Singularity Theory for Non-twist Tori

A. González A. Haro R. de la Llave

Universitat de Barcelona & University of Texas at Austin

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KAM Theory

Existence and persistence of quasi-periodic motions in Hamiltonian systems

Under twist conditions ($\simeq$ diffeomorphic frequency map):

- Persistence of quasi-periodic motions (Kolmogorov, Arnold, Moser...)
- A posteriori results of existence of invariant tori with fixed frequency, based on reducibility (de la Llave, González, Jorba, Villanueva)

Existence of Cantor families of tori under weaker conditions:

- Unfoldings of quasi-periodic tori (Broer, Huitema, Moser, Pöschel, Sevryuk, Takens, ...)
- Persistence of invariant tori under Rüssmann condition (Sevryuk, Rüssmann, Xu, You, Zhang, ...)

Non-twist tori with fixed frequency

Previous results

- Numerical results in non-twist area preserving maps:
  - del Castillo-Negrete, Greene, Morrison (1996, 1997): periodic orbits and transition to chaos (Apte, Wurm, Petrisor, Shinohara, Aizawa, ...)
  - Haro (2002): computation of non-twist tori and normal forms.
- Rigorous results:
  - Simó (1998): existence of meandering curves in area preserving non-twist maps (but the curves are twist!).
  - Delshams, de la Llave (2002): existence and persistence of non-twist curves in 2-parameter families of apm.
  - Dullin, Ivanov, Meiss (2006): normal forms for the *fold and cusp singularities* for 4D maps.

Non-twist tori with fixed frequency

This talk

Singularity Theory for non-twist tori
(arbitrary dimension and any finite-determined singularity)

- Bifurcations of invariant tori, with fixed frequency.
  - At the bifurcation points, the invariant tori are non-twist.
- Classification of the possible degeneracies of invariant tori.
- Unfolding Hamiltonian systems around non-twist tori.
- Persistence of non-twist invariant tori.
- Efficient numerical methods to study bifurcations.

Motivating example 1

A family of quadratic standard maps

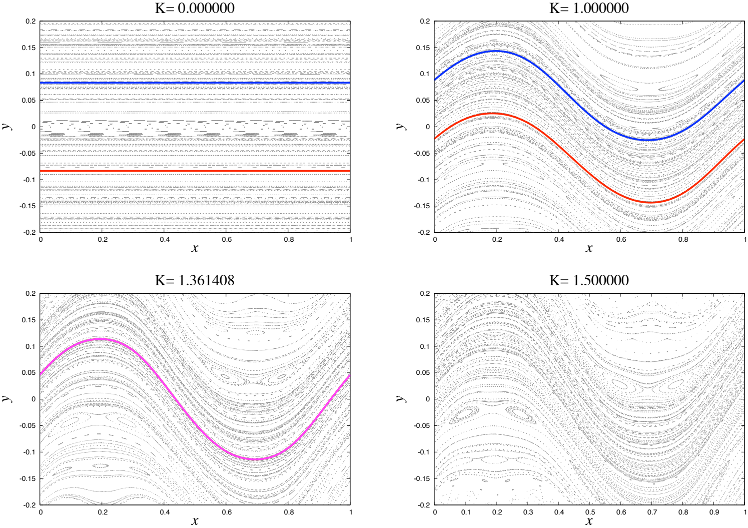
Consider the quadratic standard family of symplectomorphisms $f_\kappa : \mathbb{T} \times \mathbb{R} \to \mathbb{T} \times \mathbb{R}$, defined by:

$$\begin{cases} \bar{x} &= 0.375 + x + \bar{y}^2 \,, \\ \bar{y} &= y - \dfrac{\kappa}{2\pi} \sin(2\pi x) \,. \end{cases}$$

Problem: Look for invariant tori with frequency $\omega = \frac{3-\sqrt{5}}{2}$, with respect to parameter κ.

A collision of invariant tori

Numerical observations



Motivating example 2

Frequency map of an integrable symplectomorphism

Consider the integrable system $f_0 : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{T}^n \times \mathbb{R}^n$,

$$f_0(x, y) = \begin{pmatrix} x + \nabla W(y) \\ y \end{pmatrix},$$

where $W : \mathbb{R}^n \to \mathbb{R}$.

