Reducibility of linear equations with quasi-periodic coefficients. A survey.

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Preface

This survey deals with some aspects of the problem of reducibility for linear equations with quasi-periodic coefficients. It is a compilation of results on this problem, some already classical and some other more recent. Our motivation comes from the study of stability of quasi-periodic motions and preservation of invariant tori in Hamiltonian mechanics (where the reducibility of linear equations with quasi-periodic coefficients plays an important role), and this has influenced much of the presentation.

The first chapter is an introduction to linear equations with quasi-periodic coefficients and their reducibility. It settles notation that we will use along the survey and gives some motivations for the study of reducibility, as well as a discussion of Floquet theory for periodic systems and some results on the reducibility of linear scalar equations with quasi-periodic coefficients.

The second chapter is a survey of results on exponential dichotomy and a related spectrum for linear skew-product flows. This theory applies to general quasi-periodic equations in any dimension, and some results and definitions given there will be used in the following chapters. A reducibility result is included.

The third chapter is the longest of the whole survey and it deals with Schrödinger equation with quasi-periodic potential. It is divided into two parts. In the first one we study the classical spectral theory of self-adjoint operators both in the general case and in the case of a quasi-periodic potential. The ergodic invariants like the rotation number and the Lyapunov exponents are discussed in relation with the spectral properties of the Schrödinger operator. The second part is devoted to results on reducibility for this equation and some ideas about the proof, based on KAM techniques, are presented. Finally, some attention is paid to results on non-reducibility for this equation.

The fourth chapter tries to give an overview of the existing results on the reducibility of linear equation with quasi-periodic coefficients in dimension greater than two. In the first part of the chapter we study results on reducibility of general quasi-periodic equations, and in the second one we focus on the case when the flow is defined on a compact group, more specifically $SO(3,\mathbb{R})$, because almost all of these systems can be shown to be reducible close to constant coefficients. Finally we give some remarks on non-reducibility for these equations.

The survey is not intended to be exhaustive and it is certainly not original, neither in the results that are stated, nor in the presentation that it is chosen. References to original articles have been given when possible. There is lack of a proper exposition of the negative results on reducibility together with the techniques that are used to prove them. Some results on effective and almost reducibility of quasi-periodic systems should also be present in this survey.

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Contents

Chapter 1

Introduction. What is reducibility?

In this chapter we will introduce some basic theory on the problem of the reducibility of linear equations with quasi-periodic coefficients. Here we will settle the notation that will be used along the survey.

1.1 Linear equations with time-depending coefficients

The present work deals with linear equations with coefficients depending on time (on a way that will be specified later on). This means that we will consider systems like the following one

$$x'(t) = A(t)x(t), \tag{1.1}$$

where $'$ stands for the derivative with respect to the time t, $A(t)$ is a square matrix of dimension n depending on time and the function $x : I \subset \mathbb{R} \to \mathbb{R}^n$ is the unknown. Under fairly general conditions on the dependence of A with respect to time, which we will always assume to be fulfilled, the system (1.1) is known to have a unique solution once an initial condition is imposed. That is, for every $(t_0, x_0) \in I \times \mathbb{R}^n$ fixed, there exists a unique function $x(t; t_0, x_0)$ such that

$$x(t_0; t_0, x_0) = x_0$$

and

$$x'(t; t_0, x_0) = A(t)x(t; t_0, x_0), \qquad t \in I.$$

The linear character of equation (1.1) implies that the space of its solutions is a linear space of dimension n. There exists a n-dimensional nonsingular matrix $X(t; t_0)$ for $t, t_0 \in I$ such that each of its columns satisfies equation (1.1),

$$X'(t; t_0) = A(t)X(t; t_0), \tag{1.2}$$

with the initial condition

$$X(t_0; t_0) = Id,$$

the identity matrix. The time-dependent matrix X will be called *fundamental solution* or *fundamental matrix* of system (1.1) because, if $x(t; t_0, x_0)$ is any solution of this system, then

$$x(t; t_0, x_0) = X(t; t_0)x_0.$$

The fundamental matrix contains all the relevant information of the solutions of a linear system. In what follows we will always assume that the solutions are defined for all time, in our notation $I = \mathbb{R}$.

The simplest example of linear systems are linear equations with constant coefficients

$$x'(t) = A\ x(t) \tag{1.3}$$

where A is a constant matrix of dimension n. In this case, for any initial time t_0, the fundamental matrix is given by the exponentiation of the matrix $(t-t_0)A$

$$X(t;t_0) = \exp\left((t-t_0)A\right) = Id + \sum_{n\geq 1} \frac{(t-t_0)^n}{n!} A^n,$$

so we have a complete knowledge of the solutions of the system.

