M P E J

Mathematical Physics Electronic Journal

ISSN 1086-6655
Volume 12, 2006

Paper 5
Received: May 13, 2005, Revised: Sep 11, 2006, Accepted: Oct 16, 2006
Editor: C.E. Wayne

A Nonlinear Heat Equation with Temperature-Dependent Parameters

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Abstract

A nonlinear partial differential equation of the following form is considered:

$$u' - \operatorname{div}\Big(a(u)\nabla u\Big) + b(u)\ |\nabla u|^2 = 0,$$

which arises from the heat conduction problems with strong temperature-dependent material parameters, such as mass density, specific heat and heat conductivity. Existence, uniqueness and asymptotic behavior of initial boundary value problems under appropriate assumptions on the material parameters are established for one-dimensional case. Existence and asymptotic behavior for two-dimensional case are also proved.

Keywords: Heat equation, existence, uniqueness, asymptotic behavior.
AMS Subject Classification: 35K55, 35Q80, 35B40, 35B45.

1 Introduction

Metallic materials present a complex behavior during heat treatment processes involving phase changes. In a certain temperature range, change of temperature induces a

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phase transformation of metallic structure, which alters physical properties of the material. Indeed, measurements of specific heat and conductivity show a strong temperature dependence during processes such as quenching of steel.

Several mathematical models, as solid mixtures and thermal-mechanical coupling, for problems of heat conduction in metallic materials have been proposed, among them [6, 8, 13]. In this paper, we take a simpler approach without thermal-mechanical coupling of deformations, by considering the nonlinear temperature dependence of thermal parameters as the sole effect due to those complex behaviors.

The above discussion of phase transformation of metallic materials serves only as a motivation for the strong temperature-dependence of material properties. In general, thermal properties of materials do depend on the temperature, and the present formulation of heat conduction problem may be served as a mathematical model when the temperature-dependence of material parameters becomes important.

More specifically, in contrast to the usual linear heat equation with constant coefficients, we are interested in a nonlinear heat equation with temperature-dependent material parameters.

Existence, uniqueness and asymptotic behavior of initial boundary value problems under appropriate assumptions on the material parameters are established. For existence, we use particular compactness arguments for both $n = 1, 2$ cases succinctly. The tool of the proof is the compactness results of Lions, but to apply this theorem we need a series of bounds which we establish with a number of standard analytic techniques. For uniqueness only the case $n = 1$ can be proved. The case of $n = 2$ remains open. The asymptotic behavior in both $n = 1, 2$ is proved employing the arguments of Lions [9] and Prodi [12].

2 A nonlinear heat equation

Let $\theta(x, t)$ be the temperature field, then we can write the conservation of energy in the following form:

$$\rho\,\varepsilon' + \operatorname{div} q = 0, \tag{1}$$

where q is the heat flux, ρ the mass density, ε the internal energy density and prime denotes the time derivative.

The mass density $\rho = \rho(\theta) > 0$ may depend on temperature due to possible change of material structure, while the heat flux q is assumed to be given by the Fourier law with temperature-dependent heat conductivity,

$$q = -\kappa \nabla\theta, \qquad \kappa = \kappa(\theta) > 0. \tag{2}$$

The internal energy density $\varepsilon = \varepsilon(\theta)$ generally depends on the temperature, and the specific heat c is assume to be positive defined by

$$c(\theta) = \frac{\partial \varepsilon}{\partial \theta} > 0, \tag{3}$$

which is not necessarily a constant.

By the assumption (3), we can reformulate the equation (1) in terms of the energy ε instead of the temperature θ. Rewriting Fourier law as

$$q = -\kappa(\theta)\nabla\theta = -\alpha(\varepsilon)\nabla\varepsilon, \tag{4}$$

and observing that

$$\nabla\varepsilon = \frac{\partial\varepsilon}{\partial\theta}\nabla\theta = c(\theta)\nabla\theta,$$

we have $c(\theta)\alpha(\varepsilon) = \kappa(\theta)$, and hence

$$\alpha = \alpha(\varepsilon) > 0.$$

Now let u be defined as $u = \varepsilon(\theta)$, then the equation (1) becomes

$$\rho(u)\, u' - \mathrm{div}(\alpha(u)\nabla u) = 0.$$

Since $\rho(u) > 0$, dividing the equation by ρ, and using the relation,

$$\frac{1}{\rho(u)}\,\mathrm{div}\,(\alpha(u)\nabla u) = \mathrm{div}\,\Big(\frac{\alpha(u)\nabla u}{\rho(u)}\Big) - \Big(\nabla\frac{1}{\rho(u)}\Big)\cdot\Big(\alpha(u)\nabla u\Big),$$

we obtain

$$u' - \mathrm{div}\,\Big(\frac{\alpha(u)\nabla u}{\rho(u)}\Big) + \Big(\nabla\frac{1}{\rho(u)}\Big)\cdot\Big(\alpha(u)\nabla u\Big) = 0. \tag{5}$$

Since $c = \dfrac{\partial\varepsilon}{\partial\theta} = \dfrac{du}{d\theta}$,

$$\Big(\nabla\frac{1}{\rho(u)}\Big) = -\frac{1}{\rho(u)^2}\,\frac{d\rho(u)}{du}\,\nabla u = -\frac{1}{\rho(u)^2}\,\frac{d\rho}{d\theta}\,\frac{d\theta}{du}\,\nabla u = -\frac{1}{\rho(u)^2}\,\frac{d\rho}{d\theta}\,\frac{1}{c}\,\nabla u.$$

Substituting into equation (5) we obtain

$$u' - \mathrm{div}\,\Big(\frac{\alpha(u)\nabla u}{\rho(u)}\Big) - \Big(\frac{1}{\rho(u)^2}\,\frac{d\rho}{d\theta}\,\frac{1}{c}\,\nabla u\Big)\cdot\Big(\alpha(u)\nabla u\Big) = 0,$$

which is equivalent to

$$u' - \mathrm{div}\,(a(u)\nabla u) + b(u)\,|\nabla u|^2 = 0, \tag{6}$$

where

$$a(u) = \frac{\alpha(u)}{\rho(u)} = \frac{\alpha(u)c(u)}{\rho(u)c(u)} = \frac{k(u)}{\rho(u)c(u)} > 0 \tag{7}$$

and

$$b(u) = -\frac{\alpha(u)}{c\,\rho(u)^2}\,\frac{d\rho}{d\theta} > 0. \tag{8}$$

The positiveness of $a(u)$ and $b(u)$ is the consequence of thermodynamic considerations (see [10]), and reasonable physical experiences: the specific heat $c > 0$, the thermal conductivity $\kappa > 0$, the mass density $\rho > 0$, and the thermal expansion $d\rho/d\theta < 0$. In this paper we shall formulate the problem based on the nonlinear heat equation (6).

2.1 Formulation of the Problem

Let Ω be a bounded open set of $I\!R^n$, $n = 1, 2$, with C^1 boundary and Q be the cylinder $\Omega \times (0, T)$ of $I\!R^{n+1}$ for $T > 0$, whose lateral boundary we represent by $\Sigma = \Gamma \times (0, T)$. We shall consider the following non-linear problem:

$$\begin{cases} u' - \operatorname{div}\Big(a(u)\nabla u\Big) + b(u)\,|\nabla u|^2 = 0 \quad \text{in } Q, \\ u = 0 \quad \text{on } \Sigma, \\ u(x,0) = u_0(x) \quad \text{in } \Omega. \end{cases} \tag{9}$$

Mathematical models of semi-linear and nonlinear parabolic equations under Dirichlet or Neumann boundary conditions have been considered in several papers, among them, let us mention ([1, 2, 3]) and ([4, 5, 11, 14]), respectively.

