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EXPONENTIAL APPROXIMATION FOR HITTING TIMES IN MIXING PROCESSES

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Abstract

We present bounds for the error of the exponential approximation of the first occurrence time of a rare event in a stationary stochastic process with a finite alphabet with the α-mixing property with a summable function α or with a general ϕ-mixing property. We prove a lower bound for this error in terms of the measure of the cylinder and an upper bound as a function of the measure of the cylinder plus the decay of correlation of the process.

Keywords. *Mixing stochastic processes, occurrence time of a rare event, exponential approximation.*

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1 Introduction.

This article presents lower and upper bounds for the exponential approximation to the law of the first occurrence of a long string of symbols in a stochastic source with a finite alphabet. Our result holds for α-mixing processes with summable function α and for ϕ-mixing processes with general decreasing function ϕ. We prove that for any cylinder set A, the law of its hitting time, suitably rescaled, can be uniformly approximated by a mean one exponential law. The error in the approximation of this law is bounded from below by $C_1 \mathbb{P}\{A\}$ and it is bounded from above by $\inf_x\{x\mathbb{P}\{A\} + g(x)\}$ where $\mathbb{P}\{A\}$ is the measure of the cylinder and g is the function of the decay of correlation of the process.

Moreover, we show that the scaling factor can be written as $\xi_{A,\mu(t)} \mathbb{P}\{A\}$, where $\xi_{A,\mu(t)}$ is bounded below and above by two strictly positive constants Ξ_1 and Ξ_2, respectively, independent of A, n and t.

Important recent papers on this subject are Galves-Schmitt (1997), Hirata-Saussol-Vaienti (1998) and Collet-Galves-Schmitt (1999).

Galves-Schmitt's approach introduces a correction factor in the approximating law. This approach works for all kind of cylinders but only in the case of ϕ-mixing processes with a summable function ϕ.

The paper by Hirata, Saussol and Vaienti presents a very elegant straightforward proof. However their result is interesting for cylinders that don't allow an immediate return to themselves.

Collet, Galves and Schmitt (1999) show how this correction factor depends on the return of a rare event onto itself in the first steps.

For a brief review on the field before 1997 we refer the reader to Galves and Schmitt (1997).

In this article we present a modification of Galves-Schmitt's approach. This new technique provides sharper upper bound for this type of approximation. The improvement is obtained in part by changing the scaling factor. Our technique works for all type of cylinders and for the mentioned enlarged family of processes. This answers in part one of the questions pointed out in the paper by Hirata, Saussol and Vaienti: What is the largest class of processes for which the exponential approximation holds? We refer to Doukhan (1995) for examples and references of α-mixing processes that are not ϕ-mixing and for ϕ-mixing decaying at any rate. We recall that an irreducible and aperiodic finite state Markov chain is ϕ-mixing with exponential decay. Moreover, Gibbs states with a potential with exponential variations are exponentially ϕ-mixing. See Bowen (1975) for definitions and properties.

Kac's lemma (Kac, 1947) says that the right scaling factor for return times is $I\!P\{A\}$. This suggests that we should use it as scaling factor for hitting times which in our notation means $\xi_{A,\mu(t)} = 1$. We show that this is not always the case for hitting times. We study how $\xi_{A,\mu(t)}$ depends on A. In this sense, the use of this scaling factor provides a good approximation to the exponential law but could be difficult to calculate explicitly. To overcome this real difficulty we present in Theorem 4 a computable approximation for this parameter in terms of the short times recurrence of each cylinder. A corollary presents conditions, on the process and on the cylinders, to get the scaling parameter equal to one; and also prove that the total mass of this type of cylinders is high. This behavior was indicated in Hirata, Saussol, Vaienti (1999); Collet, Galves, Schmitt (1999) and Saussol (1999). We also present a counterexample in which the parameter is not just $\lambda_A \neq 1$ but also as small as we want, according to the process.

This paper is organized as follows. In section 3 we state the theorems. In sections 4, 5 and 6 we prove the lemmata that are needed in the proofs of the theorems and in sections 7, 8 and 9 we prove the theorems.

2 The framework.

Let $\mathcal{E}$ be a finite set. Put $\Omega = \mathcal{E}^{Z\!\!Z}$. For each $n \in Z\!\!Z$ let $X_n : \Omega \to \Re$ be the n-th coordinate projection. We denote by $T : \Omega \to \Omega$ the one-step-left shift operator.

We denote by $\mathcal{F}$ the σ-algebra over Ω generated by cylinders. Moreover we denote by $\mathcal{F}_I$ the σ-algebra generated by cylinders with coordinates in I, $I \subseteq Z\!\!Z$.

For a subset $A \subseteq \Omega$ we say that $A \in \mathcal{C}_n$ if and only if

$$A = \{X_0 = a_0, \ldots, X_{n-1} = a_{n-1}\} ,$$

with $a_i \in \mathcal{E},\ i = 1, \ldots, n$.

We consider a stationary probability measure $I\!P$ over $\mathcal{F}$. We shall assume that there are no singletons of probability 0 or 1.

Let $\alpha = (\alpha(l))_{l \geq 0}$ and $\phi = (\phi(l))_{l \geq 0}$ be two decreasing sequences converging to zero of positive real numbers. We shall say that $(\Omega, \mathcal{F}, I\!P, T)$ is *α-mixing* if, for all integers $n \geq 1$ and $l \geq 0$, the following inequality holds

$$\sup_{B\in\mathcal{F}_{\{0,.,n\}}, C\in\mathcal{F}_{\{n\geq 0\}}} \frac{\left|I\!P\left\{B\cap T^{-(n+l+1)}C\right\} - I\!P\{B\}\, I\!P\{C\}\right|}{I\!P\{B\}} = \alpha(l) \ ,$$

and *ϕ-mixing* if

$$\sup_{B\in\mathcal{F}_{\{0,.,n\}}, C\in\mathcal{F}_{\{n\geq 0\}}} \frac{\left|I\!P\left\{B\cap T^{-(n+l+1)}C\right\} - I\!P\{B\}\, I\!P\{C\}\right|}{I\!P\{B\}\, I\!P\{C\}} = \phi(l) \ ,$$

where in the above expressions the supremum is taken over the sets B and C, such that $I\!P\{B\} > 0$ in the first case and such that $I\!P\{B\}\, I\!P\{C\} > 0$ in the second one. Given $A \in \mathcal{C}_n$, we define the *entrance time* $\tau_A : \Omega \to I\!N \cup \{\infty\}$ as the following random variable defined on the probability space $(\Omega, \mathcal{F}, I\!P)$. For any $\omega \in \Omega$

$$\tau_A(\omega) = \inf\{k \geq 1 : T^k(\omega) \in A\}$$

Clearly, ϕ-mixing implies α-mixing. We recall that mixing (at any rate) implies ergodicity, and ergodicity ensures that τ_A is $I\!P$-almost surely finite (see e.g. Cornfeld, Fomin and Sinai, 1982).