Linear equations with time-dependent coefficients arise naturally in the classical problem of the stability of solutions of non-linear differential equations. Indeed, consider the system

$$x' = f(x) \tag{1.4}$$

which we have considered autonomous (that is, the right-hand side of the equation does not depend on time) for the sake of simplicity in the notation. Here $x \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a function which will be assumed to be smooth enough. Assume that we are able to compute a solution $x = x(t;x_0)$ and we ask which is the variation of the solution of the non-linear equation (1.4) when we move the initial conditions around x_0. Consider, thus, the equation

$$\frac{\partial}{\partial t} x(t;\tilde{x}) = f(x(t;\tilde{x})) \tag{1.5}$$

varying the initial condition $\tilde{x}$ in a neighbourhood of x_0. If f is smooth enough, then the solutions depend smoothly on the initial condition $\tilde{x}$ and this enables us to differentiate (1.5) with respect to this initial condition

$$\frac{\partial}{\partial \tilde{x}} \left(\frac{\partial}{\partial t} x(t;\tilde{x}) \right) = \frac{\partial}{\partial \tilde{x}} \left(f(x(t;\tilde{x})) \right). \tag{1.6}$$

Assuming again that the dependence is smooth enough, we can exchange the derivatives, obtaining a non-linear time-dependent equation

$$\frac{\partial}{\partial t} \left(\frac{\partial}{\partial \tilde{x}} x(t;\tilde{x}) \right) = \left(\frac{\partial f}{\partial \tilde{x}} (x(t;\tilde{x})) \right) \left(\frac{\partial}{\partial \tilde{x}} x(t;\tilde{x}) \right) \tag{1.7}$$

which, evaluated for $\tilde{x} = x_0$ becomes a linear equation with time-dependent coefficients

$$\left(\frac{\partial}{\partial \tilde{x}} x(t;\tilde{x}) \right)'_{|\tilde{x}=x_0} = \left(\frac{\partial f}{\partial \tilde{x}} (x(t;x_0)) \right) \left(\frac{\partial}{\partial \tilde{x}} x(t;\tilde{x}) \right)_{|\tilde{x}=x_0}. \tag{1.8}$$

This is a linear equation with varying coefficients in the notations of (1.2) if we write

$$X(t) = \left(\frac{\partial}{\partial \tilde{x}} x(t;\tilde{x}) \right)_{|\tilde{x}=x_0}$$

and

$$A(t) = \left(\frac{\partial f}{\partial \tilde{x}} (x(t;x_0)) \right).$$

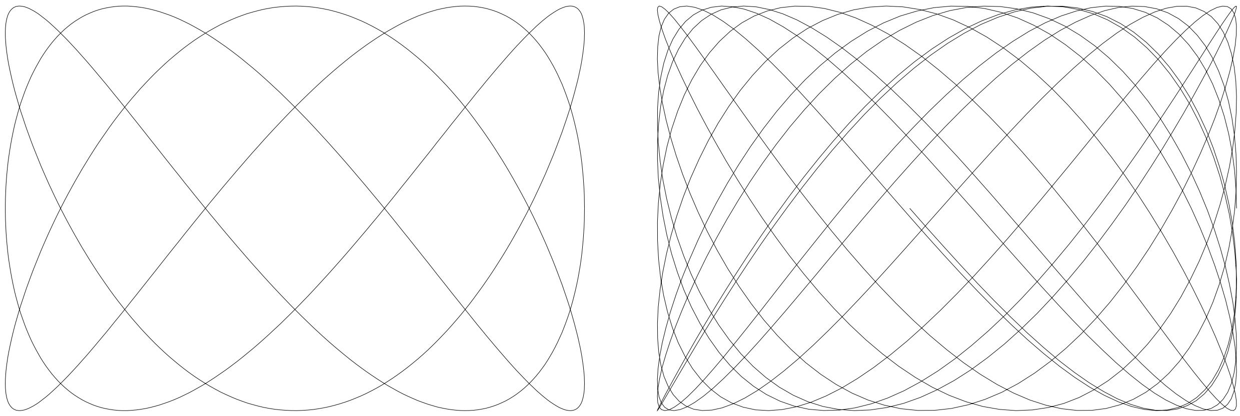

Figure 1.1: Left: Example of a periodic motion in $\mathbb{R}^2$ ($f(t) = (\cos(\frac{3t}{5}), \sin(t))$, therefore with period 10π). Right: Example of a quasi-periodic motion in $\mathbb{R}^2$ ($f(t) = \left(\cos\left(\frac{\sqrt{5}-1}{2}t\right), \sin(t)\right)$).

These are called the *first order variational equations*. As long as the function f is smooth enough we can obtain the the variational equations of higher orders in a similar way.

If x_0 is a fixed point for the non-linear system (1.4), then in the first order variational equations (1.8) the matrix A is constant, and from these equations describe how the flow defined by (1.4) sends infinitesimal variations of the initial conditions for fixed time. If the orbit is periodic, then the corresponding variational equations are periodic.