Feireisl, Petzeltová and Simondon [7] prove that with non-negative initial data, the function $a(u) \equiv 1$ and $g(u, \nabla u) \leq h(u)(1 + |\nabla u|^2)$, instead of the non-linear term $b(u)\,|\nabla u|^2$ in $(9)_1$, there exists an admissible solution positive in some interval $[0, T_{\max})$ and if $T_{\max} < \infty$ then

$$\lim_{t \to T_{\max}} \|u(t, .)\|_\infty = \infty.$$

For Problem (9) we will prove global existence, uniqueness and asymptotic behavior for the one-dimensional case ($n = 1$) and existence and asymptotic behavior for the two-dimensional case ($n = 2$), for small enough initial data.

3 Existence and Uniqueness: One-dimensional Case

In this section we investigate the existence, uniqueness and asymptotic behavior of solutions for the case $n = 1$ of Problem (9).

Let $((\cdot,\cdot))$, $\|\cdot\|$ and $(\cdot,\cdot)$, $|\cdot|$ be respectively the scalar product and the norms in $H_0^1(\Omega)$ and $L^2(\Omega)$. Thus, when we write $|u| = |u(t)|$, $\|u\| = \|u(t)\|$ it will mean the $L^2(\Omega)$, $H_0^1(\Omega)$ norm of $u(x,t)$ respectively.

To prove the existence and uniqueness of solutions for the one-dimensional case, we need the following hypotheses:

H1: $a(u)$ and $b(u)$ belongs to $C^1(I\!R)$ and there are positive constants a_0, a_1 such that,

$$a_0 \leq a(u) \leq a_1 \quad \text{and} \quad b(u)u \geq 0.$$

H2: There is positive constant $M > 0$ such that

$$\max_{s \in I\!R} \left\{ \Big|\frac{da}{du}(s)\Big|;\ \Big|\frac{db}{du}(s)\Big| \right\} \leq M.$$

H3: $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$.

Remark 1. From hypothesis (H1), we have that $b(u)u \geq 0,\ \forall u \in I\!R$. Thus from hypothesis (H2), $|b(u)| \leq M|u|$.

Theorem 1 *Under the hypotheses* (H1), (H2) *and* (H3), *there exist a positive constant* ε_0 *such that, if* u_0 *satisfies* $(|u_0| + \|u_0\| + |\Delta u_0|) < \varepsilon_0$ *then the problem* (9) *admits a unique solution* $u : Q \to I\!R$, *satisfying the following conditions:*

i. $u \in L^2(0, T; H_0^1(\Omega) \cap H^2(\Omega))$,

ii. $u' \in L^2(0, T; H_0^1(\Omega))$,

iii. $u' - \operatorname{div}(a(u)\nabla u) + b(u)\, |\nabla u|^2 = 0, \quad$ *in* $L^2(Q)$,

iv. $u(0) = u_0$.

Proof. To prove the theorem, we employ Galerkin method with the Hilbertian basis from $H_0^1(\Omega)$, given by the eigenvectors (w_j) of the spectral problem: $((w_j, v)) = \lambda_j(w_j, v)$ for all $v \in V = H_0^1(\Omega) \cap H^2(\Omega)$ and $j = 1, 2, \cdots$. We represent by V_m the subspace of V generated by vectors $\{w_1, w_2, ..., w_m\}$. We propose the following approximate problem: Determine $u_m \in V_m$, so that

$$\begin{cases} (u_m', v) + \Big(a(u_m)\nabla u_m, \nabla v\Big) + \Big(b(u_m)\, |\nabla u_m|^2, v\Big) = 0 \quad \forall v \in V_m, \\ u_m(0) = u_{0m} \to u_0 \quad \text{in} \quad H_0^1(\Omega) \cap H^2(\Omega). \end{cases} \tag{10}$$

We want to get strong convergence in $H_0^1(\Omega)$ and $L^2(\Omega)$ of u_m and ∇u_m, respectively as given later in (34). To do so, we will use the compactness results of Lions applied to a certain sequence of Galerkin approximations. First we need to establish that they converge weakly in some particular Sobolev spaces, and we do this through energy estimates.

Existence

The system of ordinary differential equations (10) has a local solution $u_m = u_m(x, t)$ in the interval $(0, T_m)$. The estimates that follow permit to extend the solution $u_m(x, t)$ to interval $[0, T[$ for all $T > 0$ and to take the limit in (10).

Estimate I: Taking $v = u_m(t)$ in equation $(10)_1$ and integrating over $(0, T)$, we obtain

$$\frac{1}{2}|u_m|^2 + a_0 \int_0^T \|u_m\|^2 + \int_0^T \int_\Omega b(u_m)u_m|\nabla u_m|^2 < \frac{1}{2}|u_0|^2, \tag{11}$$

where we have used hypothesis (H1). Taking $\hat{a}_0 = \min\{a_0, \frac{1}{2}\} > 0$, we obtain

$$|u_m|^2 + \int_0^T \|u_m\|^2 + \int_0^T \int_\Omega b(u_m)u_m|\nabla u_m|^2 < \frac{1}{2\hat{a}_0}|u_0|^2. \tag{12}$$

Thus, applying the Gronwall's inequality in (12) yields

$$(u_m) \quad \text{is bounded in} \quad L^\infty\left(0,T;L^2(\Omega)\right) \cap L^2\left(0,T;H_0^1(\Omega)\right). \tag{13}$$

Estimate II: Taking $v = u_m'$ in equation $(10)_1$ and integrating over Ω, we obtain

$$\begin{aligned} |u_m'|^2 + \frac{1}{2}\frac{d}{dt}\int_\Omega a(u_m)|\nabla u_m|^2 &= \frac{1}{2}\int_\Omega \frac{da}{du}(u_m)u_m'|\nabla u_m|^2 \\ &- \int_\Omega b(u_m)u_m'|\nabla u_m|^2. \end{aligned} \tag{14}$$

On the other hand, from hypothesis (H2), we have the following inequality,

$$\frac{1}{2}\int_\Omega \frac{da}{du}(u_m)u_m'|\nabla u_m|^2 \le \frac{1}{2}MC_0\|u_m'\|\,\|u_m\|^2, \tag{15}$$

and since $|\cdot|_{L^\infty(\Omega)} \le C_0\|\cdot\|$ and $|b(u_m)| \le M|u_m|$, we obtain

$$\left| -\int_\Omega b(u_m)u_m'|\nabla u_m|^2\right| \le MC_0^2\|u_m'\|\,\|u_m\|^3, \tag{16}$$

where $C_0 = C_0(\Omega)$ is a constant depending on Ω.

Substituting (15) and (16) into the right hand side of (14) we get

$$\begin{aligned} |u_m'|^2 + \frac{1}{2}\frac{d}{dt}\int_\Omega a(u_m)|\nabla u_m|^2 &\le \frac{1}{2}MC_0\|u_m'\|\,\|u_m\|^2 + MC_0^2\|u_m'\|\,\|u_m\|^3 \\ &\le \frac{1}{4}(MC_0)^2\|u_m'\|^2\,\|u_m\|^2 + \frac{1}{2}\|u_m\|^2 + (MC_0^2)^2\|u_m'\|^2\,\|u_m\|^4. \end{aligned} \tag{17}$$

Now taking the derivative of the equation $(10)_1$ with respect to t and making $v = u_m'$, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|u_m'|^2 &+ \int_\Omega a(u_m)|\nabla u_m'|^2 = -\int_\Omega \frac{db}{du}(u_m)|u_m'|^2|\nabla u_m|^2 \\ &- \int_\Omega \frac{da}{du}(u_m)\; u_m'|\nabla u_m|^2 - 2\int_\Omega b(u_m)\nabla u_m\nabla u_m' u_m' \\ &\le MC_0^2(3+\frac{M}{2})\|u_m'\|^2\|u_m\|^2 + \frac{1}{2}\|u_m\|^2. \end{aligned} \tag{18}$$

From the inequality (17) and (18), we have

$$\begin{aligned} &\frac{d}{dt}\left\{\frac{1}{2}|u_m'|^2 + \frac{1}{2}\int_\Omega a(u_m)|\nabla u_m|^2\right\} + |u_m'|^2 + \frac{a_0}{2}\|u_m'\|^2 + \\ &\|u_m'\|^2\left\{\frac{a_0}{2} - \alpha_0\|u_m\|^4 - \alpha_1\|u_m\|^2\right\} \le \|u_m\|^2, \end{aligned} \tag{19}$$

where we have defined $\alpha_0 = (MC_0^2)^2$ and $\alpha_1 = MC_0(\frac{3}{4}MC_0 + 3C_0)$.

