\cdot x} &= \sum_{|k|\le l}\omega a_k e^{i\omega k\cdot(x+(2l)_j)}\\ &= \sum_{|k|\le l}\omega a_k e^{i2l(\omega k)_j}e^{i\omega k\cdot x},\ j=1,2,3,\end{aligned}\tag{4.34}$$

for $(2l)_j$ denoting the vector with $2l$ as j coordinate and zeroes on the rest. Now since $e^{i\omega k\cdot x}$ are orthonormal on the image of the cube $\Pi_l = [-l,l]^3$ under ω, this implies that

$$1 = e^{i2l(\omega k)_j},\tag{4.35}$$

therefore

$$2l(\omega k)_j = 2\pi N_{k_j}, N_{k_j} \in \mathbb{Z}.\tag{4.36}$$

Therefore $\omega k \in \dfrac{\pi}{l}\mathbb{Z}^3$. The converse is clear. □

Lemma 4.10. *Let u_1 in $\mathcal{M}_l$ of the form ,*

$$u_1 = \sum_{|k|\le l} a_k e^{ik\cdot x}, \quad k \in \mathcal{K}_\omega.\tag{4.37}$$

Then the Galerkin solution in $\mathcal{M}_l$ with initial condition u_1 is of the form

$$u(t) = \sum_{|k|\le l} a_k(t)e^{ik\cdot x}, \quad a_k(t) = 0 \ for\ all\ t\ for\ k \notin \mathcal{K}_\omega.\tag{4.38}$$

Proof. In $\mathcal{M}_l$, first solve the system

$$\begin{aligned}&\partial_t a_k = -|k|^2 a_k, \quad k\notin\mathcal{K}_\omega \quad |k|\le l\\ &\partial_t a_k + \sum_j \sum_{\substack{k'+k''=k\\ |k'|\le l\\ |k''|\le l}} \left((a_{k'})_j ik''_j a_{k''} - \frac{(a_{k'})_j ik''_j a_{k''}\cdot k}{|k|^2}k\right) = -|k|^2 a_k,\ k\in\mathcal{K}_\omega,\ |k|\le l\end{aligned}\tag{4.39}$$

with initial conditions

$$\begin{aligned}a_k(0) = 0, &\quad k\notin\mathcal{K}_\omega\\ a_k(0) = a_k, &\quad k\in\mathcal{K}_\omega,\end{aligned}\tag{4.40}$$

using the a_k's from (4.37).

In particular

$$a_k(t) = 0,\ \text{for}\ \ \text{all}\ t,\ \text{for}\ k\ \text{not in}\ \mathcal{K}_\omega.\tag{4.41}$$

Now if $k \notin \mathcal{K}_\omega$, and $k' + k'' = k$, then either $k' \notin \mathcal{K}_\omega$ or $k'' \notin \mathcal{K}_\omega$. Then the unique solution of (4.39) (4.40) also solves the system

$$\begin{aligned}
&\partial_t a_k + \sum_j \sum_{\substack{k'+k''=k \\ |k'|\le l \\ |k''|\le l}} \left((a_{k'})_j i k''_j a_{k''} - \frac{(a_{k'})_j i k''_j a_{k''} \cdot k}{|k|^2} k \right) = -|k|^2 a_k,\ k \notin \mathcal{K}_\omega,\ |k| \le l \\
&\partial_t a_k + \sum_j \sum_{\substack{k'+k''=k \\ |k'|\le l \\ |k''|\le l}} \left((a_{k'})_j i k''_j a_{k''} - \frac{(a_{k'})_j i k''_j a_{k''} \cdot k}{|k|^2} k \right) = -|k|^2 a_k,\ k \in \mathcal{K}_\omega,\ |k| \le l,
\end{aligned} \tag{4.42}$$

i.e. solves the Galerkin system with initial condition u_1. □

Proposition 4.11. *Let u_1, u_2 be in $\mathcal{M}_l$ of the form*

$$u_1(x) = \sum_{|k|\le l} a_k e^{ik\cdot x}, \quad u_2(x) = \sum_{|\kappa|\le l} b_\kappa\ e^{i\kappa\cdot x}, \tag{4.43}$$

Then $u_2 = R_\omega u_1$ implies $S_l u_2 = R_\omega S_l u_1$.

Proof. Let

$$S_l u_1 = \sum_{|\kappa|\le l} a_\kappa(t)\ e^{i\kappa\cdot x}, \quad a_\kappa(0) = a_\kappa, \qquad S_l u_2 = \sum_{|\kappa|\le l} b_\kappa(t)\ e^{i\kappa\cdot x}, \quad b_\kappa(0) = b_\kappa. \tag{4.44}$$

By Lemma 4.6, $R_\omega S_l u_1$ is the unique solution of the Galerkin system on $R_\omega \mathcal{M}_l$ with initial condition $R_\omega u_1$, i.e. by (4.29) it solves

$$\partial_t \omega a_k + \sum_j \sum_{\substack{k'+k''=k \\ |k'|\le l \\ |k''|\le l}} \left((\omega a_{k'})_j i(\omega k'')_j \omega a_{k''} - \frac{(\omega a_{k'})_j i(\omega k'')_j \omega a_{k''} \cdot \omega k}{|\omega k|^2} \omega k \right) = -|\omega k|^2 \omega a_k. \tag{4.45}$$

