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On the Stability of Elliptic Equilibria

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Abstract: *We consider stability of elliptic equilibria in Hamiltonian systems in the frame of Nekhoroshev's theory, recovering the steepness assumption, in the form of convexity, from an appropriate treatment of the higher orders. The singularity of the action–angle coordinates is overcome by using Cartesian coordinates. We introduce an essential refinement of the perturbative technique used in a previous work on the subject, and obtain significant improvements of results, namely better values of the exponents controlling the stability time and the confinement around equilibrium, in case the equilibrium frequencies satisfy stronger nonresonance conditions. Within the same nonresonance assumptions the new method provides instead independent informations, namely one gets a better confinement on a reduced time scale.*

Keywords: Hamiltonian systems, Nekhoroshev theorem, elliptic equilibria.

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1. Introduction

A. Purpose and results. The aim of this paper is to improve the study of Nekhoroshev stability of elliptic equilibrium points, started in [1]. The common purpose is to prove long–time stability of elliptic equilibrium points, without assuming Diophantine–like conditions on the frequencies; this is indeed what Nekhoroshev conjectured in his celebrated 1977 paper [2]. The Hamiltonian one deals with is a convergent power series

$$h_2 + h_3 + h_4 + \cdots , \tag{1.1}$$

where any h_k is a homogeneous polynomial of degree k in the canonical coordinates $(p,q) \in \mathbb{R}^{2n}$, and $h_2 = \frac{1}{2}\sum_{j=1}^{n} \overline{\omega}_j \left(p_j^2 + q_j^2\right)$. The problem of stability arises whenever the frequencies $\overline{\omega}_j$ have opposite signs, and this happens in connections with different problems: among them, let us quote the stability of the Lagrangian equilibrium points L_4 and L_5 in the three body problem, and (after a suitable reduction) the stability of elementary configurations of a self–gravitating fluid (the so-called Riemann ellipsoids, see for example [3,4]). Because of its relevance, the problem of stability of elliptic equilibria deserved a certain attention in the literature, but while KAM–like stability was proved rather soon (see for example [5], Chapter 5), instead Nekhoroshev stability was not fully proven till very recently [1,6]. Partial achievements are indeed available: for example in [7–9] Nekhoroshev stability is proved, but (due to the use of the isochronous system h_2 as the unperturbed Hamiltonian) only in the assumption that $\overline{\omega}$ is Diophantine; on the other hand, in [10,11] such a strong assumptions is released, but then (due to the use of action–angle coordinates) the stability domain does not include a full neighbourhood of the equilibrium point.

Let us quickly illustrate the essence of [1], which is the starting point of our present study; the comparison with [6] (that we received during the preparation of this manuscript) is postponed to subsection D below. In [1] one starts with Hamiltonian (1.1), and makes essentially two assumptions: (i) $\overline{\omega}$ admits no resonances $\overline{\omega} \cdot \nu = 0$ with $\nu \in \mathbb{Z}^n \setminus \{0\}$, $|\nu| \le s-1$, for some $s \ge 5$ (here $|\nu| = \sum_j |\nu_j|$); this allows one to carry on perturbation theory up to order $s-1$ (via $s-3$ "Birkhoff steps"), giving the Hamiltonian the form

$$h(w,z) = k(I) + f(w,z) , \tag{1.2}$$

where w, z are the usual complex coordinates

$$w_j = \frac{p_j - iq_j}{i\sqrt{2}} , \qquad z_j = \frac{p_j + iq_j}{\sqrt{2}} , \qquad j = 1, \ldots, n ,$$

while $k = \overline{\omega} \cdot I + \cdots$ is a polynomial of order not exceeding $(s-1)/2$ in the n actions $I_j = iw_j z_j$, and f is a power series in z, w starting with terms of order s. (ii) The normal part k of h removes the degeneracy of the problem, namely k is convex. On the basis of these two assumptions one proves the long-time stability of the origin, more precisely stability for a time-scale

$$t \sim \exp(1/\varepsilon)^{\frac{1}{n}} , \tag{1.3}$$

where ε is (up to a constant) the initial distance of the orbit to the equilibrium point.

Reference [1] was mainly oriented to consider s as small as possible — in fact, throughout the paper $s = 5$ was used — and to reach stability times as long as possible. Concerning the confinement of the orbits around the equilibrium point, our result was instead rather poor, since within the above assumptions we could prove only $I(t) \sim \varepsilon^{1/n}$. Obtaining better confinement was indeed possible, but only on reduced time scales: for example

$$I(t) \sim \varepsilon^{\frac{1}{2}+\frac{1}{2n}} \qquad \text{for} \quad t \sim \exp(1/\varepsilon)^{\frac{1}{2n}} \ . \tag{1.4}$$

As announced in [1], for $s > 5$ one can obtain significantly better results by means of a suitably improved perturbative technique.† Such results are reported in the proposition below, where we use the notation

$$|x|^\infty = \max_{j=1,\ldots,n} |x_j| \ , \qquad x \in \mathbb{C}^n \ ,$$

and for $R > 0$

$$\mathcal{D}(R) = \{(w, z) \in \mathbb{C}^{2n} : |w|^\infty, |z|^\infty \le R\} \ .$$

By 'real' values of the coordinates w, z we mean the case $\overline{z} = iw$ (so that p, q, I, h are real), while iwz stays for the n–tuple $iw_1z_1, \ldots, iw_nz_n$; $[.]$ denotes the integer part of a positive number.

Proposition. *Assume that the Hamiltonian (1.2) converges in $\mathcal{D}(R_c)$ for some $R_c > 0$, and that k is m-convex for some $m > 0$, namely*

$$x \cdot \mathcal{K}(iwz)\, x \ge m\, x \cdot x \qquad \forall x \in \mathbb{R}^n, \quad \forall \textit{ real } (w, z) \textit{ in } \mathcal{D}(R_c) \ ,$$

$\mathcal{K}$ denoting the Hessian of k. Then there exist constants ε_* and T such that any motion of the system with real initial data (w^o, z^o) sufficiently close to the origin:

$$\varepsilon := \frac{|I^o|^\infty}{R_c^2} \le \varepsilon_*, \qquad I^o = iw^oz^o \ ,$$

satisfies

$$|I(t)|^\infty \le 2\varepsilon R_c^2 \tag{1.5}$$

for

$$|t| \le T \exp\left[\left(\frac{\varepsilon_*}{\varepsilon}\right)^{\frac{s-4}{4n}}\right] \ . \tag{1.6}$$

For any $s \ge 5$ the confinement provided by (1.5) is better than the one obtained in [1]. Stability times improve instead with respect to (1.4) if $s > 6$, and with respect to (1.3) if $s > 8$.