Now, under the condition that the following inequality,

$$\alpha_0\|u_m\|^4 + \alpha_1\|u_m\|^2 < \frac{a_0}{4}, \qquad \forall\, t \geq 0, \tag{20}$$

be valid, the coefficients of the term $\|u_m'\|$ in the relation (19) is positive and we can integrate it with respect to t,

$$|u_m'|^2 + \int_\Omega a(u_m)|\nabla u_m|^2 + 2\int_0^t |u_m'|^2 + \frac{3a_0}{2}\int_0^t \|u_m'\|^2 \leq C. \tag{21}$$

Therefore, applying the Gronwall's inequality in (21), we obtain the following estimate:

$$(u_m') \quad \text{is bounded in} \quad L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)). \tag{22}$$

Now we want to prove that the inequality (20) is valid if the initial data are sufficiently small. In other words, there is some $\varepsilon_0 > 0$ such that $(|u_0| + \|u_0\| + |\Delta u_0|) < \varepsilon_0$ and the following condition holds,

$$\frac{\alpha_0}{a_0^2}\left\{S_0 + a_1\|u_0\|^2 + \frac{1}{a_0}|u_0|^2\right\}^2 + \frac{\alpha_1}{a_0}\left\{S_0 + a_1\|u_0\|^2 + |u_0|^2\right\} < \frac{a_0}{4} \tag{23}$$

and

$$\alpha_0\|u_0\|^4 + \alpha_1\|u_0\|^2 < \frac{a_0}{4}, \tag{24}$$

where we have denoted $S_0 = \Big(M(\|u_0\| + |\Delta u_0|^2 + \|u_0\||\Delta u_0|^2\,) + a_1|\Delta u_0|\,\Big)^2$.

We shall prove this by contradiction. Suppose that (20) is false, then there is a t^* such that

$$\alpha_0\|u_m(t)\|^4 + \alpha_1\|u_m(t)\|^2 < \frac{a_0}{4} \quad \text{if} \quad 0 < t < t^* \tag{25}$$

and

$$\alpha_0\|u_m(t^*)\|^4 + \alpha_1\|u_m(t^*)\|^2 = \frac{a_0}{4}. \tag{26}$$

Integrating (19) from 0 to t^*, we obtain

$$\begin{aligned}\frac{1}{2}|u_m'(t^*)|^2 + \int_\Omega a(u_m)|\nabla u_m(t^*)|^2 &\leq \frac{1}{2}|u_m'(0)|^2 + \int_\Omega a(u_m(0))|\nabla u_m(0)|^2 \\ + \int_0^{t^*}\|u_m\|^2 &\leq \Big(M(\,\|u_0\| + |\Delta u_0|^2 + \|u_0\|\;|\Delta u_0|^2\,) + a_1|\Delta u_0|^2\,\Big)^2 \\ + a_1\|u_0\|^2 + \frac{1}{a_0}|u_0|^2,\end{aligned} \tag{27}$$

and consequently,

$$\|u_m(t^*)\|^2 \leq \frac{1}{a_0}S_0 + \frac{a_1}{a_0}\|u_0\|^2 + \frac{1}{a_0^2}|u_0|^2. \tag{28}$$

Using (23) and (24) we obtain

$$
\begin{aligned}
\alpha_0\|u_m(t^*)\|^4 \quad &+\alpha_1\|u_m(t^*)\|^2 \leq \frac{\alpha_0}{a_0^2}\left\{S_0+a_1\|u_0\|^2+\frac{1}{a_0}|u_0|^2\right\}^2 \\
&+\alpha_1\left\{\frac{1}{a_0}S_0+\frac{a_1}{a_0}\|u_0\|^2+\frac{1}{a_0}|u_0|^2\right\}<\frac{a_0}{4},
\end{aligned} \tag{29}
$$

hence, comparing with (26), we have a contradiction.

Estimate III: Taking $v=-\Delta u_m(t)$ in the equation $(10)_1$, and using hypothesis H1, we obtain

$$
\frac{d}{dt}\|u_m\|^2+\int_\Omega a(u_m)|\Delta u_m|^2 \leq a_1\|u_m\|\ |\Delta u_m|+b_1|\Delta u_m|^2\ \|u_m\|,
$$

where we have used the following inequality,

$$
\begin{aligned}
\int_\Omega b(u_m)|\nabla u_m|^2\ |\Delta u_m| \quad &\leq \hat{b}_1|\nabla u_m|_{L^\infty}\int_\Omega |\nabla u_m|\ |\Delta u_m| \\
&\leq \hat{b}_1|\nabla u_m|_{H^1(\Omega)}\ \|u_m\|\ |\Delta u_m| \\
&\leq \hat{b}_1|\Delta u_m|\ \|u_m\|\ |\Delta u_m| = \hat{b}_1\|u_m\|\ |\Delta u_m|^2.
\end{aligned}
$$

In this expression we have denoted $\hat{b}_1=\sup|b(s)|$, for all $s\in[-\frac{a_0}{4\alpha_1},\frac{a_0}{4\alpha_1}]$. Note that, from (20), we have $|u_m(t)|\leq\|u_m(t)\|<a_0/4\alpha_1$ and $b_1=\hat{b}_1+M$.

Using hypothesis (H1), we obtain

$$
\frac{d}{dt}\|u_m\|^2+\frac{a_0}{2}|\Delta u_m|^2 \leq a_1\|u_m\|\ |\Delta u_m|+b_1|\Delta u_m|^2\ \|u_m\|,
$$

or equivalently,

$$
\frac{d}{dt}\|u_m\|^2+\left(\frac{a_0}{2}-b_1\|u_m\|\right)|\Delta u_m|^2 \leq a_1\|u_m\|\ |\Delta u_m|.
$$

Similar to the choice of ε_0 for the conditions (23) and (24), ε_0 will be further restricted to guarantee that the following condition also holds,

$$
\frac{b_1}{a_0}\Big(S_0+a_1\|u_0\|^2+\frac{1}{2a_0}|u_0|^2\Big)^{1/2}<\frac{a_0}{4} \tag{30}
$$

and from (21) we obtain $\frac{a_0}{4}\leq\left(\frac{a_0}{2}-b_1\|u_m\|\right)$, it implies that

$$
\frac{d}{dt}\|u_m\|^2+\frac{a_0}{4}|\Delta u_m|^2 \leq a_1\|u_m\|\ |\Delta u_m| \leq \frac{a_0}{8}|\Delta u_m|^2+C\|u_m\|^2. \tag{31}
$$

Now, integrating from 0 to t, we obtain the estimate

$$
\|u_m\|^2+\int_0^T|\Delta u_m|^2 \leq \widehat{C}.
$$

Hence, we have

$$\begin{aligned} (u_m) &\quad \text{is bounded in} \quad L^\infty\left(0,T;H_0^1(\Omega)\right), \\ (u_m) &\quad \text{is bounded in} \quad L^2\left(0,T;H_0^1(\Omega)\cap H^2(\Omega)\right). \end{aligned} \tag{32}$$

