Singularity Theory for Non-twist Tori
From rigorous results to numerical computations

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Barcelona, 23 September 2015
KAM Theory for non-degenerate Hamiltonian systems

Existence and persistence of quasi-periodic motions in Hamiltonian systems

Classical KAM theorems guarantee persistence of Cantor families of invariant tori under perturbations of non-degenerate integrable systems.

Twist conditions (≃ diffeomorphic frequency map):
- Persistence of quasi-periodic motions
  Kolmogorov, Arnold, Moser, ...
- KAM results based on reducibility
  Celletti-Chierchia, Salamon-Zehnder, González-Jorba-de la Llave-Villanueva, ...

Rüssmann conditions (≃ weakly non-degenerate frequency map):
- Persistence of Cantor families of invariant tori
  Rüssmann, Broer, Huitema, Sevryuk, ...
KAM Theory for degenerate Hamiltonian systems

Existence and persistence of degenerate invariant tori with prescribed frequency

Non-twist tori or shearless tori are invariant tori for which the Birkhoff normal form is degenerate. (The integrable approximation violates the twist condition)

There are many numerical and experimental evidences of existence of non-twist tori, in areas such as plasma physics, celestial mechanics, oceanography.

del Castillo-Negrete, Morrison, Meiss, Apte, Dullin, Beron-Vera, Haller, ...

Rigorous results:
KAM Theory for degenerate Hamiltonian systems
Existence and persistence of degenerate invariant tori with prescribed frequency

Our contribution:

- A methodology to study non-twist tori and bifurcations of KAM tori with fixed Diophantine frequency.
- The methodology works in any dimension, and for any kind of degeneracy.
- The methodology works in the far-from-integrable regime (including meandering non-twist tori and non-twist tori close to breakdown).
- The methodology gives rise to efficient numerical algorithms (and with time and effort, to computer assisted proofs.)
Motivating examples
Example: A collision of invariant tori
A family of non-twist apm

Consider the quadratic standard family of apm in $\mathbb{T} \times \mathbb{R}$:

\[
\begin{align*}
\bar{x} &= 0.375 + x + \bar{y}^2, \\
\bar{y} &= y - \frac{\epsilon}{2\pi} \sin(2\pi x).
\end{align*}
\]

Problem:

Study the bifurcations of invariant tori with frequency $\omega = \frac{3-\sqrt{5}}{2}$, with respect to $\epsilon$. 
Motivating examples

Example: A collision of invariant tori

Numerical observations

\[ T = 0.1669253 \]

\[ T = -0.1669253 \]
Example: A collision of invariant tori

Numerical observations

\[ \varepsilon = 1.000000 \]

\[ T = 0.1123464 \]

\[ T = -0.1123464 \]
Example: A collision of invariant tori

Numerical observations

\[ \varepsilon = 1.361408 \]

\[ T = 0.6558 \cdot 10^{-8} \]

\[ T = -0.6558 \cdot 10^{-8} \]
Motivating examples

Example: A collision of invariant tori

Numerical observations

\[ \epsilon = 1.500000 \]
Motivating examples

Inspiration: Integrable systems

Frequency map of an integrable system in $\mathbb{T}^n \times \mathbb{R}^n$

Let

$$f_0(x, y) = \begin{pmatrix} x + \nabla W(y) \\ y \end{pmatrix}$$

be an integrable system, where $W : \mathbb{R}^n \to \mathbb{R}$.

- The **frequency map** is $\hat{\omega} = \nabla W : \mathbb{R}^n \to \mathbb{R}^n$.

- $\forall p \in \mathbb{R}^n$, $Z_p(\theta) = \begin{pmatrix} \theta \\ p \end{pmatrix}$ is $f_0$-invariant with frequency $\hat{\omega}(p)$:

  $$f_0(Z_p(\theta)) = Z_p(\theta + \hat{\omega}(p)).$$

- $Z_p$ is **twist** iff the torsion

  $$\bar{T}(p) = D\hat{\omega}(p) = \text{Hess } W(p)$$

  is non-degenerate. Otherwise, $Z_p$ is **non-twist**.
Inspiration: Integrable systems
Potential for frequency $\omega$ of an integrable system in $\mathbb{T}^n \times \mathbb{R}^n$

Consider the modified family (Moser, Herman):

$$f_{0,\lambda}(x, y) = \left( x + \nabla W(y) \right) + \left( \begin{array}{c} \lambda \\ 0 \end{array} \right).$$

- $\forall p \in \mathbb{R}^n$, $Z_p(\theta) = \left( \begin{array}{c} \theta \\ p \end{array} \right)$ is $f_{0,\lambda(p)}$-invariant with frequency $\omega$:

$$f_{0,\lambda(p)}(Z_p(\theta)) = Z_p(\theta + \omega),$$

where $\lambda(p) = \omega - \nabla W(p)$.