† As commented in [1], raising s within the simple technique used in [1] produces only minor improvements of results.

For the sake of completeness, we remark that, for $s < 4n$ and ε small enough, the above bounds could be improved, namely the small exponent $\frac{s-4}{4n}$ in (1.6) turns into $\frac{s-4}{4n-4}$. (This is obtained, see the remark at the end of section 2, by suitably excluding the resonance of multiplicity n.) But the improved results are not worth, in our opinion, the annoying complications one should introduce in the proof.

B. On the constants characterizing the problem. For any $R > 0$ we denote by $\mathcal{A}_R$ the set of all analytic functions $u : \mathcal{D}(R) \to \mathbb{C}$, which satisfy the reality condition $u(w, -i\bar{w}) \in \mathbb{R}$. As in [1], for functions $u \in \mathcal{A}_R$ we shall write the 'Fourier series'

$$u = \sum_{\nu \in \mathbb{Z}^n} u_\nu \ , \qquad u_\nu(w,z) = \hat{u}_\nu(I) e_\nu(w,z) \ ,$$

with $\hat{u}_\nu$ analytic in the polydisk $|I|^\infty \le R^2$ and

$$e_\nu(w,z) = \prod_{j=1}^n \eta_{\nu_j}^{|\nu_j|} \ , \qquad \eta_{\nu_j} = \begin{cases} z_j & \text{for } \nu_j > 0 \\ 1 & \text{for } \nu_j = 0 \\ w_j & \text{for } \nu_j < 0 \end{cases} \ .$$

For $u \in \mathcal{A}_R$ we shall use the norm

$$|u|_R = \sum_{\nu \in \mathbb{Z}^n} |u_\nu|_R^\infty \ , \qquad |u_\nu|_R^\infty = \sup_{(w,z)\in\mathcal{D}(R)} |u_\nu(w,z)| \ .$$

We shall denote

$$\mathcal{F} = \frac{|f|_{R_c}}{R_c^2} \ .$$

Besides R_c, m and $\mathcal{F}$, we shall characterize the Hamiltonian h by two other constants, namely

$$\Omega = \sup_{\mathrm{real}\, w,z \in \mathcal{D}(R_c)} \|\omega(iwz)\| \ , \qquad \|\,.\,\| = \text{euclidean norm} \ ,$$

where of course $\omega = \frac{\partial k}{\partial I}$, and $M \ge m$ such that

$$|\mathcal{K}(iwz)\, x|^\infty \le M |x|^\infty \qquad \forall x \in \mathbb{C}^n \ , \qquad \forall (w,z) \in \mathcal{D}(R_c) \ .$$

Possible values of the constants entering the statement are

$$\varepsilon_* = \Big(\frac{m}{2^{10} M}\Big)^{\frac{4n}{s-4}} \Big(\frac{1}{2^{17}\sqrt{n}} \frac{m R_c^2}{\mathcal{F}}\Big)^{\frac{2}{s-4}} \ , \qquad T = \frac{1}{4\sqrt{n}\,\Omega} \ . \tag{1.7}$$

C. On the proof. The proof shares with [1] the basic idea (in turn taken from [12]), namely using the Cartesian coordinates w, z to overcome the problem of the singularity of the actions at the origin (in spite of the fact that the problem is non–isochronous). This idea however is now implemented in a more subtle way. The technical novelty is as follows: let $R_0 = \sqrt{\varepsilon} R_c$; we introduce a new parameter $\mathcal{R} < R_0$, to be chosen in the geometric part of the proof depending on the resonance properties of $\omega^o = \omega(I^o)$, and we accordingly separate the initial actions I_j^o in 'inner' and 'outer' ones: that is

$$I_j^o \le \mathcal{R}^2 \quad \text{for } j \in J_{\rm in} \ , \qquad I_j^o > \mathcal{R}^2 \quad \text{for } j \in J_{\rm out} \ , \qquad J_{\rm in} \cup J_{\rm out} = \{1,\ldots,n\}$$

($J_{\rm in}$ could be empty). For $j \in J_{\rm in}$ it is mandatory to use Cartesian variables, because of the singularity $I_j = 0$. For $j \in J_{\rm out}$ one instead safely passes to action–angle variables (I_j, φ_j), namely

$$iw_j = \sqrt{I_j}\, e^{i\varphi_j} \ , \qquad z_j = \sqrt{I_j}\, e^{-i\varphi_j} \qquad \text{for } j \in J_{\rm out} \ .$$

With some abuse of notation, we shall denote by (w,z,I,φ) the new mixed coordinates, implicitly understanding that, in such a mixed combination, w_j, z_j have index $j \in J_{\rm in}$, while for I_j, φ_j one has $j \in J_{\rm out}$. The canonical transformation $(w,z,I,\varphi) \mapsto (w,z)$ will be denoted by $\mathcal{C}$. Concerning the domain of the mixed variables, we shall take a small neighbourhood of the initial datum, depending on $\mathcal{R}$ as well as on an 'extension vector' $\rho = (\rho_w, \rho_I, \rho_\varphi)$, of the form

$$\begin{aligned} D_\rho(I^o,\mathcal{R}) = \{(w,z,I,\varphi) : &|w_j|, |z_j| \le \mathcal{R} + \rho_w \ \text{for } j \in J_{\rm in}; \\ &|I_j - I_j^o| \le \rho_I,\ |{\rm Im}\,\varphi_j| \le \rho_\varphi \ \text{for } j \in J_{\rm out}\} \ . \end{aligned} \tag{1.8}$$