Observe that in this system if $\omega k \neq \kappa$ for some κ in the lattice $\frac{\pi}{l}\mathbb{Z}^3$ then $\omega a_k(t) = 0$ for all t, by Lemma 4.10. Now rename $\omega k = \kappa$, $\omega a_k(t) = c_\kappa(t)$, $\omega a_k(0) = c_\kappa(0) = b_\kappa$, $\omega k' = \kappa'$, $\omega k'' = \kappa''$ to get

$$\partial_t c_\kappa + \sum_j \sum_{\substack{\kappa'+\kappa''=\kappa \\ |\kappa'|\le l \\ |\kappa''|\le l}} \left((c_{\kappa'})_j i(\kappa'')_j c_{\kappa''} - \frac{(c_{\kappa'})_j i(\kappa'')_j c_{\kappa''} \cdot \kappa}{|\kappa|^2} \kappa \right) = -|\kappa|^2 c_\kappa, \tag{4.46}$$

with initial conditions Ra_k. This gives a permutation of the Galerkin system on $\mathcal{M}_l$ with initial condition u_2. Therefore $c_\kappa(t) = b_\kappa(t)$ for all $\kappa \in \frac{\pi}{l}\mathbb{Z}^3$. □

4.3. **Isotropic Galerkin approximations of statistical solution.** Now given v in $\widehat{\mathcal{M}_l}$, there are ω in $O(3)$ and u in $\mathcal{M}_l$ such that $v = R_\omega u$. Extend S_l from $\mathcal{M}_l$ to $\widehat{\mathcal{M}_l}$ as

$$\widehat{S}_l v = R_\omega S_l u. \tag{4.47}$$

This is well defined by Proposition 4.11: If $R_{\omega_1}u_1 = R_{\omega_2}u_2$ then a straightforward calculation shows that $R_{\omega_2^{-1}\omega_1}u_1 = u_2$, therefore $R_{\omega_2^{-1}\omega_1}S_l u_1 = S_l u_2$, or $R_{\omega_1}S_l u_1 = R_{\omega_2}S_l u_2$. In particular, $\widehat{S_l}$ satisfies

$$\widehat{S_l}R_\omega u = R_\omega S_l u \quad \forall\, u \in \mathcal{M}_l. \tag{4.48}$$

Let $v = R_\omega u$ and $w = R_{\omega_1}v = R_{\omega\omega_1}u$ where $u \in \mathcal{M}_l$. Applying R_{ω_1} to both parts of (4.47) gives $R_{\omega_1}\widehat{S_l}v = R_{\omega\omega_1}S_l u$. On the other hand, (4.47) for w can be written as $\widehat{S_l}w = R_{\omega\omega_1}S_l u$. Comparing these two equalities gives

$$R_{\omega_1}\widehat{S_l}v = \widehat{S_l}R_{\omega_1}v, \quad \forall\, v \in \widehat{\mathcal{M}_l}. \tag{4.49}$$

[VF2], p. 219 shows that

$$T_h S_l = S_l T_h. \tag{4.50}$$

Applying to both parts of (4.47) the translation operator T_h and using (2.25), (4.50) gives

$$T_h\widehat{S_l}v = R_\omega T_{\omega^{-1}h}S_l u = R_\omega S_l T_{\omega^{-1}h}u. \tag{4.51}$$

On the other hand, applying $\widehat{S_l}$ as defined in (4.47) to $T_h v = T_h R_\omega u = R_\omega T_{\omega^{-1}h}u$, obtain $\widehat{S_l}T_h v = R_\omega S_l T_{\omega^{-1}h}u$. Comparing this equality with (4.51) obtain

$$T_h\widehat{S_l}v = \widehat{S_l}T_h v \quad \forall\, v \in \widehat{\mathcal{M}_l}. \tag{4.52}$$

Define

$$\widehat{P_l}(A) = \widehat{\mu_l}(\widehat{S_l}^{-1}A), \tag{4.53}$$

for any Borel subset A of $L^2(0,T,\mathcal{H}^0(r))$ where, recall,

$$\widehat{S_l}^{-1}A = \gamma_0(A \cap \widehat{S_l}\widehat{\mathcal{M}_l}). \tag{4.54}$$

Since the measure $\widehat{\mu_l}$ is supported on $\widehat{\mathcal{M}_l}$ it is enough to consider Borel sets A satisfying $A \cap \widehat{S_l}\widehat{\mathcal{M}_l} \neq \emptyset$.

Definition (4.53) is the isotropic version of the measure P_l defined in [VF2]:

$$P_l(A) = \mu_l(S_l^{-1}A) \tag{4.55}$$

for any Borel subset A of $L^2(0,T,\mathcal{H}^0(r))$.

Lemma 4.12. *$\widehat{P_l}$ is homogeneous and isotropic.*

Proof. It suffices to show that for all $A \in \mathcal{B}(L^2(0,T;\mathcal{H}^0(r)))$:

$$\widehat{S_l}^{-1}R_\omega A = R_\omega\widehat{S_l}^{-1}A \tag{4.56}$$

and

$$\widehat{S_l}^{-1}T_h A = T_h\widehat{S_l}^{-1}A, \tag{4.57}$$

since $\widehat{P_l}(A) = \widehat{\mu_l}(\widehat{S_l}^{-1}A)$ and $\widehat{\mu_l}$ is homogeneous and isotropic.

Using (4.54), (4.49),

$$
\begin{aligned}
\widehat{S_l}^{-1} R_\omega A &= \gamma_0(R_\omega A \cap R_\omega R_{\omega^{-1}} \widehat{S_l}\widehat{\mathcal{M}_l}) \\
&= \gamma_0 R_\omega (A \cap \widehat{S_l} R_{\omega^{-1}} \widehat{\mathcal{M}_l}) \\
&= R_\omega \gamma_0 (A \cap \widehat{S_l}\widehat{\mathcal{M}_l}) \\
&= R_\omega \widehat{S_l}^{-1} A.
\end{aligned} \tag{4.58}
$$

Also, by (4.54), (4.52)

$$
\begin{aligned}
\widehat{S_l}^{-1} T_h A &= \gamma_0(T_h A \cap T_h T_{-h} \widehat{S_l}\widehat{\mathcal{M}_l}) \\
&= \gamma_0 T_h (A \cap \widehat{S_l} T_{-h} \widehat{\mathcal{M}_l}) \\
&= T_h \gamma_0 (A \cap \widehat{S_l}\widehat{\mathcal{M}_l}) \\
&= T_h \widehat{S_l}^{-1} A.
\end{aligned} \tag{4.59}
$$

□

The following relation between P_l and $\widehat{P_l}$ allows the known estimates on for P_l to be carried over to $\widehat{P_l}$. Once again, let a to be the action map of rotations on vector fields:

$$
a(\omega, u) = R_\omega u \tag{4.60}
$$

Lemma 4.13. *The equality holds:*

$$
\widehat{P_l}(A) = (P_l \times H)(a^{-1}A).
$$

where H is the Haar measure on $O(3)$, normalized.