- The potential $V : \mathbb{R}^n \to \mathbb{R}$, defined by

$$V(p) = W(p) - p^\top \omega,$$

satisfies

$$\lambda(p) = -\nabla_p V(p), \quad \bar{T}(p) = \text{Hess } V(p).$$
Motivating examples

Inspiration: Integrable systems
Invariant tori as critical points of the potential

Conclusions:

1. $Z_{p^*}$ is $f_0$-invariant with frequency $\omega$ if and only if $p^*$ is a critical point of the potential $V(p)$.

2. $Z_{p^*}$ is a twist $f_0$-invariant torus with frequency $\omega$ if and only $p^*$ is a non-degenerate critical point of $V(p)$.

3. $Z_{p^*}$ is a non-twist $f_0$-invariant torus with frequency $\omega$ if and only $p^*$ is a degenerate critical point of $V(p)$.
Motivating examples

Inspiration: Integrable systems

A degenerate case in $\mathbb{T} \times \mathbb{R}$

Let $f_\mu$ be family of integrable symplectomorphisms in $\mathbb{T} \times \mathbb{R}$ with

- frequency map $\hat{\omega}(y; \mu) = \omega + \mu + y^2$.

For fixed frequency $\omega$:

- translation map: $\lambda(p; \mu) = -\mu - p^2$;
- potential: $V(p; \mu) = \mu p + \frac{p^3}{3}$.

Then, for fixed frequency $\omega$:

- TWO invariant tori for $\mu < 0$.
- ONE NON-TWIST torus for $\mu = 0$.
- NO invariant tori for $\mu > 0$.
Motivating examples

Inspiration: Integrable systems

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\[ \mu = -0.5 \]
Motivating examples

Inspiration: Integrable systems

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\[ \mu = 0 \]
Motivating examples

Inspiration: Integrable systems

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Let $f_\mu$ be family of integrable symplectomorphisms in $\mathbb{T} \times \mathbb{R}$ with
- frequency map $\hat{\omega}(\mathbf{y}; \mu) = \omega + \mu + y^2$.

For fixed frequency $\omega$:
- translation map: $\lambda(\mathbf{p}; \mu) = -\mu - p^2$;
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Then (for fixed frequency $\omega$):
- TWO invariant tori for $\mu < 0$.
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- NO invariant tori for $\mu > 0$. 

\[ \mu = 0.5 \]
Inspiration: Integrable systems

A degenerate case in $T \times \mathbb{R}$

Let $f_\mu$ be family of integrable symplectomorphisms in $T \times \mathbb{R}$ with

- frequency map $\hat{\omega}(y; \mu) = \omega + \mu + y^2$.

For fixed frequency $\omega$:

- translation map: $\lambda(p; \mu) = -\mu - p^2$;

- potential: $V(p; \mu) = \mu p + \frac{p^3}{3}$.

Then (for fixed frequency $\omega$):

- TWO invariant tori for $\mu < 0$.
- ONE NON-TWIST torus for $\mu = 0$.
- NO invariant tori for $\mu > 0$. 

$\mu = +1.0$
Motivating examples

Natural questions
(From the motivating examples)

For a fixed Diophantine frequency $\omega$:

1. Is the potential and its properties persistent under perturbations of an integrable system?

2. Given a non-integrable system, are there sufficient conditions that guarantee the existence of a suitable potential?

3. Can the potential be used to develop a bifurcation theory of invariant tori?
Methodology
Methodology

Setting

- Let $\Omega = \begin{pmatrix} O_n & -I_n \\ I_n & O_n \end{pmatrix}$ be the standard symplectic matrix in $\mathbb{T}^n \times \mathbb{R}^n$.
- Let $P \subset \mathbb{R}^m$ be an open subset of parameters $\varepsilon$.
- Let $f_\varepsilon : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{T}^n \times \mathbb{R}^n$ a smooth family of real-analytic exact symplectomorphisms, with primitive functions $S_\varepsilon : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}$:
  \[ dS_\varepsilon = f_\varepsilon^*(y \, dx) - y \, dx. \]
- Let $\omega \in \mathbb{R}^n$ be fixed and Diophantine:
  \[ \left| \ell^\top \omega - m \right| \geq \gamma |\ell|_1^{-\tau}, \quad \forall \ell \in \mathbb{Z}^n \setminus \{0\}, \ m \in \mathbb{Z}. \]
  where $\gamma > 0$, $\tau \geq n$.