We shall use the classic probabilistic shorthand notation for events defined through random variables. We shall write $\{\tau_A = m\}$ instead of $\{\omega \in \Omega : \tau_A(\omega) = m\}$, $T^{-k}(A) = \{\omega \in \Omega : T^k(\omega) \in A\}$ and $\{X_r^s = x_r^s\} = \{X_r = x_r, \ldots, X_s = x_s\}$. As usual, the mean of a random variable X will be denoted by $I\!E(X)$. Wherever it is not ambiguous we will write C and c for different positive constants even in the same sequence of equalities/inequalities. Where a property holds for α and ϕ processes we shall replace α or ϕ by a $*$.

3 Statement of the results.

We now state our main result.

Theorem 1. *Let $(\Omega, \mathcal{F}, I\!P, T)$ be ϕ-mixing or α-mixing with α summable. Then, there exist strictly positive constants Ξ_1, Ξ_2, C_1 and C_2 such that for any $n, A \in \mathcal{C}_n$ and $t > 0$ there exists $\xi_{A,\mu(t)} \in [\Xi_1, \Xi_2]$, for which the following inequalities hold*

$$\sup_{t>0} \left| I\!P\left\{\tau_A > \frac{t}{\xi_{A,\mu(t)} I\!P\{A\}}\right\} - e^{-t}\right| \leq C_2 \inf_{\Delta\geq n} \left[2\Delta I\!P\{A\} + *(\Delta - n)\right] \ .$$

Moreover $e^{-\xi_{A,\mu(t)}\mathbb{P}\{A\}t}$ *is a left-continuous non-increasing function of* t *over each* $[k/\mathbb{P}\{A\},(k+1)/\mathbb{P}\{A\})$ *and for all* $k\in\mathbb{N}$

$$\left|\lim_{t\to\left(\frac{k+1}{\mathbb{P}\{A\}}\right)^-} e^{-\xi_{A,\mu(t)}\mathbb{P}\{A\}t}-e^{-\xi_{A,\mu((k+1)/\mathbb{P}\{A\})}\mathbb{P}\{A\}t}\right|\le C\inf_{\Delta\ge n}\left[2\Delta\mathbb{P}\{A\}+*(\Delta-n)\right] .$$

We need first to prove the following results.

Theorem 2. *Let* $(\Omega,\mathcal{F},\mathbb{P},T)$ *be* ϕ*-mixing. Then, there exist strictly positive constants* Λ_1,Λ_2,C_1 *and* C_2 *such that for any* n *and any* $A\in\mathcal{C}_n$*, there exists* $\lambda_A\in[\Lambda_1,\Lambda_2]$*, for which the following inequality holds*

$$C_1\mathbb{P}\{A\}\le\sup_{t>0}\left|\mathbb{P}\left\{\tau_A>\frac{t}{\lambda_A\mathbb{P}\{A\}}\right\}-e^{-t}\right|\le C_2\,b(A) ,$$

where $b(A)=\mathbb{P}\{A\}^\eta$, $0<\eta=\eta(\epsilon)<1$ *if* $\phi(n)\le C/n^\epsilon$ *for all* $n>n_o$ *with* $\epsilon>0$ *and* $b(A)=\phi^{1/2}(n)$ *otherwise.*

In Galves-Schmitt (1997) an upper bound of the form $\mathbb{P}\{A\}^\beta$ with $0<\beta<1$ was proved for ϕ summable processes. Therefore, Theorem 2 extends this result to *any* type of decreasing function ϕ.

Theorem 3. *Let* $(\Omega,\mathcal{F},\mathbb{P},T)$ *be* α*-mixing and let us suppose that the sequence* $\alpha:\mathbb{N}\to\mathbb{R}_+$ *is summable. Then, there exist strictly positive constants* Λ_1,Λ_2,C_1, C_2 *and* κ *such that for any* n *and any* $A\in\mathcal{C}_n$*, there exists* $\lambda_A\in[\Lambda_1,\Lambda_2]$*, for which the following inequality holds*

$$C_1\mathbb{P}\{A\}\le\sup_{t>0}\left|\mathbb{P}\left\{\tau_A>\frac{t}{\lambda_A\mathbb{P}\{A\}}\right\}-e^{-t}\right|\le C_2\mathbb{P}\{A\}^\kappa .$$

We emphasize that in both theorems the constants $\Lambda_1,\Lambda_2,C_1,C_2,\eta$ and κ are independent of n and A. Notation Λ_1,Λ_2,C_1, and C_2 stands for (possibly) different constants in each theorem.

The difference between Theorem 1 and Theorems 2 and 3 is that the first one provides a more accurate upper bound given that we use $\xi_{A,\mu(t)}$ instead of λ_A as a correction factor. We first prove Theorems 2 and 3 following Galves-Schmitt 97 approach. After this is done, we use the estimate of $\mathbb{P}\{\tau_A>1/\mathbb{P}\{A\}\}$ to prove that $1/\mathbb{P}\{A\}$ is a suitable scaling factor for the process and provide a sharper upper bound for the approximation of the exponential law.