Proof.

$$
\begin{aligned}
\widehat{P_l}(A) &= \widehat{\mu_l}(\widehat{S_l}^{-1} A) \\
&= (\mu_l \times H)(a^{-1}\widehat{S_l}^{-1} A) \\
&= (\mu_l \times H)\{(u_0, \omega) \in \mathcal{M}_l \times O(3) : R_\omega u_0 \in \widehat{S_l}^{-1} A\} \\
&= (\mu_l \times H)\{(u_0, \omega) \in \mathcal{M}_l \times O(3) : \widehat{S_l} R_\omega u_0 \in A\} \\
&= (\mu_l \times H)\{(u_0, \omega) \in \mathcal{M}_l \times O(3) : R_\omega S_l u_0 \in A\}, \text{ by (4.48)} \\
&= (P_l \times H)\{(S_l u_0, \omega) \in C^1(0, T; \mathcal{M}_l) \times O(3) : R_\omega S_l u_0 \in A\} \\
&= (P_l \times H)(a^{-1}A).
\end{aligned} \tag{4.61}
$$

□

5. Convergence of isotropic Galerkin approximations

5.1. **Galerkin Approximation of Homogeneous Statistical Solutions.** This section summarizes the Galerkin approximation in [VF2]. Recall that the following are shown in Chapter VII there:

Given any μ homogeneous probability measure on $\mathcal{H}^0(r)$, there exist for each l homogeneous probability measures μ_l on $\mathcal{H}^0(r)$, supported on $\mathcal{M}_l$, converging to μ in characteristic i.e.:

$$\int_{\mathcal{H}^0(r)} e^{i<u,\nu>} \ \mu_l(du) \to \int_{\mathcal{H}^0(r)} e^{i<u,\nu>} \ \mu(du), \ \ l \to \infty, \tag{5.1}$$

for any test function ν.

5.1.1. *Convergence.* The probability measures P_l, defined by (4.55) are homogeneous in x if the initial μ is homogeneous, and converge weakly to a homogeneous probability measure P on $L^2(0,T;\mathcal{H}^0(r))$. Weak convergence relies on the following three uniform estimates on P_l: There are constants $C, C(N)$ independent of l such that for each t in $[0,T]$,

$$\int \left(|u(t,x)|^2 + \int_0^t |\nabla u|^2(\tau,x) \ d\tau \right) \ P_l(du) \leq C \int |u_0(x)|^2 \ \mu_l(du_0), \tag{5.2}$$

$$\int \|u|_{B_N}\|_{-s} \ P_l(du) \leq C(N), \tag{5.3}$$

$$\int \|\partial_t u|_{B_N}\|_{-s} \ P_l(du) \leq C(N) \tag{5.4}$$

for $\|v|_{B_N}\|_{-s}$ the dual norm (3.2), and with $s > 11/2$.

5.1.2. *P is a homogeneous statistical solution of the Navier-Stokes system.* As already remarked, the weak limit P is supported on $L^2(0,T;\mathcal{H}^1(r)) \cap BV^{-s} \cap \mathcal{G}_{NS}$ so that each $u(.,.)$ in the support of P satisfies the weak form (3.1) of Navier-Stokes equations, and the right limit in time $\gamma_t u$ exists for each t with respect to the Φ^{-s} norm. This extra regularity of P relies on the estimate

$$\int \|u\|_{BV^{-s}} \ P(du) \leq \infty, \quad s > 11/2 \tag{5.5}$$

for $\|u\|_{BV^{-s}}$ the norm (3.3). In addition, P satisfies energy estimate (3.13).

5.1.3. *P solves the initial value problem.* The measure defined by $P(\gamma_0^{-1}A)$, for A Borel of $\mathcal{H}^0(r)$, is the initial measure μ. The proof of this relies on the convergence (5.1).

5.2. **Homogeneous and Isotropic Statistical Solutions via isotropic Galerkin Approximations.** In the setting of isotropic measures, the construction of the previous subsection can be repeated as follows:

5.2.1. *Approximation of initial measure.* Given initial homogeneous and isotropic $\widehat{\mu}$ on $\mathcal{H}^0(r)$, construct its (merely) homogeneous approximation μ_l, with $\mu_l \to \widehat{\mu}$ in characteristic as in the previous section. Then use Definitions 4.1 and 4.2 to obtain from the μ_l's probability measures $\widehat{\mu_l}$ that are now homogeneous and isotropic and supported by $\widehat{\mathcal{M}_l}$. Convergence in characteristic still holds, thanks to the following:

Lemma 5.1. *$\widehat{\mu_l}$ converges to $\widehat{\mu}$ in characteristic as $l \to \infty$.*