Goal: Finding $f_\varepsilon$-invariant tori, with frequency $\omega$. 
Methodology

Step 1: KAM theory for families of invariant tori

Using **Newton-like methods** for families of symplectic deformations, find

$$(p; \varepsilon) \in U \times P \rightarrow K(p; \varepsilon) = (\lambda(p; \varepsilon), K(p; \varepsilon))$$

such that

$$f_{\varepsilon} \circ K(p; \varepsilon)(\theta) + \begin{pmatrix} \lambda(p; \varepsilon) \\ 0 \end{pmatrix} = K(p; \varepsilon)(\theta + \omega),$$

$$\langle K(p; \varepsilon)(\theta) - \begin{pmatrix} \theta \\ p \end{pmatrix} \rangle = 0.$$

For each $\varepsilon$, $p \rightarrow K(p; \varepsilon)$ is a **Lagrangian deformation**.

**Non-degeneracy conditions for Lagrangian deformations are mild.**
Methodology

Step 1: KAM theory for families of invariant tori

The **torsion** of $K_{(p;\epsilon)}$ is the symmetric matrix $\bar{T}(p;\epsilon) = \left\langle T_{(p;\epsilon)}(\theta) \right\rangle$, where

$$T_{(p;\epsilon)}(\theta) = N_{(p;\epsilon)}(\theta + \omega)^\top \Omega D_zf_{\epsilon}(K_{(p;\epsilon)}(\theta)) N_{(p;\epsilon)}(\theta),$$

$$N_{(p;\epsilon)}(\theta) = JD_\theta K_{(p;\epsilon)}(\theta) \left(D_\theta K_{(p;\epsilon)}(\theta)^\top D_\theta K_{(p;\epsilon)}(\theta)\right)^{-1}.$$ 

The torus is twist if the torsion is non-degenerate.
Methodology

Step 2: Symplectic Geometry of families of invariant tori

Define the potential (depending on $\varepsilon$),

$$V(p; \varepsilon) = -p^\top \lambda(p; \varepsilon) - \left\langle S_\varepsilon \circ K_{(p;\varepsilon)}(\theta) \right\rangle.$$

Properties of the potential:

- $\lambda(p; \varepsilon) = -\nabla_p V(p; \varepsilon)$.
- $\tilde{T}(p; \varepsilon) W_1(p; \varepsilon) = W_2(p; \varepsilon) \text{Hess } V(p; \varepsilon)$ where
  $$W_1(p; \varepsilon) = D_p C(p; \varepsilon), \text{ with } C(p; \varepsilon) = \left\langle K^y_{(p;\varepsilon)}(\theta)^\top D_\theta K^x_{(p;\varepsilon)}(\theta) \right\rangle^\top,$$
  $$W_2(p; \varepsilon) = \left\langle N^y_{(p;\varepsilon)}(\theta) - D_\theta K^y_{(p;\varepsilon)}(\theta) \mathcal{R}_\omega T_{(p;\varepsilon)}(\theta) \right\rangle.$$
Methodology

Step 3: Singularity Theory for critical points of the potential

For $\varepsilon_*$ fixed:

- $K_{(p^*; \varepsilon_*)}$ is $f_{\varepsilon_*}$-invariant with frequency $\omega$ iff $p_*$ is a critical point of the potential $V(p; \varepsilon_*)$.