Assuming

$$\rho_w \le \mathcal{R} \ , \qquad \rho_I \le \frac{1}{2}\mathcal{R}^2 \ , \qquad \rho_\varphi \le \frac{1}{2} \ , \qquad \mathcal{R} \le \frac{1}{2} R_0 \ , \tag{1.9}$$

one immediately finds

$$\mathcal{C}\big(D_\rho(I^o,\mathcal{R})\big) \subset \mathcal{D}(S) \ , \qquad S = \max(\mathcal{R} + \rho_w, \sqrt{R_0^2 + \rho_I}\, e^{\rho_\varphi}) < 2\,R_0 \ . \tag{1.10}$$

At the same time, the outer actions are well bounded away from zero:

$$|I_j| \ge \frac{1}{2}\mathcal{R}^2 \qquad \text{for } j \in J_{\rm out} \ . \tag{1.11}$$

We shall denote by $A_\sigma(I^o,\mathcal{R})$ the set of all analytic functions $D_\sigma(I^o,\mathcal{R}) \to \mathbb{C}$, which are real for 'real' (z,w). The new Hamiltonian $\widetilde{h} = h \circ \mathcal{C}$ belongs to $A_\rho(I^o,\mathcal{R})$ and has the form

$$\widetilde{h}(w,z,I,\varphi) = k(I) + \widetilde{f}(w,z,I,\varphi) \ , \qquad \widetilde{f} = f \circ \mathcal{C} \ . \tag{1.12}$$

We shall work on $\widetilde{h}$ by the traditional tools of Nekhoroshev theory; as in [1,12] we shall use our favourite Lie series method, namely the 'vector field' version originally described in [13]. The use of Cartesian coordinates for the inner variables eliminates the problem of the singularity of the actions at $I = 0$, while for the outer variables, because of (1.11), the singularity problem does not arise. The essential gain with respect to [1], where all variables are treated as inner ones ($\mathcal{R} = R_0$), is the freedom in the choice of $\mathcal{R}$: the crucial point is that the ratio $\mathcal{R}/R_0$ is not fixed, but adapts to the resonance properties of the initial frequency ω^o, and in particular, for low resonant initial data, it can be much smaller than one. In such a situation, the maximal oscillation of the actions $I_1, \ldots, I_n$ in $D_\rho(I^o, \mathcal{R})$, which is of order $\mathcal{R}^2$, is much less than R_0^2; this makes easier the control of the small divisors, and allows one to base the analysis of resonances on the actual initial frequency ω^o, not on $\overline{\omega}$ as in [1]. This makes the difference.

D. A comment. As already remarked, Nekhoroshev stability of elliptic fixed points has been recently studied also by Niedermann, in [6]. The results by Niedermann concern explicitly the case $s = 6$, and are similar to the present ones, since for such value of s he obtains (in our notations) $t \sim \exp(1/\varepsilon)^{\frac{1}{2n}}$; the confinement is also similar. Although not explicitly stated, it is quite clear that Niedermann's results generalize to any $s \geq 5$, and are very similar to the present ones. On the contrary, they apparently do not cover the results of [1] (better times with worse confinement), although, perhaps, these results could possibly be reproduced by modifying some choices inside the proof. Concerning the technique, our papers share the basic idea, namely using the Cartesian variables to overcome the singularity of the actions. Besides this, they are essentially different: indeed, [6] follows the method by Lochak [10,11] (often referred to as method of "simultaneous approximation"), while we follow the traditional approach by Nekhoroshev [2], with Pöschel's improved geometrical construction [14].

As is known, in the traditional Nekhoroshev approach one constructs normal forms around resonances of any multiplicity (i.e., number of independent resonance relations satisfied by ω) between zero and $n-1$; Lochak's method is based instead on the construction of a reduced set of normal forms, around resonances of maximal multiplicity $n-1$, that is around periodic orbits. As a general rule, the latter method leads in a rapid brilliant way to stability results for exponentially long times; on the other hand the traditional method, thanks to the details of the normal forms, can provide more informations on the actual behavior of the system during the exponentially long times. For the problem at hand of the stability of elliptic equilibria, the traditional approach that here we follow can be useful to investigate, besides the overall stability times, which particular energy exchanges among the n oscillators are allowed or forbidden, for a given initial datum and for a given time scale. Among the possible applications, let us mention the ergodic problem for the Fermi–Pasta–Ulam system: the latter is in fact a case in which the frequencies $\overline{\omega}_1, \ldots, \overline{\omega}_n$ have the same sign, so that the (perpetual) stability of the equilibrium point is trivial, and understanding the details of the energy exchanges is the only relevant question.†

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† In this connection, we would like to make a further comment on a claim in ref. [6]. Apparently, one there suggests that the technique there employed, namely Lochak's method implemented in Cartesian coordinates, could be conveniently used to reproduce, possibly in a simpler way, the results worked out in [12] concerning the stability of gyroscopic rotations. There is however a point that needs to be stressed: in the study of the fast rigid body (in particular, but not only, in the case of gyroscopic rotations), the most relevant question to be looked at is the motion of the angular momentum m in space. But the qualitative features of this motion depend in a crucial way on the resonance properties of the initial datum: if no resonances are present, then the unit vector $\mu = m/|m|$ moves (nearly) regularly along a well defined curve on the unit sphere, like in the Lagrange top; instead in case of resonance, one 'slow angle' appears in the normal form and couples with the degree of freedom describing the orientation of m, possibly giving rise to

2. Proof of the proposition

A. Functions in mixed variables. Consider a function $u = \sum_\nu u_\nu \in \mathcal{A}_{R_c}$, and let $\widetilde{u} \in A_\rho(I^o, \mathcal{R})$ denote its pull–back under $\mathcal{C}$,

$$\widetilde{u} = \sum_{\nu \in \mathbb{Z}^n} \widetilde{u}_\nu \ , \qquad \widetilde{u}_\nu = u_\nu \circ \mathcal{C} \ .$$

For any $\sigma = (\sigma_w, \sigma_I, \sigma_\varphi) \le \rho$ (the inequality is intended to work separately on the separate entries) we denote

$$|\widetilde{u}|_\sigma = \sum_{\nu \in \mathbb{Z}^n} |\widetilde{u}_\nu|_\sigma^\infty \ , \qquad |\widetilde{u}_\nu|_\sigma^\infty = \sup_{(w,z,I,\varphi) \in D_\sigma(I^o, \mathcal{R})} |\widetilde{u}_\nu(w,z,I,\varphi)| \ .$$