For Hamiltonian dynamical systems (see [2], [59] or [17] for an introduction), most of the stable motions are quasi-periodic and from the properties of the variational equations that they define we can explore the phase structure of the system around these orbits. Our main motivation will be, therefore, to give tools to study the behaviour of linear equations with quasi-periodic coefficients. In the following sections we make the statements in this paragraph a bit more precise.

1.2 Quasi-periodic motions and quasi-periodic functions

Let us begin with one example of quasi-periodic motions. Consider a mechanical system consisting of d punctual masses (normalized to be one) attached with d different springs to the origin. The motion on each spring is assumed to follow Hooke's law, so that if $(x_1, \dots, x_d)$ are the positions of the d different springs with respect to the origin, then the following d equations are satisfied

$$x_k'' + \omega_k^2 x_k = 0, \qquad k = 1, \dots, d,$$

where ω_k are the constants associated to each spring. These equations can be solved in terms of trigonometric functions:

$$x_k(t) = x_k(0)\cos(\omega_k t) + \frac{1}{\omega_k} x_k'(0)\sin(\omega_k t), \qquad k = 1, \dots, d.$$

Note that the motion of each spring is periodic, and it has period $2\pi/\omega_k$, but the motion of $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ is not periodic unless all of the *frequencies* ω_k are integer multiples of a fixed frequency. This kind of motion will receive the name of *quasi-periodic motion* (see figure 1.1), and leads very naturally to the definition of quasi-periodic functions:

Definition 1.2.1 *Let $f : \mathbb{R} \to \mathbb{R}$ be a function. We shall say that f is* quasi-periodic *whenever there exist real constants $\omega_1, \ldots, \omega_d$ and a continuous function $F : \mathbb{R}^d \to \mathbb{R}$, 2π-periodic in each variable such that*

$$f(t) = F(\omega_1 \cdot t, \ldots, \omega_d \cdot t), \qquad t \in \mathbb{R}.$$

The constants $\omega_1, \ldots, \omega_d$ are called the basic frequencies *of f and we will call F the* lift *of f.*

Remark 1.2.2 *From now on we will denote by $\mathbb{T}$ the quotient space $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$, so in the above notations we will write $F : \mathbb{T}^d \to \mathbb{R}$. The elements of $\mathbb{T}^d$ will be called* angles *and they will be denoted by θ or ϕ.*

Remark 1.2.3 *Note that without imposing extra conditions on the frequency vector ω neither the frequency ω nor the lift of a quasi-periodic function are uniquely determined. Indeed, consider the following example*

$$f(t) = \cos(t) + \cos(2t).$$

Then we can write

$$f(t) = F(t, 2t),$$

where

$$F(\theta_1, \theta_2) = \cos(\theta_1) + \cos(\theta_2),$$

but also, using trigonometric relations

$$f(t) = G(t),$$

where

$$G(\theta) = \cos(\theta) + 2\cos^2(\theta) - 1.$$

The reason for this is that we must impose that the components of the frequency vector are rationally independent. This will be assumed in the sequel. With this hypothesis all possible choices of basic frequencies have the same number of components.

Remark 1.2.4 *The continuity of the lift is necessary, because otherwise all functions are quasi-periodic. A weaker condition for F like $L^2(\mathbb{T}^d)$ gives also problems, because the lift can be defined to coincide with $f(t)$ along the trajectory ωt on $\mathbb{T}^d$ (which is a set of zero measure) and to be zero outside this trajectory.*

We can also produce many quasi-periodic functions having the same frequency vector than another quasi-periodic function. Indeed, fixed $F : \mathbb{T}^d \to \mathbb{R}$ and $\omega \in \mathbb{R}^d$, for each $\phi \in \mathbb{T}^d$ (which we will refer as *initial phase*) we consider

$$f_\phi(t) = F(\omega t + \phi).$$

In general, we will assume that the initial phase is zero, because changing the lift we can get this.

As we have seen, the useful notion of regularity for a quasi-periodic function has more to do with the lift than with the quasi-periodic function itself. In fact, in the third chapter we will give an example of a quasi-periodic function f which is analytic, but whose lift F to $\mathbb{T}^d$ is only continuous. As a general rule, the regularity properties (continuity, C^r character, analyticity) will refer to the lift of the quasi-periodic function.