- If $W_1(p^*; \varepsilon_*)$ and $W_2(p^*; \varepsilon_*)$ are invertible:

  $$\dim \ker \bar{T}(p^*, \varepsilon_*) = \dim \ker \text{Hess}_p V(p^*, \varepsilon_*)$$
KAM theory for translated tori
Translated tori
The functional equation

Given:
- a homotopic to the identity real-analytic symplectomorphism $f : \mathbb{T}^n \times \mathbb{R}^n \rightarrow \mathbb{T}^n \times \mathbb{R}^n$;
- a Diophantine vector $\omega \in \mathbb{R}^n$, and denote $R_\omega(\theta) = \theta + \omega$;

for each height $p \in \mathbb{R}^n$ we define the error function

$$E(p; K, \Lambda) \overset{\text{def}}{=} \left( f \circ K - K \circ R_\omega + \Lambda \right)$$

for any
- homotopic to the zero-section torus $K : \mathbb{T}^n \rightarrow \mathbb{T}^n \times \mathbb{R}^n$
- translation $\Lambda = \begin{pmatrix} \lambda \\ \sigma \end{pmatrix} \in \mathbb{R}^{2n}$

Our goal
To solve $E(p; K, \Lambda) = 0$ by a Newton-Nash-Moser-Zehnder method.
Newton method

The linearized equation

Let \((K, \Lambda)\) be an approximate solution: \(E(p; K, \Lambda) = (e, \nu)^\top\).

1. A Newton step involves solving a linear system for \((\Delta K, \Delta \Lambda)\):

\[
\begin{align*}
Df(K(\theta))\Delta K - \Delta K \circ R_\omega + \Delta \Lambda &= -e \\
\langle \Delta \rangle &= -\nu
\end{align*}
\]  

(1)

2. The approximately symplectic matrix

\[
M(\theta) = \begin{pmatrix} DK(\theta) & N(\theta) \end{pmatrix},
\]

with \(N(\theta) = JDK(\theta)(DK(\theta)^\top DK(\theta))^{-1}\), almost reduces \(Df(K(\theta))\):

\[
M(\theta + \omega)^{-1}Df(K(\theta))M(\theta) \simeq (I_{2n} + \hat{T}(\theta)),
\]

where

\[
\hat{T}(\theta) = \begin{pmatrix} 0_n & T(\theta) \\ 0_n & 0_n \end{pmatrix} \quad \text{and} \quad T(\theta) = N(\theta + \omega)^\top \Omega Df(K(\theta)) N(\theta).
\]
Newton method
Homological equations

3. We write $\Delta K(\theta) = M(\theta)\xi(\theta)$, so the Newton equation (1) is approximately equivalent to

$$(I + \hat{T})\xi - \xi \circ R_\omega + (M \circ R_\omega)^{-1} \Delta \lambda = -(M \circ R_\omega)^{-1} e$$

$$\langle M\xi \rangle = -\nu$$

(2)

4. $L_\omega$ is the operator acting on functions $\xi : \mathbb{T}^n \to \mathbb{R}$ by

$$L_\omega \xi \overset{\text{def}}{=} \xi - \xi \circ R_\omega,$$

We define its “inverse”, as the “small divisors" operator $R_\omega$ acting on functions $\eta : \mathbb{T}^n \to \mathbb{R}$ by

$$R_\omega \eta \overset{\text{def}}{=} \xi \text{ is the zero-average solution of } L_\omega \xi = \eta - \langle \eta \rangle.$$

Notice that $R_\omega L_\omega \xi = \xi - \langle \xi \rangle$, $L_\omega R_\omega \eta = \eta - \langle \eta \rangle$. 
Newton method

Approximate solution of Newton step

Define $M_+ = M \circ R_\omega$, i.e. $M_+(\theta) = M(\theta + \omega)$.

Let

$$\eta = -M_+^{-1} e$$

be the new r.h.s., and $\xi_0 = \langle \xi \rangle$.

$$\xi = (I - R_\omega \hat{T}) \xi_0 - R_\omega (M_+^{-1} - \hat{T} R_\omega M_+^{-1}) \Delta \Lambda + R_\omega (\eta - \hat{T} R_\omega \eta)$$

where $\xi_0$ and $\Delta \Lambda$ are computed from

$$\begin{pmatrix} \langle \hat{T} \rangle \\ \langle M(I - R_\omega \hat{T}) \rangle - \langle M R_\omega (M_+^{-1} - \hat{T} R_\omega M_+^{-1}) \rangle \end{pmatrix} \begin{pmatrix} \xi_0 \\ \Delta \Lambda \end{pmatrix} = \begin{pmatrix} \tilde{\eta} \\ \tilde{\nu} \end{pmatrix}$$

where $\tilde{\eta} = \langle \eta - \hat{T} R_\omega \eta \rangle$, $\tilde{\nu} = -\nu - \langle M R_\omega (\eta - \hat{T} R_\omega \eta) \rangle$. 
Newton method