Let now $\widetilde{U} = (\widetilde{U}^{w_1}, \dots, \widetilde{U}^{\varphi_n})$ denote the Hamiltonian vector field of $\widetilde{u}$. We shall write

$$\widetilde{U} = \sum_{\nu \in \mathbb{Z}^n} \widetilde{U}_\nu \ ,$$

where $\widetilde{U}_\nu$ is the Hamiltonian vector field of $\widetilde{u}_\nu$, with components $\widetilde{U}_\nu^{w_j}, \dots, \widetilde{U}_\nu^{\varphi_j}$. For vector fields we shall use the norm

$$\begin{aligned} \|\widetilde{U}\|_\sigma &= \sum_{\nu \in \mathbb{Z}^n} \|\widetilde{U}_\nu\|_\sigma^\infty \ , \\ \|\widetilde{U}_\nu\|_\sigma^\infty &= \max\Big(\frac{1}{\rho_w} |\widetilde{U}_\nu^{w_j}|_\sigma^\infty, \frac{1}{\rho_w} |\widetilde{U}_\nu^{z_j}|_\sigma^\infty, \frac{1}{\rho_I} |\widetilde{U}_\nu^{I_j}|_\sigma^\infty, \frac{1}{\rho_\varphi} |\widetilde{U}_\nu^{\varphi_j}|_\sigma^\infty \Big) \end{aligned}$$

(pay attention to the normalization with ρ, not with σ) where the index j is intended to run in $J_{\rm in}$ for the w, z components, in $J_{\rm out}$ for the I, φ components.

Lemma 1. *Let $\widetilde{F}$ be the Hamiltonian vector field of $\widetilde{f}$ in (1.12). If $R_0 < \frac{1}{4} R_c$ and*

$$\rho_w = \mathcal{R} \ , \qquad \rho_I = \frac{1}{2} \mathcal{R}^2 \ , \qquad \rho_\varphi = \frac{1}{2} \ , \tag{2.1}$$

then one has

$$|\widetilde{f}|_\rho \le \Big(\frac{4R_0}{R_c}\Big)^s R_c^2 \mathcal{F} \ , \qquad \|\widetilde{F}\|_\rho \le \frac{4R_c^2}{\mathcal{R}^2} \Big(\frac{4R_0}{R_c}\Big)^s \mathcal{F} \ . \tag{2.2}$$

The easy proof is deferred to section 3. (Here and in the following, the choice (2.1) of ρ, in place of the inequalities (1.9), has no deep meaning, and is made only to simplify some analytic expressions.)

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chaotic motions of μ on the sphere (see [15] for an illustration). The method of simultaneous approximation treats every initial datum as maximally resonant, so it ignores such a distinction. This problem is typical of degenerate systems.

B. Resonant regions. As in [1], we use the Nekhoroshev decomposition of the frequency space into resonant regions, with the improved estimates provided by Pöschel [14]. For any given $N \geq 1$ we consider all the d-dimensional sublattices Λ of $\mathbb{Z}^n$, $0 \leq d \leq n$, called N*-lattices*, such that: (i) Λ is generated by d vectors $\nu_1, \dots \nu_d \in \mathbb{Z}^n$ with $|\nu_j| \leq N$; (ii) Λ is maximal (it is not properly contained in any sublattice of the same dimension). We denote by 0 the zero–dimensional lattice, constituted by the null vector alone. The cells of any lattice $\Lambda \neq 0$ have a minimal d–dimensional euclidean volume, which we denote $\|\Lambda\|$; we put $\|0\| = 1$. Moreover, we denote by $P_\Lambda \omega$ the orthogonal projection of a vector ω onto a lattice $\Lambda \neq 0$.

The definition of the resonant regions depends on the cutoff N and on two positive parameters b and δ. Following [14], for any N–lattice Λ of dimension $d \geq 1$ one defines

$$\delta_\Lambda \;=\; (b\,N)^{d-1}\,\frac{\delta}{\|\Lambda\|}\;; \tag{2.3}$$

for any N-lattice Λ of dimension $d = 0, \dots, n$, one then defines the resonant region B_Λ as the set of all points $\omega \in \mathbb{R}^n$ such that

$$\begin{aligned} \|P_\Lambda \omega\| \;&<\; \delta_\Lambda \\ \|P_{\Lambda'} \omega\| \;&\geq\; \delta_{\Lambda'} \qquad \text{for any } N\text{-lattice } \Lambda' \text{ of dimension } d+1 \end{aligned}$$

(only the former condition for $d = n$, only the latter for $d = 0$). For any given $N \geq 1$ the resonant regions cover $\mathbb{R}^n$, and the following lemma holds [14]:

Lemma 2. *Consider any $N \geq 1$ and assume $b > \sqrt{2}$. Then for any N–lattice $\Lambda \neq \mathbb{Z}^n$, and for any $\omega \in B_\Lambda$, one has*

$$|\omega \cdot \nu| \geq \gamma_\Lambda \qquad\qquad \forall \nu \in \mathbb{Z}^n \setminus \Lambda\,,\; |\nu| \leq N\;,$$

with

$$\gamma_0 \;=\; \delta\;, \qquad \gamma_\Lambda \;=\; (b - \sqrt{2})\,N\,\delta_\Lambda \quad \textit{if} \quad \Lambda \neq 0\,,\mathbb{Z}^n$$

C. The normal form. We shall construct a normal form for the mixed variables Hamiltonian $\widetilde{h}$, adapted to any given N–lattice Λ, up to a small remainder. We assume that the frequency $\omega = \frac{\partial k}{\partial I}$ satisfies a nonresonance condition of the form

$$|\omega \cdot \nu| \geq \alpha \qquad\qquad \forall \nu \in \mathbb{Z}^n \setminus \Lambda\;, \quad |\nu| \leq N\;, \tag{2.4}$$

in the whole domain $D_\rho \equiv D_\rho(I^o, \mathcal{R})$ of $\widetilde{h}$. By $\Pi_\Lambda : A_\rho \to A_\rho$ we shall denote the projection $\Pi_\Lambda \widetilde{u} = \sum_{\nu \in \Lambda} \widetilde{u}_\nu$. In the analytic lemma below the constants N and α are free parameters, to be fixed, like the parameter $\mathcal{R}$ determining the separation of coordinates between inner and outer ones, only in the geometric part of the proof.