Real analytic functions $F : \mathbb{T}^d \to \mathbb{R}$ deserve a special attention. As F, considered as a function defined on $\mathbb{R}^d$, is real analytic, at any point $x \in \mathbb{R}^d$, there exists a complex neighbourhood $U_x \subset \mathbb{C}^d$ such that F can be extended to an analytic function on U_x. As this can be done for all $x \in \mathbb{R}^d$ and the function F is periodic in all the components, these neighbourhoods can be made uniform: there exists a positive value ρ such that F is analytic in the complex set defined by the condition

$$|\mathrm{Im}\ x| < \rho,$$

where Im z denotes the imaginary part of a complex number z. By obvious reasons, the above set is called *analyticity strip* and the number ρ *width of the analyticity strip.*

Quasi-periodic functions are a generalization of periodic functions (the latter correspond to the case with one frequency) so it is natural to look for an analog of the Fourier series for quasi-periodic functions. This is achieved resorting again to the lift $F : \mathbb{T}^d \to \mathbb{R}$ of the quasi-periodic function f with frequency ω. At a formal level, we can associate a Fourier series with d angular variables to F. This series is

$$\sum_{\mathbf{k}\in\mathbb{Z}^d} \hat{F}_{\mathbf{k}} \exp(i\langle \mathbf{k}, \theta\rangle), \tag{1.9}$$

where $\mathbf{k} = (k_1, \ldots, k_d)$ is called a *multi-integer* or *multi-index*, and $\langle\cdot,\cdot\rangle$ denotes the scalar product for vectors in $\mathbb{R}^d$

$$\langle u, v\rangle = \sum_{j=1}^{d} u_j \cdot v_j.$$

As it happens when $d = 1$, the regularity properties of F impose a certain rate of convergence of the above sum, so that it is not only formal. The coefficients $\hat{F}_{\mathbf{k}}$ are the *Fourier coefficients* of F and they can be computed via the formula

$$\hat{F}_{\mathbf{k}} = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} F(\theta) \exp(-i\langle \mathbf{k}, \theta\rangle) d\theta,$$

where the integration is taken with respect to the Lebesgue measure on the torus. The first Fourier coefficient $\hat{F}_{\mathbf{0}}$ has a special meaning and it is called the *average* of F. It satisfies the following property: if the function $F : \mathbb{T}^d \to \mathbb{R}$ is continuous (or just Riemann integrable) then, for any frequency vector $\omega \in \mathbb{R}^d$, which we again assume to be rationally independent, and initial phase ϕ, the limit

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T F(\omega t + \phi) dt$$

exists and agrees with the average of F. This value will be usually denoted by $[f]$.

This result (called the *theorem on averages for quasi-periodic functions*) suggests that, for a quasi-periodic function f with lift F, we can obtain the Fourier coefficients of F only knowing that f is a quasi-periodic function which has frequency ω and initial phase $\phi \in \mathbb{T}^d$. It also suggests that these Fourier coefficients have a meaning for the quasi-periodic function. The latter is immediate, because if $f(t) = F(\omega t + \phi)$ and if the Fourier series (1.9) converges uniformly to F (this can be granted if F is smooth enough), then we can write

$$f(t) = F(\omega t + \phi) = \sum_{\mathbf{k}\in\mathbb{Z}^d} \hat{F}_{\mathbf{k}} \exp(i\langle \mathbf{k}, \omega t + \phi\rangle).$$

To obtain the Fourier coefficients of F from f we only have to apply the theorem on averages stated in the previous paragraph to conclude that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(t) \exp(-i\langle \mathbf{k}, \omega t + \phi\rangle) dt = \hat{F}_{\mathbf{k}}.$$

Note that we can recover the Fourier coefficients of F from f if we know the frequencies. The problem of determining the Fourier coefficients and the frequencies of a quasi-periodic function only from a knowledge of f is not so trivial, but very interesting for the applications (see [54], [55], [32] and references therein).

1.3 Linear equations with quasi-periodic coefficients and reducibility

We have already given some motivations to study equations of the type

$$x' = A(t)x, \tag{1.10}$$

where $x \in \mathbb{R}^n$ and $A(t)$ is a matrix depending quasi-periodically on time. This means that there exists a frequency vector $\omega \in \mathbb{R}^d$ and a lift of A, which we will denote as $\tilde{A}$, such that

$$A(t) = \tilde{A}(\omega t),$$

and $\tilde{A}$ is defined on the d-dimensional torus $\mathbb{T}^d$. The quasi-periodicity of A makes it possible to *lift* the equation (1.10) to a system of linear equations on $\mathbb{R}^n \times \mathbb{T}^d$ simply writing

$$x' = \tilde{A}(\theta)x, \qquad \theta' = \omega, \tag{1.11}$$

where the equation (1.10) is obtained when the initial value for θ is zero. It will turn out that for most properties of the quasi-periodic equation (1.10) we will have to resort to the lifted system (1.11).

There is a useful formalism to study linear equations with quasi-periodic coefficients. Note that the fundamental matrix belongs always to the group $GL(n, \mathbb{R})$ of linear invertible transformations from $\mathbb{R}^n$ to itself. This means that the matrix generating the differential equation, in our notations $A(t)$, belongs to the infinitesimal Lie algebra $gl(n, \mathbb{R})$, of linear transformations of $\mathbb{R}^n$ to itself. See, for instance, [66] or any textbook on differential geometry, for an exposition of Lie groups and Lie algebras. Now we can consider the matrix equation for (1.11), so that the lifted system becomes

$$X' = \tilde{A}(\theta)X, \qquad \theta' = \omega, \tag{1.12}$$

where $X(t) \in GL(n, \mathbb{R})$ and $A(t) \in gl(n, \mathbb{R})$, so that the pair (X, θ) belongs to $GL(n, \mathbb{R}) \times \mathbb{T}^d$.