We now present the theorem that gives an estimation of $\xi_{A,\mu(t)}$. Let s be a positive integer. Denoting by $I\!P_A$ the conditional probability on A we define

$$\zeta_{A,s} = I\!P_A \left\{ \tau_A > \frac{n}{s} \right\} = \frac{I\!P \left\{ \left(\tau_A > \frac{n}{s} \right) \cap A \right\}}{I\!P \{A\}} .$$

Theorem 4. *Let $(\Omega, \mathcal{F}, I\!P, T)$ be exponentially α-mixing. Let s be a positive integer. Then, there exist strictly positive constants Ψ_1, Ψ_2, C_1, C_2 and c such that for any $n \in I\!N$ and any $A \in \mathcal{C}_n$, we have that $\zeta_{A,s} \in [\Psi_1, \Psi_2]$ and the following inequality hold*

$$C_1 I\!P \{A\} \leq \sup_{t>0} \left| I\!P \left\{ \tau_A > \frac{t}{\zeta_{A,s} I\!P \{A\}} \right\} - e^{-t} \right| \leq C_2 e^{-cn} .$$

4 Estimates of the probability of a cylinder.

Lemma 1. *Let the process be ϕ-mixing with any decreasing function ϕ or α-mixing with α summable. There exist strictly positive constants C, $\bar{C}$, and Γ such that for any fixed positive integer n and any $A \in \mathcal{C}_n$ the following inequalities hold*

$$I\!P \{A\} \leq C e^{-\Gamma n} , \tag{1}$$

and

$$\sum_{k=1}^{n} I\!P \left\{ A \cap T^{-k} A \right\} \leq \bar{C} I\!P \{A\} . \tag{2}$$

Proof. Take $A = \{X_1^n = a_1^n\}$. Using the α-mixing property we have

$$I\!P \{A\} \leq [\alpha(n_0 - 1) + \rho]^{[\frac{n}{n_0}]+1} ,$$

where

$$\rho = \sup\{I\!P(a_i) : a_i \in \mathcal{E}\}.$$

Since, by hypothesis, $\rho < 1$ and the function $\alpha(l) \to 0$, there exists an integer n_0 such that

$$(\rho + \alpha(n_0 - 1)) < 1 .$$

This proves (1). To prove (2), in the α-mixing case we remark that

$$I\!P\left\{A\cap T^{-k}A\right\}\leq\alpha\left(\left[\frac{k}{2}\right]\right)I\!P\left\{A\right\}+I\!P\left\{A^{(k-[\frac{k}{2}])}\right\}I\!P\left\{A\right\} ,$$

where $A^{(l)}=\{X_{n-l+1}^{n}=a_{n-l+1}^{n}\}$. Then, since α is summable and using (1)

$$\sum_{k=1}^{n}I\!P\left\{A\cap T^{-k}A\right\}\leq 2I\!P\left\{A\right\}\sum_{k=0}^{[\frac{n}{2}]}\alpha\left(k\right)+2I\!P\left\{A\right\}\sum_{k=1}^{n-[\frac{n}{2}]}I\!P\left\{A^{(k)}\right\}\leq CI\!P\left\{A\right\}.$$

For the general ϕ-mixing case we refer the reader to Lemma 1 on Galves-Schmitt (1997).

5 Bounds for the parameter λ_A.

For any positive integer k let us define

$$N_k=\sum_{l=1}^{k}1\!\!1_{T^{-l}(A)} , \tag{3}$$

where $1\!\!1_A$ is the indicator function of the set A. For any $\omega\in\Omega$, $N_k(\omega)$ is the number of times the process visits A, during the first k steps. We remark that

$$\{\tau_A\leq k\}=\{N_k\geq 1\} .$$

Lemma 2. *For any real number $t\geq 1$ the following inequality holds*

$$I\!P\left\{\tau_A\leq t\right\}\leq t\,I\!P\left\{A\right\} . \tag{4}$$

Proof.

$$I\!P\left\{\tau_A\leq t\right\}=\sum_{l=1}^{[t]}I\!P\left\{\tau_A=l\right\}\leq\sum_{l=1}^{[t]}I\!P\left\{T^{-l}A\right\}=[t]I\!P\left\{A\right\} , \tag{5}$$

where we use the invariance of the measure $I\!P$ with respect to the transformation T in the last equality.

Lemma 3. *Let the process be ϕ-mixing or α-mixing with α summable. For any positive integer n, any cylinder $A \in \mathcal{C}_n$ and $t > 0$, the following holds*

$$I\!P\{\tau_A \le t\} \ge \frac{[t]^2 I\!P\{A\}^2}{t\, I\!P\{A\} + C' t\, I\!P\{A\} + C(t\, I\!P\{A\})^2 + 2Kt\, I\!P\{A\}}, \qquad (6)$$

where $C' > 0, C > 0$ and $K > 0$ are constants independent of n, A and t.

Proof. Let $N = N_{[t]}$. We first remark that

$$(I\!E(N))^2 = \left(I\!E(N 1\!\!1_{\{N\ge 1\}})\right)^2 \le I\!E(N^2) I\!P\{N \ge 1\} ,$$

where the last inequality follows from the Schwarz inequality. Therefore

$$I\!P\{\tau_A \le t\} = I\!P\{N \ge 1\} \ge \frac{(I\!E(N))^2}{I\!E(N^2)} .$$

By definition $I\!E(N) = [t] I\!P\{A\}$.

To obtain an upper bound for the denominator $I\!E(N^2)$, we decompose it as follows. Assume first $[t] > n$, then the definition of N gives

$$I\!E(N^2) = \sum_{l=1}^{[t]} I\!E\left(1\!\!1_{T^{-l}(A)}\right) + 2\sum_{l=1}^{n} ([t]-l)\, I\!E\left(1\!\!1_A\, 1\!\!1_{T^{-l}(A)}\right) + 2\sum_{l=n+1}^{[t]-1} ([t]-l)\, I\!E\left(1\!\!1_A\, 1\!\!1_{T^{-l}(A)}\right). \qquad (7)$$

The first term of this decomposition is $I\!E(N)$.

Using Lemma 1 in the second term, we get

$$\sum_{l=1}^{n}([t]-l) I\!E\left(1\!\!1_A\, 1\!\!1_{T^{-l}(A)}\right) \le [t] \sum_{l=1}^{n} P\{A \cap T^{-l}(A)\} \le C' t\, I\!P\{A\} .$$

The α-mixing property provides an upper bound for the expectations inside the third term

$$I\!E\left(1\!\!1_A\, 1\!\!1_A(T^l)\right) \le I\!P\{A\}^2 + \alpha(l-n-1) I\!P\{A\}.$$

Therefore the third term is bounded above by

$$2 I\!P\{A\} \left(I\!P\{A\} \sum_{l=1}^{[t]-1} l + t \sum_{l=1}^{+\infty} \alpha(l) \right) .$$

Now we remark that

$$\sum_{l=1}^{[t]-1} l = \frac{1}{2}[t]([t]-1) \ .$$

Finally we use the hypothesis that the series $\sum_{l=1}^{+\infty} \alpha(l)$ is convergent to get the upper bound

$$2 \sum_{l=n+1}^{[t]} ([t]-l) \, I\!\!E \left(1\!\!1_A \, 1\!\!1_{T^{-l}(A)} \right) \leq (t I\!\!P\{A\})^2 + 2Kt \, I\!\!P\{A\},$$

where $K = \sum_{l=1}^{+\infty} \alpha(l) < +\infty$.