Lemma 3 *Let $\widetilde{h}$, $\mathcal{R}$, D_ρ, M be as above, with ρ as in (2.1). Given $N > 1$ and any N–lattice $\Lambda \in \mathbb{Z}^n$, assume the nonresonance condition (2.4) is satisfied with some $\alpha > 0$. If $r \in \mathbb{N}$ is such that*

$$2^7\,r\,\frac{C}{\alpha}\,\|\widetilde{F}\|_\rho \leq 1\;, \qquad r \leq \frac{1}{6}N\;, \tag{2.5}$$

where

$$C = 1 + \frac{8M\mathcal{R}^2}{\alpha} \ , \tag{2.6}$$

then there exists a real analytic canonical transformation $\Phi : D_{\frac{1}{2}\rho} \to D_\rho$, $(w,z,I,\varphi) = \Phi(w',z',I',\varphi')$, *bounded by*

$$|\zeta_j' - \zeta_j| \le \frac{2C}{\alpha}\, \|\widetilde{F}\|_\rho\, \rho_\zeta \ , \qquad \zeta = w,z,I,\varphi \tag{2.7}$$

($\rho_z \equiv \rho_w$), *which gives the Hamiltonian* $h' = \widetilde{h} \circ \Phi$ *the normal form*

$$h' = k + g + e^{-r} f' \ , \qquad g = \Pi_\Lambda g \ , \tag{2.8}$$

g, f' *and the Hamiltonian vector field* F' *of* f' *satisfying the estimates*

$$|g|_{\frac{1}{2}\rho} \le 2|\widetilde{f}|_\rho \ , \qquad |f'|_{\frac{1}{2}\rho} \le |\widetilde{f}|_\rho \ , \qquad \|F'\|_{\frac{1}{2}\rho} \le \|\widetilde{F}\|_\rho \ .$$

The proof of the lemma requires only minor changes with respect to [1], so we shall skip most details. As usual, we construct the diffeomorphism Φ as a composition of r elementary steps. The single step is described in the following

Lemma 4. *Let* k, $\mathcal{R}$, ρ, N, α *and* Λ *be as in lemma 3; consider*

$$\mathcal{H}(w,z,I,\varphi) = k(I) + u(z,w,I,\varphi) + v(z,w,I,\varphi)$$

with $u, v \in A_\sigma$, $\sigma < \rho$, *and denote by* U, V *the Hamiltonian vector fields of* u *and* v *respectively. If* $x > 0$ *is such that* $x\rho < \sigma$, *and* v *is small, precisely*

$$C\,\|V\|_\sigma \le \frac{1}{8}\alpha x \ , \qquad C \text{ as in (2.6)} \ ,$$

then there exists a real analytic canonical transformation $\Psi : D_{\sigma - x\rho} \to D_\sigma$, *bounded by*

$$|(\Phi - \mathrm{Id})^{\zeta_j}|^\infty_{\sigma - x\rho} \le \frac{2C}{\alpha}\|V\|_\sigma\, \rho_\zeta \ , \qquad \zeta = z,w,I,\varphi$$

such that the new Hamiltonian $\mathcal{H}' = \mathcal{H} \circ \Psi$ *belongs to* $A_{\sigma - x\rho}$ *and has the form*

$$\mathcal{H}'(w,z,I,\varphi) = k(I) + u'(z,w,I,\varphi) + v'(z,w,I,\varphi) \ ,$$

with

$$u' = u + \Pi_\Lambda v$$

and

$$\begin{aligned} |v'|_{\sigma - x\rho} &\le \frac{2}{x\alpha}(\|U\|_\sigma + (1+C)\|V\|_\sigma)|v|_\sigma + e^{-\frac{1}{2}Nx}|v|_\sigma \\ \|V'\|_{\sigma - x\rho} &\le \frac{4C}{x\alpha}(\|U\|_\sigma + 2\|V\|_\sigma)\|V\|_\sigma + e^{-\frac{1}{2}Nx}\|V\|_\sigma \ , \end{aligned} \tag{2.9}$$

V' denoting the Hamiltonian vector field of v'.

Proof of lemma 4: We generate Ψ by the Lie method, that is as the time–one map Φ_1^X of a Hamiltonian vector field X of Hamiltonian χ. Specifically, if χ satisfies the equation

$$\{k,\chi\} \;=\; \sum_{\nu\in\mathbb{Z}^n\setminus\Lambda,|\nu|\le N} v_\nu \;, \tag{2.10}$$

then $\widetilde{h}\circ\Phi_1^X$ has the required form, with

$$v' \;=\; R_1^X(u+v) \;+\; R_2^X(k) \;+\; (1-\Pi_\Lambda)\,v^{>N} \;; \tag{2.11}$$

here $v^{>N} = \sum_{|\nu|>N} v_\nu$ is the 'ultraviolet' part of v, while $R_p^X = \sum_{j=p}^\infty \frac{1}{j!}L_X^j$ denotes the p–th remainder of the Lie series, L_X being the Lie derivative (working on functions and fields) associated to X.

Equation (2.10) is trivially solved by

$$\chi = \sum_{\nu\in\mathbb{Z}^n\setminus\Lambda,|\nu|\le N} \frac{v_\nu}{i\omega\cdot\nu} \;, \tag{2.12}$$

and the following rather standard lemmas are easily proved, see section 3:

Lemma 5. *One has*

$$|\chi|_\sigma \le \frac{1}{\alpha}|v|_\sigma \;, \qquad \|X\|_\sigma \le \frac{C}{\alpha}\|V\|_\sigma \;, \tag{2.13}$$

with C is as in (2.6).

