M P E J

Mathematical Physics Electronic Journal

ISSN 1086-6655
Volume 14, 2008

Paper 2
Received: Oct 4, 2006, Revised: Oct 19, 2007, Accepted: Aug 8, 2008
Editor: R. Benguria

On the essential spectrum of the Jansen-Hess operator for two-electron ions

D. H. Jakubassa-Amundsen

Mathematics Institute, University of Munich
Theresienstr. 39, 80333 Munich, Germany

E-mail: dj@mathematik.uni-muenchen.de

Abstract

Based on the HVZ theorem, the absence of embedded single-particle eigenvalues and dilation analyticity of the pseudorelativistic no-pair Jansen-Hess operator, it is proven that for subcritical central charge Z there is no singular continuous spectrum in $\mathbb{R}\backslash[\Sigma_0, 2m]$. Moreover, for the two-particle Brown-Ravenhall operator, the absence of eigenvalues above $2m$ is shown (if $Z \leq 50$).

1 Introduction

We consider two interacting electrons of mass m in a central Coulomb field, generated by a point nucleus of charge number Z which is fixed at the origin. The Jansen-Hess operator that is used for the description of this system, results from a block-diagonalization of the Coulomb-Dirac operator up to second order in the fine structure constant $e^2 \approx 1/137.04$ [8, 16]. Convergence of this type of expansion has recently been proven for $Z < 52$ [24, 11], and numerical higher-order investigations have established the Jansen-Hess operator as a very good approximation (see e.g. [23, 30]).

Based on the work of Lewis, Siedentop and Vugalter [19] the essential spectrum of the two-particle Jansen-Hess operator $h^{(2)}$ was localized in $[\Sigma_0, \infty)$ with $\Sigma_0 - m$ being the ground-state energy of the one-electron ion [15]. A more detailed information on the essential spectrum exists only for the single-particle Jansen-Hess operator, for which, in case of sufficiently small central potential strength γ, the absence of the singular continuous spectrum σ_{sc} and of embedded eigenvalues was proven [13]. These results were obtained with the help of scaling properties and dilation analyticity of this operator, combined with the virial theorem, methods which, initiated by Aguilar and Combes, are well-known from the analysis of the Schrödinger operator [1],[22, p.231] and of the single-particle Brown-Ravenhall operator h_1^{BR} [9, 2]. For more than one electron the absence of σ_{sc} was only shown in the Schrödinger case [3, 25], the basic ingredient (apart from the dilation analyticity of the operator) being the relative compactness of the Schrödinger potential with respect to the Laplace operator. Such a compactness property does, however, not exist for Dirac-type operators. Instead, the proof (of Theorem 1, section 4) can be based on the HVZ theorem combined with the absence of embedded eigenvalues for the single-particle operators. The absence of eigenvalues above m for the Brown-Ravenhall operator, the Jansen-Hess operator and the exact single-particle block-diagonalized Dirac operator is stated in Proposition 1 (section 3). For the two-particle operators the virial theorem is formulated and a modification of the proof by Balinsky and Evans [2] is tested on h_2^{BR} to show the absence of eigenvalues in $[2m, \infty)$ (Proposition 2, section 3).

Let us now define our operators in question. The two-particle pseudorelativistic no-pair Jansen-Hess operator, acting in the Hilbert space $\mathcal{A}(L_2(\mathbb{R}^3) \otimes \mathbb{C}^2)^2$ where $\mathcal{A}$ denotes antisymmetrization with respect to particle exchange, is given (in relativistic units, $\hbar = c = 1$) by [16]

$$h^{(2)} \;=\; h_2^{BR} \;+\; \sum_{k=1}^{2} b_{2m}^{(k)} \;+\; c^{(12)}. \tag{1.1}$$

The term up to first order in e^2 is the (two-particle) Brown-Ravenhall operator

[4, 9, 14]

$$h_2^{BR} \;=\; \sum_{k=1}^{2} \left(T^{(k)} + b_{1m}^{(k)}\right) \;+\; v^{(12)},$$

$$T^{(k)} :=\; E_{p_k} :=\; \sqrt{p_k^2 + m^2}, \qquad b_{1m}^{(k)} \;\sim\; -P_0^{(12)}\, U_0^{(k)}\, \frac{\gamma}{x_k}\, U_0^{(k)-1}\, P_0^{(12)}, \quad (1.2)$$

$$v^{(12)} \;\sim\; P_0^{(12)}\, U_0^{(1)} U_0^{(2)}\, \frac{e^2}{|\mathbf{x}_1 - \mathbf{x}_2|}\, (U_0^{(1)} U_0^{(2)})^{-1}\, P_0^{(12)},$$

where the index m refers to the particle mass, $\mathbf{p}_k = -i\boldsymbol{\nabla}_k$ is the momentum and $\mathbf{x}_k$ (with $x_k := |\mathbf{x}_k|$) the location of particle k relative to the origin. $\gamma = Ze^2$ is the central field strength, and $v^{(12)}$ the electron-electron interaction. $U_0^{(k)}$ denotes the unitary Foldy-Wouthuysen transformation,

$$U_0^{(k)} \;=\; A(p_k) \;+\; \beta^{(k)} \boldsymbol{\alpha}^{(k)} \mathbf{p}_k g(p_k),$$

$$A(p) := \left(\frac{E_p + m}{2E_p}\right)^{\frac{1}{2}}, \qquad g(p) := \frac{1}{\sqrt{2E_p(E_p + m)}} \quad (1.3)$$

and the inverse $U_0^{(k)-1} = U_0^{(k)*} = A(p_k) + \boldsymbol{\alpha}^{(k)} \mathbf{p}_k g(p_k) \beta^{(k)}$ with $\boldsymbol{\alpha}^{(k)},\ \beta^{(k)}$ Dirac matrices [26]. Finally, $P_0^{(12)} = P_0^{(1)} P_0^{(2)}$ where $P_0^{(k)} := \frac{1+\beta^{(k)}}{2}$ projects onto the upper two components of the four-spinor of particle k (hence reducing the four-spinor space to a two-spinor space). In (1.2) and in the equations below, the notation l.h.s. $\sim$ r.h.s. means that the l.h.s. is defined by the nontrivial part (i.e. the upper block) of the r.h.s. (see e.g. [9, 16]).

The remaining potentials in (1.1) which are of second order in the fine structure constant consist of the single-particle contributions

$$b_{2m}^{(k)} \;\sim\; P_0^{(12)}\, U_0^{(k)}\, \frac{\gamma^2}{8\pi^2} \left\{ \frac{1}{x_k}\, (1 - \tilde{D}_0^{(k)})\, V_{10,m}^{(k)} \;+\; h.c. \right\}\, U_0^{(k)-1}\, P_0^{(12)}, \quad k = 1, 2,$$

$$\tilde{D}_0^{(k)} :=\; \frac{\boldsymbol{\alpha}^{(k)} \mathbf{p}_k + \beta^{(k)} m}{E_{p_k}}, \qquad V_{10,m}^{(k)} :=\; 2\pi^2 \int_0^\infty dt\; e^{-tE_{p_k}}\, \frac{1}{x_k}\, e^{-tE_{p_k}}, \quad (1.4)$$

where $\tilde{D}_0^{(k)}$ has norm unity, $V_{10,m}^{(k)}$ is bounded and $h.c.$ stands for hermitean conjugate (such that $b_{2m}^{(k)}$ is a symmetric operator). The two-particle interaction is given by

$$c^{(12)} \sim P_0^{(12)} U_0^{(1)} U_0^{(2)} \frac{1}{2} \sum_{k=1}^{2} \left\{ \frac{e^2}{|\mathbf{x}_1 - \mathbf{x}_2|}\, (1 - \tilde{D}_0^{(k)})\, F_0^{(k)} + h.c. \right\} \left(U_0^{(1)} U_0^{(2)}\right)^{-1} P_0^{(12)},$$

$$F_0^{(k)} :=\; -\frac{\gamma}{2} \int_0^\infty dt\; e^{-tE_{p_k}} \left(\frac{1}{x_k} \;-\; \tilde{D}_0^{(k)}\, \frac{1}{x_k}\, \tilde{D}_0^{(k)} \right)\, e^{-tE_{p_k}}. \quad (1.5)$$

For later use, we also provide the kernel of the bounded operator $F_0^{(k)}$ in momentum space,

$$k_{F_0^{(k)}}(\mathbf{p},\mathbf{p}') \;=\; -\frac{\gamma}{(2\pi)^2}\,\frac{1}{|\mathbf{p}-\mathbf{p}'|^2}\,\frac{1}{E_p+E_{p'}}\left(1\,-\,\tilde{D}_0^{(k)}(\mathbf{p})\,\tilde{D}_0^{(k)}(\mathbf{p}')\right). \qquad (1.6)$$

$h^{(2)}$ is a well-defined operator in the form sense for $\gamma < 0.98$ (which follows from the form boundedness of the Jansen-Hess potential with respect to the kinetic energy with relative bound less than one; see section 2 for the improvement of the bound 0.89 given in [16]), and is self-adjoint by means of its Friedrichs extension.

2 Dilation analyticity

For a one-particle function $\varphi \in L_2(\mathbb{R}^3)\otimes\mathbb{C}^2$ and $\theta := e^\xi \in \mathbb{R}_+$ we define the unitary group of dilation operators d_θ by means of [1]

$$d_\theta\varphi(\mathbf{p}) :=\; \theta^{-3/2}\,\varphi(\mathbf{p}/\theta) \qquad (2.1)$$

with the property

$$d_{\theta_1}d_{\theta_2}\varphi(\mathbf{p}) \;=\; (\theta_1\theta_2)^{-3/2}\,\varphi(\mathbf{p}/\theta_1\theta_2) \;=\; d_\theta\,\varphi(\mathbf{p}) \qquad (2.2)$$

where $\theta := \theta_1\theta_2 = e^{\xi_1+\xi_2}$. For a two-particle function $\psi \in \mathcal{A}(L_2(\mathbb{R}^3)\otimes\mathbb{C}^2)^2$ we have $d_\theta\,\psi(\mathbf{p}_1,\mathbf{p}_2) = \theta^{-3}\psi(\mathbf{p}_1/\theta,\mathbf{p}_2/\theta)$.

Let $\mathcal{O}_\theta := d_\theta\mathcal{O}d_\theta^{-1}$ be the dilated operator $\mathcal{O}$ (e.g. $h_\theta^{(2)} := d_\theta h^{(2)} d_\theta^{-1}$). From the explicit structure of the summands of $h^{(2)}$ in momentum space one derives the following scaling properties, using the form invariance $(\psi, h^{(2)}\psi) = (d_\theta\psi, h_\theta^{(2)} d_\theta\psi)$ for $\psi \in \mathcal{A}(H_{1/2}(\mathbb{R}^3)\otimes\mathbb{C}^2)^2$, the form domain of $h^{(2)}$ (see [9, 13], [12, p.42,73]),

$$T_\theta^{(k)}(m) \;=\; \sqrt{p_k^2/\theta^2+m^2} \;=\; \frac{1}{\theta}\,\sqrt{p_k^2+m^2\theta^2} \;=\; \frac{1}{\theta}\,T^{(k)}(m\cdot\theta) \qquad (2.3)$$

$$h_{2,\theta}^{BR}(m) \;=\; \frac{1}{\theta}\,h_2^{BR}(m\cdot\theta), \qquad h_\theta^{(2)}(m) \;=\; \frac{1}{\theta}\,h^{(2)}(m\cdot\theta)$$

where we have indicated explicitly the mass dependence of the operators.