Remark 1.3.1 *The same construction can be done if $G \subset GL(n, \mathbb{R})$ is a certain Lie subgroup (for instance $SL(n, \mathbb{R})$ or $SO(n, \mathbb{R})$, see [66]).*

Remark 1.3.2 *Although we will mainly focus on the properties of linear equations with quasi-periodic coefficients in $\mathbb{R}$, it is clear that all the above construction can be performed to deal with the case of complex coefficients.*

Thus a natural question arises: which tools have to be used to describe the behaviour of linear equations with quasi-periodic coefficients ?

So far, we have only given one example of linear equation with quasi-periodic coefficients, which is the trivial case of a linear equation whose matrix does not depend on time. In that case, both the qualitative and the quantitative behaviour of the system could be obtained, because we could directly integrate the system. One would like to have always such a strong knowledge of the properties of the solutions of a linear equation with quasi-periodic coefficients, but this seems difficult if we cannot integrate the system of equations (which is the usual situation). Therefore, we would like to know which are the systems whose qualitative behaviour can be *reduced* to the behaviour of a system with constant coefficients.

In more precise terms this reduction can be expressed in the following way

Definition 1.3.3 *An equation (1.10) is said to be (*Lyapunov-Perron*)* reducible *whenever there exists a linear time-varying change of variables*

$$x = Z(t)y,$$

called Lyapunov-Perron transformation*, such that it is non-singular for all $t \in \mathbb{R}$, Z, Z^{-1} and Z' are bounded in $\mathbb{R}$, and which transforms the system into an equation like*

$$y' = By,$$

where B is a constant matrix.

This notion is also valid for general linear systems like (1.1), and it implies that, whenever such a system is Lyapunov-Perron reducible to a constant coefficients system like (1.3), then many properties of the original system (such as the growth of the solutions or their boundness) are the same as those of the reduced system with constant coefficients.

Lyapunov-Perron reducibility, without requiring additional conditions on the transformation, is not fine enough to study linear equations with quasi-periodic coefficients, because the *rotational behaviour* is not taken into account. For instance, if all the solutions of (1.10) and its derivatives are bounded, then the system is Lyapunov-Perron reducible and any two such systems can be reduced to the same system with constant coefficients. Therefore additional conditions must be imposed.

For the rest of the survey, reducibility will mean Lyapunov-Perron reducibility to constant coefficients by means of a quasi-periodic transformation. We shall usually impose that the minimum number of basic frequencies of the transformation is the same that for the original system, but not in general that the basic frequencies are the same. The two following examples of linear equations with quasi-periodic coefficients together with the discussion of their reducibility problems will, hopefully, make subsequent discussion clearer.

1.4 Example 1: One frequency and many dimensions. Floquet theory for periodic systems

In this section we will deal with linear equations with quasi-periodic coefficients having only one frequency, in the previous notation $d = 1$. Therefore, our linear equation will be periodic and, in this case, the easy arguments of Floquet theory will guarantee reducibility.

Indeed, consider the equation

$$x' = A(t)x, \tag{1.13}$$

where $x \in \mathbb{R}^n$ and A is a matrix depending periodically on time with period T. This means that $A(t+T) = A(t)$ for all $t \in \mathbb{R}$. Let $X(t)$ be a fundamental matrix for the periodic system (1.13) being the identity at $t = 0$, and consider the map $\mathcal{P} = X(T)$ (which due to the periodicity, is an endomorphism of $\mathbb{R}^d$, called the *Poincaré map* for the system (1.13)).

Let B be a matrix satisfying that

$$\mathcal{P} = X(T) = \exp(TB).$$

Such a matrix can be always found if we don't require B to be real (if $\mathcal{P}$ has negative eigenvalues, then B must necessarily be complex). We shall call B a *Floquet matrix* of (1.13). Floquet matrices for a periodic system are *not* uniquely determined.