Lemma 6. *If $\|X\|_\sigma \le \frac{1}{8}x$, then Φ_1^X maps $D_{\sigma-x\rho}$ into $D_{\sigma'}$, with*

$$\sigma' = \sigma - x\rho + \|X\|_\rho < \sigma \;, \tag{2.14}$$

and one has

$$\begin{aligned} |R_1^X(u+v)|_{\sigma-x\rho} &\le \frac{2}{x}|\chi|_\sigma\|U+V\|_\sigma \\ |R_2^X k|_{\sigma-x\rho} &\le \frac{4}{x}\|X\|_\sigma|v|_\sigma \\ \|R_1^X(U+V)\|_{\sigma-x\rho} &\le \frac{4}{x}\|X\|_\sigma\|U+V\|_\sigma \\ \|R_2^X K\|_{\sigma-x\rho} &\le \frac{4}{x}\|X\|_\sigma\|V\|_\sigma \;, \end{aligned} \tag{2.15}$$

where of course K denotes the Hamiltonian vector field of k.

On the basis of these lemmas the proof of lemma 4 is immediate, and practically reduces to the estimate of the ultraviolet part of v entering (2.11). For this one easily finds

$$\begin{aligned} |(1-\Pi_\Lambda)\,v^{>N}|_{\sigma-x\rho} &\le |v^{>N}|_{\sigma-x\rho} \le e^{-\frac{1}{2}Nx}|v|_\sigma \\ \|(1-\Pi_\Lambda)\,V^{>N}\|_{\sigma-x\rho} &\le \|V^{>N}\|_{\sigma-x\rho} \le e^{-\frac{1}{2}Nx}\|V\|_\sigma \;; \end{aligned} \tag{2.16}$$

indeed, for each $\nu \in \mathbb{Z}^n$, if the domain reduces from σ to $\sigma - x\rho$, then any Fourier component v_ν of v reduces in norm by (at least) a factor

$$\Big(\frac{\mathcal{R} + \sigma_w - x\rho_w}{\mathcal{R} + \sigma_w}\Big)^{|\nu_j|} \leq \Big(1 - x\frac{\rho_w}{\mathcal{R} + \rho_w}\Big)^{|\nu_j|}$$

for each index $j \in J_{\rm in}$, and by a factor $e^{-|\nu_j| x\rho_\varphi}$ for each $j \in J_{\rm out}$. With the choices (2.1) of ρ, the inequalities (2.16) are immediate. The conclusion of the proof of lemma 4 is straightforward. ∎

We can now proceed to the

Proof of lemma 3. We perform r perturbative steps based on lemma 4, with equal domain reduction $x\rho$, $x = \frac{1}{2r}$. We assume inductively that after l steps, $0 \leq l \leq r-1$, we deal with a Hamiltonian

$$h_l = k + g_l + f_l \ , \qquad h_l \in A_{\rho - lx\rho} \ ,$$

with

$$g_l = \Pi_\Lambda(f_0 + \cdots f_{l-1})$$

and

$$|f_l|_{\rho - lx\rho} \leq e^{-l}|\widetilde{f}|_\rho \ , \qquad \|F_l\|_{\rho - lx\rho} \leq e^{-l}\|\widetilde{F}\|_\rho$$

(the assumption is trivially satisfied for $l = 0$ by $g_0 = 0$, $h_0 = \widetilde{h}$). One easily sees that, for r satisfying (2.5), the induction can proceed one step further. Indeed, from the former of (2.5) one knows that lemma 4 can be applied, with of course $u = g_l$, $v = f_l$, and (2.9) holds. Now, from the inductive assumption one has

$$\|G_l\|_{\rho - lx\rho} + 2\|F_l\|_{\rho - lx\rho} \leq 2\|\widetilde{F}\|_\rho \ ,$$

so that

$$\|F_{l+1}\|_{\rho - (l+1)x\rho} \leq \Big(\frac{8C}{x\alpha}\|\widetilde{F}\|_\rho + e^{-\frac{1}{2}Nx}\Big)\|F_l\|_{\rho - lx\rho} \ ;$$

using (2.5) the claimed estimate on F_{l+1} is immediate. In the same way one works out the estimate on f_{l+1}. Finally, the overall canonical transformation $\Phi : D_{\frac{1}{2}\rho} \to D_\rho$ is easily estimated by summing a geometric series. ∎

D. The confinement of the actions. We use here the normal form provided by lemma 3, together with the convexity of k, to produce bounds on the variation of the actions. In lemma 7 below we refer to the Nekhoroshev–Pöschel decomposition of the frequency space illustrated above, see lemma 2; the constants $N > 1$, $b > \sqrt{2}$ and $\delta > 0$ determining the decomposition are still free, and will be conveniently fixed later. The same holds for the constant $\mathcal{R}$ distinguishing inner and outer variables.

Lemma 7. *Consider any real motion of the Hamiltonian system (1.2), and let the initial frequency ω^o belong to B_Λ for some N–lattice Λ. Assume the following technical inequalities are satisfied:*

$$\begin{aligned} 10M\mathcal{R}^2 &\leq (b-\sqrt{2})\delta_\Lambda && (2.17a)\\ \|\widetilde{F}\|_\rho &\leq \min\Big(\frac{b-\sqrt{2}}{2^6}, \frac{1}{2\sqrt{n}}\Big)\delta_\Lambda && (2.17b)\\ 24\delta_\Lambda &\leq m\mathcal{R}^2 && (2.17c)\\ \mathcal{R} &\leq \frac{1}{2}R_0 && (2.17d)\\ m\,(\mathcal{R}^2\|\widetilde{F}\|_\rho + 5|\widetilde{f}|_\rho) &\leq \frac{5}{2}\delta_\Lambda^2\ , && (2.17e) \end{aligned}$$

and let $N \geq 6$, $r = [\frac{1}{6}N] \geq 1$. Then one has

$$|I(t) - I^o|^\infty \leq \frac{1}{2}\mathcal{R}^2$$

for

$$|t| \leq \Omega^{-1}\frac{e^r}{4\sqrt{n}}\ . \qquad (2.18)$$

Proof. We first show that the assumptions of lemma 3 are satisfied, with

$$\alpha = \frac{1}{2}\gamma_\Lambda = \frac{1}{2}(b-\sqrt{2})N\delta_\Lambda\ .$$

Indeed, according to the definition (1.8) of D_ρ, in such a set one has $|I - I^o|^\infty \leq 5\mathcal{R}^2$, and so for $\nu \in \mathbb{Z}\setminus\Lambda$, $|\nu| \leq N$,

$$|\omega\cdot\nu| \geq \gamma_\Lambda - 5NM\mathcal{R}^2\ ;$$

using (2.17a) one immediately finds (2.4). Concerning the former of (2.5), this follows from (2.17b), after observing that, because of (2.17a) and $N \geq 6$, it is $C \leq \frac{4}{3}$. The latter of (2.5) is trivially guaranteed by the choice of r.