Limit of the approximate solutions

From the estimates (13), (22) and (32), we can take the limit of the nonlinear system (10). In fact, there exists a subsequence of $(u_m)_{m\in N}$, which we denote as the original sequence, such that

$$\begin{aligned} u_m' &\longrightarrow u' \quad \text{weak star} && \text{in } L^\infty\left(0,T;L^2(\Omega)\right), \\ u_m' &\longrightarrow u' \quad \text{weak} && \text{in } L^2\left(0,T;H_0^1(\Omega)\right), \\ u_m &\longrightarrow u \quad \text{weak star} && \text{in } L^\infty\left(0,T;H_0^1(\Omega)\right), \\ u_m &\longrightarrow u \quad \text{weak} && \text{in } L^2\left(0,T;H_0^1(\Omega)\cap H^2(\Omega)\right). \end{aligned} \tag{33}$$

Thus, by compact injection of $H_0^1(\Omega\times(0,T))$ into $L^2(\Omega\times(0,T))$ it follows by compactness arguments of Aubin-Lions [9], we can extract a subsequence of $(u_m)_{m\in N}$, still represented by $(u_m)_{m\in N}$ such that

$$\begin{aligned} u_m &\longrightarrow u \quad \text{strong} \quad \text{in } L^2\left(0,T;H_0^1(\Omega)\right) \quad \text{and} \quad \text{a.e.} \quad \text{in } Q, \\ \nabla u_m &\longrightarrow \nabla u \quad \text{strong} \quad \text{in } L^2(Q) \quad \text{and} \quad \text{a.e.} \quad \text{in } Q. \end{aligned} \tag{34}$$

Let us analyze the nonlinear terms from the approximate system (10). From the first term, we know that

$$\int_\Omega |a(u_m)\nabla u_m|^2 \le a_1^2 \int_\Omega |\nabla u_m|^2 \le a_1^2 C, \tag{35}$$

and since $u_m \to u$ a.e. in Q and $a(x,.)$ is continuous, we get

$$a(u_m) \longrightarrow a(u) \quad \text{and} \quad \nabla u_m \longrightarrow \nabla u \quad \text{a.e. in } Q. \tag{36}$$

Hence, we also have

$$|a(u_m)\nabla u_m|^2 \longrightarrow |a(u)\nabla u|^2 \quad \text{a.e. in } Q. \tag{37}$$

From (35) and (37), and Lions' Lemma, we obtain

$$a(u_m)\nabla u_m \longrightarrow a(u)\nabla u \quad \text{weak in } L^2(Q). \tag{38}$$

From the second term, we know that

$$\begin{aligned} &\int_0^T \int_\Omega |b(u_m)|\nabla u_m|^2|^2 \le b_1^2 \int_0^T |\nabla u_m|^2_{L^\infty(\Omega)} \int_\Omega |\nabla u_m|^2 \\ &\le b_1^2 C_0 \int_0^T \|u_m\|^2 |\nabla u_m|^2_{H^1} \le b_1^2 C_0 \|u_m\|^2 \int_0^T |\Delta u_m|^2 \le C, \end{aligned} \tag{39}$$

where C_0 has been defined in (16).

By the same argument that leads to (37), we get

$$|b(u_m)|\nabla u_m|^2|^2 \longrightarrow |b(u)|\nabla u|^2|^2 \quad \text{a.e. in} \quad Q \tag{40}$$

Hence, from (39) and (40) we obtain the convergence,

$$b(u_m)|\nabla u_m|^2 \longrightarrow b(u)|\nabla u|^2 \quad \text{weak in } L^2(Q). \tag{41}$$

Taking into account (33), (38) and (41) into $(10)_1$, there exists a function $u(x,t)$ defined over $\Omega \times [0,T[$ with value in $I\!R$ satisfying (9). Moreover, from the convergence results obtained, we have that $u_m(0) = u_{0m} \to u_0$ in $H_0^1(\Omega) \cap H^2(\Omega)$ and the initial condition is well defined.

Hence, we conclude that equation (9) holds in the sense of $L^2(0,T;L^2(\Omega))$.

Uniqueness

Let $w(x,t) = u(x,t) - v(x,t)$, where $u(x,t)$ and $v(x,t)$ are solutions of Problem (9). Then we have

$$\begin{cases} w' - \operatorname{div}(a(u)\nabla w) - \operatorname{div}(a(u) - a(v))\nabla v \\ \qquad + b(u)\,(|\nabla u|^2 - |\nabla v|^2) + (b(u) - b(v))|\nabla v|^2 = 0 \quad \text{in } Q, \\ w = 0 \quad \text{on } \Sigma, \\ w(x,0) = 0 \quad \text{in } \Omega. \end{cases} \tag{42}$$

Multiplying by $w(t)$, integrating over Ω, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|w|^2 + \int_\Omega a(u)|\nabla w|^2 \leq \int_\Omega |\frac{da}{du}(\widehat{u})|\,|w|\,|\nabla v|\,|\nabla w| \\ &+ \int_\Omega |b(u)|\Big(|\nabla u| + |\nabla v|\Big)\,|\nabla w|\,|w| + \int_\Omega |\nabla v|^2|\,|\frac{\partial b}{\partial u}(\overline{u})|\,|w|^2 \\ &\leq M\int_\Omega |\nabla w|\,|w|\,|\nabla v| + C_0\int_\Omega \Big(|\nabla u| + |\nabla v|\Big)\,|\nabla w|\,|w| + M\int_\Omega |\nabla v|^2\,|w|^2 \\ &\leq M|\nabla v|_{L^\infty(\Omega)}\,\|w\|\,|w| + C_0\Big(|\nabla u|_{L^\infty(\Omega)} + |\nabla v|_{L^\infty(\Omega)}\Big)\,\|w\|\,|w| \\ &+M|\nabla v|^2_{L^\infty(\Omega)}\,|w|^2 \leq \frac{a_0}{2}\|w\|^2 + \overline{C}(|\Delta u|^2 + |\Delta v|^2)|w|^2, \end{aligned} \tag{43}$$

where we have used

(a) The generalized mean-value theorem, i.e.,
$$|a(u) - a(v)| = |\frac{da}{du}(\widehat{u})(u-v)| \leq |\frac{da}{du}(\widehat{u})|\,|w|, \qquad u \leq \widehat{u} \leq v,$$

(b) $\Big|b(u)\,(|\nabla u|^2 - |\nabla v|^2)\Big| \leq |b(u)|\,|\nabla w|\Big(|\nabla u| + |\nabla v|\Big),$

(c) $|\nabla v|_{L^\infty(\Omega)} \leq \|\nabla v\|_{H^1(\Omega)} \leq \|v\|_{H^2(\Omega)} \leq |\Delta v|_{L^2(\Omega)}$.

The last inequality is valid only for one-dimensional case.

Integrating (43) from 0 to t, we obtain

$$\frac{1}{2}|w|^2 + \frac{a_0}{2}\int_0^t \|w\|^2 \leq \frac{1}{2}|w(0)|^2 + \overline{C}\int_0^t (|\Delta u|^2 + |\Delta v|^2)|w|^2,$$

where $\overline{C}$ denotes a different positive constant. Since $w(0) = u_0 - v_0 = 0$, using the Gronwall's inequality, we obtain

$$|w|^2 + \int_0^t \|w\|^2 = 0,$$

which implies the uniqueness, $w(x,t) = u(x,t) - v(x,t) = 0$ and the theorem is proved. □

Asymptotic behavior

In the following we shall prove that the solution $u(x,t)$ of Problem (9) decays exponentially when time $t \to \infty$, using the same procedure developed in Lions [9] and Prodi [12]. Thus, we will initially show the exponential decay of the energy associated with the approximate solutions $u_m(x,t)$ of Problema (10).