A mild non degeneracy condition

Non degeneracy condition

In order to perform the Newton step, it is sufficient that the $4n \times 4n$ matrix

$$Q(K) \overset{\text{def}}{=} \begin{pmatrix}
\langle \hat{T} \rangle & \langle M_+^{-1} - \hat{T} R_\omega M_+^{-1} \rangle \\
\langle M(I - R_\omega \hat{T}) \rangle & \langle MR_\omega (M_+^{-1} - \hat{T} R_\omega M_+^{-1}) \rangle
\end{pmatrix}$$

be invertible.
Newton method
A mild non degeneracy condition

Integrable case
Non degeneracy condition is satisfied, since

\[ Q(Z_p) = \begin{pmatrix}
  O_n & \langle T \rangle & I_n & O_n \\
  O_n & O_n & O_n & I_n \\
  I_n & O_n & O_n & O_n \\
  O_n & I_n & O_n & O_n
\end{pmatrix} \]
Translated tori theorem

Informal statement

Theorem (Translated tori theorem)

Let \((p_0; K_0, \Lambda_0)\) be an approximate solution of the functional equation \(E(p; K, \Lambda) = 0\).

- If \(Q(K_0)\) is invertible and \(E(p_0; K_0, \Lambda_0)\) is small enough,
- then, for each \(p\) close to \(p_0\) there exist a unique \(\Lambda(p) \in \mathbb{R}^{2n}\) and a unique \(K_p : \mathbb{T}^n \to \mathbb{T}^n \times \mathbb{R}^n\) s.t.

\[
\begin{align*}
  f(K_p(\theta)) & = K_p(\theta + \omega) - \Lambda(p), \\
  \left\langle K_p(\theta) - \begin{pmatrix} \theta \\ p \end{pmatrix} \right\rangle & = 0.
\end{align*}
\]
Translating tori theorem

Informal statement (cont.)

Theorem (TTT cont.)

Moreover:

- \( \Lambda(p) \) and \( K_p \) are smooth with respect to \( p \).

- \( \Lambda(p) \) and \( K_p \) depend smoothly on parameters of \( f \).

If \( f \) is exact w.r.t. to the Liouville form \( \alpha = y \, dx \), that is \( f^* \alpha - \alpha = dS \) for a primitive function \( S : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R} \):

- The translation is horizontal:

\[
\Lambda(p) = \begin{pmatrix} \lambda(p) \\ 0 \end{pmatrix},
\]

- \( \lambda(p) \) is a gradient: \( \lambda(p) = -\nabla_p V(p) \).
Applications

- **Close-to-integrable systems:**
  1. Persistence of invariant tori with fixed frequency under weak non-degeneracy conditions.
  2. Persistence of invariant tori for perturbations of isochronous systems.
  3. Robust bifurcation theory for non-twist tori.

- **Numerical computations in far-to-integrable systems:**
  1. A fold singularity.
  2. Birth of meandering.
  3. Continuation of non-twist tori.
Bifurcation theory for non-twist tori
A persistence result of non-twist tori

Hypotheses

Theorem (Persistence theorem of non-twist tori)

Given:

- \( \omega \in \mathbb{R}^n \), a Diophantine frequency vector.
- \( f_0 \), a real-analytic integrable symplectic map with frequency map \( \hat{\omega}(y) = \omega + \nabla V(y) \), s.t. \( y^* = 0 \) is a finitely determined singularity of \( V(y) \).
- \( f_\mu \), a \( \mu \)-family of real-analytic integrable symplectic map with frequency map \( \hat{\omega}_\mu(y) = \omega + \nabla V_\mu(y) \), s.t. \( V_\mu(y) \) is a stable unfolding of the singularity at zero.
- \( f_{\mu, \varepsilon} \), a perturbed family of exact symplectomorphisms:

\[
f_{\mu, \varepsilon}(x, y) = \left( x + \omega + \nabla V_\mu(y) \right)_y + \varepsilon g(x, y, \varepsilon).
\]
A persistence result of non-twist tori

Theorem (Theses)

Then:

- **(KAM)** For $(\mu, \varepsilon)$ sufficiently small, there exists a potential $V_{\mu,\varepsilon}(p)$ defined around $0 \in \mathbb{R}^n$ and an $n$-dimensional torus $K_{p,\mu,\varepsilon} : \mathbb{T}^n \to \mathbb{T}^n \times \mathbb{R}^n$, depending smoothly on $(p, \mu, \varepsilon)$, such that:

  $$f_{\mu,\varepsilon} \circ K_{p,\mu,\varepsilon}(\theta) = K_{p,\mu,\varepsilon}(\theta + \omega) + \left( \begin{array}{c} \nabla_p V_{\mu,\varepsilon}(p) \\ 0 \end{array} \right).$$

- **(ST)** For $\varepsilon$ sufficiently small, there exist $\mu(\varepsilon)$ and $p(\varepsilon)$ such that $V_{\mu(\varepsilon),\varepsilon}$ has a singularity at $p(\varepsilon)$ that is of the same type (in the sense of Singularity Theory) of the singularity of $V_0$ at zero.
Numerical examples
Example 1: The collision of invariant tori

Applying the methodology

- **Modified family:**

  \[ f_{(\varepsilon, \lambda)}(x, y) = \left( 0.375 + x + \left( y - \frac{\varepsilon}{2\pi} \sin(2\pi x) \right)^2 + \lambda \right). \]

- The primitive function of \( f_{\varepsilon, \lambda} \) is

  \[ S_\varepsilon(x, y) = \frac{\varepsilon}{4\pi^2} \cos(2\pi x) + \frac{2}{3} \left( y - \frac{\varepsilon}{2\pi} \sin(2\pi x) \right)^3. \]

- The **co-rank** of the torsion of a torus is either 0 (twist) or 1 (non-twist).
Example 1: The collision of invariant tori

Applying the methodology

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Example 1: The collision of invariant tori

Applying the methodology

- Modified family:

\[
f_{(\varepsilon,\lambda)}(x, y) = \left(0.375 + x + \left(y - \frac{\varepsilon}{2\pi} \sin(2\pi x)\right)^2 + \lambda, y - \frac{\varepsilon}{2\pi} \sin(2\pi x)\right).
\]

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S_{\varepsilon}(x, y) = \frac{\varepsilon}{4\pi^2} \cos(2\pi x) + \frac{2}{3} \left(y - \frac{\varepsilon}{2\pi} \sin(2\pi x)\right)^3.
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- The co-rank of the torsion of a torus is either 0 (twist) or 1 (non-twist).
Example 1: The collision of invariant tori
A fold singularity

\[ \varepsilon = 0.000000 \]

\[ T = 0.1669253 \]

\[ T = -0.1669253 \]
Example 1: The collision of invariant tori
A fold singularity

\[ T = 0.1123464 \]
\[ T = -0.1123464 \]
Example 1: The collision of invariant tori

A fold singularity

\[
\varepsilon = 1.361408
\]

\[
T = 0.6558 \cdot 10^{-8}
\]

\[
T = -0.6558 \cdot 10^{-8}
\]
Example 1: The collision of invariant tori

A fold singularity

\( \varepsilon = 1.500000 \)
Example 2: The birth of meandering tori

Another non-twist family

Consider a quadratic standard family of symplectomorphisms \( f_\varepsilon : \mathbb{T} \times \mathbb{R} \to \mathbb{T} \times \mathbb{R} \), defined by:

\[
\begin{align*}
\bar{x} &= x + (\bar{y} + 0.1)(\bar{y} - 0.2), \\
\bar{y} &= y - \frac{\varepsilon}{2\pi} \sin(2\pi x).
\end{align*}
\]

Problem: Look for invariant tori with frequency \( \omega = \frac{\sqrt{5} - 1}{32} \), with respect to parameter \( \varepsilon \).
A fold singularity
The birth of meandering tori

$\varepsilon = 0.420000$
A fold singularity
The birth of meandering tori

\[ \varepsilon = 0.430396 \]

\[ T = -1.6261 \cdot 10^{-7} \]

\[ T = 1.6261 \cdot 10^{-7} \]
A fold singularity
The birth of meandering tori

$\varepsilon = 0.440000$

$T = -0.0211805$
$T = 0.0211805$
A fold singularity
The birth of meandering tori

\( \varepsilon = 0.450000 \)

\[ T = -0.0309003 \]
\[ T = 0.0309003 \]
Example 3: Continuation of a non-twist torus

The problem

Consider a quadratic standard family of symplectomorphisms \( f_{\mu, \varepsilon} : \mathbb{T} \times \mathbb{R} \rightarrow \mathbb{T} \times \mathbb{R} \), defined by:

\[
\begin{align*}
\bar{x} &= \omega + \mu + x - \bar{y}^2, \\
\bar{y} &= y - \frac{\varepsilon_1}{2\pi} \sin(2\pi x) - \frac{\varepsilon_2}{4\pi} \cos(4\pi x).
\end{align*}
\]

with \( \omega = \frac{\sqrt{5} - 1}{2} \).