So we can use lemma 3, and profit of the normal form (2.8), as far as $\xi'(t) := (w'(t), z'(t), I'(t), \varphi'(t))$ remains in $D_{\frac{1}{2}\rho}$. We now show that for t as in (2.18), if $\xi'(t)$ does not escape $D_{\frac{1}{2}\rho}$, one has

$$|I'(t) - I'(0)|^\infty \leq 6\,\frac{\delta_\Lambda}{m}\ ; \qquad (2.19)$$

in turn this inequality, using (2.17c), implies (by the usual consistency argument) that $\xi'(t)$ cannot escape $D_{\frac{1}{2}\rho}$ for t as above. To prove (2.19) one uses energy conservation applied to the normal form Hamiltonian (2.8), together with the m–convexity of k. Precisely, denoting

$$\Delta I' = I'(t) - I'(0) \ , \qquad \Delta g = g(\xi'(t)) - g(\xi'(0)) \ , \qquad \Delta f' = f'(\xi'(t)) - f'(\xi'(0)) \ ,$$

one immediately finds

$$\frac{1}{2}m\|\Delta I'\|^2 \leq |\omega(I'(0))\cdot \Delta I'| + |\Delta g| + e^{-r}|\Delta f'| \ .$$

We now treat explicitly the case $\Lambda \neq 0$, but the conclusions hold for $\Lambda = 0$, too. One writes

$$\begin{aligned} |\omega(I'(0))\cdot\Delta I'| &\leq |\omega^o\cdot\Delta I'| + \sqrt{n}M|I'(0)-I^o|^\infty \|\Delta I'\| \\ &\leq \|P_\Lambda\omega^o\|\,\|\Delta I'\| + \Omega\,\|(1-P_\Lambda)\Delta I'\| + \sqrt{n}M|I'(0)-I^o|^\infty \|\Delta I'\| \ . \end{aligned}$$

But for assumption $\|P_\Lambda\omega^o\| \leq \delta_\Lambda$, while the second term, in force of the normal form (2.8), is bounded by

$$\Omega\|(1-P_\Lambda)\Delta I'\| \leq 4\sqrt{n}\Omega\, e^{-r}\,|t|\mathcal{R}^2\|\widetilde{F}\|_\rho \leq \mathcal{R}^2\|\widetilde{F}\|_\rho \ .$$

Using (2.7) to estimate $|I'(0)-I^o|^\infty$, and profiting once more of (2.17), one finally gets $|\omega(I'(0))\cdot\Delta I'| \leq 2\delta_\Lambda\|\Delta I'\| + \mathcal{R}^2\|\widetilde{F}\|_\rho$. This leads to the inequality

$$\frac{m}{2}\|\Delta I'\|^2 \leq 2\delta_\Lambda\|\Delta I'\| + \mathcal{R}^2\|\widetilde{F}\|_\rho + 5|\widetilde{f}|_\rho$$

(use was made of $|\Delta g| \leq 2|g|_{\frac{1}{2}\rho} \leq 4|\widetilde{f}|_\rho$, $e^{-r}|\Delta f'| \leq 2e^{-r}|\widetilde{f}|_\rho \leq |\widetilde{f}|_\rho$). Solving the inequality, and using again (2.17), inequality (2.19) follows. One has finally

$$|I(t)-I(0)|^\infty_\rho \leq |\Delta I'|^\infty_\rho + |I(t)-I'(t)|^\infty_\rho + |I'(0)-I(0)|^\infty_\rho \leq 8\frac{\delta_\Lambda}{m} \ \leq \ \frac{1}{3}\mathcal{R}^2 \ . \qquad (2.20)$$

(use was made of (2.7) and (2.17c)). ■

E. Conclusion of the proof. To conclude the proof of the proposition we must show that one can conveniently choose $\mathcal{R}$, N, δ and b (in turn determining δ_Λ according to section 2.B) in such a way that the technical inequalities (2.17) are satisfied, and moreover that the confinement provided by lemma 7 corresponds to (1.5) and (1.6) in the proposition, the constants being as in (1.7).

Inequality (2.17c) is satisfied as an equality by choosing $\mathcal{R}$ dependent on Λ, namely

$$\mathcal{R}^2 = 24\frac{\delta_\Lambda}{m} \ .$$

Inequality (2.17a) is then satisfied, also as an equality, by the choice of b:

$$b - \sqrt{2} = 240\, M/m \ .$$

Inequality (2.17d), recalling the expression (2.3) of δ_Λ and $R_0 = \sqrt{\varepsilon} R_c$, turns into

$$24 (bN)^{d-1} \frac{\delta}{\|\Lambda\|} \le \frac{1}{4} \varepsilon\, m\, R_c^2 \ ,$$

and is satisfied, in the worst case $d = n$ and $\|\Lambda\| = 1$, by the choice of δ:

$$96\,(bN)^{n-1} \delta = \varepsilon\, m\, R_c^2 \ .$$

Inequalities (2.17b,e), using lemma 1 and the expression (2.3) of δ_Λ, are both satisfied if

$$4\sqrt{n}\, m\, R_c^2\, (16\varepsilon)^{\frac{s}{2}}\, \mathcal{F} \le (bN)^{2d-2} \frac{\delta^2}{\|\Lambda\|^2} \ .$$

In the worst case one has $\|\Lambda\| \le N^d$; after the above choice of δ, one then gets

$$2^{17} \sqrt{n}\, \frac{\mathcal{F}}{m R_c^2}\, (16\varepsilon)^{\frac{s-4}{2}} \le (bN)^{-2n} \ .$$

This is satisfied by appropriately choosing N,

$$N = \varepsilon_1 \Big(\frac{1}{16\varepsilon} \Big)^{\frac{s-4}{4n}} \ , \qquad \varepsilon_1 = \frac{m}{240 M} \Big(\frac{1}{2^{17}\sqrt{n}} \frac{m R_c^2}{\mathcal{F}} \Big)^{\frac{1}{2n}} \ . \qquad (2.21)$$

Correspondingly, using $r = [\frac{1}{6} N]$, one gets (1.6), with T and ε_* as in (1.7). The bound (1.5) follows from (2.20), after the above choice of δ. The proof of the proposition is thus complete. ∎

Remark: as remarked in the introduction, results somehow improve if one excludes the resonance of multiplicity n, and correspondingly considers, as the worst case, $d = n - 1$. The improvement is that $\frac{s-4}{4n}$ in (2.21), and thus in (1.6), is replaced by $\frac{s-4}{4n-4}$. But this requires imposing an extra consistency condition, namely $(bN)^{n-1}\delta < \Omega$. One easily sees that such a condition can be satisfied for any positive Ω by keeping ε small, only if s is not too large, precisely if $s < 4n$.