Let us extend θ to a domain $\mathcal{D}$ in the complex plane,

$$\mathcal{D} :=\; \{\theta\in\mathbb{C}:\; \theta = e^\xi,\; |\xi| < \xi_0\}, \qquad (2.4)$$

with $0 < \xi_0 < \frac{1}{2}$ to be fixed later. The definition of the dilated operators with the scaling properties (2.3) is readily extended to $\theta \in \mathcal{D}$.

In order to establish the existence of $h_\theta^{(2)}$ for $\theta \in \mathcal{D}$ as a form sum it suffices for $T_\theta := T_\theta^{(1)} + T_\theta^{(2)}$ to show the $|T_\theta|$-form boundedness of the potential of $h_\theta^{(2)}$ with relative bound smaller than one. For the single-particle contributions this was shown earlier for potential strength $\gamma < 1.006$ [13].

Let us start by noting that the m-dependent factors appearing in the potential terms of $h^{(2)}$ are all of the form $E_p^\lambda,\ (E_p+m)^\lambda,\ \ \lambda \in \mathbb{R},$ as well as $\frac{1}{E_p+E_{p'}}$ (see e.g. (1.3), (1.6)). This assures that $h^{(2)}\psi$ is an analytic function of m for $m \neq 0$.

For $\theta \in \mathcal{D}$ we basically have to replace m by $m \cdot \theta$. We can use estimates of the type [13]

$$\begin{aligned} 1-\xi_0 \ &\leq\ \left|\frac{1}{\theta}\right| \ \leq\ 1+2\xi_0 \\ (1-\xi_0)\ E_p \ &\leq\ |E_\theta(p)| \ \leq\ (1+2\xi_0)\ E_p \end{aligned} \tag{2.5}$$

where $E_\theta(p) := \sqrt{p^2+m^2\theta^2}$. From these relations one derives the relative boundedness of the following dilated operators with respect to those for $\theta = 1$,

$$\begin{aligned} |A_\theta(p)|^2 \ &\leq\ \frac{1+2\xi_0}{1-\xi_0}\ A^2(p) \\ |\frac{p}{\theta}\ g_\theta(p)|^2 \ &\leq\ \frac{1}{(1-\xi_0)^4}\ p^2\ g^2(p) \\ \left|\frac{1}{E_\theta(p)+E_\theta(p')}\right| \ &\leq\ \frac{1}{(1-\xi_0)^3}\ \frac{1}{E_p+E_{p'}}. \end{aligned} \tag{2.6}$$

As a consequence, the dilated Foldy-Wouthuysen transformation is bounded, $|U_\theta^{(k)}| \leq |A_\theta(p_k)| + |\frac{p_k}{\theta}\, g_\theta(p_k)| \leq \tilde{c}$, and also $|\tilde{D}_{0,\theta}^{(k)}| \leq \frac{1}{|E_\theta(p_k)|}\,(p_k + m|\theta|) \leq \tilde{c}$ with some constant $\tilde{c}$.

In order to show the relative form boundedness of $h_\theta^{(2)}$, we write $h^{(2)} = T + W$ and introduce the respective massless ($m = 0$) operators $T_0 = p_1 + p_2$ and W_0,

$$|(\psi, W_\theta \psi)| \ \leq\ |\frac{1}{\theta}\,(\psi, W_0\psi)| \ +\ |(\psi, \left(W_\theta - \frac{1}{\theta}\,W_0\right)\psi)|. \tag{2.7}$$

The form boundedness of W_0 with respect to T_0 follows from the previous single-particle [27, 6, 5] and two-particle [16] $m = 0$ estimates. For the single-particle contributions we profit from [5] $(\psi, (p_k + b_1^{(k)} + b_2^{(k)})\,\psi) \geq (1 - \frac{\gamma}{\gamma_{BR}} + d\gamma^2)(\psi, p_k\psi)$ together with [13] $b_1^{(k)} + b_2^{(k)} < 0$ for $\gamma \leq \frac{4}{\pi}$. Note that (e.g. for $k = 1$) $\psi = \psi_{\mathbf{x}_2}(\mathbf{x}_1)$ acts as a one-particle function depending parametrically on the coordinates of the second particle. For the two-particle terms, use is made of $(U_0^{(k)*}\psi_0, p_k U_0^{(k)*}\,\psi_0) = (\psi, p_k\,\psi)$ where $\psi_0 := \binom{\psi}{0}$ denotes a two-particle spinor whose lower components are zero by the action of $P_0^{(12)}$, showing that the four-spinor estimates from [16] are applicable. Thus,

$$|(\psi, W_0\,\psi)| \;\leq\; \sum_{k=1}^{2} |(\psi, (b_1^{(k)} + b_2^{(k)})\,\psi)| \;+\; |(\psi, c_0^{(12)}\,\psi)| \;+\; |(\psi, v_0^{(12)}\,\psi)|$$

$$\leq \left(\frac{\gamma}{\gamma_{BR}} - d\gamma^2 + \gamma\,\frac{e^2\pi^2}{4} \;+\; \frac{e^2}{2\gamma_{BR}} \right) (\psi, T_0\,\psi) \;=:\; \tilde{c}_0\,(\psi, T_0\,\psi), \tag{2.8}$$

where $\gamma_{BR} = 2/(\frac{2}{\pi} + \frac{\pi}{2}) \approx 0.906$ is the maximum potential strength for which h_1^{BR} is bounded from below, and $d = \frac{1}{8}\left(\frac{\pi}{2} - \frac{2}{\pi}\right)^2$.

For the proof of the form boundedness with respect to $|T_\theta|$, we can estimate for $|\text{Im}\;\xi| < \frac{\pi}{4}$

$$\text{Re}\;\sqrt{p_k^2 + m^2\theta^2} \;\geq\; p_k \tag{2.9}$$

(setting $\theta = e^{u+iv}$ and $E_\theta = x+iy$ leads for $z := x^2$ to $4z^2 - 4(p_k^2 + m^2 e^{2u}\cos 2v)z - m^4 e^{4u}\sin^2 2v = 0$, with a solution that can be estimated by $z \geq p_k^2$ upon dropping all terms $\sim m$), such that

$$|\theta| \cdot |(\psi, T_\theta^{(k)}\psi)| \;\geq\; |\text{Re}\;(\psi, \sqrt{p_k^2 + m^2\theta^2}\;\psi)| \;\geq\; (\psi, T_0^{(k)}\psi). \tag{2.10}$$

The uniform boundedness of the single-particle remainder in (2.7), $\frac{1}{|\theta|}\,|(\psi, (b_{1m\cdot\theta}^{(k)} - b_1^{(k)})\,\psi)| + \frac{1}{|\theta|}\,|(\psi, (b_{2m\cdot\theta}^{(k)} - b_2^{(k)})\,\psi)|$ was proven in [13] based on the respective results for $\theta = 1$ [28, 5].

For the proof of the uniform boundedness of $(\psi, (c^{(12)}(m\cdot\theta) - c_0^{(12)})\,\psi)$ and $(\psi, (v^{(12)}(m\cdot\theta) - v_0^{(12)})\,\psi)$ we proceed in a similar way. Since $c^{(12)}$ and $v^{(12)}$ are analytic functions of m, the mean value theorem can be applied in the form $|f(m\cdot\theta) - f(0)| \;\leq m(\,|\frac{\partial f}{\partial m}(\tilde{m}_1\cdot\theta)| \,+\, |\frac{\partial f}{\partial m}(\tilde{m}_2\cdot\theta)|\,)$ with $0 \leq \tilde{m}_1, \tilde{m}_2 \leq m$ (adapted to complex-valued functions [13]). The kernel of $v^{(12)}$ is given by $K_{v^{(12)}}(\mathbf{p}_1, \mathbf{p}_2; \mathbf{p}_1', \mathbf{p}_2') := U_0^{(2)} U_0^{(1)} k_{v^{(12)}} U_0^{(1')*} U_0^{(2')*}$ with

$$k_{v^{(12)}} \;:=\; \frac{e^2}{2\pi^2}\;\frac{1}{|\mathbf{p}_1 - \mathbf{p}_1'|^2}\;\delta(\mathbf{p}_2' - \mathbf{p}_2 + \mathbf{p}_1' - \mathbf{p}_1), \tag{2.11}$$

such that one gets

$$\left|(K_{v^{(12)}}(m\cdot\theta) - K_{v_0^{(12)}})(\mathbf{p}_1, \mathbf{p}_2; \mathbf{p}_1', \mathbf{p}_2')\right| \;\leq\; m\,k_{v^{(12)}}$$

$$\cdot \left(\left| \frac{\partial}{\partial m}\left(U_0^{(1)} U_0^{(2)} U_0^{(1')*} U_0^{(2')*} \right)(\tilde{m}_1\cdot\theta) \right| \;+\; (\tilde{m}_1 \mapsto \tilde{m}_2) \right), \tag{2.12}$$

where $(\tilde{m}_1 \mapsto \tilde{m}_2)$ means the first term in the second line of (2.12) repeated with $\tilde{m}_1$ replaced by $\tilde{m}_2$, and $U_0^{(k')}$ is $U_0^{(k)}$ with $\mathbf{p}_k$ replaced by $\mathbf{p}_k'$. Further,

$$\left| \frac{\partial}{\partial m}\,(U_0^{(1)} \cdots U_0^{(2')*}) \right| \;\leq\; \left| \frac{\partial U_0^{(1)}}{\partial m} \right| \cdot \left| U_0^{(2)}\,U_0^{(1')*} U_0^{(2')*} \right| \;+\; ...$$

$$+\ \left|U_0^{(1)}U_0^{(2)}\,U_0^{(1')*}\right|\cdot\left|\frac{\partial U_0^{(2')*}}{\partial m}\right|. \tag{2.13}$$

From the boundedness of $U_0^{(k)}$ and of θ one gets the estimate (noting that $U_0^{(k)}$ is only a function of $m/p_k =: \xi$)

$$\left|\frac{\partial}{\partial m}\,U_0^{(k)}(\xi\cdot\theta)\right| \;=\; \frac{|\theta|}{p_k}\,\left|\frac{\partial}{\partial(\xi\cdot\theta)}\,U_0^{(k)}(\xi\cdot\theta)\right|$$

$$\leq\; \frac{|\theta|}{p_k}\,\frac{c}{1+\xi}\;\leq\;\frac{\tilde{c}}{p_k+m}\;\leq\;\frac{\tilde{c}}{p_k} \tag{2.14}$$

with some constants $c, \tilde{c}$ independent of m. With this estimate the boundedness of $v^{(12)}(m\cdot\theta)-v_0^{(12)}$ is readily shown (see e.g. [16] and Appendix A, where a sketch of the boundedness proof for $c^{(12)}(m\cdot\theta)-c_0^{(12)}$ is given).