\( \mu \) is the unfolding parameter.

Fixed \( \phi \), take \( \varepsilon_1 = \cos(2\pi \phi)\varepsilon \), \( \varepsilon_2 = \sin(2\pi \phi)\varepsilon \), with \( \varepsilon \) as the continuation parameter.

Parameter \( \phi \) breaks reversibility.

Problem: Look for non-twist invariant tori with frequency \( \omega = \frac{\sqrt{5} - 1}{2} \) (for suitable \( \mu \)), with respect to parameter \( \varepsilon \), from \( \varepsilon = 0 \) (and \( \mu = 0 \)).
Example 3: Continuation of a non-twist torus

Case $\varphi = \frac{1}{8}$
Example 3: Continuation of a non-twist torus and its adapted symplectic frame

\[ \varepsilon = 0.000, \mu = 0.0000000000 \]

NF = 1024
Example 3: Continuation of a non-twist torus and its adapted symplectic frame

\[ \varepsilon = 0.500, \mu = 0.0006691132 \]

**NF = 1024**
Example 3: Continuation of a non-twist torus and its adapted symplectic frame

\[ \varepsilon = 1.000, \mu = 0.0026359979 \]

NF = 1024
Example 3: Continuation of a non-twist torus and its adapted symplectic frame

\[ \varepsilon = 1.500, \mu = 0.0057799523 \]

\[ \alpha \]

\[ \theta \]

\[ x \]

\[ y \]

\( NF = 1024 \)
Numerical examples

Example 3: Continuation of a non-twist torus and its adapted symplectic frame

$\varepsilon = 2.000$, $\mu = 0.0099248657$

$\text{NF} = 1024$
Example 3: Continuation of a non-twist torus and its adapted symplectic frame

\[ \varepsilon = 2.500, \mu = 0.0150071316 \]

**NF** = 1024
Example 3: Continuation of a non-twist torus and its adapted symplectic frame

\[ \varepsilon = 2.800, \mu = 0.0187418638 \]

NF = 32768
Example 3: Continuation of a non-twist torus and its adapted symplectic frame

\( \varepsilon = 2.860, \mu = 0.0195948602 \)

\( \text{NF} = 524288 \)
Example 3: Continuation of a bunch of non-twist tori

Taking $\varphi = 0.000, 0.001, 0.002, \ldots, 1.000$. 
Contributions and perspectives
Contributions

- The infinite dimensional problem of finding invariant tori for Hamiltonian systems is reduced to the finite dimensional problem of finding critical points of the potential.

- Singularity Theory for invariant tori
  - Non-twist tori correspond to degenerate critical points.
  - Applicable to any finite-determined singularity of the frequency map.

- Persistence results for small twist systems.
  - Applicable to perturbation of isochronous systems, introducing a suitable Melnikov potential.

See
Perspectives

- Study the **connections with Symplectic Geometry**.
  - The theory is developed using such language.

- Develop **efficient numerical algorithms** to compute and **validate** invariant tori and their bifurcations.
  - The method is designed for the cases on which numerical evidence of bifurcations is known but the system is not close to integrable.
  - It is useful for studying of the breakdown of degenerate invariant tori.

- Apply the techniques to **“real” problems** in plasma physics, oceanography (LCS?), celestial mechanics, etc.
A road map

- Symplectic deformations and moment maps
- Potential for invariant fibered Lagrangian deformations
- Persistence of invariant tori for close-to-integrable systems
- Non-twist tori for close-to-integrable systems
- Existence of invariant fibered Lagrangian deformations
- Singularity Theory for invariant tori
- Computational methods for invariant tori bifurcations
- Parametric KAM Theorem
- Approx. reducibility of approx. invariant tori
- Reducibility of invariant tori