Proof. For $\widehat{\mu_l}$ defined as above,

$$\begin{aligned} \chi_{\widehat{\mu_l}}(\nu) &= \int_{\mathcal{H}^0(r)} e^{iu\cdot\nu}\ \widehat{\mu_l}(du) \\ &= \int_{\mathcal{H}^0(r)} \int_{O(3)} e^{iR_\omega u\cdot\nu}\ d\omega\ \mu_l(du) \\ &= \int_{O(3)} \int_{\mathcal{H}^0(r)} e^{iu\cdot R_{\omega^{-1}}\nu}\ \mu_l(du)\ d\omega \\ &= \int_{O(3)} \chi_{\mu_l}(R_{\omega^{-1}}\nu)\ d\omega. \end{aligned} \tag{5.6}$$

Since $|\chi_{\mu_l}((R_{\omega^{-1}}\nu)| \le 1$ and $\int_{O(3)} 1\ d\omega = 1$, (5.1) implies, via the Lebesgue Dominated Convergence Theorem,

$$\chi_{\widehat{\mu_l}}(\nu) = \int_{O(3)} \chi_{\mu_l}(R_{\omega^{-1}}\nu)\ d\omega \to \int_{O(3)} \chi_{\widehat{\mu}}(R_{\omega^{-1}}\nu)\ d\omega \tag{5.7}$$

as $l \to \infty$. Therefore, since

$$\int_{O(3)} \chi_{\widehat{\mu}}(R_{\omega^{-1}}\nu)\ d\omega = \int_{O(3)} \int_{\mathcal{H}^0(r)} e^{iR_\omega u\cdot\nu}\ \widehat{\mu}(du)\ d\omega = \chi_{\widehat{\mu}}(\nu), \tag{5.8}$$

$\chi_{\widehat{\mu_l}}(\nu) \to \chi_{\widehat{\mu}}(\nu)$ as $l \to \infty$. □

5.2.2. *Convergence to a homogeneous and isotropic measure.* Now construct the homogeneous and isotropic measures $\widehat{P}_l$ according to (4.53). To prove weak convergence the following is needed

Lemma 5.2. *The following estimates hold:*

$$\int \|u|_{B_N}\|_{-s}\ \widehat{P}_l(du) \le C(N), \tag{5.9}$$

$$\int \|\partial_t u|_{B_N}\|_{-s}\ \widehat{P}_l(du) \le C(N). \tag{5.10}$$

Proof. By Lemma 4.13 and relations (3.21),

$$\begin{aligned} \int \|\partial_t u|_{B_N}\|_{-s}\ \widehat{P}_l(du) &= \int \int_{O(3)} \|R_\omega \partial_t u|_{B_N}\|_{-s}\ d\omega\ P_l(du) \\ &= \int \|\partial_t u|_{B_N}\|_{-s}\ P_l(du). \end{aligned} \tag{5.11}$$

[VF2] Chapter VII, Theorem 3.1 proves that the right hand side of (5.11) is bounded by a constant $C(N)$ that depends on N but does not depend on l. This proves (5.10). The bound (5.9) is proved similarly. □

The following is also needed to repeat the proof of the weak convergence of the $\widehat{P}_l$'s.

Lemma 5.3. *With pointwise averages defined as in (3.8),(3.9),(3.10), for any t in $[0,T]$,*

$$\int \left(|u|^2(t,x) + \int_0^t \left(|u|^2(\tau,x) + |\nabla u|^2(\tau,x) \right) d\tau \right) \widehat{P}_l(du) \leq C \int |u|^2(x)\ \widehat{\mu}_l(du). \tag{5.12}$$

Proof. Using an equality similar to (3.9), (3.10) one can show that

$$\int |u|^2(t,x)\ \widehat{\mu}_l(du) = \int_{O(3)} \int |u|^2(t,x)\ R^*_\omega \mu_l(du)\ d\omega, \tag{5.13}$$

$$\begin{aligned} &\int \left(|u|^2(t,x) + \int_0^t \left(|u|^2(t,x) + |\nabla u|^2(t,x) \right) d\tau \right) \widehat{P}_l(du) \\ &= \int_{O(3)} \int \left(|u|^2(t,x) + \int_0^t \left(|u|^2(t,x) + |\nabla u|^2(t,x) \right) d\tau \right) R^*_\omega P_l(du)\ d\omega. \end{aligned} \tag{5.14}$$

Applying to (5.13), (5.14) Lemma 3.10 and taking into account the inequality for Galerkin approximations of the homogeneous statistical solution

$$\int \left(|u|^2(t,x) + \int_0^t \left(|u|^2(t,x) + |\nabla u|^2(t,x) \right) d\tau \right) P_l(du) \leq C \int |u|^2(x)\ \mu_l(du) \tag{5.15}$$

that was proved in [VF2] Chapter VII, Lemma 2.4, yields (5.12). □

Lemma 5.4. *The measures $\widehat{\mu}_l$ satisfy*

$$\int |u|^2(x)\ \widehat{\mu}_l(du) \leq \int |u|^2(x)\ \mu_l(du) \tag{5.16}$$

Proof. This follows from (5.13) and Lemma 3.10. □

With Lemmas 5.2, 5.3, 5.4 established, there are no further obstacles in repeating the arguments in [VF2] to show that the family $\widehat{P}_l$ is weakly compact. The argument for this in [VF2] is that the measures P_l are supported on the space Ω of elements in $L^2(0,T;\mathcal{H}^0(r))$ with the following norm finite:

$$\sum_{N=1}^{\infty} \frac{1}{2^N C(N)} (\|u\|_{L^2(0,T;\mathcal{H}^{-s}(B_N))} + \|\frac{\partial u}{\partial t}\|_{L^1(0,T;\mathcal{H}^{-s}(B_N))} + \|u\|_{L^2(0,T;\mathcal{H}^1(r_1))}) < \infty, \quad s > 11/2, \quad r < r_1 < -\frac{3}{2}. \tag{5.17}$$

Then Ω is compactly embedded in $L^2(0,T;\mathcal{H}^0(r))$, [VF2], Chapter VII, Lemma 5.2, and

$$\sup_l \int \|u\|_\Omega \; P_l(du) < \infty, \tag{5.18}$$

[VF2], proof of Theorem 6.1 in Chapter VII. From the estimates of Lemmas 5.2, 5.3 for the measures $\widehat{P}_l$, the energy conservation (5.12), and the uniform estimate (5.4), it becomes clear that the all measures $\widehat{P}_l$ are supported on Ω and the uniform estimate (5.18), with P_l changed to $\widehat{P}_l$, still holds.