Thus we obtain

$$|(\psi, W_\theta\psi)|\;\leq\;\tilde{c}_0\;|(\psi, T_\theta\psi)|\;+C(\psi,\psi) \tag{2.15}$$

with $\tilde{c}_0$ from (2.8) and some constant C. We have $\tilde{c}_0 < 1$ for $\gamma < 0.98$ $(Z \leq 134)$. This holds for all $\theta \in \mathcal{D}$. Besides this T_0- and T_θ-form boundedness with $\tilde{c}_0 < 1$, (2.3) assures that for ψ in the form domain of T_0, $(\psi, h_\theta^{(2)}\psi)$ is an analytic function in $\mathcal{D}$. Thus $h_\theta^{(2)}$ satisfies the criterions for being a dilation analytic family in the form sense [9],[22, p.20].

We remark that for the Brown-Ravenhall operator the relative form boundedness of the potential and hence the dilation analyticity hold for $\gamma < \gamma_{BR} - \frac{e^2}{2} \approx 0.902$ (see (2.8)).

3 Absence of embedded eigenvalues

The scaling properties of the pseudorelativistic operator under dilations allows one to show that under certain restrictions on the potential strength γ there are no eigenvalues embedded in the essential spectrum. For different methods, see [7],[22, p.222]. We start by stating some results for the single-particle case.

Proposition 1 *Let* $h_1^{BR} = T^{(k)} + b_{1m}^{(k)}$ *be the single-particle Brown-Ravenhall operator,* $b_m^{(k)} = h_1^{BR} + b_{2m}^{(k)}$ *the single-particle Jansen-Hess operator and* $h_{ex} \sim P_0^{(k)}U^{(k)}(\boldsymbol{\alpha}^{(k)}\mathbf{p}_k + \beta^{(k)}m - \gamma/x_k)U^{(k)-1}P_0^{(k)}$ *the exact block-diagonalized Dirac operator (projected onto the positive spectral subspace) from [24]. Then*

(i) For $\gamma < \gamma_{BR}$ $(Z \leq 124)$, h_1^{BR} *has no eigenvalues in* $[m, \infty)$.

(ii) For $\gamma < 0.29 \quad (Z < 40),\quad b_m^{(k)}$ has no eigenvalues in $[m, \infty)$, and for $\gamma < 1.006$ the eigenvalues do not accumulate in $\mathbb{R}_+ \backslash \{m\}$.

(iii) For $Z \leq 35,\ \ h_{ex}$ has no eigenvalues in (m, ∞).

Item (i) was proven in [2] (for $\gamma \leq \frac{3}{4}$) and in [12]. The first part of (ii) was proven in [13], and all proofs rely on the virial theorem. Item (iii) was derived in [17] with the help of a Mourre-type estimate [7].

The second statement of (ii) results from $\sigma_{ess}(b_{m,\theta}^{(k)}) = \sigma_{ess}(T_\theta^{(k)})$, based on the compactness of the difference of the resolvents of $b_{m,\theta}^{(k)}$ and $T_\theta^{(k)}$ [13] for any $\theta \in \mathcal{D}$. We have $\sigma(T_\theta^{(k)}) \cap \mathbb{R} = \{m\}$ if Im $\xi \neq 0$ [29, 10], so there are at most isolated eigenvalues of $b_{m,\theta}^{(k)}$ in $\mathbb{R}_+ \backslash \{m\}$. Since $b_{m,\theta}^{(k)}$ is a dilation analytic family these eigenvalues are invariant when $\theta \to 1$ (see also section 4).

Let us now turn to the two-particle operator. The virial theorem for the one-particle case [2] is easily generalized to two-particle operators obeying the scaling properties (2.3). Assuming that ψ is an eigenfunction of the Jansen-Hess operator $h^{(2)}$ to some eigenvalue λ and that $\theta \in \mathcal{D} \cap \mathbb{R}_+$, the virial theorem reads

$$\lim_{\theta \to 1} \,(\psi_\theta, \frac{h^{(2)}(m \cdot \theta) - h^{(2)}(m)}{\theta - 1}\, \psi) \;=\; \lambda\, \|\psi\|^2, \tag{3.1}$$

where the mass dependence of $h^{(2)}$ is indicated explicitly. By the mean value theorem, the operator on the l.h.s. is transformed into $m\,(\frac{dh^{(2)}(m)}{dm})(m \cdot \tilde{\theta})$ for some $\tilde{\theta}$ on the line between 1 and θ. Since this operator can be bounded independently of $\tilde{\theta}$ (see section 2) and $\|\psi_\theta\| \;=\|\psi\|$, the theorem of dominated convergence applies and the limit $\theta \to 1$ can be carried out. We get, making use of the symmetry property of ψ under particle exchange,

$$\frac{\lambda}{2m}\, \|\psi\|^2 \;=\; (\psi, \frac{m}{E_{p_1}}\, \psi) \,+\, (\psi, \left(\frac{db_{1m}^{(1)}}{dm} + \frac{db_{2m}^{(1)}}{dm} + \frac{1}{2}\, \frac{dv^{(12)}}{dm} + \frac{1}{2}\, \frac{dc^{(12)}}{dm} \right)\, \psi). \tag{3.2}$$

This equation has to be combined with the eigenvalue equation which we take in the following form,

$$\lambda\, (F\psi, \psi) \;=\; (F\psi, \left(\sum_{k=1}^{2} (E_{p_k} + b_{1m}^{(k)} + b_{2m}^{(k)}) + v^{(12)} + c^{(12)} \right)\, \psi), \tag{3.3}$$

$$F\,\psi := \; c_0\, (1 - \frac{m}{E_{p_1}})\, \frac{1}{E_{p_1} + E_{p_2} - m}\, \psi \;=\; F^*\, \psi.$$

In the single-particle case, F is taken in such a way that the negative contribution (termed β_{10}) to the linear term $\frac{db_{1m}}{dm}$ can be eliminated [2]. Here, only a partial compensation is possible because one cannot avoid that $F(b_{1m}^{(1)} + b_{1m}^{(2)})$

is a two-particle operator (which cannot be split into single-particle terms). The symmetric (with respect to particle exchange) energy denominator $(E_{p_1} + E_{p_2} - m)^{-1}$ assures that the operator $Fh^{(2)}$ appearing on the r.h.s. of (3.3) is bounded. $c_0 \in \mathbb{R}_+$ is a parameter to be determined later. Let us now restrict ourselves to the Brown-Ravenhall operator. Then we have

Proposition 2 *Let h_2^{BR} be the two-particle Brown-Ravenhall operator and assume $\gamma \leq \gamma_c$ with $\gamma_c = 0.37 \quad (Z \leq 50)$. Then there are no eigenvalues in $[2m, \infty)$.*

Note that, with $\sigma_{ess}(h_2^{BR}) = [\Sigma_0, \infty)$ and $\Sigma_0 < 2m$ [21], no information on embedded eigenvalues is provided for the subset $[\Sigma_0, 2m)$. This corresponds to the multi-particle Schrödinger case where the virial-theorem method proves the absence of eigenvalues only in the subset $[0, \infty)$ of the essential spectrum [22, p.232].

Proof. Defining $d\boldsymbol{\pi} := d\mathbf{p}_1 d\mathbf{p}_2 d\mathbf{p}_1' d\mathbf{p}_2'$ we have from [2]

$$(\psi, \frac{db_{1m}^{(1)}}{dm}\,\psi) \;=\; \beta_{10}(m) \;+\; \beta_{11}(m), \tag{3.4}$$

$$\beta_{10}(m) := \;\mathrm{Re}\;(\psi, \left(\frac{1}{E_{p_1}} \;-\; \frac{m}{E_{p_1}^2}\right) b_{1m}^{(1)}\,\psi)$$

$$\beta_{11}(m) := \;\frac{\gamma}{2\pi^2}\int_{\mathbb{R}^{12}} d\boldsymbol{\pi}\;\overline{g(p_1)\boldsymbol{\sigma}^{(1)}\mathbf{p}_1\,\hat{\psi}(\mathbf{p}_1,\mathbf{p}_2)}\;\frac{1}{|\mathbf{p}_1 - \mathbf{p}_1'|^2}$$
$$\cdot\left(\frac{1}{E_{p_1}} \;+\; \frac{1}{E_{p_1'}}\right)\;g(p_1')\,\boldsymbol{\sigma}^{(1)}\mathbf{p}_1'\;\delta(\mathbf{p}_2 - \mathbf{p}_2')\;\hat{\psi}(\mathbf{p}_1',\mathbf{p}_2').$$

Subtraction of the real part of (3.3) from (3.2), while dropping the second-order terms $b_{2m}^{(k)}$ and $c^{(12)}$, results in

$$0 \;=\; M_0 \;+\; \gamma\,M_1 \;+\; e^2\,M_2, \tag{3.5}$$

$$\begin{aligned}
M_0 \;&:=\; (\psi, \Bigg((1 - \frac{\lambda}{2m})\,(1 \;-\; \frac{2c_0 m(1 - \frac{m}{E_{p_1}})}{E_{p_1} + E_{p_2} - m})\\
&\qquad -\,(1 \;-\; \frac{m}{E_{p_1}})\,(1 \;+\; \frac{c_0(E_{p_1} + E_{p_2} - 2m)}{E_{p_1} + E_{p_2} - m})\Bigg)\;\psi)\\
\gamma\,M_1 \;&:=\; \beta_{10}(m) \;-\; \mathrm{Re}\;(\psi, c_0(1 \;-\; \frac{m}{E_{p_1}})\;\frac{1}{E_{p_1} + E_{p_2} - m}\\
&\qquad \cdot(b_{1m}^{(1)} + b_{1m}^{(2)})\;\psi) \;+\; \beta_{11}(m)\\
e^2\,M_2 \;&:=\; \mathrm{Re}\;(\psi, (1 \;-\; \frac{m}{E_{p_1}})\,(\frac{1}{E_{p_1}} \;-\; \frac{c_0}{E_{p_1} + E_{p_2} - m})\;v^{(12)}\;\psi)
\end{aligned}$$

$$-\int_{\mathbb{R}^{12}} d\boldsymbol{\pi}\, \overline{g(p_1)\, \boldsymbol{\sigma}^{(1)}\mathbf{p}_1\, U_0^{(2)*}\, \hat{\psi}_0(\mathbf{p}_1,\mathbf{p}_2)}\, \left(\frac{1}{E_{p_1}} + \frac{1}{E_{p_1'}}\right)$$

$$\cdot k_{v^{(12)}} \left(g(p_1')\, \boldsymbol{\sigma}^{(1)}\mathbf{p}_1'\, U_0^{(2')*}\, \hat{\psi}_0(\mathbf{p}_1',\mathbf{p}_2')\right).$$

In the expression for the electron-electron interaction term, $e^2 M_2$, it is used that $b_{1m}^{(1)}$ and $v^{(12)}$ have the same structure. Indeed, the kernel of $b_{1m}^{(1)}$ is given by $U_0^{(1)} k_{b_{1m}} U_0^{(1')*}$ with

$$k_{b_{1m}} := -\frac{\gamma}{2\pi^2}\, \frac{1}{|\mathbf{p}_1 - \mathbf{p}_1'|^2}\, \delta(\mathbf{p}_2 - \mathbf{p}_2'), \tag{3.6}$$

as compared to the kernel of $v^{(12)}$ defined above (2.11). Due to the symmetry upon particle exchange, the kernel of $\frac{1}{2}\frac{dv^{(12)}}{dm}$ in (3.2) can be replaced by $U_0^{(2)}\, \frac{d}{dm}(U_0^{(1)} k_{v^{(12)}} U_0^{(1)*})\, U_0^{(2)*}$. Therefore (3.4), with $k_{v^{(12)}}$ substituted for $k_{b_{1m}}$, is applicable. As in section 2, $\psi_0 = \binom{\psi}{0}$ is a two-particle spinor with the lower components set equal to zero.