Theorem 2 *Let $u(x,t)$ be the solution of Problem* (9)*. Then there exist positive constants $\widehat{s}_0$ and $\widehat{C} = \widehat{C}\{\|u_0\|, |\Delta u_0|\}$ such that*

$$\|u\|^2 + |u'|^2 \leq \widehat{C}\exp^{-\widehat{s}_0\, t}. \tag{44}$$

Proof. To prove the theorem, complementary estimates are needed.

Estimate I′: Consider the approximate system (10). Using the same argument as Estimate I, i.e, taking $v = u_m(t)$, we have

$$\frac{d}{dt}|u_m|^2 + \int_\Omega a(u_m)|\nabla u_m|^2 + \int_\Omega b(u_m)u_m|\nabla u_m|^2 = 0. \tag{45}$$

Integrating (46) from $(0,T)$, we obtain

$$\frac{1}{2}|u_m|^2 + a_0\int_0^T \|u_m\|^2 + \int_0^T\int_\Omega b(u_m)u_m|\nabla u_m|^2 < \frac{1}{2}|u_0|^2. \tag{46}$$

From H1 hypothesis, (45) and (46) we conclude

$$a_0\|u_m\|^2 \leq 2|u_m'|\,|u_m| \leq 2|u_m'|\,|u_0|. \tag{47}$$

Estimate II$'$: Taking derivative of the system (10) with respect to t and making $v = u'_m$, we obtain

$$\begin{aligned}\frac{d}{dt}|u'_m|^2 + a_0\|u'_m\|^2 &\leq M\Big(\int_\Omega |u'_m|^2|\nabla u_m| + \int_\Omega |u'_m|^2\ |\nabla u_m|^2\Big)\\ +2M\int_\Omega |\nabla u_m||\nabla u'_m||u'_m| &\leq C_1\Big(\|u'_m\|^2\|u_m\| + \|u'_m\|^2\|u_m\|^2\Big),\end{aligned} \tag{48}$$

where $C_1 = C_1(M,\Omega)$.

Hence,

$$\frac{d}{dt}|u'_m|^2 + \frac{a_0}{2}\|u'_m\|^2 + \|u'_m\|^2\left(\frac{a_0}{2} - C_1\|u_m\| - C_1\|u_m\|^2\right) \leq 0. \tag{49}$$

Using (47) then we can write the inequality (49) in the form,

$$\frac{d}{dt}|u'_m|^2 + \frac{a_0}{2}\|u'_m\|^2 + \|u'_m\|^2\Big(\frac{a_0}{2} - C_1\Big(\frac{2|u_0|}{a_0}\Big)^{1/2}|u'_m|^{1/2} - C_1\frac{2|u_0|}{a_0}|u'_m|\Big) \leq 0 \tag{50}$$

Let $v = u'_m(0)$ in (10). Then

$$|u'_m(0)|^2 \leq C\left(\|u_0\| + |\Delta u_0| + |\Delta u_0|^2 + |\Delta u_0|^3\right)|u'_m(0)|,$$

where $C = C(a_1, M, \Omega)$, and

$$|u'_m(0)|^2 \leq \Big(C\left(\|u_0\| + |\Delta u_0| + |\Delta u_0|^2 + |\Delta u_0|^3\right)\Big)^2. \tag{51}$$

We define the operator

$$\begin{aligned}J(u_0) = &\left(\frac{2|u_0|}{a_0}\right)^{1/2}\Big(C\left(\|u_0\| + |\Delta u_0| + |\Delta u_0|^2 + |\Delta u_0|^3\right)\Big)^{1/2} +\\ &\frac{2|u_0|}{a_0}\ C\left(\|u_0\| + |\Delta u_0| + |\Delta u_0|^2 + |\Delta u_0|^3\right).\end{aligned} \tag{52}$$

Then we have shown that

$$\left(\frac{2|u_0|}{a_0}\right)^{1/2}|u'_m(0)|^{1/2} + \frac{2|u_0|}{a_0}\ |u'_m(0)| \leq J(u_0). \tag{53}$$

If we choose C_1 a positive constant and u_0 small enough, so that

$$C_1 J(u_0) < \frac{a_0}{4}, \tag{54}$$

the following inequality holds,

$$C_1\left(\frac{2|u_0|}{a_0}\right)^{1/2}|u'_m|^{1/2} + C_1\frac{2|u_0|}{a_0}|u'_m| < \frac{a_0}{4}, \qquad \forall\, t \geq 0. \tag{55}$$

Indeed, we can prove this by contradiction. Suppose that there is a t^* such that

$$C_1\Big(\frac{2|u_0|}{a_0}\Big)^{1/2}|u_m'(t^*)|^{1/2}+C_1\frac{2|u_0|}{a_0}|u_m'(t^*)|=\frac{a_0}{4}. \tag{56}$$

Integrating (50) from 0 to t^*, we obtain $|u'(t^*)|^2\le |u'(0)|^2$.

From (54) and (55), we conclude that

$$\begin{aligned} &C_1\Big(\frac{2|u_0|}{a_0}\Big)^{1/2}|u_m'(t^*)|^{1/2}+C_1\frac{2|u_0|}{a_0}|u_m'(t^*)| \\ &\le C_1\Big(\frac{2|u_0|}{a_0}\Big)^{1/2}|u_m'(0)|^{1/2}+C_1\frac{2|u_0|}{a_0}|u_m'(0)|\le C_1J(u_0)<\frac{a_0}{4}. \end{aligned} \tag{57}$$

Therefore, we have a contradiction by (56).

From (50), (55) and using the Poincaré inequality, we obtain

$$\frac{d}{dt}|u_m'|^2+s_0|u_m'|^2\le 0 \tag{58}$$

where $s_0=(a_0c_0)/2$ and c_0 is a positive constant such that $\|\cdot\|_{H_0^1(\Omega)}\ge c_0|\cdot|_{L^2(\Omega)}$. Consequently, we have

$$\frac{d}{dt}\Big\{\exp^{s_0t}\ |u_m'|^2\Big\}\le 0 \tag{59}$$

and hence

$$\begin{aligned} |u_m'|^2\le |u_m'(0)|^2\ \exp^{-s_0t} &\le \Big(C\,(\|u_0\|+|\Delta u_0|+|\Delta u_0|^2+|\Delta u_0|^3)\Big)^2\exp^{-s_0t} \\ &\le \widetilde{C}\Big(\|u_0\|,|\Delta u_0|\Big)\ \exp^{-s_0t}, \end{aligned} \tag{60}$$

where we have used the inequality (51).

We also have from (47) that

$$\|u_m\|^2\le \frac{2}{a_0}|u_0|\ |u_m'|\le \frac{2}{a_0}\overline{C}\Big(\|u_0\|,|\Delta u_0|\Big)|u_0|\ \exp^{-s_0t/2}\,. \tag{61}$$

Defining $\widehat{s}_0=s_0/2$ and $\widehat{C}=\widetilde{C}+\dfrac{2}{a_0}\overline{C}$ then the result follows from (60), (61) inequality and of the Banach-Steinhaus theorem. □

4 Existence: Two-dimensional Case

In this section we investigate the existence and asymptotic behavior of solutions for the case $n=2$ of Problem (9). In order to prove these results we need the following hypotheses:

H1: Let $a(u)$ belongs to $C^2[0,\infty)$ and $b(u)$ belongs to $C^1[0,\infty)$ and there are positive constants a_0, a_1 such that,

$$a_0 \leq a(u) \leq a_1 \quad \text{and} \quad b(u)u \geq 0.$$

H2: There is a positive constant $M > 0$ such that

$$\max_{s\in I\!R} \left\{ \Big|\frac{da}{du}(s)\Big|;\ \Big|\frac{db}{du}(s)\Big|;\ \Big|\frac{d^2a}{du^2}(s)\Big| \right\} \leq M.$$

H3: $\dfrac{da}{du}(0) = 0$.
H4: $u_0 \in H_0^1(\Omega) \cap H^3(\Omega)$.