Let $\widehat{Q}$ be the limit of some weakly convergent subsequence of $\widehat{P}_l$'s, as $l \to \infty$.

Lemma 5.5. *If $\widehat{P}_l$ are homogeneous and isotropic and $\widehat{P}_l \Rightarrow \widehat{Q}$ weakly on $L^2(0,T;\mathcal{H}^0(r))$, then $\widehat{Q}$ is homogeneous and isotropic.*

Proof. If $\widehat{P}_l \Rightarrow \widehat{Q}$ weakly then by definition

$$\int f(u) \; \widehat{P}_l(du) \to \int f(u) \; \widehat{Q}(du). \tag{5.19}$$

for any f continuous and bounded on $L^2(0,T;\mathcal{H}^0(r))$. Since the measures $\widehat{P}_l$ are homogeneous and isotropic, by definitions 2.2, 2.4

$$\begin{aligned} \int f(u) \; \widehat{P}_l(du) &= \int f(T_h u) \; \widehat{P}_l(du) \\ &= \int f(R_\omega u) \; \widehat{P}_l(du), \end{aligned} \tag{5.20}$$

for any $h \in \mathbb{R}^3$ and $\omega \in O(3)$. These equalities and

$$\begin{aligned} \int f(T_h u) \; \widehat{P}_l(du) &\to \int f(T_h u) \; \widehat{Q}(du), \\ \int f(R_\omega u) \; \widehat{P}_l(du) &\to \int f(R_\omega u) \; \widehat{Q}(du), \end{aligned} \tag{5.21}$$

imply

$$\begin{aligned} \int f(u) \; \widehat{Q}(du) &= \int f(T_h u) \; \widehat{Q}(du) \\ &= \int f(R_\omega u) \; \widehat{Q}(du). \end{aligned} \tag{5.22}$$

□

5.2.3. *Extra regularity for right t-limits and the initial condition.* That the support of $\widehat{Q}$ is in addition in $L^2(0,T;\mathcal{H}^1(r)) \cap BV^{-s}$ (where right limits with respect to time are well defined by [VF2], Chapter VII, Lemma 8.2) uses only the estimate

$$\int \|u\|_{BV^{-s}} \; \widehat{Q}(du) \leq \infty \tag{5.23}$$

for $\|u\|_{BV^{-s}}$ the norm (3.3), which follows as in the proof of Lemma 5.2. Define therefore γ_0 by (3.6) and think of $\gamma_0^* \widehat{Q}$ as the initial value of $\widehat{Q}$.

That the initial value $\gamma_0^* P$ of the the homogeneous statistical solution P of Theorem 3.4 is the initial measure μ is shown in [VF2] as Theorem 10.1, Lemma 10.1, Lemma 10.2, and Theorem 10.2 of Chapter VII there. Of these, Lemma 10.1, Lemma 10.2, and Theorem 10.2 of Chapter VII are valid verbatim for $\widehat{Q}$. Theorem 10.1 uses only the convergence in characteristic of the μ_l's to μ. The analogous convergence of the $\widehat{\mu}_l$'s to $\widehat{\mu}$ was established here as Lemma 5.1.

6. The support of the measure $\widehat{Q}$.

This section uses the approach of [VF2] Chapter VII, Section 7, to show that the homogeneous and isotropic measure $\widehat{Q}$ is supported by generalized solutions of the Navier-Stokes equations. To realize this approach, some subtle points regarding the definition of the equations for isotropic Galerkin approximations in the x-representation need to be addressed first.

6.1. **Equations for isotropic Galerkin approximations in the x-representation.** Section 4 defined isotropic Galerkin approximations by introducing and investigating the Galerkin equations in terms of Fourier coefficients. Here, a complete description of the Galerkin equations in the x-representation is given, beginning with a more precise determination of the domain of their definition.

In addition to the sets $\mathcal{M}_l$ and $\widehat{\mathcal{M}_l}$ defined by (4.1), (4.3), introduce the set of periodic vector fields $\mathcal{N}_l$ with the cube of periods Π_l defined by (4.13):

$$\mathcal{N}_l = \{u(x) = (u_1, u_2, u_3) \in L^2(\Pi_l) : \text{ div } u(x) = 0\}, \tag{6.1}$$

for div$u$ understood in the weak sense, see (2.3). Also define the space

$$\widehat{\mathcal{N}_l} = \cup_{\omega \in O(3)} R_\omega \mathcal{N}_l \tag{6.2}$$

which, of course, is not linear. Since $\widehat{\mathcal{N}_l} \subset \mathcal{H}^0(r)$ for $r < -3/2$, $\widehat{\mathcal{N}_l}$ is a metric space with the metric generated by the norm of $\mathcal{H}^0(r)$. Now use the set $C^1(0, T; \widehat{\mathcal{N}_l})$ to define the Galerkin equation, recalling that this equation was defined in (4.17) only for $u \in C^1(0, T; \mathcal{M}_l) \subset C^1(0, T; \mathcal{N}_l)$. To extend this definition from $C^1(0, T; \mathcal{M}_l)$ to $C^1(0, T; \widehat{\mathcal{M}_l})$, first extend the operator p_l to $\mathcal{N}_l$ as in (4.16):

$$\begin{gathered} p_l : \mathcal{N}_l \to \mathcal{M}_l; \\ \mathcal{N}_l \ni u(x) = \sum_{k \in \frac{\pi}{l}\mathbb{Z}^3} a_k e^{ikx} \mapsto p_l u(x) = \sum_{k \in \frac{\pi}{l}\mathbb{Z}^3, |k| \le l} a_k e^{ikx} \in \mathcal{M}_l. \end{gathered} \tag{6.3}$$