For a symmetric integral operator $\mathcal{O}$ with kernel $K + K^*$, we use the Lieb and Yau formula, derived from the Schwarz inequality, in the following form [20] (see also [14])

$$|(\psi, \mathcal{O}\,\psi)| \le \int_{\mathbb{R}^6} d\mathbf{p}_1 d\mathbf{p}_2\, |\hat{\psi}(\mathbf{p}_1,\mathbf{p}_2)|^2\, \left(I_1(\mathbf{p}_1,\mathbf{p}_2) + I_2(\mathbf{p}_1,\mathbf{p}_2)\right) \tag{3.7}$$

$$I_1(\mathbf{p}_1,\mathbf{p}_2) := \int_{\mathbb{R}^6} d\mathbf{p}_1' d\mathbf{p}_2'\, |K(\mathbf{p}_1,\mathbf{p}_2;\mathbf{p}_1',\mathbf{p}_2')|\, \frac{f(p_1)}{f(p_1')}\, \frac{g(p_2)}{g(p_2')}$$

and I_2 results from the replacement of $K(\mathbf{p}_1,\mathbf{p}_2;\mathbf{p}_1',\mathbf{p}_2')$ by $K^*(\mathbf{p}_1',\mathbf{p}_2';\mathbf{p}_1,\mathbf{p}_2)$. f and g are suitable nonnegative convergence generating functions such that $I_1,\ I_2$ exist as bounded functions for $\mathbf{p}_1,\mathbf{p}_2 \in \mathbb{R}^3$. In order to get rid of the particle mass m, we introduce the new variables $\mathbf{p}_i =: m\mathbf{q}_i,\ \ \mathbf{p}_i' =: m\mathbf{q}_i',\ \ i = 1,2$. With $s := 1 - 2c_0(1 - \frac{1}{\sqrt{q_1^2+1}})/(\sqrt{q_1^2+1} + \sqrt{q_2^2+1} - 1)$ we estimate

$$0 \le M_0 + \gamma\, |M_1| + e^2\, |M_2| \tag{3.8}$$

$$\le m^6 \int_{\mathbb{R}^6} d\mathbf{q}_1 d\mathbf{q}_2\, |\hat{\psi}(m\mathbf{q}_1, m\mathbf{q}_2)|^2\, s\left(1 - \frac{\lambda}{2m} + \phi(q_1,q_2)\right).$$

For $c_0 < 2$ (or $c_0 \le 2$ if $q_2 \ne 0$) we have $s > 0$ and then

$$\phi(q_1,q_2) := \frac{1}{s}\, \{-(1 - \frac{1}{\sqrt{q_1^2+1}}) \left(1 + c_0\, \frac{\sqrt{q_1^2+1} + \sqrt{q_2^2+1} - 2}{\sqrt{q_1^2+1} + \sqrt{q_2^2+1} - 1}\right)$$

$$+ \gamma\, q_1^2\, \tilde{M}_1 + e^2\, q_1^2\, \tilde{M}_2\} \tag{3.9}$$

where $q_1^2 \tilde{M}_i,\ \ i = 1,2,$ result from the estimates of M_i and are given in Appendix B. From (3.8) it follows that if $\phi(q_1,q_2) < 0$, we need $1 - \frac{\lambda}{2m} > 0$ which confines

λ to $\lambda < 2m$. A numerical investigation shows that the supremum of $s\,\phi(q_1,q_2)$ is attained for $q_1,q_2 \to \infty$ with $q_1 \ll q_2$. Then $s \to 1$ and from the explicit expression (see Appendix B) it follows that

$$\sup_{q_1,q_2\geq 0} s\,\phi(q_1,q_2) \;=\; \lim_{\substack{q_1\to\infty\\ q_2\gg q_1}} \phi(q_1,q_2) \;=\; -(1+c_0) + \gamma\,(4+2c_0) + 4\,e^2. \quad (3.10)$$

For the optimum choice $c_0 = 2$, we obtain $\sup\limits_{q_1,q_2\geq 0} s\,\phi(q_1,q_2) = 0$ for $\gamma =: \gamma_c = 0.37$. □

The proof of Proposition 2 can readily be extended to the Jansen-Hess operator $h^{(2)}$. However, the so obtained critical potential strength γ_c is expected to be rather small. Recall that the inclusion of the second-order term in the single-particle case leads to a reduction from $\gamma_c \approx 0.906$ to 0.29.

4 Absence of the singular continuous spectrum

Dilation analyticity is a powerful tool to show the absence of the singular continuous spectrum [1, 3]. Based on the additional fact that there are no eigenvalues of the single-particle operator above m, we shall prove

Theorem 1 *Let $h_2^{BR} = \sum\limits_{k=1}^{2}(T^{(k)} + b_{1m}^{(k)}) + v^{(12)}$ be the two-particle Brown-Ravenhall operator and $h^{(2)} = h_2^{BR} + \sum\limits_{k=1}^{2} b_{2m}^{(k)} + c^{(12)}$ the two-particle Jansen-Hess operator. Let $\Omega := \mathbb{R}_+\backslash[\Sigma_0, 2m]$ where $\Sigma_0 - m$ is the lowest bound state of the respective single-particle operator. Then*

(i) For $\gamma < \gamma_{BR}$, $\sigma_{sc}(h_2^{BR}) = \emptyset$ in Ω.

(ii) For $\gamma < 0.29$, $\sigma_{sc}(h^{(2)}) = \emptyset$ in Ω.

Proof. Let $h_\theta^{(2)}$ be the dilated operator from (2.3). In order to show that there are only discrete points of the spectrum $\sigma(h_\theta^{(2)})$ on the real line outside $[\Sigma_0, 2m]$ (note that $\Sigma_0 - m < m$ [30]), we follow closely the proof of the hard part of the HVZ theorem for $\theta = 1$ [15]. We introduce the two-cluster decompositions of $h_\theta^{(2)}$,

$$h_\theta^{(2)} \;=\; T_\theta + a_{j,\theta} + r_{j,\theta}, \qquad j = 0,1,2, \quad (4.1)$$

where $a_{j,\theta}$ collects all interactions not involving particle j ($j = 1,2$ denote the two electrons and $j = 0$ refers to the nucleus) and $r_{j,\theta}$ is the remainder. Explicitly,

$$a_{1,\theta} \;=\; b_{1m,\theta}^{(2)} + b_{2m,\theta}^{(2)}, \qquad a_{2,\theta} \;=\; b_{1m,\theta}^{(1)} + b_{2m,\theta}^{(1)}, \qquad a_{0,\theta} \;=\; v_\theta^{(12)}. \quad (4.2)$$

We note that due to the symmetry of $h_\theta^{(2)}$ upon electron exchange we have $\sigma(T_\theta + a_{1,\theta}) = \sigma(T_\theta + a_{2,\theta})$ and thus need not consider $j = 1$ and $j = 2$ separately.

Let us shortly investigate the spectrum of $T_\theta + a_{1,\theta}$. First we prove

$$\sigma(T_\theta + a_{1,\theta}) \;=\; \sigma(T_\theta^{(1)}) \;+\; \sigma(b_{m,\theta}^{(2)}) \tag{4.3}$$

by showing the sectoriality of $T_\theta + a_{1,\theta}$. According to [25] $b_{m,\theta}^{(2)} = T_\theta^{(2)} + b_{1m,\theta}^{(2)} + b_{2m,\theta}^{(2)}$ is sectorial if there exists a vertex $z_0 \in \mathbb{C}$, a direction $\beta \in [0, 2\pi)$ and an opening angle $\phi \in [0, \pi)$ such that

$$(\psi, b_{m,\theta}^{(2)}\,\psi) \subset \;\{z \in \mathbb{C} :\; |\arg\,(e^{-i\beta}(z - z_0))| \;\leq\; \frac{\phi}{2}\} \tag{4.4}$$

for $\psi \in \mathcal{A}(H_{1/2}(\mathbb{R}^3) \otimes \mathbb{C}^2)^2$ with $\|\psi\| = 1$.

Clearly, $T_\theta^{(2)}$ is sectorial for $\theta = e^\xi \in \mathcal{D}$ because it is given by the set $\{(p\,e^{-2i\,\mathrm{Im}\,\xi} + m^2)^{\frac{1}{2}} : \, p \in \mathbb{R}_+\}$ which lies in the sector defined by $z_0 = 0, \;\; \beta = 0$ and $\phi = 2|\mathrm{Im}\,\xi| \,\leq 2\xi_0$.

The $|T_\theta^{(2)}|$-form boundedness of the potential part of $b_{m,\theta}^{(2)}$ was proven in the following form (with $\varphi := \psi_{\mathbf{x}_1}(\mathbf{x}_2)$; see section 2),

$$\begin{aligned} |(\varphi, (b_{1m,\theta}^{(2)} + b_{2m,\theta}^{(2)})\,\varphi)| \;&\leq\; \frac{1}{|\theta|}\,(\varphi, (b_1^{(2)} + b_2^{(2)})\,\varphi) \;+\; C\,(\varphi,\varphi) \\ &\leq\; \frac{1}{|\theta|}\,c_0\,(\varphi, p_2\,\varphi) \;+\; C\,(\varphi,\varphi), \end{aligned} \tag{4.5}$$

where $\frac{1}{|\theta|} \leq 1 + 2\xi_0$, and $c_0 = \frac{\gamma}{\gamma_{BR}} - d\gamma^2 < 1$ if $\gamma < 1.006$. In turn, from (2.10), $\frac{1}{|\theta|}(\varphi, p_2\varphi) \;\leq |(\varphi, T_\theta^{(2)}\varphi)|$. Moreover, using estimates similar to those given in [13], we even obtain (for $\xi_0 < \frac{\pi}{4}$)

$$\begin{aligned} |(\varphi, (b_{1m,\theta}^{(2)} + b_{2m,\theta}^{(2)})\,\varphi)| \;&\leq\; c_1\;\mathrm{Re}\;(\varphi, T_\theta^{(2)}\,\varphi) \;+\; C\,(\varphi,\varphi) \\ c_1 :=&\; \frac{c_0}{1 - \xi_0}. \end{aligned} \tag{4.6}$$

Since $c_0 < 1$ we have $c_1 < 1$ for sufficiently small ξ_0. According to [18, Thm 1.33, p.320] (4.6) guarantees that $b_{m,\theta}^{(2)}$ as form sum is also sectorial, with the opening angle ϕ given by

$$0 \;<\; \tan\frac{\phi}{2} \;=\; \frac{\tan|\mathrm{Im}\;\xi| \;+ c_1}{1 - c_1} \;<\infty, \tag{4.7}$$

and some vertex $z_0 < 0$ which has to be sufficiently small (one has the estimate [18, eq.(VI–1.47)] Re $(\varphi, b_{m,\theta}^{(2)}\varphi) \;\geq -C(\varphi,\varphi)$ with the constant C from (4.6)).