Theorem 3 *Under the hypotheses* (H1) - (H4)*, there exists a positive constant* ε_0 *such that, if* u_0 *satisfies* $(|\Delta u_0| + \|u_0\|_{H^3(\Omega)}) < \varepsilon_0$*, then Problem* (9) *admits a solution* $u : Q \to I\!R$*, satisfying the following conditions:*

i. $u \in L^\infty(0,T; H_0^1(\Omega) \cap H^2(\Omega))$,

ii. $u' \in L^2(0,T; H_0^1(\Omega) \cap H^2(\Omega))$,

iii. $u' - \operatorname{div}(a(u)\nabla u) + b(u)\, |\nabla u|^2 = 0 \quad in \quad L^2(Q)$,

iv. $u(0) = u_0$.

Proof. To prove the theorem, we employ Galerkin method with the Hilbertian basis from $H_0^1(\Omega)$, given by the eigenvectors (w_j) of the spectral problem: $((w_j, v)) = \lambda_j(w_j, v)$, for all $v \in V = H_0^1(\Omega) \cap H^2(\Omega))$ and $j = 1, 2, \cdots$. We represent by V_m the subspace of V generated by vectors $\{w_1, w_2, ..., w_m\}$. Let $u_m(x,t)$ be the local solution of the approximate problem (10). With similar arguments for the one-dimensional case, in order to extend the local solution to the interval $(0,T)$ independent of m, the following a priori estimates are needed.

Estimate I: Taking $v = u_m(t)$ in the equation $(10)_1$ and integrating over $(0,T)$, we obtain

$$\frac{1}{2}|u_m|^2 + a_0 \int_0^T \|u_m\|^2 + \int_0^T \int_\Omega b(u_m)u_m|\nabla u_m|^2 < \frac{1}{2}|u_0|^2. \tag{62}$$

Using the hypothesis (H1), we have

$$(u_m) \quad \text{is bounded in} \quad L^\infty\left(0,T; L^2(\Omega)\right) \cap L^2\left(0,T; H_0^1(\Omega)\right). \tag{63}$$

Estimate II: Taking $v = -\Delta u_m(t)$ in $(10)_1$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_m\|^2 + \int_\Omega a(u_m)|\Delta u_m|^2 = -\int_\Omega b(u_m)|\nabla u_m|^2\Delta u_m + \int_\Omega \frac{da}{du}|\nabla u_m|^2\Delta u_m \tag{64}$$

From hypothesis (H1) and using Sobolev embedding theorem we have the inequality

$$\begin{aligned}\Big|\int_\Omega b(u_m)|\nabla u_m|^2\Delta u_m\Big| &\leq M\int_\Omega |u_m||\nabla u_m|^2|\Delta u_m| \\ &\leq M|u_m|_{L^6(\Omega)}\,|\nabla u_m|_{L^6(\Omega)}\,|\nabla u_m|_{L^6(\Omega)}\,|\Delta u_m|_{L^2(\Omega)} \\ &\leq MC_1^3\|u_m\|\,\|u_m\|^2_{H^2(\Omega)}\,|\Delta u_m| \leq C_0|\Delta u_m|^4,\end{aligned}\tag{65}$$

where $C_0 = MC_1^3C_2C_3 > 0$, since $C_1,\ C_2,\ C_3$ are positive constants satisfying the following inequalities:

$$|u_m|_{L^6} \leq C_1\|u_m\|_{H^1},\quad \|u_m\|_{H^2} \leq C_2|\Delta u_m| \quad\text{and}\quad \|u_m\|_{H^1} \leq C_3|\Delta u_m|.\tag{66}$$

We also have that

$$\begin{aligned}\Big|\int_\Omega \frac{da}{du}|\nabla u_m|^2\Delta u_m\Big| &\leq M\int_\Omega |\nabla u_m|^2|\Delta u_m| \leq M|\nabla u_m|_{L^4(\Omega)}|\nabla u_m|_{L^4(\Omega)}|\Delta u_m| \\ &\leq MC_4^2\|u_m\|_{H^2(\Omega)}\,\|u_m\|_{H^2(\Omega)}\,|\Delta u_m| \leq MC_4^2C_2^2|\Delta u_m|^3 \\ &\leq C_5|\Delta u_m|^3 \leq \frac{C_5^2}{a_0}|\Delta u_m|^4 + \frac{a_0}{4}|\Delta u_m|^2.\end{aligned}\tag{67}$$

where $C_5 = MC_4^2C_2^2 > 0$ and C_4 is a positive constant satisfying the inequality: $|u_m|_{L^4(\Omega)} \leq C_4\|u_m\|_{H^1(\Omega)}$.

Substituting (65) and (67) in the equality (64) and using the hypothesis (H1), we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_m\|^2 + \frac{3a_0}{4}|\Delta u_m|^2 \leq \Big(C_0 + \frac{C_5^2}{a_0}\Big)|\Delta u_m|^4.\tag{68}$$

Consider now $v = -\Delta u_m'$ in $(10)_1$. Integrating in Ω, we obtain

$$\begin{aligned}\|u_m'\|^2 &+\frac{1}{2}\frac{d}{dt}\int_\Omega a(u_m)|\Delta u_m|^2 = -\int_\Omega b(u_m)|\nabla u_m|^2\Delta u_m' \\ &+\int_\Omega \frac{da}{du}|\nabla u_m|^2\Delta u_m' + \frac{1}{2}\int_\Omega \frac{da}{du}\,u_m'|\Delta u_m|^2.\end{aligned}\tag{69}$$

In the following, we shall get estimates for the first, the second and the third terms on the hand right side of (69).

For the first term, from hypothesis (H2) and using Sobolev embedding theorem, we obtain

$$\begin{aligned}\Big|\int_\Omega b(u_m)|\nabla u_m|^2\Delta u_m'\Big| &\leq M\int_\Omega |u_m||\nabla u_m|^2|\Delta u_m'| \\ &\leq M|u_m|_{L^6(\Omega)}|\nabla u_m|_{L^6(\Omega)}\,|\nabla u_m|_{L^6(\Omega)}\,|\Delta u_m'|_{L^2(\Omega)} \\ &\leq MC_1^3\|u_m\|\,\|u_m\|^2_{H^2(\Omega)}\,|\Delta u_m'| \leq C_0|\Delta u_m|^3\,|\Delta u_m'| \\ &\leq \frac{C_0^2}{2}|\Delta u_m|^4 + \frac{1}{2}|\Delta u_m|^2\,|\Delta u_m'|^2,\end{aligned}\tag{70}$$

where $C_0 = MC_1^3C_2C_3 > 0$, was defined in (65).

For the second term on the hand right side of (69), we have

$$\begin{aligned}\Big|\int_\Omega \frac{da}{du}|\nabla u_m|^2\Delta u_m'\Big| &\le M\int_\Omega |\nabla u_m|^2\ |\Delta u_m'| \\ &\le M|\nabla u_m|^2_{L^4(\Omega)}\ |\Delta u_m'|_{L^2(\Omega)} \\ &\le MC_4^2\|u_m\|^2_{H^2(\Omega)}\ |\Delta u_m'|_{L^2(\Omega)} \le C_5|\Delta u_m|^2\ |\Delta u_m'| \\ &\le \frac{2C_5^2}{a_0}|\Delta u_m|^4 + \frac{a_0}{8}|\Delta u_m'|^2,\end{aligned} \tag{71}$$

where C_4 and C_5 are positive constants defined in (66) and (67).