3. Proof of technical lemmas

A. Proof of lemma 1. The former of (2.2) trivially follows from (1.10). Concerning the latter, one writes $|\widetilde{F}_\nu^{w_j}|_\rho^\infty = |\frac{\partial \widetilde{f}_\nu}{\partial z_j}|_\rho^\infty \le |\frac{\partial f_\nu}{\partial z_j}|_{2R_0}^\infty$, and so by Cauchy inequality, also using $\rho_w = \mathcal{R}$,

$$\frac{1}{\rho_w} |\widetilde{F}_\nu^{w_j}|_\rho^\infty \le \frac{|f_\nu|_{4R_0}^\infty}{2 R_0 \mathcal{R}} \ .$$

The same holds for $\widetilde{F}^{z_j}$. For the I_j component one writes

$$\widetilde{F}_\nu^{I_j} = -\frac{\partial \widetilde{f}_\nu}{\partial \varphi_j} = \frac{\partial f_\nu}{\partial w_j}\frac{\partial w_j}{\partial \varphi_j} + \frac{\partial f_\nu}{\partial z_j}\frac{\partial z_j}{\partial \varphi_j} \ ;$$

but $|\frac{\partial w_j}{\partial \varphi_j}|, |\frac{\partial w_j}{\partial \varphi_j}| \le |\sqrt{I_j}|\, e^{\rho_\varphi}$, so that

$$\frac{1}{\rho_I}|\widetilde{F}_\nu^{I_j}|_\rho^\infty \le \frac{4|f|_{4R_0}^\infty}{\mathcal{R}^2} \ .$$

In a very similar way one finds

$$\frac{1}{\rho_\varphi}|\widetilde{F}_\nu^{\varphi_j}|_\rho^\infty \le \frac{3|f|_{4R_0}^\infty}{R_0\mathcal{R}} \ .$$

The I_j component dominates. By summing over ν the conclusion is immediate. ∎

B. Proof of Lemma 5. The former of (2.13) is trivial. For the latter one must estimate the different components of $X_\nu = \frac{v_\nu}{i\omega\cdot\nu}$, recalling of course that ω in the denominator is not constant but depends on $I_1,\ldots,I_n$, with $I_j = iw_jz_j$ for $j\in J_{\rm in}$. For the I_j component one immediately finds $|X_\nu^{I_j}|_\sigma^\infty \le \alpha^{-1}|V_\nu^{I_j}|_\sigma^\infty$. For the φ_j component one has instead

$$X_\nu^{\varphi_j} = \frac{V_\nu^{\varphi_j}}{i\omega\cdot\nu} - \frac{v_\nu}{i(\omega\cdot\nu)^2}\sum_{l=1}^n \nu_l\,\frac{\partial\omega_l}{\partial I_j} \ .$$

But

$$\nu_l\,v_\nu = \begin{cases} z_l\frac{\partial v_\nu}{\partial z_l} - w_l\frac{\partial v_\nu}{\partial w_l} & \text{for } l\in J_{\rm in} \\ -i\frac{\partial v_\nu}{\partial \varphi_l} = iV_\nu^{I_l} & \text{for } l\in J_{\rm out} \end{cases} \ ,$$

and one easily concludes

$$\frac{1}{\rho_\varphi}|X_\nu^{\varphi_j}|_\sigma^\infty \ \le\ \frac{\|V_\nu\|_\sigma^\infty}{\alpha} + \frac{2M}{\alpha^2}\max\big(2(\mathcal{R}+\rho_w)\rho_w,\rho_I\big)\,\|V_\nu\|_\sigma^\infty \ \le\ \frac{C}{\alpha}\|V_\nu\|_\sigma^\infty \ .$$

In a very similar way one works on the components w_j, z_j of X_ν, getting the estimate $\frac{1}{\rho_w}|X_\nu^w|_\sigma^\infty \le \frac{C}{\alpha}\|V_\nu\|_\sigma^\infty$; the conclusion of the proof is immediate. ∎

C. Proof of lemma 6. We limit ourselves to a sketch. Inequality (2.14) is trivial. The estimates (2.15) on the remainders are based on the standard inequalities

$$|L_X\,y|_{\sigma-x\rho} \le \frac{1}{x}\|X\|_{\sigma-x\rho}\,|y|_\sigma$$
$$\|L_X\,Y\|_{\sigma-x\rho} \le \frac{1}{x}(\|X\|_{\sigma-x\rho}\,\|Y\|_\sigma + \|X\|_\sigma\,\|Y\|_{\sigma-x\rho}) \ ,$$

where y is any function in A_σ and Y is the corresponding vector field (see for example appendix B of [15]). From these inequalities one gets by induction (see again [15])

$$\frac{1}{l!}|L_X^l\, y|_{\sigma-x\rho} \le \frac{1}{e}\Big(\frac{e}{x}\|X\|_\sigma\Big)^l |y|_\sigma\ , \qquad \frac{1}{l!}|L_X^l\, Y|_{\sigma-x\rho} \le \frac{1}{2}\Big(\frac{4}{x}\|X\|_\sigma\Big)^l \|Y\|_\sigma$$

and the conclusion is easy; as usual $\mathcal{R}_2^X k$ is estimated as $\mathcal{R}_1^X v$, and similarly for $\mathcal{R}_2^X K$. ∎

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