As we have just shown, $T_\theta^{(1)}$ is sectorial with maximum opening angle $\phi = 2\xi_0$ and $b_{m,\theta}^{(2)}$ is sectorial with maximum opening angle $\phi_0 =: \phi(\xi_0)$ (obtained upon replacing $|\text{Im}\ \xi|$ by ξ_0 in (4.7) since tan and arctan are monotonically increasing functions). Let us take $\xi_0 < \frac{1}{2}$ such that $\phi + \phi_0 < \pi$. This is done in the following way. Choose some ξ_0. If $2\xi_0 + \phi_0 < \pi$, we are done. If not, since $0 < \phi_0 < \pi$ there is $\delta > 0$ such that $\phi_0 < \delta < \pi$. Then define $\xi_1 := \frac{1}{2}(\pi - \delta) < \xi_0$. From (4.7) and the monotonicity of tan and arctan we have $\phi_0 > \phi(\xi_1)$ and thus $2\xi_1 + \phi(\xi_1) < \pi$.

Writing $\psi \in \mathcal{A}(H_{1/2}(\mathbb{R}^3) \otimes \mathbb{C}^2)^2$ in the form domain of $T_\theta + a_{1,\theta}$ as a finite linear combination of product states $\varphi^{(1)}\varphi^{(2)}$ with $\varphi^{(k)}$ relating to particle k, we have

$$(\varphi^{(1)}\varphi^{(2)}, (T_\theta + a_{1,\theta})\, \varphi^{(1)}\varphi^{(2)}) \;=\; (\varphi^{(1)}, T_\theta^{(1)}\, \varphi^{(1)})(\varphi^{(2)}, \varphi^{(2)})$$

$$+\; (\varphi^{(2)}, b_{m,\theta}^{(2)}\, \varphi^{(2)})\, (\varphi^{(1)}, \varphi^{(1)}). \tag{4.8}$$

Thus, the necessary assumptions for Proposition 4 of [25] (which is based on a lemma of Ichinose) are satisfied, which guarantees that $T_\theta + a_{1,\theta}$ is sectorial, as well as the validity of (4.3).

Now we derive the explicit form of $\sigma(T_\theta + a_{1,\theta})$. Since $T_\theta^{(1)} = \sqrt{p_1^2/\theta^2 + m^2}$, $p_1 \geq 0$, it follows that $\sigma(T_\theta^{(1)}) = \sigma_{ess}(T_\theta^{(1)})$ is for each $\theta \in \mathcal{D}$ a curve in the complex plane intersecting $\mathbb{R}$ only in the point m.

Concerning the spectrum of $b_{m,\theta}^{(2)}$ we know that $\sigma_{ess}(b_{m,\theta}^{(2)}) = \sigma_{ess}(T_\theta^{(2)})$. Thus we get from (4.3)

$$\sigma(T_\theta + a_{1,\theta}) = \{\sqrt{p_1^2/\theta^2 + m^2} : p_1 \geq 0\} + \left(\{\sqrt{p_2^2/\theta^2 + m^2} : p_2 \geq 0\} \cup \sigma_d(b_{m,\theta}^{(2)})\right)$$

$$=\; \sigma_{ess}(T_\theta + a_{1,\theta}). \tag{4.9}$$

Any $\lambda_2^{(\theta)} \in \sigma_d(b_{m,\theta}^{(2)})$ is a discrete eigenvalue of finite multiplicity. Therefore, since $b_{m,\theta}^{(2)}$ is a dilation analytic operator in $\mathcal{D}$ it follows from [18, p.387],[22, p.22] that $\lambda_2^{(\theta)}$ is an analytic function of θ in $\mathcal{D}$ (as long as it remains an isolated eigenvalue). If $\theta \in \mathbb{R} \cap \mathcal{D}$, $\lambda_2^{(\theta)} = \lambda_2 \in \sigma_d(b_m^{(2)})$ because d_θ is unitary for real θ. By means of the identity theorem of complex analysis one has $\lambda_2^{(\theta)} = \lambda_2$ for all $\theta \in \mathcal{D}$ [1]. Conversely, assume there exists $\tilde{\lambda}_2^{(\theta)} \in \sigma_d(b_{m,\theta}^{(2)})$ in $\mathbb{C}\backslash\mathbb{R}$ (called 'resonance' [22, p.191]) for a given $\theta \in \mathcal{D}$. Then from the group property (2.2), a further dilation by any $\tilde{\theta} \in \mathbb{R}$ leaves $\tilde{\lambda}_2^{(\theta)}$ invariant. Thus $\tilde{\lambda}_2^{(\theta)}$ is invariant in the subset of $\mathcal{D}$ in which it is analytic.

As a consequence [29], resonances are only possible in the sector bounded by $\sigma(T_\theta^{(2)})$ and $[m, \infty)$. The curve $\{\sqrt{p^2/\theta^2 + m^2} : p \geq 0\}$ lies in the closed half plane below (respectively above) the real axix, if $\theta = e^\xi$ with Im $\xi > 0$

(respectively Im $\xi < 0$). Therefore, if e.g. Im $\xi > 0$, no elements of $\sigma_d(b^{(2)}_{m,\theta})$ lie in the upper half plane (they would be isolated for all θ with Im $\xi \geq 0$, but such elements have to be real). Moreover, they can at most accumulate at m. (If they did accumulate at some $z_0 \in \sigma(T_\theta^{(2)})\backslash\{m\}$ then, for $\theta_0 = e^{\xi + i\delta}$ $(\delta > 0)$ they would, due to their θ-invariance, still accumulate at $z_0 \notin \sigma(T_{\theta_0}^{(2)})$ which is impossible.) Likewise, real elements of $\sigma_d(b^{(2)}_{m,\theta})$ can only accumulate at m.

From (4.9) it follows that $\sigma(T_\theta + a_{1,\theta})$ consists of a system of parallel curves each starting at $m + \lambda_2^{(\theta)}$ for any $\lambda_2^{(\theta)} \in \sigma_d(b^{(2)}_{m,\theta})$ (with $e^{-i\,\mathrm{Im}\,\xi}\mathbb{R}_+ + \lambda_2^{(\theta)}$ as asymptote), supplied by an area in the complex plane bounded to the right by such a curve starting at the point $2m$. (This curve has the asymptote $m + e^{-i\,\mathrm{Im}\,\xi}\mathbb{R}_+$; the left boundary is a line starting at $2m$ with $e^{-i\,\mathrm{Im}\,\xi}\mathbb{R}_+$ as asymptote.) We note that in the Schrödinger case this area degenerates to one straight line [3]. From the discussion above, $\sigma(T_\theta + a_{1,\theta}) \cap \mathbb{R}$ consists of $\{2m\}$ plus isolated points which can at most accumulate at $2m$.

As concerns the spectrum $\sigma(T_\theta + v_\theta^{(12)})$, (2.3) leads to

$$T_\theta + v_\theta^{(12)} \;=\; \frac{1}{\theta}\sqrt{p_1^2 + m^2\theta^2} \;+\; \frac{1}{\theta}\sqrt{p_2^2 + m^2\theta^2} \;+\; \frac{1}{\theta}\, v^{(12)} \tag{4.10}$$

with $\sigma(T_\theta)$ as discussed above. For the two-particle potential we have

$$(\psi, v_\theta^{(12)}\,\psi) \;=\; \frac{1}{\theta}\,(\psi, v^{(12)}\,\psi) \;\subset\; e^{-i\,\mathrm{Im}\,\xi}(\mathbb{R}_+ \cup \{0\}) \tag{4.11}$$

since $v^{(12)} \geq 0$. This shows that $T_\theta + v_\theta^{(12)}$ is sectorial and that $\sigma(T_\theta + v_\theta^{(12)}) \cap \mathbb{R} \subset \{2m\}$.

When considering the essential spectrum of $h_\theta^{(2)}$ one can drop the two-particle interaction $c_\theta^{(12)}$. In the case $\theta = 1$ (and $\gamma < 0.66$) it was shown for $\tilde{h}_\theta^{(2)} := h_\theta^{(2)} - c_\theta^{(12)}$ that $\sigma_{ess}(h^{(2)}) = \sigma_{ess}(\tilde{h}^{(2)})$ since the difference of the resolvents of $h^{(2)}$ and $\tilde{h}^{(2)}$ is compact [12, 15]. The required estimates hold also for $\theta \in \mathcal{D}$ due to the dilation analyticity of the involved operators and because the constituents $A_\theta(p_k)$, $g_\theta(p_k)$ and $E_\theta(p_k)$ of $h_\theta^{(2)}$ are estimated by their respective expressions for $\theta = 1$ according to (2.5) and (2.6). For the sake of demonstration we provide in Appendix C the proof of the relative operator boundedness of the potential terms of $h_\theta^{(2)}$ with respect to T_θ (which is required for the compactness proof and which necessitates the bound $\gamma_c = 0.66$).

Let us now assume that $\lambda \in \sigma_{ess}(\tilde{h}_\theta^{(2)})$ with $\lambda \in \mathbb{R}$, $\lambda > 2m$. Then there exists a Weyl sequence $(\psi_n)_{n\in\mathbb{N}} \in \mathcal{A}(C_0^\infty(\mathbb{R}^6\backslash B_n(0)) \otimes \mathbb{C}^4)$ with $\psi_n \overset{w}{\rightharpoonup} 0$, $\|\psi_n\| = 1$ and $\|(\tilde{h}_\theta^{(2)} - \lambda)\psi_n\| \to 0$ as $n \to \infty$. Using that $|(\psi_n, (\tilde{h}_\theta^{(2)} - \lambda)\psi_n)| \leq \|\psi_n\|\;\|(\tilde{h}_\theta^{(2)} - \lambda)\psi_n\|$, one also has

$$\lim_{n\to\infty}(\psi_n, (\tilde{h}_\theta^{(2)} - \lambda)\,\psi_n) \;=\; 0. \tag{4.12}$$

The existence of a Weyl sequence supported outside balls $B_n(0)$ with radius n centered at the origin follows from the respective fact for $\theta = 1$ [15], from the dilation analyticity of $\tilde{h}_\theta^{(2)}$ for $\theta \in \mathcal{D}$ and from ψ_n being an analytic vector.

For the subsequent argumentation we need the localization formula

$$(\psi_n, \tilde{h}_\theta^{(2)}\, \psi_n) \;=\; \sum_{j=0}^{2} (\phi_j \psi_n,\, (T_\theta + a_{j,\theta})\, \phi_j \psi_n) \;+\; O(\frac{1}{n})\; \|\psi_n\|^2 \tag{4.13}$$

where $(\phi_j)_{j=0,1,2}$ is the Ruelle-Simon partition of unity subordinate to the two-cluster decompositions (4.1) with the property $\sum\limits_{j=0}^{2} \phi_j^2 = 1$ (see e.g. [7, p.33],[19]). This formula was derived in [15] for $\theta = 1$. Its proof involves the uniform boundedness of $\sum\limits_{j=0}^{2} (\phi_j \psi_n, [\tilde{h}^{(2)}, \phi_j] \psi_n)$ and of $(\phi_j \psi_n, r_j\, \phi_j \psi_n)$, the bound being of $O(\frac{1}{n})$. For $\theta \in \mathcal{D}$, the required estimates can be carried out in the same way by using (2.5) and (2.6) as in the compactness proof mentioned above.