In the ϕ-mixing case we simply observe that

$$I\!\!E \left(1\!\!1_A \, 1\!\!1_{T^{-l}(A)} \right) \leq (1 + \phi(l-n)) \ \ I\!\!P\{A\}^2 .$$

Therefore the third term is bounded above by

$$2\frac{1}{2}[t]([t]-1)\,(1+\phi(0))\, I\!\!P\{A\}^2 \leq C \ (t \, I\!\!P\{A\})^2 \ .$$

For $t \leq n$ the proof is the same with the third term absent. This concludes the proof of Lemma 3.

Let f_A be a positive integer such that

$$f_A I\!\!P\{A\} \leq \frac{1}{2} \ . \tag{8}$$

and let us define

$$\lambda_{A,f_A} = \frac{-\log I\!\!P\{\tau_A > f_A\}}{f_A I\!\!P\{A\}} \ . \tag{9}$$

Lemma 4. *There exists a positive integer n_0, such that, for all $n \geq n_0$ and $A \in \mathcal{C}_n$, the following inequalities hold*

$$\Lambda_1 \leq \lambda_{A,f_A} \leq \Lambda_2 \ ,$$

where Λ_1 and Λ_2 are two positive constants independent of n, A and f_A.

Proof. To obtain the lower bound, we first remark that

$$\frac{\theta}{2} \leq 1 - e^{-\theta} \leq \theta \ , \tag{10}$$

where the right inequality holds for all $\theta \geq 0$ and the left holds for all $\theta \in [0,1]$ or equivalently for all $1 - e^{-\theta} \leq 1 - e^{-1} > 1/2$. Let us take $\theta = \theta(A) = -\log I\!\!P\{\tau_A > f_A\}$ and remark that $I\!\!P\{\tau_A \leq f_A\} = 1 - e^{-\theta}$. Using (10) and Lemma 3, we get

$$\frac{\theta(A)}{f_A I\!\!P\{A\}} \geq \frac{1}{1 + C' + C f_A I\!\!P\{A\} + 2K} \geq \frac{1}{1 + C' + C + 2K} = \Lambda_1 \ . \quad (11)$$

where C', C and K are the same constants that appear in Lemma 3.

On the other hand, it follows from (10), (8) and Lemma 2 that

$$-\log I\!\!P\{\tau_A > f_A\} \leq 2 I\!\!P\{\tau_A \leq f_A\} \leq 2 f_A I\!\!P\{A\} \ ,$$

for all $n \geq n_0$, since $\theta \in [0, \log 2] \subset [0,1]$. Therefore, we can take $\Lambda_2 = 2$. This ends the proof of Lemma 4.

6 The asymptotic independence property.

Lemma 5. *Suppose that the process is $*$-mixing. Let $A \in \mathcal{C}_n$ and $t, s \in I\!\!N$ such that $\max\{t,s\} \geq n$. For all $\Delta \in I\!\!N$ with $n \leq \Delta \leq \max\{t,s\}$, the following inequality holds*

$$|I\!\!P\{\tau_A > t + s\} - I\!\!P\{\tau_A > t\}\, I\!\!P\{\tau_A > s\}| \leq [2\Delta I\!\!P\{A\} + *(\Delta - n)] \ .$$

Proof. We use Galves-Schmitt approach. We introduce a gap Δ from $t+1$ to $t+\Delta$. The idea is that this gap should be large enough for the process to lose memory but small enough in order to keep the probability of the event close to the original one. Without loss of generality we can assume that $t \leq s$.

$$\begin{aligned}
& |I\!\!P\{\tau_A > t + s\} - I\!\!P\{\tau_A > t\}\ I\!\!P\{\tau_A > s\}| \\
\leq \ & \left| I\!\!P\{\tau_A > t + s\} - I\!\!P\left\{\tau_A > t \cap \tau_A \circ T^{t+\Delta} > s - \Delta\right\}\right| \\
+ \ & \left| I\!\!P\left\{\tau_A > t \cap \tau_A \circ T^{t+\Delta} > s - \Delta\right\} - I\!\!P\{\tau_A > t\}\, I\!\!P\left\{\tau_A \circ T^{t+\Delta} > s - \Delta\right\}\right| \\
+ \ & |I\!\!P\{\tau_A > t\}\, I\!\!P\{\tau_A > s - \Delta\} - I\!\!P\{\tau_A > t\}\, I\!\!P\{\tau_A > s\}| \ . \qquad (12)
\end{aligned}$$

We use the stationarity of the measure and Lemma 2 to get an upper bound for the first and third terms. We use the mixing property on the second. Therefore equation (12) is bounded above by

$$I\!\!P\{\tau_A \leq \Delta\} + *(\Delta - n) + I\!\!P\{\tau_A \leq \Delta\} \leq 2\Delta I\!\!P\{A\} + *(\Delta - n) \ .$$

We shall need the following iterated version of Lemma 5.

Lemma 6. *For all $A \in \mathcal{C}_n$ let us define $\theta = \theta(A) = -\log I\!P\{\tau_A > s\}$. Then, for any integer $k \geq 1$ and any $s > n$, the following inequality holds*

$$\left| I\!P\{\tau_A > k\, s\} - e^{-\theta k} \right| \leq \inf_{n \leq \Delta \leq s} \left\{ \frac{2\Delta I\!P\{A\} + *(\Delta - n)}{I\!P\{\tau_A \leq s\}} \right\} .$$

Proof. It is enough to prove, by induction, that

$$\left| I\!P\{\tau_A > k\, s\} - e^{-\theta k} \right| \leq [2\Delta I\!P\{A\} + *(\Delta - n)] \left[1 + e^{-\theta} + \ldots + e^{-\theta(k-2)} \right] ,$$

holds for any integer $k \geq 1$.