Observations:

- For $p \in \mathbb{R}^n$, the torus $K_p(\theta) = \begin{pmatrix} \theta \\ p \end{pmatrix}$ is invariant for f_0 with frequency $\hat{\omega}(p) = \nabla W(p)$:

$$f_0(K_p(\theta)) = K_p(\theta + \hat{\omega}(p)) .$$

- K_p is twist if $\mathrm{D}\hat{\omega}(p) = \mathrm{Hess}\, W(p)$ is non-degenerate.

Integrable symplectomorphisms

Invariant tori with frequency ω

Problem: Look for invariant tori with frequency ω.

- Define the counterterm $\lambda(p) = \nabla W(p) - \omega$.
- Define the modified family

$$f_{\lambda(p)}(x, y) = f_0(x, y) - \begin{pmatrix} \lambda(p) \\ 0 \end{pmatrix} .$$

- For $p \in \mathbb{R}^n$, the torus $K_p(\theta) = \begin{pmatrix} \theta \\ p \end{pmatrix}$ is invariant for $f_{\lambda(p)}$ with frequency ω:

$$f_0(K_p(\theta)) - \begin{pmatrix} \lambda(p) \\ 0 \end{pmatrix} = K_p(\theta + \omega) .$$

Integrable symplectomorphisms

Invariant tori as critical points of the potential A

- Define the potential $A(p) = W(p) - p^\top \omega$.
- Obviously: $\lambda(p) = \nabla_p A(p)$.

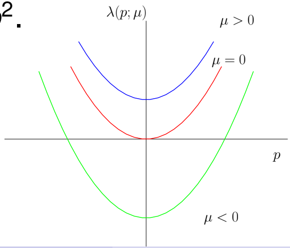
Then:

- K_{p_*} is an f_0-invariant torus with frequency ω if and only if p_* is a critical point of the potential $A(p)$.
- K_{p_*} is a twist f_0-invariant torus with frequency ω if and only p_* is a non-degenerate critical point of the potential $A(p)$.

Integrable symplectomorphisms

The simplest degenerate example

- Assume $A(p) = \dfrac{p^3}{3}$ $\qquad (\hat{\omega}(y) = \omega + y^2)$
  - $p = 0$ is a degenerate critical point of $A(p)$.
  - The torus $K_0(\theta) = \begin{pmatrix} \theta \\ 0 \end{pmatrix}$ with frequency ω is non-twist.
- Unfolding: $A(p; \mu) = \mu p + \dfrac{p^3}{3}$ $\qquad (\hat{\omega}(y; \mu) = \omega + \mu + y^2)$
  - Counterterm: $\lambda(p; \mu) = \mu + p^2$.



Methodology

Setting

- $\mathbb{T}^n \times U$ is an annulus, where $U \subset \mathbb{R}^n$ is open simply connected.
- $P \subset \mathbb{R}^m$ is an open subset of parameters.
- $f_\kappa : \mathbb{T}^n \times U \to \mathbb{T}^n \times \mathbb{R}^n$ is a smooth family of exact symplectomorphisms, with $\kappa \in P$:
  - $\mathrm{D}f_\kappa(x,y)^\top J \,\mathrm{D}f_\kappa(x,y) = J$, where $J = \begin{pmatrix} O_n & -I_n \\ I_n & O_n \end{pmatrix}$;
  - There exists a primitive function $S_\kappa : \mathbb{T}^n \times U \to \mathbb{R}$, such that $dS_\kappa = f_\kappa^*(y\,\mathrm{d}x) - y\,\mathrm{d}x$.
- Let $\omega \in \mathbb{R}^n$ be fixed and Diophantine:

$$\left|\ell^\top \omega - m\right| \geq \gamma\, |\ell|_1^{-\tau}, \qquad \forall \ell \in \mathbb{Z}^n \setminus \{0\},\ m \in \mathbb{Z}.$$

Methodology

Modified family and invariant Lagrangian deformations

1) Consider the modified family

$$\mathbf{f}_{(\kappa,\lambda)}(x,y) = f_\kappa(x,y) - \begin{pmatrix} \lambda \\ 0 \end{pmatrix},$$

where $\lambda \in \mathbb{R}^n$ is the counterterm.

2) Use KAM techniques to find a family

$$(p;\kappa) \in U \times P \to (\lambda(p;\kappa), K_{(p;\kappa)})$$

such that:

$$\begin{aligned} \mathbf{f}_{(\kappa,\lambda(p;\kappa))} \circ K_{(p;\kappa)}(\theta) &= K_{(p;\kappa)}(\theta+\omega)\,, \\ \left\langle K_{p;\kappa}(\theta) - \begin{pmatrix} \theta \\ p \end{pmatrix} \right\rangle &= 0\,. \end{aligned}$$

Methodology

Potential and countertem

3) Define the *potential*

$$A(p;\kappa) = p^\top \lambda(p;\kappa) + \left\langle S_\kappa \circ K_{(p;\kappa)}(\theta) \right\rangle ,$$

where S_κ is the primitive function of f_κ.