The combination of (4.12) and (4.13) leads to two real equations,

$$\lambda \;=\; \lim_{n\to\infty}\; \mathrm{Re}\; \sum_{j=0}^{2} (\phi_j \psi_n,\, (T_\theta + a_{j,\theta})\, \phi_j \psi_n) \tag{4.14}$$

$$0 \;=\; \lim_{n\to\infty}\; \mathrm{Im}\; \sum_{j=0}^{2} (\phi_j \psi_n,\, (T_\theta + a_{j,\theta})\, \phi_j \psi_n).$$

Since $\sigma(T_\theta + a_{j,\theta})$ for $\theta \in \mathcal{D}\backslash\mathbb{R}$ lies in the closed half plane of $\mathbb{C}$ (the lower one if Im $\xi > 0$) for all j, the second equation requires $(\phi_j \psi_n, (T_\theta + a_{j,\theta}) \phi_j \psi_n)$ to be real. From (4.9) and the discussion below we have $\sigma(T_\theta + a_{1,\theta}) \cap \mathbb{R} \subset [0, 2m]$ since $b_m^{(2)}$ does not have eigenvalues above m (for $\gamma < 0.29$ according to Proposition 1). Dilation analyticity of $b_{m,\theta}^{(2)}$ assures that also $b_{m,\theta}^{(2)}$ does not have such eigenvalues. Moreover, from the discussion below (4.11) we know that $\sigma(T_\theta + a_{0,\theta}) \cap \mathbb{R} \subset [0, 2m]$. Therefore,

$$\lambda \;\leq\; \lim_{n\to\infty} \sum_{j=0}^{2} 2m\; (\phi_j \psi_n, \phi_j \psi_n) \;=\; 2m \tag{4.15}$$

in contradiction to the assumption $\lambda > 2m$.

In order to show that there are only discrete eigenvalues of $h_\theta^{(2)}$ on the real line below Σ_0 we recall that for $\theta = 1$, the HVZ theorem states that $\sigma_{ess}(h^{(2)}) = [\Sigma_0, \infty)$ where $\Sigma_0 = \inf \sigma(T + a_1)$. Thus $h^{(2)}$ has at most isolated eigenvalues (of finite multiplicity) below Σ_0.

Due to the sectoriality of $T_\theta + a_{j,\theta}$ and the fact that $\sigma(T_\theta + v_\theta^{(12)}) \cap \mathbb{R} \subset \sigma(T_\theta + a_{1,\theta}) \cap \mathbb{R}$, there is no spectrum of $T_\theta + a_{j,\theta}$ in the half plane of $\mathbb{C}$ bounded to the right by $\{z \in \mathbb{C}\,|\,\mathrm{Re}\, z = \Sigma_0\}$. Therefore, $\sigma(T_\theta + a_{j,\theta}) \cap \mathbb{R} \subset [\Sigma_0, \infty)$. Then, repeating the above argumentation for $\lambda \in \sigma_{ess}(h_\theta^{(2)})$ with $\lambda < \Sigma_0$, one gets from (4.14)

$$\lambda \;\geq\; \lim_{n\to\infty} \sum_{j=0}^{2} \Sigma_0 \, (\phi_j \psi_n, \phi_j \psi_n) \;=\; \Sigma_0, \tag{4.16}$$

a contradiction. This proves that $\sigma_{ess}(h_\theta^{(2)}) \cap \mathbb{R}$ is confined to the interval $[\Sigma_0, 2m]$.

The basic ingredient of the proof of Theorem 1 is the invariance of the resolvent form under dilations with $\theta \in \mathcal{D} \cap \mathbb{R}$,

$$(\psi, \frac{1}{h^{(2)} - z}\, \psi) \;=\; (d_\theta \psi, \frac{1}{h_\theta^{(2)} - z}\, d_\theta \psi) \qquad \text{for } z \in \mathbb{C}\backslash\mathbb{R}. \tag{4.17}$$

where we restrict ourselves to analytic vectors $\psi \in \mathcal{A}(N_{\xi_0} \otimes \mathbb{C}^2)^2$ with $N_{\xi_0} := \{\varphi \in H_{1/2}(\mathbb{R}^3) : d_\theta \varphi$ is analytic in $\mathcal{D}\}$. For $z \in \mathbb{C}\backslash\sigma(h_\theta^{(2)})$, the analyticity of $(h_\theta^{(2)} - z)^{-1}$ and of the function $d_\theta \psi$ allows for the extension of the r.h.s. of (4.17) to complex $\theta \in \mathcal{D}$. The identity theorem of complex analysis then guarantees the equality (4.17) for all $\theta \in \mathcal{D}$. Since N_{ξ_0} is dense in $H_{1/2}$ [22, p.187], (4.17) holds for all ψ in $\mathcal{A}(H_{1/2}(\mathbb{R}^3) \otimes \mathbb{C}^2)^2$.

From (4.17) it follows that $(\psi, \frac{1}{h^{(2)}-z}\, \psi)$ has continuous boundary values as $\mathrm{Im}\; z \to 0$ for $\mathrm{Re}\;\, z \notin [\Sigma_0, 2m] \cup (\sigma_d(h_\theta^{(2)}) \cap \mathbb{R})$. Therefore the singular continuous spectrum is absent for $\mathbb{R}\backslash([\Sigma_0, 2m] \cup \sigma_d(h_\theta^{(2)})$ [22, p.137,187]. Since the elements of $\sigma_d(h_\theta^{(2)})$ are discrete, there is no singular continuous spectrum of $h^{(2)}$ outside $[\Sigma_0, 2m]$.

For the Brown-Ravenhall operator h_2^{BR} the proof is exactly the same, because one only has to drop the second-order potential terms. The less restrictive bound on γ follows from Proposition 1 (i). □

Appendix A (Boundedness of $c^{(12)}(m \cdot \theta) - c_0^{(12)}$)

From (1.5) and (1.6) one derives for the $k = 1$ contribution to the kernel of this operator [16], using the mean value theorem,

$$\left|(K^{(1)}_{c^{(12)}(m\cdot\theta)} - K^{(1)}_{c_0^{(12)}})(\mathbf{p}_1, \mathbf{p}_2; \mathbf{p}_1', \mathbf{p}_2')\right| \;\leq\; \frac{\gamma e^2 m}{(2\pi)^4}\, \frac{1}{|\mathbf{p}_2 - \mathbf{p}_2'|^2}\, \frac{1}{|\mathbf{p}_2 - \mathbf{p}_2' + \mathbf{p}_1 - \mathbf{p}_1'|^2}$$

$$\cdot \left| \frac{\partial}{\partial m}\, \{U_0^{(1)} U_0^{(2)}\, [\frac{1}{E_{|\mathbf{p}_2 - \mathbf{p}_2' + \mathbf{p}_1|} + E_{p_1'}} \Big(1 + \tilde{D}_0^{(1)}(\mathbf{p}_1') - \tilde{D}_0^{(1)}(\mathbf{p}_2 - \mathbf{p}_2' + \mathbf{p}_1) \right. \tag{A.1}$$

$$-\tilde{D}_0^{(1)}(\mathbf{p}_2-\mathbf{p}_2'+\mathbf{p}_1)\tilde{D}_0^{(1)}(\mathbf{p}_1')\Big) \;+\; h.c.\,]\;\; U_0^{(1')*}U_0^{(2')*}\,\}(\tilde{m}_1\cdot\theta) \;+\; (\tilde{m}_1\mapsto\tilde{m}_2)\Big|$$

where *h.c.* denotes the hermitean conjugate of the first term together with the replacement $(\mathbf{p}_1,\mathbf{p}_2)\leftrightarrows(\mathbf{p}_1',\mathbf{p}_2')$. (The second contribution ($k=2$) to the kernel arises from particle exchange and is therefore bounded by the same constant.) After carrying out the derivative, the modulus of each of the resulting terms is estimated separately, using the boundedness of the dilated $U_0^{(k)}$, $\tilde{D}_0^{(k)}$ and the estimate (2.6) for the dilated energy denominator. According to the Lieb and Yau formula (3.7), $c^{(12)}(m\cdot\theta)-c_0^{(12)}$ is bounded if the integral

$$I(\mathbf{p}_1,\mathbf{p}_2) := \int_{\mathbb{R}^6} d\mathbf{p}_1' d\mathbf{p}_2' \,\left|(K^{(1)}_{c^{(12)}(m\cdot\theta)}-K^{(1)}_{c_0^{(12)}})(\mathbf{p}_1,\mathbf{p}_2;\mathbf{p}_1',\mathbf{p}_2')\right| \,\frac{f(p_1)g(p_2)}{f(p_1')g(p_2')} \tag{A.2}$$

is bounded for all $\mathbf{p}_1,\mathbf{p}_2\in\mathbb{R}^3$, where $f,g\geq 0$ are suitably chosen functions. The derivative of the operator $\tilde{D}_{0,\theta}^{(k)}$ can be estimated by $\frac{c}{p_k}$ because $\tilde{D}_0^{(k)}$ is bounded and its m-dependence enters only via m/p_k (see (2.14)). Finally, since from (2.5), $\left|\frac{m\theta}{E_\theta(p')}\right| \leq m|\theta|\,\frac{1}{(1-\xi_0)E_{p'}} \leq (1-\xi_0)^{-2}$, the derivative of the energy denominator is estimated by $\left|\frac{\partial}{\partial m}\,\frac{1}{E_\theta(p)+E_\theta(p')}\right| \leq \frac{c}{p}\,\frac{1}{p+p'}$.

For reasons of convergence we have to keep, however, the m-dependence of the energy denominator in those contributions to (A.1) which contain the factor $1/p_2'$ from the estimate of the derivatives. This can be handled in the following way: Let $f(m)g(m)-f(0)g(0)=[f(m)-f(0)]g(m)+f(0)[g(m)-g(0)]$ and interpret g as the energy denominator and f as the adjacent factors inside the curly bracket in (A.1). Then, while estimating the derivative of $f(m)$ by an m-independent function (in general setting $m=0$), the energy denominator can be estimated, using (2.6), by

$$\left|\frac{1}{E_\theta(|\mathbf{p}_2-\mathbf{p}_2'+\mathbf{p}_1|)+E_\theta(p_1')}\right| \;\leq\; \frac{1}{(1-\xi_0)^3}\;\frac{1}{E_{|\mathbf{p}_2-\mathbf{p}_2'+\mathbf{p}_1|}+E_{p_1'}}$$

$$\leq\; \frac{c}{|\mathbf{p}_2-\mathbf{p}_2'+\mathbf{p}_1|+p_1'+1} \tag{A.3}$$

which relies on $m\neq 0$.