For each $\omega \in O(3)$ the operator p_l induces operator

$$p_{l,\omega} = R_\omega p_l R_\omega^{-1} \ : \ R_\omega \mathcal{N}_l \to R_\omega \mathcal{M}_l. \tag{6.4}$$

The family of operators $p_{l,\omega}$, $\omega \in O(3)$ defines the operator

$$\widehat{p_l} \ : \ \widehat{\mathcal{N}_l} \to \widehat{\mathcal{M}_l} \tag{6.5}$$

as follows: Since for each $u \in \widehat{\mathcal{N}_l}$ there exist $\omega \in O(3)$ and $v \in \mathcal{N}_l$ such that $u = R_\omega v$, define $\widehat{p_l}u = p_{l,\omega}u = R_\omega p_l v$. This is well defined by the obvious extension of Proposition 4.4 to infinite series.

Similarly, the projection operator $\pi_l : L^2(\Pi_l) \to \mathcal{N}_l$ of periodic vector fields onto solenoidal periodic vector fields yields an operator on $R_\omega L^2(\Pi_l)$ by

$$\pi_{l,\omega} = R_\omega \pi_l R_\omega^{-1} \; : \; R_\omega L^2(\Pi_l) \to R_\omega \mathcal{N}_l, \tag{6.6}$$

and the family $\pi_{l,\omega},\ \omega \in O(3)$ defines the operator

$$\widehat{\pi_l} \; : \; \cup_{\omega \in O(3)} R_\omega L^2(\Pi_l) \to \widehat{\mathcal{N}_l} \tag{6.7}$$

by

$$\cup_{\omega \in O(3)} R_\omega L^2(\Pi) \ni u = R_{\omega_0} v \to \widehat{\pi_l}u = \pi_{l,\omega_0}u = R_{\omega_0}\pi_l v. \tag{6.8}$$

The remaining operators Δ and ∇ in equation (4.17), are already well defined on the larger space $C^1(0,T;\mathcal{H}^2(r)) \supset C^1(0,T;\widehat{\mathcal{M}_l})$ and therefore need not be redefined specifically for $C^1(0,T;\widehat{\mathcal{M}_l})$.

Thus, the Galerkin equation on the set $C^1(0,T;\widehat{\mathcal{M}_l})$ is now defined as follows:

$$\partial_t u - \Delta u + \widehat{p_l}\widehat{\pi_l}[(u,\nabla)u] = 0, \text{ where } u = u(t,x) \in C^1(0,T;\widehat{\mathcal{M}_l}) \tag{6.9}$$

Equation (6.9) is the x-representation of the Galerkin equation that was written in terms of Fourier coefficients and was studied in Section 4. In particular, the resolving operator $\widehat{S_l}v$ of the Cauchy problem for this equation was defined in (4.47).

6.2. **Definition and estimates on a functional.** Let

$$v(t,x) \in G^\infty \equiv C_0^\infty((0,T)\times\mathbb{R}^3) \cap C(0,T;\mathcal{H}^0(r)) \tag{6.10}$$

be a vector field and $B \subset \mathbb{R}^3$ be a ball with center at the origin, satisfying

$$\text{supp } v(t,x) \subset B \quad \forall\, t \in [0,T] \quad . \tag{6.11}$$

Then there exists $l_0 > 0$ such that for each $l \geq l_0$

$$B \subset \bigcap_{\omega \in O(3)} \omega\Pi_l. \tag{6.12}$$

From now on consider only $l \geq l_0$. For every such l and for each $\omega \in O(3)$ extend $v(t,x)|_{(0,T)\times\omega\Pi_l}$ from $(0,T)\times\omega\Pi_l$ into $(0,T)\times\mathbb{R}^3$ as a periodic in x vector field $v_{l,\omega}(t,x)$ with cube of periods $\omega\Pi_l$. Denote the family of functions $v_{l,\omega}(t,x),\ \omega \in O(3)$ by $\widehat{v}_{l,\cdot}(t,x)$.

Note that (6.1) implies

$$R_\omega \mathcal{N}_l = \{v(x) = (v_1,v_2,v_3) \in L^2(\omega\Pi_l) : \text{ div } v(x) = 0\}. \tag{6.13}$$

For $u,v \in L^2(0,T;R_\omega\mathcal{N}_l)$, define

$$[u,v]_{l,\omega} = \int_0^T \int_{\omega\Pi_l} u(t,x)\cdot v(t,x)\,dx\,dt. \tag{6.14}$$

Finally, for $v \in G^\infty$ as above define, for each $j = 1, 2, 3$, the functional $F_{j,v}$ on $C^1(0,T;\widehat{\mathcal{M}_l})$ as follows: If $u \in C^1(0,T;\widehat{\mathcal{M}_l})$, with $u = (u_1, u_2, u_3) = \omega\tilde{u}$ for $\omega \in O(3)$, $\tilde{u} \in C^1(0,T;\mathcal{M}_l)$, then

$$F_{j,v}(u) = \left[(I - p_{l,\omega})\pi_{l,\omega}(u_j u), v_{l,\omega}\right]_{l,\omega}, \tag{6.15}$$

for I the identity operator.