Then, it is satisfied that

$$X_T(t) = X(t+T) = X(t)\exp(TB), \tag{1.14}$$

for all real t. Indeed, the equality is true for $t = 0$, and X_T satisfies the differential equation

$${X_T}'(t) = A(t+T)X(t+T) = A(t)X_T(t)$$

so it is a fundamental solution for (1.13). Therefore the identity (1.14) holds for all t. To construct the reduction transformation, let

$$Z(t) = X(t)\exp(-tB).$$

This is a periodic transformation with period T

$$Z(t+T) = X(t+T)\exp(-(t+T)B) =$$

$$= X(t)\exp(TB)\exp(-TB)\exp(-tB) = X(t)\exp(-tB) = Z(t)$$

which, by means of the change of variables $x = Z(t)y$, reduces the system to

$$y' = By.$$

Remark 1.4.1 *If we want the reduced matrix (the Floquet matrix) to be real, then the reducing transformation cannot be always T-periodic but just $2T$-periodic, because*

$$X(2T) = X(T)^2$$

and for a matrix of this form (the square of a non-singular matrix) there exists always a real logarithm. This phenomenon of period-doubling *happens in general quasi-periodic equations. We need to impose that the frequency vector of the reducing transformation is $\omega/2$ instead of ω if we don't want to complexify the system.*

We have thus proved

Theorem 1.4.2 (Floquet's theorem) *Let equation (1.13) be periodic with period T. Then there exists a $2T$-periodic change of variables $x = Z(t)y$ and a real constant matrix B such that*

$$y' = By.$$

That is, all periodic systems are reducible by means of a periodic transformation with double the period of the original equation.

Remark 1.4.3 *As we have seen, the Floquet matrix of a periodic system is not uniquely determined (because the exponential is not one-to-one). In practical situations, however, it can be useful to impose extra conditions so that it is uniquely determined. For instance, it can be chosen to minimize a certain norm or, if the system depends on external parameters, to be continuous on these parameters. All these choices depend on the properties of our system.*

Remark 1.4.4 *From the proof of Floquet's theorem, we get a representation of a fundamental solution as*

$$X(t) = Z(t)\exp(Bt),$$

where Z is a $2T$-periodic matrix and B is a constant matrix. This is called a Floquet representation *of the solutions of (1.13) and can be extended to general quasi-periodic equations. We shall say that the quasi-periodic system (1.10) admits a Floquet representation, whenever there exists a fundamental matrix $X(t)$, a quasi-periodic matrix $Z(t)$ with frequency vector $\omega/2$ and a constant matrix B such that*

$$X(t) = Z(t)\exp(tB).$$

If we want to consider Floquet representations for the lifted system (which, recall, represents a family of quasi-periodic linear equations parameterized by the phase ϕ), then we write

$$X(t;\phi) = \tilde{Z}\left(\frac{\omega}{2}t + \phi\right)\exp(tB)\tilde{Z}(\phi)^{-1},$$

where $\tilde{Z}$ is the lift of the quasi-periodic function Z. This last representation is equivalent to the fulfillment of the following equation for Z,

$$\partial_{\frac{\omega}{2}}\tilde{Z}(\theta) = \tilde{A}(\theta)\tilde{Z}(\theta) - \tilde{Z}(\theta)B, \qquad \theta \in \mathbb{T}^d, \tag{1.15}$$

where $\partial_\omega\cdot = \langle \nabla_\theta \cdot, \omega\rangle$. An equation of this type is called homological equation *and will be often found along this survey.*

1.5 Example 2: Many frequencies and one dimension. Small divisor problems

We have seen that periodic linear equations are reducible, and the reason is that the flow is *exactly* the same after the period. When we have more than two frequencies, the flow is *never* the same for any shift of time, but it gets closer and closer to previous values. In view of the previous section one could think that this recurrence of the flow is enough to prove reducibility. We will see in this section that this is not so easy by means of a simple example.

Among linear equations with quasi-periodic coefficients and more than two frequencies the simplest are scalar equations, in our notations $x(t) \in \mathbb{R}$,

$$x'(t) = a(t)x(t), \tag{1.16}$$

where $a : \mathbb{R} \to \mathbb{R}$ is a quasi-periodic function with d frequencies. This equation can be directly integrated, but as we want to illustrate the reducibility problem, we proceed in a more qualitative way.

To render equation (1.16) to constant coefficients we merely have to set

$$z(t) = \exp(g(t)),$$

where

$$g(t) = \int_0^t (a(s) - [a])\, ds, \tag{1.17}$$

and $[a]$ is the average of the quasi-periodic function a. Indeed, if $x = z(t)y$, then

$$y' = a(t)x\exp(-g(t)) - x\exp(g(t))g'(t) = (a(t) - a(t) + [a])\, y,$$

so the transformed system is

$$y' = [a]y.$$

One may think that the addition of the term $-[a]$ in the integrand of (1.17) is quite artificial. However, we haven't yet checked whether the transformation defined by z is quasi-periodic or not. This is equivalent to see whether g is quasi-periodic or not. That the addition of the average is necessary is clear from the theorem on averages, because, as