The result is trivially true for $k = 1$. Let us assume that it holds for $k \geq 1$. By the triangle inequality

$$\begin{aligned} \left| I\!P\{\tau_A > (k+1)\, s\} - e^{-\theta(k+1)} \right| &\leq \left| I\!P\{\tau_A > (k+1)\, s\} - I\!P\{\tau_A > k\, s\} e^{-\theta} \right| \\ &+ e^{-\theta} \left| I\!P\{\tau_A > k\, s\} - e^{-\theta k} \right| . \end{aligned} \tag{13}$$

By Lemma 5

$$\left| I\!P\{\tau_A > (k+1)\, s\} - I\!P\{\tau_A > k\, s\} e^{-\theta} \right| \leq 2\Delta I\!P\{A\} + *(\Delta - n) . \tag{14}$$

Bounding the first term on the right hand side of (13) as in (14) and using the hypothesis that the result holds for k, we obtain the corresponding inequality for $k+1$. This concludes the proof of Lemma 6.

7 Proof of Theorems 2 and 3.

It is enough to prove that the theorems hold for any cylinder $A \in \mathcal{C}_n$ with $n \geq n_0$. Let us use the shorthand notation $\lambda_A = \lambda_{A, f_A}$.

Proof of the lower bound.

The proof of the lower bound follows the proof of proposition 2.3 in Hirata-Saussol-Vaienti (1998).

Since $I\!P\{\tau_A > t\}$ is constant in $[[t], [t] + 1) = [k, k+1)$ we get that for any $0 < \delta < 1$

$$\begin{aligned}\sup_{t\in[k,k+\delta)}\left|\mathbb{P}\{\tau_A > t\} - e^{-t\lambda_A \mathbb{P}\{A\}}\right| &\geq \frac{1}{2}\left|e^{-k\lambda_A \mathbb{P}\{A\}} - e^{-(k+\delta)\lambda_A \mathbb{P}\{A\}}\right| \\ &\geq e^{-k\lambda_A \mathbb{P}\{A\}}\frac{1-e^{-\delta\Lambda_1 \mathbb{P}\{A\}}}{2} \\ &\geq C\,\delta\,\Lambda_1 \mathbb{P}\{A\}\, e^{-k\Lambda_2 \mathbb{P}\{A\}}.\end{aligned}$$

In particular

$$\sup_{t>0}\left|\mathbb{P}\{\tau_A > t\} - e^{-t\lambda_A \mathbb{P}\{A\}}\right| \geq C\mathbb{P}\{A\}.$$

Proof of the upper bound.

Let us take f_A as in (8). Fix $t > 0$ and write $t = k\ [f_A] + r$ where $k = k(A) \geq 0$ is the integer part of $t/[f_A]$ and $0 \leq r < [f_A]$. Let us write t_A for $k\ [f_A]$

$$\begin{aligned}\left|\mathbb{P}\{\tau_A > t\} - e^{-\lambda_{A,f_A}\mathbb{P}\{A\}t}\right| &\leq |\mathbb{P}\{\tau_A > t\} - \mathbb{P}(\tau_A > t_A)| \\ &+ \left|\mathbb{P}\{\tau_A > t_A\} - e^{-\lambda_{A,f_A}\mathbb{P}\{A\}t_A}\right| \\ &+ \left|e^{-\lambda_{A,f_A}\mathbb{P}\{A\}t_A} - e^{-\lambda_{A,f_A}\mathbb{P}\{A\}t}\right| . \quad (15)\end{aligned}$$

By stationarity

$$|\mathbb{P}\{\tau_A > t\} - \mathbb{P}\{\tau_A > t_A\}| \leq r\ \mathbb{P}\{A\} \leq f_A \mathbb{P}\{A\}\ . \quad (16)$$

Lemma 6 and (11) give an upper bound for the second term in (15). We recall that in order to use Lemma 6 we just need to choose Δ and f_A such that $n \leq \Delta \leq f_A$.

$$\left|\mathbb{P}\{\tau_A > t_A\} - e^{-\lambda_{A,f_A}\mathbb{P}\{A\}t_A}\right| \leq C \inf_{n\leq\Delta\leq f_A}\frac{[2\Delta\mathbb{P}\{A\} + *(\Delta - n)]}{f_A\mathbb{P}\{A\}}\ . \quad (17)$$

Elementary calculus provides a first upper bound for the third term in (15)

$$\left|e^{-\lambda_{A,f_A}\mathbb{P}\{A\}t_A} - e^{-\lambda_{A,f_A}\mathbb{P}\{A\}t}\right| \leq \lambda_A\mathbb{P}\{A\}\, r \leq \Lambda_2 f_A\mathbb{P}\{A\}\ . \quad (18)$$

For Theorem 3 we note that the summability of α implies that $\alpha(x) \leq C/x$. Choose $f_A = \left(\frac{1}{\mathbb{P}\{A\}}\right)^{3/4}$, define $\lambda_A = \lambda_{A,f_A}$ and choose $\Delta = \left(\frac{1}{\mathbb{P}\{A\}}\right)^{1/2} + n$. Summing the three terms (16), (17) and (18), and noting that there is a positive constant C such that $n\,\mathbb{P}\{A\}^{3/4} \leq C\,\mathbb{P}\{A\}^{1/4}$, we get the desired upper bound.

This proves the upper bound in Theorem 3.

For Theorem 2, if $\phi(x) \leq C/x^{\epsilon}$ with $\epsilon > 0$, choose $f_A = \frac{1}{\mathbb{P}\{A\}^{\gamma}}$, with $\gamma = \frac{2+\epsilon}{2+2\epsilon}$, and $\Delta = \frac{1}{\mathbb{P}\{A\}^{\beta}} + n$, with $\beta = \frac{1}{1+\epsilon}$. We get

$$\frac{\Delta \mathbb{P}\{A\} + \phi(\Delta - n)}{f_A \mathbb{P}\{A\}} + f_A \mathbb{P}\{A\} \quad \leq \quad C \mathbb{P}\{A\}^{\epsilon/2(\epsilon+1)} \ .$$