Lemma 6.1. *For each $u \in C^1(0,T;\widehat{\mathcal{M}_l})$ the functional (6.15) satisfies, for all j, the estimate :*

$$|F_{j,v}(u)| \le C(1+3l^2)^{-1}\|u\|_{L^2(0,T;\mathcal{H}^0(r))}\|v\|_{C(0,T;H^{s-r+1}(B))}, \tag{6.16}$$

where $s > 3/2$, $r < -3/2$, and $C > 0$ does not depend on l, u, or v.

Proof. Using the decomposition in Fourier series one sees that the operators $p_{l,\omega}$ and $\pi_{l,\omega}$ commute: $p_{l,\omega}\pi_{l,\omega} = \pi_{l,\omega}p_{l,\omega}$. In addition, the operators $p_{l,\omega}$ and $\pi_{l,\omega}$ are symmetric. Therefore, with $\operatorname{div} v_{l,\omega} = 0$, obtain:

$$\begin{aligned} F_{j,v}(u) &= \left[(I - p_{l,\omega})(u_j u), \pi_{l,\omega}v_{l,\omega}\right]_{l,\omega} \\ &= \left[u_j u, (I - p_{l,\omega})v_{l,\omega}\right]_{l,\omega}. \end{aligned} \tag{6.17}$$

Using the Sobolev Embedding Theorem: $H^s(\omega\Pi_l) \subset C(\omega\Pi_l)$ for $s > 3/2$, and (6.17),

$$\begin{aligned} |F_{j,v}(u)| &\le C_1\|u\|^2_{L^2(0,T;H^0(\omega\Pi_l))} \sup_{[0,T]\times\omega\Pi_l} |(I - p_{l,\omega})v_{l,\omega}| \\ &\le C\|u\|^2_{L^2(0,T;H^0(\omega\Pi_l))}\|(I - p_{l,\omega})v_{l,\omega}\|_{C(0,T;H^s(\omega\Pi_l))}, \end{aligned} \tag{6.18}$$

with $s > 3/2$. Clearly, for periodic vector fields with cube of periods $\omega\Pi_l$

$$\|u\|^2_{L^2(0,T;H^0(\omega\Pi_l))} \le (1+3l^2)^{-r}\|u\|^2_{L^2(0,T;\mathcal{H}^0(r))}, \quad \text{for} \quad r < 0. \tag{6.19}$$

Decomposition in Fourier series yields:

$$\begin{aligned} \|(I - p_{l,\omega})v_{l,\omega}\|_{C(0,T;H^s(\omega\Pi_l))} &= \sup_{t\in[0,T]} \sum_{m\in\frac{\pi}{l}\omega\mathbb{Z}^3, m>l} |\hat{v}(m)|^2(1+|m|^2)^s \\ &\le (1+3l^2)^{r-1} \sup_{t\in[0,T]} \|v_{l,\omega}(t,\cdot)\|^2_{H^{s-r+1}(\omega\Pi_l)} \\ &\le (1+3l^2)^{r-1}\|v\|_{C(0,T;H^{s-r+1}(B))}, \end{aligned} \tag{6.20}$$

where the last inequality holds by (6.12). Now (6.16) follows from (6.18), (6.19), (6.20). $\square$

6.3. **The main result.** The goal of this subsection is to prove

Theorem 6.2. *Let $\widehat{Q}$ be a weak limit of isotropic Galerkin approximations $\widehat{P_l}$ of statistical solution as $l \to \infty$. Then $\widehat{Q}$ is supported on weak solutions of Navier-Stokes equations, i.e. on the set $\mathcal{G}_{NS}$ of Definition 3.1*

Proof. Let the function $\varphi_R(\lambda) \in C^\infty(\mathbb{R}_+)$ satisfy

$$\varphi_R(\lambda) = \begin{cases} 1, & \lambda \le R \\ 0, & \lambda \ge R+1. \end{cases} \tag{6.21}$$

For v, ψ in G^∞ (see (6.10)), construct the families $\widehat{v}_{l,\cdot} = v_{l\omega}(t,x)$, $\widehat{\psi}_{l,\cdot} = \psi_{l\omega}(t,x)$ as explained immediately after relation (6.12). Analogously to (6.15), define the following functionals on $C^1(0,T;\widehat{\mathcal{M}_l})$: If $u = \omega\tilde{u}$ for $\omega \in O(3)$ and $\tilde{u} \in C^1(0,T;\mathcal{M}_l)$, then

$$u \to [u, \widehat{\psi}_{l,\cdot}]_{l,\cdot} = [u, \psi_{l,\omega}]_{l,\omega}, \tag{6.22}$$

and

$$u \to L_l(u, \widehat{v}_l) = [u, \partial_t v_{l,\omega} + \Delta v_{l,\omega}]_{l,\omega} + \sum_{j=1}^{3} [p_{l,\omega}\pi_{l,\omega}(u_j u), \frac{\partial v_{l,\omega}}{\partial x_j}]_{l,\omega}. \tag{6.23}$$

By the definition (4.53) of the isotropic Galerkin approximations $\widehat{P}_l$ of a statistical solution and taking into account that $\widehat{S}_l v$ is the solution operator of the Galerkin equations (6.9), the measure $\widehat{P}_l$ is supported by solutions of (6.9) belonging to $C^1(0,T;\widehat{\mathcal{M}_l})$. Therefore, comparing (6.9) to (6.23),

$$\int_{C^1(0,T;\widehat{\mathcal{M}_l})} \varphi_R(\|u\|_{L^2(0,T;\mathcal{H}^0(r))}) L_l(u, \widehat{v}_l) e^{i[u,\widehat{\psi}_l]_{l,\cdot}} \widehat{P}_l(du) = 0 \tag{6.24}$$

for each $v, \psi \in G^\infty$ and $l \ge l_0$, with l_0 defined by (6.12), and with the ball B now containing both the support of v and ψ.