The following holds:

$$\lambda(p;\kappa) = \nabla_p A(p;\kappa)$$

Hence, fixed $\kappa = \kappa_*$:

$K_{(p_*;\kappa_*)}$ is f_{κ_*}-invariant with frequency ω if and only if p_* is a critical point of the potential $A(p;\kappa_*)$.

Methodology

Potential and torsion

4) The torsion of $K_{(p;\kappa)}$ is the $n \times n$ symmetric matrix

$$\bar{T}(p;\kappa) = \left\langle N_{(p;\kappa)}(\theta+\omega)^\top \, J \, \mathrm{D}_z \mathbf{f}_{(\kappa,\lambda(p;\kappa))}(K_{(p;\kappa)}(\theta)) \, N_{(p;\kappa)}(\theta) \right\rangle ,$$

$$\text{where} \quad N_{(p;\kappa)}(\theta) = J \, \mathrm{D}_\theta K_{(p;\kappa)}(\theta) \left(\mathrm{D}_\theta K_{(p;\kappa)}(\theta)^\top \mathrm{D}_\theta K_{(p;\kappa)}(\theta) \right)^{-1} .$$

There exist $n \times n$ matrices $W_1(p;\kappa)$ and $W_2(p;\kappa)$ such that:

$$\bar{T}(p;\kappa)\, W_1(p;\kappa) = W_2(p;\kappa)\, \mathrm{Hess} A(p;\kappa) \, .$$

Hence, if $W_1(p_*;\kappa_*)$ and $W_2(p_*;\kappa_*)$ are invertible:

The co-rank of p_* as a critical point of $A(p;\kappa_*)$ equals the co-rank of the torsion $\bar{T}(p_*;\kappa_*)$.

Example 1

Applying the methodology

- Modified family:

$$\mathbf{f}_{(\kappa,\lambda)}(x,y) = \begin{pmatrix} 0.375 + x + \left(y - \frac{\kappa}{2\pi}\sin(2\pi x)\right)^2 - \lambda \\ y - \frac{\kappa}{2\pi}\sin(2\pi x) \end{pmatrix} .$$

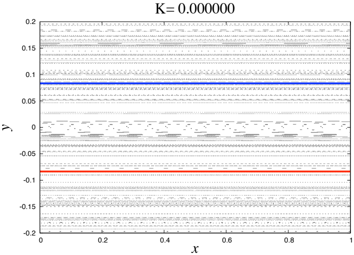
- The primitive function of f_κ is

$$S_\kappa(x,y) = \frac{\kappa}{4\,\pi^2}\cos(2\pi x) + \frac{2}{3}\left(y - \frac{\kappa}{2\pi}\sin(2\pi x)\right)^3 .$$

- The co-rank of the torsion of a torus is either 0 (twist) or 1 (non-twist).