For the third term on the hand right side of (69), we have

$$\begin{aligned}\frac{1}{2}\Big|\int_\Omega \frac{da}{du}\ u_m'|\Delta u_m|^2\Big| &\le \frac{M}{2}\int_\Omega |u_m'||\Delta u_m|^2 \le \frac{M}{2}|u_m'|_{C^0(\Omega)}\ |\Delta u_m|^2 \\ &\le \frac{M}{2}C_6|u_m'|_{H^2(\Omega)}\ |\Delta u_m|^2 \le \frac{M}{2}C_6C_2|\Delta u_m'|\ |\Delta u_m|^2 \\ &\le C_7|\Delta u_m'|\ |\Delta u_m|^2 \le \frac{2C_7^2}{a_0}|\Delta u_m|^4 + \frac{a_0}{8}|, \Delta u_m'|^2,\end{aligned} \tag{72}$$

where C_6 is the positive constant of embedding between spaces, $H^2(\Omega) \hookrightarrow C^0(\Omega)$, i.e, $\|u\|_{C^0(\Omega)} \le C_6\|u\|_{H^2(\Omega)}$ and $C_7 = \frac{M}{2}C_6C_2 > 0$. Substituting (70), (71) and (72) in (69), we obtain

$$\begin{aligned}\|u_m'\|^2 + \frac{1}{2}\frac{d}{dt}\int_\Omega a(u_m)|\Delta u_m|^2 &\le \Big(\frac{C_0^2}{2} + \frac{2C_5^2 + 2C_7^2}{a_0}\Big)|\Delta u_m|^4 + \frac{a_0}{4}|\Delta u_m'|^2 \\ &+\frac{1}{2}|\Delta u_m|^2|\Delta u_m'|^2.\end{aligned} \tag{73}$$

On the other hand, taking derivative in the equation $(10)_1$ with respect to t and taking $v = -\Delta u_m'$ and integrating in Ω, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u_m'\|^2 &+ \int_\Omega a(u_m)|\Delta u_m'|^2 = -\int_\Omega \frac{da}{du}\ u_m'\Delta u_m\Delta u_m' \\ &-\int_\Omega \frac{d^2a}{du^2}\ u_m'|\nabla u_m|^2\Delta u_m' - 2\int_\Omega \frac{da}{du}\nabla u_m\nabla u_m'\Delta u_m' \\ &-\int_\Omega \frac{db}{du}\ u_m'|\nabla u_m|^2\Delta u_m' - 2\int_\Omega b(u_m)\nabla u_m\nabla u_m'\Delta u_m'.\end{aligned} \tag{74}$$

Again, we shall estimate the five terms on the hand right side of (74) individually. For the first term, from hypothesis (H2) and using Sobolev embedding theorem we obtain

$$\begin{aligned} \Big|\int_\Omega \frac{da}{du} u_m' \Delta u_m \Delta u_m'\Big| &\le M|u_m'|_{C^0(\Omega)}|\Delta u_m|\,|\Delta u_m'| \le MC_6C_2|\Delta u_m|\,|\Delta u_m'|^2 \\ &\le \frac{2M^2C_6^2C_2^2}{a_0}|\Delta u_m|^2|\Delta u_m'|^2 + \frac{a_0}{8}|\Delta u_m'|^2. \end{aligned} \tag{75}$$

For the second term, we have

$$\begin{aligned} \Big|\int_\Omega \frac{d^2a}{du^2}\, u_m'|\nabla u_m|^2\Delta u_m'\Big| &\le M|u_m'|_{L^6(\Omega)}|\nabla u_m|^2_{L^6(\Omega)}|\Delta u_m'|_{L^2(\Omega)} \\ &\le MC_2^2C_1C_3|\Delta u_m|^2|\Delta u_m'|^2. \end{aligned} \tag{76}$$

For the third term,

$$\begin{aligned} &\Big|\int_\Omega \frac{da}{du}\nabla u_m \nabla u_m' \Delta u_m'\Big| \le M|\nabla u_m|_{L^4(\Omega)}|\nabla u_m'|^2_{L^4(\Omega)}|\Delta u_m'|_{L^2(\Omega)} \\ &\le MC_4^2C_2^2|\Delta u_m|\,|\Delta u_m'|^2 \le \frac{a_0}{16}|\Delta u_m'|^2 + \frac{4M^2}{a_0}C_4^4C_2^4|\Delta u_m|^2|\Delta u_m'|^2. \end{aligned} \tag{77}$$

For the fourth term,

$$\Big|\int_\Omega \frac{db}{du}\, u_m'|\nabla u_m|^2\Delta u_m'\Big| \le MC_2^2C_1C_3|\Delta u_m|^2|\Delta u_m'|^2. \tag{78}$$

For the last term of (74), we have

$$\begin{aligned} \Big|\int_\Omega b(u_m)\nabla u_m \nabla u_m' \Delta u_m'\Big| &\le MC_4^2C_2^2|\Delta u_m|\,|\Delta u_m'|^2 \\ &\le \frac{a_0}{16}|\Delta u_m'|^2 + \frac{4M^2}{a_0}C_4^4C_2^4|\Delta u_m|^2\,|\Delta u_m'|^2. \end{aligned} \tag{79}$$

Substituting the estimates (75)-(79) in (74), we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_m'\|^2 + \int_\Omega a(u_m)|\Delta u_m'|^2 \le C_8|\Delta u_m|^2|\Delta u_m'|^2 + \frac{3a_0}{8}|\Delta u_m'|^2, \tag{80}$$

where $C_8 = 2\Big(\frac{M^2C_6^2C_2^2}{a_0} + MC_2^2C_1C_3 + \frac{8M^2C_4^4C_2^4}{a_0}\Big)$.

Adding the estimates (68), (73) and (80), we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\Big(\|u_m\|^2 + \|u_m'\|^2 \quad &+ \int_\Omega a(u_m)|\Delta u_m|^2\Big) + \frac{3a_0}{4}|\Delta u_m|^2 + \frac{3a_0}{8}|\Delta u_m'|^2 \\ &\le C_9|\Delta u_m|^4 + C_8|\Delta u_m|^2|\Delta u_m'|^2, \end{aligned} \tag{81}$$

where $C_9 = \Big(\frac{3C_5^2 + 2C_7^2}{a_0} + \frac{C_0^2}{2} + C_0\Big)$.