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t a(s)ds = [a],$$

if we replace $[a]$ by any other constant in (1.17) the function is unbounded and, therefore, not quasi-periodic. Note that g satisfies the equation

$$g'(t) = a(t) - [a], \tag{1.18}$$

so if g has to be quasi-periodic (with a certain regularity), then it can be written by means of the Fourier series of the lift $\tilde{g}$,

$$g(t) = \tilde{g}(\omega t + \phi) = \sum_{\mathbf{k}\in\mathbb{Z}^d} \tilde{g}_{\mathbf{k}} \exp(i\langle \mathbf{k}, \omega t + \phi\rangle)$$

and also a has such an expression

$$a(t) = \tilde{a}(\omega t + \phi) = \sum_{\mathbf{k}\in\mathbb{Z}^d} \tilde{a}_{\mathbf{k}} \exp(i\langle \mathbf{k}, \omega t + \phi\rangle).$$

With these definitions, we can lift equation (1.18) to $\mathbb{T}^d$, with

$$\partial_\omega \tilde{g}(\theta), \omega\rangle = \tilde{a}(\theta) - [a], \tag{1.19}$$

for all $\theta \in \mathbb{T}^d$. Note that

$$\partial_\omega \tilde{g}(\theta) = \sum_{\mathbf{k}\in\mathbb{Z}^d} \tilde{g}_{\mathbf{k}} (i\langle \mathbf{k}, \omega\rangle) \exp(i\langle \mathbf{k}, \theta\rangle),$$

so the formal solution of the homological equation (1.19) in terms of the Fourier coefficients is given by

$$\tilde{g}_{\mathbf{k}} = -i\frac{\tilde{a}_{\mathbf{k}}}{\langle \mathbf{k}, \omega\rangle}$$

for $\mathbf{k} \in \mathbb{Z}^d$ different from zero and

$$\tilde{g}_0 = 0,$$

provided none of the quotients $\langle \mathbf{k}, \omega \rangle$ vanishes. If there is a non-trivial multi-integer $\mathbf{k} \in \mathbb{Z}^d$, such that

$$\langle \mathbf{k}, \omega \rangle = 0$$

then we shall say that the frequency vector ω is *resonant*. In this case the expression $\langle \mathbf{k}, \omega \rangle = 0$ is called a *resonance* and the value

$$|\mathbf{k}| = |k_1| + \cdots + |k_d|$$

is called the *order of the resonance*. If we assume the frequency vector to be rationally independent, then no resonances occur but, is non-resonance enough to prove that the function g is quasi-periodic or do we need to impose additional conditions?

Recall that, up to now, we have only solved formally the homological equation (1.19). If we want to prove that g is actually quasi-periodic we will have to check that the coefficients $(g_{\mathbf{k}})_{\mathbf{k} \in \mathbb{Z}^d}$ are the Fourier coefficients of a smooth enough function. This is not immediate, because we need to know the asymptotic behaviour of the Fourier coefficients of $\tilde{a}$ and of the quotients $1/\langle \mathbf{k}, \omega \rangle$. A problem of this type is called a *small divisors problem*, for obvious reasons. So far we have been quite loose in the smoothness properties, but now we will have to be more accurate.

Assume that $\tilde{a}$ is of class $C^r(\mathbb{T}^d)$, where r can be a finite number (satisfying a certain lower bound that we will give in a moment) or ∞. We know that writing

$$\mathcal{D}_\alpha = \left\{ f \in L^2(\mathbb{T}^d); \sup_{\mathbf{k} \in \mathbb{Z}^d} |\mathbf{k}|^\alpha |f_{\mathbf{k}}| < +\infty \right\},$$

then

$$\mathcal{D}_{r+d+1} \subset C^r(\mathbb{T}^d) \subset \mathcal{D}_r.$$

The analytic case is treated considering

$$\mathcal{A}_\rho = \left\{ f \in L^2(\mathbb{T}^d); \sup_{\mathbf{k} \in \mathbb{Z}^d} \exp(\rho |\mathbf{k}|) |f_{\mathbf{k}}| < +\infty \right\}.$$

The value ρ appearing in the definition of the set A_ρ is the width of the analyticity strip that we are considering, so functions in this set are analytic in the set $|\mathrm{Im}\ \theta| < \rho$.

To measure the quotients which appear in the definition of $g_{\mathbf{k}}$ we shall impose that the frequency vector ω is not too close to resonances. Of course that we cannot expect that the quotients

$$\frac{1}{\langle \mathbf{k}, \omega \rangle}$$

are uniformly bounded for $\mathbf{k} \in \mathbb{Z}^d$, because the values $\langle \mathbf{k}, \omega \rangle$, for $\mathbf{k} \in \mathbb{Z}^d$ are dense in $\mathbb{R}$ (provided that at least two components of ω are rationally independent). We shall only impose that these quotients can be controlled by powers of $|\mathbf{k}| = |k_1| + \cdots + |k_d|$. This is called a *Diophantine condition on* ω: we will assume that there exist positive constants κ and τ such that

$$|\langle \mathbf{k}, \omega \rangle| \geq \frac{\kappa}{|\mathbf{k}|^\tau}, \tag{1.20}$$

for all $\mathbf{k} \in \mathbb{Z}^d$, $\mathbf{k} \neq 0$. It can be shown that almost all frequency vectors $\omega \in \mathbb{R}^d$ (with respect to the Lebesgue measure) satisfy some Diophantine condition with suitable positive constants κ and τ (for which one has always the lower bound $\tau \geq d-1$). On the other hand, the set of frequencies in $\mathbb{R}^d$ which do not satisfy any Diophantine condition is dense and open. Sometimes we will write the set of frequency vectors satisfying a Diophantine condition (1.20) with constants κ and τ as $DC(\kappa, \tau)$.