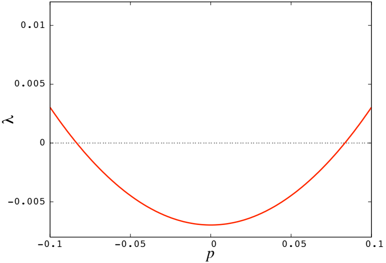
A fold singularity

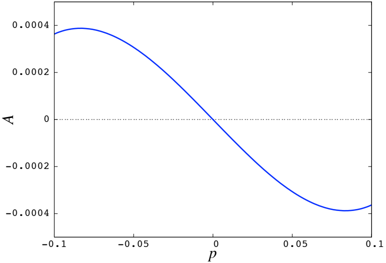
A collision of invariant tori



$T = \quad 0.1669253$

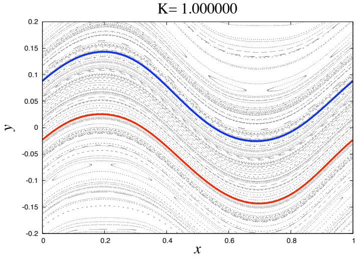
$T = -0.1669253$





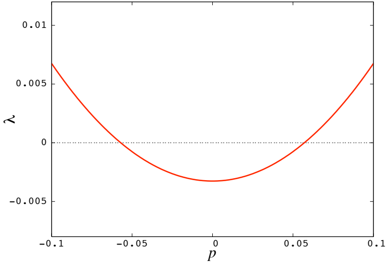
A fold singularity

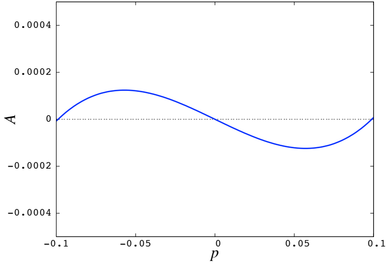
A collision of invariant tori



$T = \quad 0.1123464$

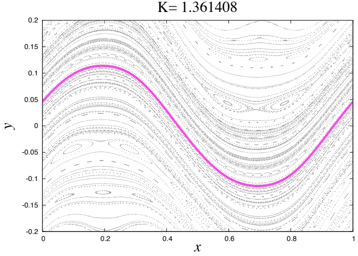
$T = -0.1123464$



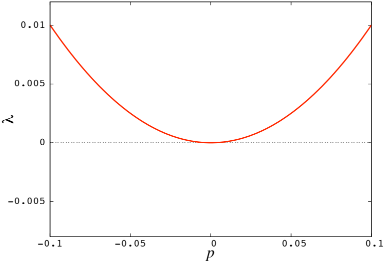


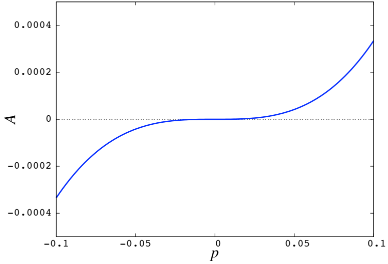
A fold singularity

A collision of invariant tori



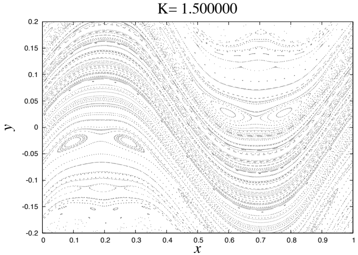
$$T = \quad 0.6558 \cdot 10^{-8}$$
$$T = -0.6558 \cdot 10^{-8}$$

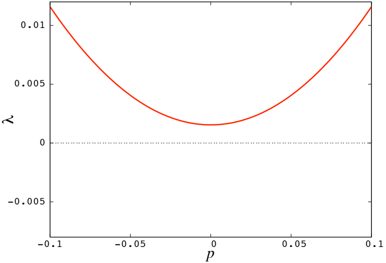


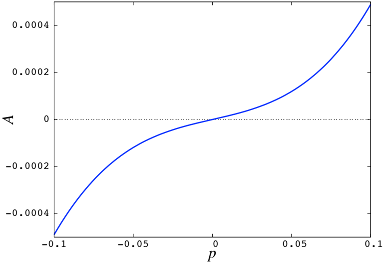


A fold singularity

A collision of invariant tori







Example 2

Another family of quadratic standard maps

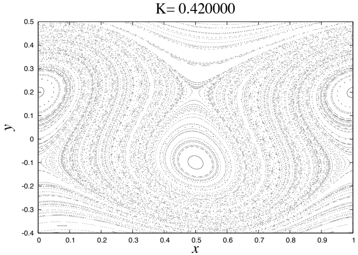
Consider a quadratic standard family of symplectomorphisms $f_\kappa : \mathbb{T} \times \mathbb{R} \to \mathbb{T} \times \mathbb{R}$, defined by:

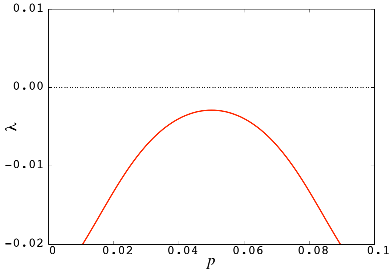
$$\begin{cases} \bar{x} &= x + (\bar{y} + 0.1)(\bar{y} - 0.2)\,, \\ \bar{y} &= y - \dfrac{\kappa}{2\pi} \sin(2\pi x)\,. \end{cases}$$

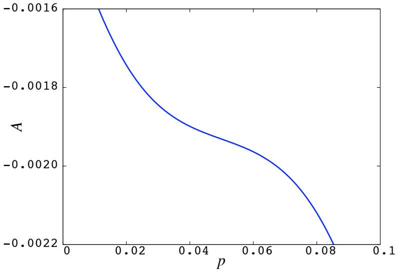
Problem: Look for invariant tori with frequency $\omega = \frac{\sqrt{5}-1}{32}$, with respect to parameter κ.