Let

$$[u,v] = \int_0^T \int_{\mathbb{R}^3} u(t,x) \cdot v(t,x)\ dxdt. \tag{6.25}$$

Since v and ψ satisfy (6.11), then

$$[u, \widehat{v}_{l,\cdot}]_{l,\cdot} = [u,v], \quad [u, \widehat{\psi}_{l,\cdot}]_{l,\cdot} = [u,\psi] \quad \forall\ u \in C^1(0,T;\widehat{\mathcal{M}_l}), \tag{6.26}$$

where $[u, \widehat{\psi}_{l,\cdot}]_{l,\cdot}$, $[u,\psi]$ are defined by (6.22), (6.25) respectively. Therefore,

$$L_l(u, \widehat{v}_{l,\cdot}) = L(u,v) - \sum_{j=1}^{3} F_{j,\frac{\partial v}{\partial x_j}}(u) \quad \forall\ u \in C^1(0,T;\widehat{\mathcal{M}_l}), \tag{6.27}$$

where $L_l(u, \widehat{v}_{l,\cdot})$, $L(u,v)$, and $F_{j,v}(u)$ are defined by (6.23), (3.1), and (6.15) respectively.

Substitution of (6.26), (6.27) into (6.24) yields:

$$\begin{aligned} &\int \varphi_R(\|u\|_{L^2(0,T;\mathcal{H}^0(r))}) L(u,v) e^{i[u,\psi]} \widehat{P}_l(du) \\ &\qquad - \sum_{j=1}^{3} \int \varphi_R(\|u\|_{L^2(0,T;\mathcal{H}^0(r))}) F_{j,\frac{\partial v}{\partial x_j}}(u) e^{i[u,\psi]} \widehat{P}_l(du) = 0. \end{aligned} \tag{6.28}$$

Since the integrand of the first integral in (6.28) can be extended to a bounded continuous functional on $L^2(0,T;\mathcal{H}^0(r))$, the of weak convergence $\widehat{P}_l \to \widehat{Q}$ on this space gives:

$$\int \varphi_R(\|u\|_{L^2(0,T;\mathcal{H}^0(r))})L(u,v)e^{i[u,\psi]}\widehat{P}_l(du) \to \int \varphi_R(\|u\|_{L^2(0,T;\mathcal{H}^0(r))})L(u,v)e^{i[u,\psi]}\widehat{Q}(du), \tag{6.29}$$

as $l \to \infty$. Moreover, by Lemma 6.1, for $j = 1, 2, 3$,

$$\begin{aligned} &\left| \int \varphi_R(\|u\|_{L^2(0,T;\mathcal{H}^0(r))})F_{j,\frac{\partial v}{\partial x_j}}(u)e^{i[u,\psi]}\widehat{P}_l(du) \right| \\ &\leq C\frac{\|\nabla v\|_{L^2(0,T;H^{s-r+1}(B))}}{(1+3l^2)} \int \varphi_R(\|u\|_{L^2(0,T;\mathcal{H}^0(r))})\|u\|_{L^2(0,T;\mathcal{H}^0(r))}\widehat{P}_l(du), \end{aligned} \tag{6.30}$$

therefore

$$\left| \sum_{j=1}^{3} \int \varphi_R(\|u\|_{L^2(0,T;\mathcal{H}^0(r))})F_{j,\frac{\partial v}{\partial x_j}}(u)e^{i[u,\psi]}\widehat{P}_l(du) \right| \to 0, \ l \to \infty. \tag{6.31}$$

Equations (6.28), (6.29), (6.31) imply

$$\int \varphi_R(\|u\|_{L^2(0,T;\mathcal{H}^0(r))})L(u,v)e^{i[u,\psi]}\widehat{Q}(du) = 0 \quad \forall\ v, \psi \in G^\infty. \tag{6.32}$$

With this, repeat the arguments from [VF2], p. 243-244, to derive from (6.32) the assertion of the theorem. □

As a final result obtain the following theorem:

Theorem 6.3. *$\widehat{Q}$ is a homogeneous and isotropic statistical solution of the Navier-Stokes equations with initial condition $\widehat{\mu}$.*

7. Comparison of $\widehat{P}$ and $\widehat{Q}$

To conclude, observe that the isotropic solutions of section 3 and the isotropic solutions constructed in sections 4, 5, and 6 coincide in the following sense:

Let P be a homogeneous statistical solution with initial condition $\widehat{\mu}$ according to Theorem 3.4, and let $\widehat{P}$ its isotropic average according to (3.37). (Whether $\widehat{\mu}$ is only homogeneous or not is irrelevant to the point about to be made.) The construction of P is via Galerkin approximations, therefore P is the weak limit of a sequence of homogeneous P_l's. Construct the corresponding $\widehat{P}_l$ for each l according to (4.53).

Probability measures on metric spaces are determined by their integrals on bounded and continuous functions, see [B], page 8. For the probability measures $\widehat{P}$ and $\widehat{Q}$, calculate for each

f continuous and bounded on $L^2(0,T;\mathcal{H}^0(r))$:

$$\begin{aligned}
\int_{L^2(0,T;\mathcal{H}^0(r))} f(u)\ \widehat{P}(du) &= \int_{O(3)} \int_{L^2(0,T;\mathcal{H}^0(r))} f(R_\omega u)\ P(du)\ d\omega \qquad \text{(by (3.37))} \\
&= \lim_{l\to\infty} \int_{O(3)} \int_{L^2(0,T;\mathcal{H}^0(r))} f(R_\omega u)\ P_l(du)\ d\omega \\
&\qquad\qquad \text{(by weak and dominated convergence)} \qquad (7.1) \\
&= \lim_{l\to\infty} \int_{L^2(0,T;\mathcal{H}^0(r))} f(u)\ \widehat{P}_l(du) \qquad \text{(by Lemma 4.13)} \\
&= \int f(u)\ \widehat{Q}(du).
\end{aligned}$$

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