Otherwise, if $\phi(x)$ decrease slower than $1/x^{\epsilon}$ for all $\epsilon > 0$, we choose Δ such that $\Delta \mathbb{P}\{A\} = \phi(\Delta - n)$ and f_A such that $\Delta / f_A = f_A \mathbb{P}\{A\}$. (We recall that this choice for Δ and f_A guarantees the condition of Lemma 6, $n \leq \Delta \leq f_A$.) Notice that $\Delta - n > n$, since otherwise $\phi(n) < \phi(\Delta - n) = \Delta \mathbb{P}\{A\} < 2n\mathbb{P}\{A\} < Cne^{-cn}$ which contradicts the decay of ϕ. This gives the bound

$$\frac{\Delta \mathbb{P}\{A\} + \phi(\Delta - n)}{f_A \mathbb{P}\{A\}} + f_A \mathbb{P}\{A\} \leq \phi^{1/2}(\Delta - n) \leq \phi^{1/2}(n) \ .$$

Put again $\lambda_A = \lambda_{A,f_A}$. This proves the upper bound in Theorem 2. (We recall that these choices of Δ and f_A, satisfy the condition on Lemma 6 and that $f_A \mathbb{P}\{A\} < 1/2$.)

This, together with Lemma 4, concludes the proof of Theorems 2 and 3.

8 Proof of Theorem 1.

Notice that it is enough to consider $\Delta \in [n, \frac{1}{\mathbb{P}\{A\}}]$ since otherwise the bound is trivial. Then, let Δ be the integer in the interval $[n, \frac{1}{\mathbb{P}\{A\}}]$ which minimizes the quantity $\Delta P(A) + *(\Delta - n)$.

For $t \leq \Delta$ let us define $\xi_{A,\mu(t)} = \lambda_{A,\frac{1}{\mathbb{P}\{A\}}}$. For t such that, $\Delta < t < \frac{1}{\mathbb{P}\{A\}}$, let us write $t = \frac{1}{k}\ \mu \frac{1}{\mathbb{P}\{A\}}$, with $k \in \mathbb{N}$ and $1 \leq \mu < \frac{k}{k-1}$. Define $\xi_{A,\mu(t)} = \lambda_{A,\mu\frac{1}{\mathbb{P}\{A\}}}$. Finally, for $t \geq \frac{1}{\mathbb{P}\{A\}}$, let us write $t = k\ \mu \frac{1}{\mathbb{P}\{A\}}$, with $k \in \mathbb{N}$ and $1 \leq \mu < \frac{k+1}{k}$. Define $\xi_{A,\mu(t)} = \lambda_{A,\mu\frac{1}{\mathbb{P}\{A\}}}$.

To prove that $\xi_{A,\mu(t)}$ is bounded away from zero and infinity (since the choice $f_A = \mu\frac{1}{I\!P\{A\}}$ with $1 \le \mu < 2$, no longer satisfies condition (8)) we just need to apply Theorems 2 and 3 to estimate $I\!P\left\{\tau_A > \mu\frac{1}{I\!P\{A\}}\right\}$.

For $t \le \Delta$ we have

$$\begin{aligned}\left|I\!P\{\tau_A > t\} - e^{-t\xi_{A,\mu(t)}I\!P\{A\}}\right| &= \left|I\!P\{\tau_A \le t\} - \left(1 - e^{-t\xi_{A,\mu(t)}I\!P\{A\}}\right)\right| \\ &\le I\!P\{\tau_A \le \Delta\} + 1 - e^{-\Delta\lambda_{A,\frac{1}{I\!P\{A\}}}I\!P\{A\}} \\ &\le \Delta I\!P\{A\} + \Xi_2\Delta I\!P\{A\}\ .\end{aligned}$$

For $t \ge \frac{1}{I\!P\{A\}}$ by Lemma 6 and Theorem 2 and 3

$$\begin{aligned}\left|I\!P\{\tau_A > t\} - e^{-tI\!P\{A\}\xi_{A,\mu(t)}}\right| &= \left|I\!P\left\{\tau_A > k\mu\frac{1}{I\!P\{A\}}\right\} - I\!P\left\{\tau_A > \mu\frac{1}{I\!P\{A\}}\right\}^k\right| \\ &\le \frac{2\Delta I\!P\{A\} + *(\Delta - n)}{I\!P\left\{\tau_A \le \mu\frac{1}{I\!P\{A\}}\right\}} \\ &\le C\left(2\Delta I\!P\{A\} + *(\Delta - n)\right)\ .\end{aligned}$$

For t such that, $\Delta < t < \frac{1}{I\!P\{A\}}$, we apply the Mean Value Theorem to the function $f(x) = x^{\frac{1}{k}}$. After this, we apply Lemma 6 with gap of length Δ (since $t > \Delta$) and $s = \frac{\mu}{k}\frac{1}{I\!P\{A\}}$.

$$\begin{aligned}&\left|I\!P\{\tau_A > t\} - e^{-t\xi_{A,\mu(t)}I\!P\{A\}}\right| \\ = &\left|I\!P\left\{\tau_A > \frac{\mu}{k}\frac{1}{I\!P\{A\}}\right\} - I\!P\left\{\tau_A > \mu\frac{1}{I\!P\{A\}}\right\}^{1/k}\right| \\ \le &\left|I\!P\left\{\tau_A > \frac{\mu}{k}\frac{1}{I\!P\{A\}}\right\}^k - I\!P\left\{\tau_A > \mu\frac{1}{I\!P\{A\}}\right\}\right| \\ &\times \frac{1}{k}\max\left\{I\!P\left\{\tau_A > \frac{\mu}{k}\frac{1}{I\!P\{A\}}\right\}^k ; I\!P\left\{\tau_A > \mu\frac{1}{I\!P\{A\}}\right\}\right\}^{\frac{1}{k}-1} \\ \le &\frac{2\Delta I\!P\{A\} + \alpha(\Delta - n)}{I\!P\left\{\tau_A \le \frac{\mu}{k}\frac{1}{I\!P\{A\}}\right\}}\frac{1}{k}C \\ \le &C\left(2\Delta I\!P\{A\} + \alpha(\Delta - n)\right)\ .\end{aligned} \tag{19}$$

where the last inequality holds since, by an application of Lemma 3, $I\!P\left\{\tau_A \le \frac{\mu}{k}\frac{1}{I\!P\{A\}}\right\} \ge C\frac{\mu}{k}$, and the maximum is bounded (above and below) by an application of Theorems 2 and 3.