A fold singularity

The birth of meandering tori

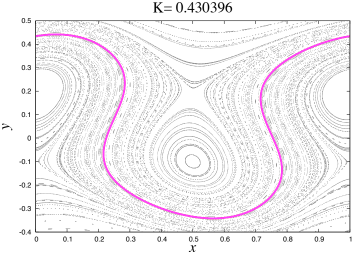






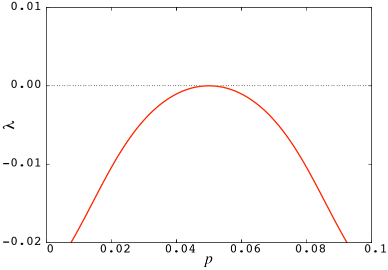
A fold singularity

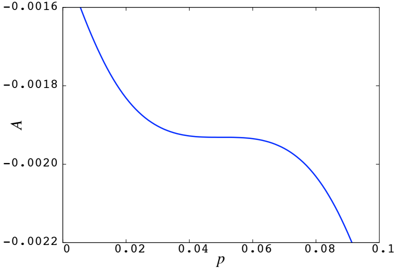
The birth of meandering tori



$$T = -1.6261 \cdot 10^{-7}$$

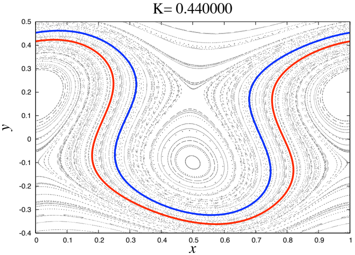
$$T = \quad 1.6261 \cdot 10^{-7}$$





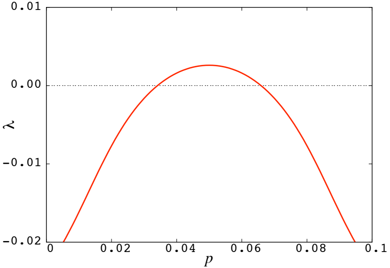
A fold singularity

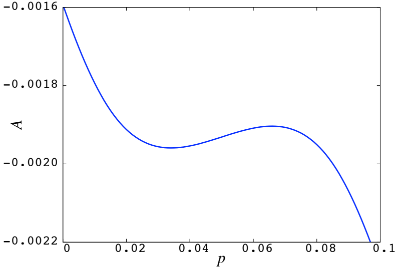
The birth of meandering tori



$$T = -0.0211805$$

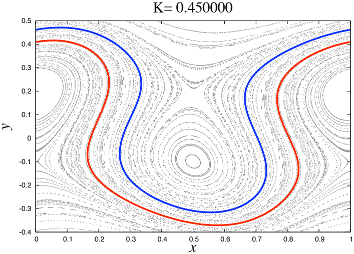
$$T = 0.0211805$$





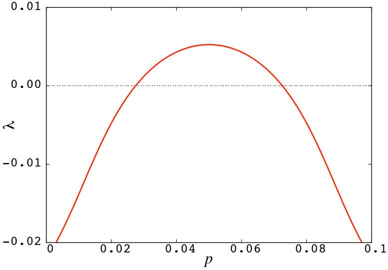
A fold singularity

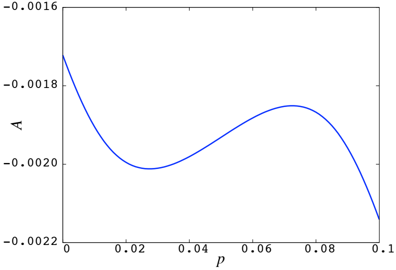
The birth of meandering tori



$T = -0.0309003$

$T = 0.0309003$





Conclusions

- The infinite dimensional problem of finding invariant tori for Hamiltonian systems is reduced to the finite dimensional problem of finding critical points of the potential.
- Non-twist tori correspond to degenerate critical points of the potential.
- The Singularity Theory for invariant tori is provided by the Singularity Theory of the critical points of the potential.

Conclusions

- The method yields very *efficient numerical algorithms* to compute invariant tori and its bifurcations.
- The method is suitable to *validate* numerical computations;
- We are able to study any finite-determined singularity of the frequency map.
- The method can be also applied for obtaining small twist theorems.
- Our method is designed for the cases on which numerical evidence of bifurcations is known but the system is not close to integrable.

Thanks!