For the sake of demonstration we select the contribution to (A.1) which contains the derivative of $U_0^{(2')*}$, leading to the estimate $\frac{\tilde{c}}{p_2'}$ according to (2.14). Absorbing the bounds of $U_0^{(1)}$, $U_0^{(2)}$, $U_0^{(1')*}$ and $\tilde{D}_0^{(1)}$ into the generic constant c, we get for the respective contribution, say $\tilde{I}$, to (A.2),

$$\tilde{I}(\mathbf{p}_1,\mathbf{p}_2) \;\leq\; m\cdot c\int_{\mathbb{R}^6} d\mathbf{p}_1' d\mathbf{p}_2'\;\frac{1}{|\mathbf{p}_2-\mathbf{p}_2'|^2}\;\frac{1}{|\mathbf{p}_2-\mathbf{p}_2'+\mathbf{p}_1-\mathbf{p}_1'|^2}$$

$$\cdot\,\frac{1}{|\mathbf{p}_2-\mathbf{p}_2'+\mathbf{p}_1|+p_1'+1}\cdot\frac{1}{p_2'}\;\frac{f(p_1)g(p_2)}{f(p_1')g(p_2')}. \tag{A.4}$$

We choose $f(p) = p^{\frac{1}{2}}$ and $g(p) = p$. Making the substitution $\mathbf{q} := \mathbf{p}_2' - \mathbf{p}_2$ for $\mathbf{p}_2'$ and defining $\boldsymbol{\xi}_2 := \mathbf{q} - \mathbf{p}_1$ we have

$$\tilde{I}(\mathbf{p}_1, \mathbf{p}_2) \;\le\; m \cdot c\, p_1^{\frac{1}{2}} \int_{\mathbb{R}^3} d\mathbf{q}\, \frac{1}{q^2}\, \frac{p_2}{|\mathbf{q} + \mathbf{p}_2|^2} \int_{\mathbb{R}^3} d\mathbf{p}_1'\, \frac{1}{|\boldsymbol{\xi}_2 + \mathbf{p}_1'|^2}\, \frac{1}{\xi_2 + p_1' + 1}\, \frac{1}{p_1'^{1/2}}. \tag{A.5}$$

For $\xi_2 = 0$, the second integral is bounded. For $\xi_2 \neq 0$, let $y := p_1'/\xi_2$. Then the second integral turns into [12, Appendix A]

$$\frac{2\pi}{\xi_2} \int_0^\infty \frac{dp_1'}{\xi_2 + p_1' + 1}\, p_1'^{1/2}\, \ln \frac{\xi_2 + p_1'}{|\xi_2 - p_1'|}$$

$$= \; 2\pi \int_0^\infty \frac{dy}{y^{\frac{1}{2}}}\, \ln \frac{1+y}{|1-y|} \cdot \frac{\xi_2^{\frac{1}{2}}\, y}{\xi_2(1+y)+1} \;\le\; \frac{\tilde{c}}{1 + \xi_2^{1/2}} \tag{A.6}$$

since the last factor can be estimated by $\frac{c}{1+\xi_2^{1/2}}$ and the remaining integral is convergent. Therefore we get with the substitution $\mathbf{q}_2 := \mathbf{q}/p_2$,

$$\tilde{I}(\mathbf{p}_1, \mathbf{p}_2) \;\le\; m\, c\, \tilde{c} \int_{\mathbb{R}^3} d\mathbf{q}_2\, \frac{1}{q_2^2}\, \frac{1}{|\mathbf{q}_2 + \mathbf{e}_{p_2}|^2} \cdot \frac{1}{p_1^{-\frac{1}{2}} + |\mathbf{q}_2 p_2/p_1 \,-\, \mathbf{e}_{p_1}|^{\frac{1}{2}}} \tag{A.7}$$

where $\mathbf{e}_{p_i}$ is the unit vector in the direction of $\mathbf{p}_i$, $\;i = 1, 2$. The last factor is bounded for $p_1 < \infty$, and the remaining integral is finite. For $p_1 \to \infty$, one gets at most an additional square-root singularity, which is integrable. Thus $\tilde{I}$ is finite.

The contribution to (A.1) arising from the derivative of $U_0^{(2)}$ which is estimated by $\frac{c}{p_2}$, is handled by the same integrals if one chooses $g(p) = p^2$ instead of $g(p) = p$. For the boundedness of the remaining contributions to (A.1) one can use similar techniques as for the proof of the p-form boundedness of $c^{(12)}$ [16]. One must, however, take care to use the same convergence generating functions in the corresponding hermitean conjugate term entering into the r.h.s. of (A.1). (For example, in the estimates of the derivative of $\tilde{D}_0^{(1)}$, one should take $f(p) = p^{\frac{3}{2}}$ and $g(p) = 1$.)

Appendix B (Estimates for γM_1 and $e^2 M_2$)

From (3.4) and (3.5) we have

$$|\gamma M_1| \;\le\; \frac{1}{2} \Big| \int_{\mathbb{R}^{12}} d\boldsymbol{\pi}\, \overline{\hat{\psi}(\mathbf{p}_1, \mathbf{p}_2)} \Big\{ (1 - \frac{m}{E_{p_1}})\, (\frac{1}{E_{p_1}} - \frac{c_0}{E_{p_1} + E_{p_2} - m})\, b_{1m}^{(1)}$$

$$+\, b_{1m}^{(1)}\, (1 - \frac{m}{E_{p_1}})\, (\frac{1}{E_{p_1}} - \frac{c_0}{E_{p_1} + E_{p_2} - m}) \,-\, c_0\, (1 - \frac{m}{E_{p_1}})\, \frac{1}{E_{p_1} + E_{p_2} - m}\, b_{1m}^{(2)}$$

$$- \; c_0 \, b_{1m}^{(2)} \, (1 \, - \, \frac{m}{E_{p_1}}) \;\; \frac{1}{E_{p_1} + E_{p_2} - m} \Big\} \, \hat{\psi}(\mathbf{p}_1, \mathbf{p}_2) \Big| \; + \; |\beta_{11}(m)|. \qquad \text{(B.1)}$$

Each of the four terms in curly brackets is estimated separately by its modulus. For the sake of demonstration we select the second term. With $k_{b_{1m}}$ from (3.6) and the Lieb and Yau formula (3.7), we get

$$T_b := \; \frac{1}{2} \Big| \int_{\mathbb{R}^{12}} d\boldsymbol{\pi} \; \overline{\hat{\psi}_0(\mathbf{p}_1, \mathbf{p}_2)} \, U_0^{(1)} \, k_{b_{1m}} U_0^{(1')*} \left(1 \, - \, \frac{m}{E_{p_1'}}\right)$$

$$\cdot \left(\frac{1}{E_{p_1'}} \, - \, \frac{c_0}{E_{p_1'} + E_{p_2'} - m} \right) \, \hat{\psi}_0(\mathbf{p}_1', \mathbf{p}_2') \Big| \qquad \text{(B.2)}$$

$$\leq \; \frac{1}{2} \int_{\mathbb{R}^6} d\mathbf{p}_1 d\mathbf{p}_2 \, \left| U_0^{(1)*} \, \hat{\psi}_0(\mathbf{p}_1, \mathbf{p}_2) \right|^2 \cdot I_b.$$

Taking $f(p_1) = \frac{p_1^{5/2}}{\sqrt{p_1^2 + m^2} + m}$, $\;g = 1$, and estimating $|\frac{1}{E_{p_1'}} - \frac{c_0}{E_{p_1'} + E_{p_2'} - m}| \leq \frac{1}{E_{p_1'}}$ (which holds for $c_0 \leq 2$) we obtain in the new variables $\mathbf{q}_i, \mathbf{q}_i'$ after performing the angular integration in the variable $\mathbf{q}_1'$ [12, Appendix A],

$$I_b := \; \frac{\gamma}{2\pi^2} \int_{\mathbb{R}^6} d\mathbf{p}_1' d\mathbf{p}_2' \, (1 \, - \, \frac{m}{E_{p_1'}}) \, \left| \frac{1}{E_{p_1'}} \, - \, \frac{c_0}{E_{p_1'} + E_{p_2'} - m} \right|$$

$$\cdot \, \frac{1}{|\mathbf{p}_1 - \mathbf{p}_1'|^2} \, \delta(\mathbf{p}_2 - \mathbf{p}_2') \, \frac{f(p_1)}{f(p_1')} \qquad \text{(B.3)}$$

$$\leq \; \frac{\gamma}{\pi} \, q_1^{3/2} \, \frac{1}{\sqrt{q_1^2 + 1} + 1} \int_0^\infty dq_1' \; \ln \, \frac{q_1 + q_1'}{|q_1 - q_1'|} \; q_1'^{\frac{1}{2}} \, \frac{1}{q_1'^2 + 1}.$$

In order to get an analytical estimate of (B.3) we use

$$\frac{1}{q_1'^2 + 1} \; \leq \; \begin{cases} 1, & q_1' \leq 1 \\ \frac{1}{q_1'^2}, & q_1' > 1 \end{cases} \qquad \text{(B.4)}$$

such that, upon substituting $q_1' =: q_1 \, z$ [12, Appendix A],

$$I_b \; \leq \; \frac{\gamma}{\pi} \, \frac{q_1^3}{\sqrt{q_1^2 + 1} + 1} \left[\int_0^{1/q_1} dz \, z^{\frac{1}{2}} \, \ln \frac{1 + z}{|1 - z|} \; + \; \frac{1}{q_1^2} \int_{1/q_1}^\infty \frac{dz}{z^{3/2}} \; \ln \, \frac{1 + z}{|1 - z|} \right]$$

$$= \; \frac{\gamma}{\pi} \, q_1^2 \, \frac{1}{\sqrt{q_1^2 + 1} + 1} \, \left[q_1 F_{1/2}(\frac{1}{q_1}) \, + \, \frac{1}{q_1} \, G_{-3/2}(\frac{1}{q_1}) \right], \qquad \text{(B.5)}$$

$$F_{1/2}(a) := \; \frac{2}{3} \, \left[a^{3/2} \ln \left| \frac{1 + a}{1 - a} \right| \, + \, 4\sqrt{a} \, - \, 2 \arctan \sqrt{a} \, - \, \ln \left| \frac{1 + \sqrt{a}}{1 - \sqrt{a}} \right| \right]$$

$$G_{-3/2}(a) := \; 2\pi \, - \, 2 \ln \left| \frac{\sqrt{a} + 1}{\sqrt{a} - 1} \right| \, - \, 4 \arctan \sqrt{a} \, + \, \frac{2}{\sqrt{a}} \, \ln \left| \frac{1 + a}{1 - a} \right|.$$

For the first contribution to $|\gamma M_1|$, the same functions f, g have to be taken, and the approximation $\sqrt{q_1'^2+1} \leq q_1'+1$ is made to allow for an analytic evaluation of the corresponding integral. For the third and fourth contribution to $|\gamma M_1|$ we use instead $f=1,\;\; g(p_2)=p_2^{3/2}$ and the additional estimate (for $c \geq 0$)

$$\frac{1}{\sqrt{q'^2+1}+c} \;\leq\; \begin{cases} \dfrac{1}{1+c}, & q' \leq 1+c \\[2ex] \dfrac{1}{q'}, & q' > 1+c \end{cases} . \tag{B.6}$$