Moreover if $t \in [k/\mathbb{P}\{A\}, (k+1)/\mathbb{P}\{A\})$ with $k \in \mathbb{N}$ then $t = k\mu \frac{1}{\mathbb{P}\{A\}}$ and $e^{-\xi_{A,\mu(t)}\mathbb{P}\{A\}t}$ is a left-continuous non-increasing function of t.

$$
\begin{aligned}
& \left| \lim_{t \to (k+1)/\mathbb{P}\{A\}} e^{-\xi_{A,\mu(t)}\mathbb{P}\{A\}t} - e^{-\xi_{A,\mu((k+1)/\mathbb{P}\{A\})}\mathbb{P}\{A\}t} \right| \\
= & \left| \mathbb{P}\left\{ \tau_A > \frac{k+1}{k}\frac{1}{\mathbb{P}\{A\}} \right\}^k - \mathbb{P}\left\{ \tau_A > \frac{1}{\mathbb{P}\{A\}} \right\}^{k+1} \right| \\
\leq & \left| \mathbb{P}\left\{ \tau_A > \frac{k+1}{k}\frac{1}{\mathbb{P}\{A\}} \right\}^k - \mathbb{P}\left\{ \tau_A > (k+1)\frac{1}{\mathbb{P}\{A\}} \right\} \right| \\
+ & \left| \mathbb{P}\left\{ \tau_A > (k+1)\frac{1}{\mathbb{P}\{A\}} \right\} - \mathbb{P}\left\{ \tau_A > \frac{1}{\mathbb{P}\{A\}} \right\}^{k+1} \right| \\
\leq & \ 2\left(2\Delta\mathbb{P}\{A\} + *(\Delta - n)\right) .
\end{aligned}
$$

This together with Lemma 1 concludes the proof.

Corollary. *For any exponentially decreasing α-mixing process, the rate of convergence in Theorem 1 is $n\mathbb{P}\{A\}$.*

Proof. It is enough to compute the minimum on the upper bound in Theorem 2 and apply Lemma 1

9 Behavior of $\xi_{A,\mu(t)}$.

We prove now Theorem 4 which gives an estimate of the parameter $\xi_{A,\mu(t)}$ (and λ_{A,f_A}). We recall here briefly that λ_{A,f_A} was defined in (9) and for a given $t = \mu k \frac{1}{\mathbb{P}\{A\}}$, with $k \in \mathbb{N}$ and $1 \leq \mu < \frac{k+1}{k}$, we define $\xi_{A,\mu(t)} = \lambda_{A,\mu\frac{1}{\mathbb{P}\{A\}}}$.

Proof of Theorem 4. Let f_A be as in (8). For any positive integer $k \leq f_A$, let us define $\sigma_{A,k} = \mathbb{P}\{\tau_A \leq k\}/k\mathbb{P}\{A\}$ and $\sigma_{A,f} = \mathbb{P}\{\tau_A \leq f_A\}/f_A\mathbb{P}\{A\}$. Then

$$
\begin{aligned}
\left| \mathbb{P}\{\tau_A > t\} - e^{-\zeta_{A,s}\mathbb{P}\{A\}t} \right| \leq & \left| \mathbb{P}\{\tau_A > t\} - e^{-\lambda_{A,f_A}\mathbb{P}\{A\}t} \right| \\
+ & \left| e^{-\lambda_{A,f_A}\mathbb{P}\{A\}t} - e^{-\sigma_{A,f}\mathbb{P}\{A\}t} \right| \\
+ & \left| e^{-\sigma_{A,f}\mathbb{P}\{A\}t} - e^{-\zeta_{A,s}\mathbb{P}\{A\}t} \right| .
\end{aligned} \tag{20}
$$

First note that $\theta \leq 1 - e^{-\theta} + \theta^2/2$ for all $\theta \geq 0$. Then, using the right hand side inequality of (10) we have that $\theta \leq 1 - e^{-\theta} + 2\left(1 - e^{-\theta}\right)^2$ for all $\theta \in [0,1]$. Put $\theta = -\log I\!P\{\tau_A > k\}$ with $k \leq f_A$. As in the proof of Lemma 4 we have that $0 \leq \theta \leq 1$. Using this together with Lemma 2 we have,

$$\sigma_{A,k} \leq \frac{-\log I\!P\{\tau_A > k\}}{kI\!P\{A\}} \leq \sigma_{A,k} + 2kI\!P\{A\} ,$$

for all $k \leq f_A$. Thus

$$|\sigma_{A,k} - \lambda_{A,k}| \leq CkI\!P\{A\} \leq Cf_AI\!P\{A\} .$$

By the Mean Value Theorem,

$$\begin{aligned} \left|e^{-\sigma_{A,f}I\!P\{A\}t} - e^{-\zeta_{A,s}I\!P\{A\}t}\right| &\leq |\sigma_{A,f} - \zeta_{A,s}| \; I\!P\{A\}t \, e^{-\min\{\sigma_{A,f},\zeta_{A,s}\}I\!P\{A\}t} \\ &\leq |\sigma_{A,f} - \zeta_{A,s}| \; . \end{aligned}$$

Using the fact that by stationarity,

$$\begin{aligned} I\!P\{\tau_A = j+1\} &= I\!P\{\tau_A > j\} - I\!P\{\tau_A > j+1\} \\ &= I\!P\{\tau_A \circ T > j\} - I\!P\{\tau_A > j+1\} \\ &= I\!P\left\{T^{-1}A \bigcap \tau_A \circ T > j\right\} \\ &= I\!P\left\{A \bigcap \tau_A > j\right\} , \end{aligned}$$

for all positive integer j, we have

$$\begin{aligned} &|\zeta_{A,s} - \sigma_{A,f}| \\ = \; & \left|\frac{I\!P\{A \bigcap \tau_A > \frac{n}{s}\}}{I\!P\{A\}} - \frac{I\!P\{\tau_A \leq f_A\}}{f_AI\!P\{A\}}\right| \\ = \; & \frac{1}{f_AI\!P\{A\}}\left|\sum_{j=1}^{f_A}\left[I\!P\left\{\tau_A = \frac{n}{s}+1\right\} - I\!P\{\tau_A = j\}\right]\right| \\ \leq \; & \frac{1}{f_AI\!P\{A\}}\left[\sum_{j=\frac{n}{s}+1}^{f_A} I\!P\left\{\tau_A \leq j - \frac{n}{s} - 1 \cap \tau_A \circ T^{j-\frac{n}{s}-1} = \frac{n}{s}+1\right\} + \frac{n}{s}I\!P\{A\}\right] \\ \leq \; & \frac{1}{f_AI\!P\{A\}}\sum_{j=\frac{n}{s}+1}^{f_A} I\!P\left\{\tau_A \leq j - \frac{n}{s} - 1 \cap T^{-j}A^{(\frac{n}{2s})}\right\} + \frac{n}{sf_A} \end{aligned} \qquad (21)$$