From estimate (81), we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\|u_m\|^2+\|u_m'\|^2+\int_\Omega a(u_m)|\Delta u_m|^2\Big)+\frac{a_0}{2}|\Delta u_m|^2\\
&+|\Delta u_m|^2\Big(\frac{a_0}{4}-C_9|\Delta u_m|^2\Big)+\frac{a_0}{8}|\Delta u_m'|^2+|\Delta u_m'|^2\Big(\frac{a_0}{4}-C_8|\Delta u_m|^2\Big)\leq 0.
\end{aligned}
\tag{82}
$$

On the other hand, making $t=0$ in the equation $(10)_1$, taking $v=-\Delta u_m'(0)$, integrating in Ω and using the hypothesis (H3), we have

$$
\begin{aligned}
\|u_m'(0)\|^2 &= -\int_\Omega a\left(u_{0m}\right)\Delta u_{0m}\Delta u_m'(0)+\int_\Omega \frac{da}{du}(u_{0m})(\nabla u_{0m})^2\Delta u_m'(0)\\
&-\int_\Omega b\left(u_{0m}\right)\ |\nabla u_{0m}|^2\Delta u_m'(0)=\int_\Omega \frac{da}{du}(u_{0m})\nabla u_{0m}\Delta u_m'(0)\nabla u_m'(0)\\
&+\int_\Omega a\left(u_{0m}\right)\nabla\left(\Delta u_{0m}\right)\nabla u_m'(0)-\int_\Omega \frac{d^2a}{du^2}(u_{0m}|\nabla u_{0m}|^2\nabla u_{0m}\nabla u_m'(0)\\
&-\int_\Omega \frac{da}{du}(u_{0m})\nabla(|\nabla u_{0m}|^2)\nabla u_m'(0)-\int_\Omega \frac{db}{du}\left(u_{0m}\right)|\nabla u_{0m}|^2\nabla u_{0m}\nabla u_m'(0)\\
&-\int_\Omega b(u_{0m})\nabla\left(|\nabla u_{0m}|^2\right)\nabla u_m'(0).
\end{aligned}
\tag{83}
$$

From (83), we obtain the following estimate for the $u_m'(0)$ term,

$$
\|u_m'(0)\|^2\leq C_{10}\Big(\|u_{0m}\|_{H^3(\Omega)}\ |\Delta u_{0m}|+\|u_{0m}\|_{H^3(\Omega)}+|\Delta u_{0m}|^3\Big)\|u_m'(0)\|.
$$

Or equivalently

$$
\|u_m'(0)\|^2\leq C_{10}^2\Big(\|u_{0m}\|_{H^3(\Omega)}\ |\Delta u_{0m}|+\|u_{0m}\|_{H^3(\Omega)}+|\Delta u_{0m}|^3\Big)^2,
\tag{84}
$$

where $C_{10}=\max\Big\{MC_4^2C_2+2MC_4C_2+MC_1^2C_3C_2,\ M,\ 2MC_1C_2\Big\}$.

There is $\varepsilon_0>0$ such that for $(|\Delta u_0|+\|u_0\|_{H^3(\Omega)})<\epsilon_0$, we have

$$
\begin{aligned}
&|\Delta u_0|^2<\frac{1}{(C_8+C_9)}\frac{a_0}{4},\\
&\|u_0\|+C_{10}^2\Big(\|u_0\|_{H^3(\Omega)}\ |\Delta u_0|+\|u_0\|_{H^3(\Omega)}+|\Delta u_0|^3\Big)^2+a_1|\Delta u_0|^2\\
&\qquad<a_0\frac{1}{(C_8+C_9)}\frac{a_0}{4}.
\end{aligned}
\tag{85}
$$

Therefore, we can confirm that

$$
|\Delta u_m|^2<\frac{1}{(C_8+C_9)}\frac{a_0}{4},\qquad \forall\, t\geq 0.
\tag{86}
$$

Indeed, by presuming absurdity, there is a t^* by (86) such that

$$
\begin{aligned}
&|\Delta u_m(t)|^2 < \frac{1}{(C_8+C_9)}\frac{a_0}{4}, \quad \text{if} \quad 0<t<t^* \quad \text{and} \\
&|\Delta u_m(t^*)|^2 = \frac{1}{(C_8+C_9)}\frac{a_0}{4}.
\end{aligned} \tag{87}
$$

Then, by integrating (82) from 0 to t^* and using hypothesis (H1), we obtain

$$
\|u_m(t^*)\|^2 + \|u_m'(t^*)\|^2 + a_0|\Delta u_m(t^*)|^2 \le \|u_{0m}\|^2 + \|u_m'0\|^2 + a_1|\Delta u_{0m}|^2. \tag{88}
$$

From (84), (85) and (88) we obtain

$$
\frac{1}{2}\Big(\|u_m(t^*)\|^2 + \|u_m'(t^*)\|^2 + a_0|\Delta u_m(t^*)|^2\Big) < \frac{a_0}{2}\frac{1}{(C_8+C_9)}\frac{a_0}{4}. \tag{89}
$$

Therefore, we conclude that

$$
|\Delta u_m(t^*)|^2 < \frac{1}{(C_8+C_9)}\frac{a_0}{4}.
$$

This leads to a contradiction by $(87)_2$.

Since (86) is valid, the terms $\Big(\frac{a_0}{4} - C_8|\Delta u_m|^2\Big)$ and $\Big(\frac{a_0}{4} - C_9|\Delta u_m|^2\Big)$ on the left hand-side of (82) are also positive. Hence, by integrating (82) from 0 to T, we obtain

$$
\frac{1}{2}\|u_m\|^2 + \|u_m'\|^2 + \frac{a_0}{2}|\Delta u_m|^2 + \frac{a_0}{2}\int_0^T |\, \Delta u_m|^2 + \frac{a_0}{8}\int_0^T |\Delta u_m'|^2 \le C. \tag{90}
$$

Therefore,

$$
\begin{aligned}
(u_m) \quad &\text{is bounded in} \quad L^\infty\left(0,T; H_0^1(\Omega)\cap H^2(\Omega)\right), \\
(u_m') \quad &\text{is bounded in} \quad L^2\left(0,T; H_0^1(\Omega)\cap H^2(\Omega)\right) \cap L^\infty\left(0,T; H_0^1(\Omega)\right).
\end{aligned} \tag{91}
$$

The limit of the approximate solutions can be obtained following the same arguments for (33), (34), (38) and (41), i.e, we obtain the solution in $L^2(Q)$.□

Asymptotic behavior

In the following we shall prove that the solution $u(x,t)$ of Problem (9), in the case $n=2$, also decays exponentially when time $t\to\infty$.

Theorem 4 *Let $u(x,t)$ be the solution of Problem* (9). *Then there exist positive constants S_0 and $C = C\{\|u_0\|, |\Delta u_0|, \|u_0\|_{H^3(\Omega)}\}$ such that*

$$
\|u\|^2 + \|u'\|^2 + \int_\Omega a(u)|\Delta u|^2 \le C\exp^{-S_0\,t}. \tag{92}
$$

Proof. Let $H(t) = \frac{1}{2}\Big(\|u_m\|^2 + \|u_m'\|^2 + \int_\Omega a(u_m)|\Delta u_m|^2\Big)$.

From (82), we obtain

$$\frac{d}{dt}H(t) + a_0|\Delta u_m|^2 + \frac{a_0}{4}|\Delta u_m'|^2 \leq 0. \tag{93}$$

We also have

$$\begin{aligned} H(t) &\leq \frac{1}{2}C_3^2\Big(|\Delta u_m|^2 + |\Delta u_m'|^2\Big) + \frac{a_1}{2}|\Delta u_m|^2 \\ &\leq \hat{C}_1\Big(|\Delta u_m|^2 + |\Delta u_m'|^2\Big) \leq \hat{C}_2\Big(\frac{a_0}{2}|\Delta u_m|^2 + \frac{a_0}{8}|\Delta u_m'|^2\Big), \end{aligned} \tag{94}$$

where $\hat{C}_1 = \max\Big\{C_3^2, \frac{a_1}{2}\Big\}$ and $\hat{C}_2 = \frac{8\hat{C}_1}{a_0}$. Using (93) and (94), we conclude that

$$\frac{1}{2}\frac{d}{dt}H(t) + \hat{C}_2 H(t) \leq 0,$$

which implies that

$$H(t) \leq C\exp^{-S_0 t},$$

where $S_0 = -2\hat{C}_2$ and the positive constant $C = C\{\|u_0\|, |\Delta u_0|, \|u_0\|_{H^3(\Omega)}\}$ is determined by (84). The result follows from the Banach-Steinhaus theorem. $\square$

Acknowledgement: We would like to thank the reviewer who carefully went through our paper and gave valuable comments and suggestions.

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