For the estimate of $|\beta_{11}(m)|$ we define $\psi_1 := g(p_1)\boldsymbol{\sigma}^{(1)}\mathbf{p}_1\,\psi$, take $f(p_1) = p_1^{3/2}$, $g=1$ and use again (B.6). With $|U_0^{(1)*}\hat{\psi}_0(\mathbf{p}_1,\mathbf{p}_2)|^2 = |\hat{\psi}(\mathbf{p}_1,\mathbf{p}_2)|^2$ and $|\hat{\psi}_1(\mathbf{p}_1,\mathbf{p}_2)|^2 = \frac{q_1^2}{2\sqrt{q_1^2+1}(\sqrt{q_1^2+1}+1)}\,|\hat{\psi}(\mathbf{p}_1,\mathbf{p}_2)|^2$ we then obtain

$$\begin{aligned} q_1^2\,\tilde{M}_1 = \frac{q_1^2}{l\,(l+1)} \Biggl\{ \left| \frac{1}{l} - \frac{c_0}{l+\sqrt{q_2^2+1}-1} \right| \frac{q_1(q_1+2)}{l+1} \\ + \frac{l}{2\pi}\left(q_1 F_{1/2}(\frac{1}{q_1}) + \frac{1}{q_1}\,G_{-3/2}(\frac{1}{q_1})\right) + c_0\,\frac{q_2}{l+\sqrt{q_2^2+1}-1} \\ + \frac{c_0}{2\pi}\left(\frac{q_2}{l}\,F_{-1/2}(\frac{l}{q_2}) + G_{-3/2}(\frac{l}{q_2})\right) \\ + \frac{1}{2\pi}\left(\frac{2\pi q_1}{l} + q_1 F_{-1/2}(\frac{1}{q_1}) + G_{-3/2}(\frac{1}{q_1})\right) \Biggr\} \end{aligned} \tag{B.7}$$

where $l = \sqrt{q_1^2+1}$ and

$$F_{-1/2}(a) := \;2\sqrt{a}\,\ln\left|\frac{1+a}{1-a}\right| + 4\,\arctan\sqrt{a} \,-\, 2\,\ln\left|\frac{\sqrt{a}+1}{\sqrt{a}-1}\right|. \tag{B.8}$$

For estimating $e^2 M_2$ the same techniques are used, except for the simpler estimate $\frac{1}{\sqrt{q_1'^2+1}} \leq \frac{1}{q_1'}$ in the last contribution (which has little effect on γ_c due to the smallness of e^2). This results in

$$\begin{aligned} q_1^2\tilde{M}_2 \;=\; \frac{q_1^2}{l\,(l+1)} \Biggl\{ \left| \frac{1}{l} - \frac{c_0}{l+\sqrt{q_2^2+1}-1} \right| \frac{q_1(q_1+2)}{l+1} \\ + \frac{l}{2\pi}\left(q_1 F_{1/2}(\frac{1}{q_1}) + \frac{1}{q_1}\,G_{-3/2}(\frac{1}{q_1})\right) + \left(\frac{q_1}{l}+1\right) \Biggr\}. \end{aligned} \tag{B.9}$$

Appendix C (Relative boundedness of the potential of $\mathbf{h}_\theta^{(2)}$)

First we estimate with the help of (2.9),

$$\|\theta\, T_\theta\, \psi\|^2 \;=\; \|(\sqrt{p_1^2+m^2\theta^2} \,+\, \sqrt{p_2^2+m^2\theta^2}\,)\,\psi\|^2 \tag{C.1}$$

$$\geq\; (\psi, (\mathrm{Re}\,\sqrt{p_1^2+m^2\theta^2} \,+\, \mathrm{Re}\,\sqrt{p_2^2+m^2\theta^2}\,)^2\,\psi) \;\geq\; (\psi, (p_1+p_2)^2\,\psi),$$

such that $\|T_0\psi\| \;\leq\; |\theta|\,\|T_\theta\psi\|$. Next we decompose for $h_\theta^{(2)} = T_\theta + W_\theta$ analogously to (2.7),

$$\|W_\theta\,\psi\| \;\leq\; \frac{1}{|\theta|}\,\|W_0\psi\| \;+\; \|(W_\theta \,-\, \frac{1}{\theta}\,W_0)\,\psi\|. \tag{C.2}$$

The boundedness of the second term in (C.2) follows immediately from the method of proof of the form boundedness of $W_\theta - \frac{1}{\theta}W_0$ (see e.g. Appendix A). For the first term we estimate, using $\|p_1\psi\|^2 \;=\frac{1}{2}(\psi,(p_1^2+p_2^2)\,\psi) \;\leq \frac{1}{2}(\psi,(p_1+p_2)^2\,\psi)$,

$$\|W_0\,\psi\| \;\leq\; \|\sum_{k=1}^{2}(b_1^{(k)}+b_2^{(k)})\,\psi\| \;+\; \|v_0^{(12)}\psi\| \;+\; \|c_0^{(12)}\psi\| \tag{C.3}$$

$$\leq\; \sqrt{c_w}\;\|T_0\psi\| \;+\; \frac{1}{\sqrt{2}}\,\sqrt{c_v}\,\|T_0\psi\| \;+\; 2\,\frac{1}{\sqrt{2}}\,\sqrt{c_s}\;\|T_0\psi\| \;=:\tilde{c}_1\;\|T_0\psi\|$$

where $c_v = 4e^4$, $c_w = (\frac{4}{3}\gamma+\frac{2}{9}\gamma^2)^2$ and $c_s = (\frac{2\gamma}{\pi}\,[\pi^2/4-1])^2\,c_v$ are calculated in [12, p.72]. With the inequality below (C.1) this guarantees the relative T_θ-boundedness of W_θ. We have $\tilde{c}_1 < 1$ for $\gamma \leq 0.66$.

Acknowledgment

I would like to thank L.Erdös for valuable comments and H.Kalf, H.Siedentop, M.Huber and S.Morozov for helpful discussions. Partial support by the EU network Analysis and Quantum (contract HPRN-CT-2002-00277) is gratefully acknowledged.

References

[1] Aguilar J. and Combes J.M.: A class of analytic perturbations for one-body Schrödinger Hamiltonians, Commun. Math. Phys. 22, 269-279 (1971).

[2] Balinsky A.A. and Evans W.D.: On the virial theorem for the relativistic operator of Brown and Ravenhall, and the absence of embedded eigenvalues, Lett. Math. Phys. 44, 233-248 (1998).

[3] Balslev E. and Combes J.M.: Spectral properties of many-body Schrödinger operators with dilation-analytic interactions, Commun. Math. Phys. 22, 280-294 (1971).

[4] Brown G.E. and Ravenhall D.G.: On the interaction of two electrons, Proc. Roy. Soc. London A208, 552-559 (1951).

[5] Brummelhuis R., Siedentop H. and Stockmeyer E.: The ground-state energy of relativistic one-electron atoms according to Jansen and Hess, Doc. Math. 7, 167-182 (2002).

[6] Burenkov V.I. and Evans W.D.: On the evaluation of the norm of an integral operator associated with the stability of one-electron atoms, Proc. Roy. Soc. (Edinburgh) 128A, 993-1005 (1998).

[7] Cycon H.L., Froese R.G., Kirsch W. and Simon B.: *Schrödinger Operators with Application to Quantum Mechanics and Global Geometry*, Text and Monographs in Physics, 1st Edition, Berlin, Springer, 1987

[8] Douglas M. and Kroll N.M.: Quantum electrodynamical corrections to the fine structure of helium, Ann. Phys. (N.Y.) 82, 89-155 (1974).

[9] Evans W.D., Perry P. and Siedentop H.: The spectrum of relativistic one-electron atoms according to Bethe and Salpeter, Commun. Math. Phys. 178, 733-746 (1996).

[10] Herbst I.W.: Spectral theory of the operator $(p^2+m^2)^{\frac{1}{2}} - Ze^2/r$, Commun. Math. Phys. 53, 285-294 (1977).

[11] Huber M. and Stockmeyer E.: Perturbative implementation of the Furry picture, arXiv:math-ph/0607009 (2006).

[12] Jakubassa-Amundsen D.H.: Spectral theory of the atomic Dirac operator in the no-pair formalism, Ph.D. Thesis, University of Munich (2004).

[13] Jakubassa-Amundsen D.H.: The essential spectrum of relativistic one-electron ions in the Jansen-Hess model, Math. Phys. Electron. J. 8, No.3, 30pp. (2002).

[14] Jakubassa-Amundsen D.H.: The HVZ theorem for a pseudo-relativistic operator, Ann. Henri Poincaré 8, 337-360 (2007).

[15] Jakubassa-Amundsen D.H.: Localization of the essential spectrum for relativistic N-electron ions and atoms, Doc. Math. 10, 417-445 (2005).

[16] Jakubassa-Amundsen D.H.: Pseudorelativistic operator for a two-electron ion, Phys. Rev. A71, 032105, 8pp. (2005).

[17] Jakubassa-Amundsen D.H.: On the point spectrum of a relativistic electron in an electric and magnetic field, J. Phys. A41, 275304, 17pp. (2008).

[18] Kato T.: *Perturbation Theory for Linear Operators*, Berlin, Springer, 1980

[19] Lewis R.T., Siedentop H. and Vugalter S.: The essential spectrum of relativistic multi-particle operators, Ann. Inst. Henri Poincaré 67, No.1, 1-28 (1997).

[20] Lieb E.H. and Yau H.-T.: The stability and instability of relativistic matter, Commun. Math. Phys. 118, 177-213 (1988).

[21] Morozov S. and Vugalter S.: Stability of atoms in the Brown-Ravenhall model, Ann. Henri Poincaré 7, 661-687 (2006).

[22] Reed M. and Simon B.: *Analysis of Operators*, Vol. IV of Methods of Modern Mathematical Physics, New York, Academic Press, 1978

[23] Reiher M. and Wolf A.: Exact decoupling of the Dirac Hamiltonian. II.The generalized Douglas-Kroll-Hess transformation up to arbitrary order, J. Chem. Phys. 121, 10945-10956 (2004).

[24] Siedentop H. and Stockmeyer E.: The Douglas-Kroll-Hess method: Convergence and block-diagonalization of Dirac operators, Ann. Henri Poincaré 7, 45-58 (2006).

[25] Simon B.: Quadratic form techniques and the Balslev-Combes theorem, Commun. Math. Phys. 27, 1-9 (1972).

[26] Thaller B.: *The Dirac Equation*, Berlin, Springer, 1992

[27] Tix C.: Strict positivity of a relativistic Hamiltonian due to Brown and Ravenhall, Bull. London Math. Soc. 30, 283-290 (1998).

[28] Tix C.: Self-adjointness and spectral properties of a pseudo-relativistic Hamiltonian due to Brown and Ravenhall, Preprint mp-arc/97-441 (1997)

[29] Weder R.A.: Spectral properties of the Dirac Hamiltonian, Ann. Soc. Scient. Bruxelles 87 (III), 341-355 (1973).

[30] Wolf A., Reiher M. and Hess B.A.: The generalized Douglas-Kroll transformation, J. Chem. Phys. 117, 9215-9226 (2002).