$$
\begin{aligned}
&\leq \frac{1}{f_A \mathbb{P}\{A\}} \sum_{j=\frac{n}{s}+1}^{f_A} \mathbb{P}\left\{\tau_A \leq j - \frac{n}{s} - 1\right\} \left(\mathbb{P}\left\{A^{(\frac{n}{2s})}\right\} + \alpha(\frac{n}{2s}) \right) + \frac{n}{s f_A} \\
&\leq f_A \left(\mathbb{P}\left\{A^{(\frac{n}{2s})}\right\} + \alpha(\frac{n}{2s}) \right) + \frac{n}{s f_A} \\
&\leq C e^{-cn} .
\end{aligned}
$$

First and second inequalities are just by stationarity and inclusion of sets. We recall that the notation $A^{(l)}$ was defined in the proof of Lemma 1. We note that in (21) we can't take A instead of $A^{(n/2s)}$ since in that case we would not have gap between $\{\tau_A \leq j - \frac{n}{s} - 1\}$ and $\{T^{-j}A\}$. The third inequality is by the α-mixing property. Fourth is by Lemma 2. The last inequality follows choosing

$$
f_A = \min \left\{ \frac{1}{\sqrt{\mathbb{P}\left\{A^{(\frac{n}{2s})}\right\}}} \; ; \; \frac{1}{\sqrt{\alpha(\frac{n}{2s})}} \right\} ,
$$

together with an application of Lemma 1 and the exponential decay of α.

Finally, we note that the proof of Theorem 3 with this choice of f_A, and $\Delta = -\log \mathbb{P}\{A\}/c$, where c is such that $\alpha(m) \leq e^{-cm}$ for all positive integers m, gives and exponential upper bound for (20).

We remark that these choices satisfy $f_A \mathbb{P}\{A\} < 1/2$ and $\Delta < f_A$.

Moreover $\zeta_{A,s} \leq 1$ and since $|\xi_A - \zeta_{A,s}| \leq C e^{-cn}$ and $\xi_A \geq \Xi_1 > 0$, there exists $\Psi_1 \leq \zeta_{A,s}$. This concludes the proof.

Let $s \in \mathbb{N}$. Define $\mathcal{B}_n = \mathcal{B}_n(s)$ as the set of $A \in \mathcal{C}_n$ which recur before time n/s, namely, $A \in \mathcal{B}_n$ if and only if there is an integer $1 \leq j \leq n/s$ such that $A \cap T^{-j} A \neq \emptyset$ for some $1 \leq j \leq n/s$.

Corollary. *Assume that the process is stationary and exponentially α-mixing and let $A \in \mathcal{E}^n \backslash \mathcal{B}_n$. Then in Theorem 3 we get $\lambda_A = 1$, namely, there are positive numbers c, C_1 and C_2 such that the following inequality holds*

$$
C_1 \, \mathbb{P}(A) \leq \sup_{t \geq 0} \left| \mathbb{P}\left\{ \tau_A > \frac{t}{\mathbb{P}\{A\}} \right\} - e^{-t} \right| \leq C_2 \, e^{-cn} .
$$

Proof. If $A \in \mathcal{B}_n$ then $\zeta_{A,s} = 1$.

Now we prove that the cylinders for which the last corollary holds, are typical in the following sense

Lemma 7. *Let the process be α-mixing (at any rate). There exist $s \in I\!N$ and two positive constants C and c such that*

$$I\!P\{\mathcal{B}_n(s)\} \leq Ce^{-cn} .$$

Proof. Let s be a positive integer. Take $A = \{X_1^n = a_1^n\} \in \mathcal{B}_n(s)$. We first observe that there is an integer k, with $1 \leq k \leq [n/s] - 1$ such that

$$a_k^n = a_1^{n-k+1} .$$

this implies that

$$a_{jk+1}^{(j+1)k} = a_1^k ,$$

for all $0 \leq j < [n/k] - 1$. Let $m(k) = [n/k]$, therefore

$$\begin{aligned}
&\sum_{a_1^n \in \mathcal{B}_n(s)} I\!P\{X_1^n = a_1^n\} \\
\leq\ & \sum_{k=1}^{n/s} \sum_{b_1^k \in \mathcal{E}^k} I\!P\{X_1^k = X_{k+1}^{2k} = \cdots = X_{(m(k)-1)k+1}^{m(k)k} = b_1^k, X_{m(k)k+1}^{m(k)k+r} = b_1^r\} \\
\leq\ & \sum_{k=1}^{n/s} \sum_{b_1^k \in \mathcal{E}^k} \left(\alpha(0) + I\!P\{X_1^k = b_1^k\}\right) I\!P\{X_{k+1}^{2k} = \cdots = X_{(m(k)-1)k+1}^{m(k)k} = b_1^k\} \\
\leq\ & \sum_{k=1}^{n/s} \sum_{b_1^k \in \mathcal{E}^k} \left(\alpha(0) + I\!P\{X_1^k = b_1^k\}\right) Ce^{-c\, n(s-1)/s} \\
\leq\ & C\alpha(0)e^{n\left[\frac{\ln|\mathcal{E}| - c\,(s-1)}{s}\right]} + Ce^{-c\,n} ,
\end{aligned}$$

where we use the α-mixing property and Lemma 1 together with the fact that $n - k \geq n - n/s$. Taking a large enough s we get the exponential decay.

Counterexample. We now provide an example of a process and a sequence of cylinders A_n such that $\lambda_{A_n} < 1$.

Let $\mathcal{E} = \{0, 1\}$ and $A = A_n = \{X_0 = 1, \ldots, X_{n-1} = 1\}$ $\forall n \in I\!N$. Suppose that X_n are i.i.d. with $I\!P\{X_n = 1\} = p$. Then $\zeta_{A,s} = 1 - p$.

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