Assume now that $\tilde{a} \in C^r(\mathbb{T}^d)$ and that $\omega \in DC(\kappa, \tau)$. Then, we know that $\tilde{a} \in \mathcal{D}_r$, so that

$$\sup_{|\mathbf{k}|>0} |\tilde{g}_{\mathbf{k}}| = \sup_{|\mathbf{k}|>0} \frac{|\tilde{a}_{\mathbf{k}}|}{|\langle \mathbf{k}, \omega \rangle|} \leq \sup_{|\mathbf{k}|>0} \frac{C_1 \kappa |\mathbf{k}|^r}{|\mathbf{k}|^\tau},$$

so $\tilde{g} \in \mathcal{D}_{r-\tau}$. This implies that, whenever $\tilde{a} \in C^r(\mathbb{T}^d)$, with $r \geq \tau + d + 1$, and $\omega \in DC(\kappa, \tau)$, g is quasi-periodic and its lift if of class $C^{r-1-\tau-d}(\mathbb{T}^d)$. If $\tilde{a}$ is analytic, then we can proceed in the same way, but using the sets $\mathcal{A}_\rho$ instead of $\mathcal{D}_r$ and reducing ρ instead of r, which means reducing the domain of analyticity.

We have therefore shown that, even in the simple case of one-dimensional linear equations with quasi-periodic coefficients, the question of reducibility is not trivial. In fact, for general quasi-periodic functions (whose lift is not smooth, for instance), the series $(g_{\mathbf{k}})_{\mathbf{k}}$ may diverge, thereby obstructing reducibility.

Remark 1.5.1 *Note that the reducibility of linear scalar equations with quasi-periodic coefficients, under appropriate smoothness hypothesis on the lift and Diophantine conditions on ω applies also to the reducibility of diagonal systems with quasi-periodic coefficients.*

Remark 1.5.2 *For equations having more dimensions, the terms of the homological equation (1.15) do not commute between them, so we cannot solve the equation directly. This is also related to the fact that, when dimension grows, there appear more frequencies in the system, which are called* internal frequencies *of the system (in opposition to the* external frequencies *ω), and that can be resonant with the external frequencies and between them. These frequencies are not known a priori, but if the system is reducible, they are related to the imaginary parts of the reduced (Floquet) matrix.*

Chapter 2

Reducibility through the Sacker-Sell spectrum

This chapter is devoted to the theory developed by Johnson, Sacker and Sell to study systems of linear differential equations with quasi-periodic coefficients. The idea is to introduce a spectral theory valid for a large variety of linear differential systems and that generalizes many concepts from the theory of linear equations with constant coefficients.

The spectral invariants provided by this theory can be used to grant reducibility of some linear systems. The disadvantages are, however, that it can be difficult to check the assumptions on the spectrum and that they do not cover all reducible systems. On the other hand these methods are not perturbative, but rather *topological*, so we can deal with the 'far-from-constant' case.

The present chapter is divided into two parts. The first one introduces the basic objects that we shall use for the presentation of the Sacker-Sell spectrum, in a slightly more general context, which will be useful in the next chapters and are of interest for their own sake. In the second section we introduce the Sacker-Sell spectrum, together with some of its basic properties. We mainly focus on the case of linear equations with quasi-periodic coefficients discussing the relation of the mentioned spectrum with Lyapunov characteristic numbers and the reducibility of these equations.

This chapter includes some definitions and concepts that will be useful in the rest of the present survey. In the following chapters we will make free use of the concepts of flows, vector bundles, exponential dichotomy, stable and unstable bundles, Lyapunov exponents and Sacker-Sell spectrum.

2.1 Linear skew-product flows and vector bundles

In this first section we will introduce some geometrical objects that arise naturally when studying linear equations with quasi-periodic coefficients. Once a linear equation with quasi-periodic coefficients is fixed,

$$x' = A(t)x, \tag{2.1}$$

it is natural to consider its lift to $\mathbb{T}^d$

$$x' = \tilde{A}(\theta)x, \qquad \theta' = \omega \tag{2.2}$$

so that it now takes place in $\mathbb{R}^n \times \mathbb{T}^d$ if $x \in \mathbb{R}^n$ and d is the number of frequencies of the quasi-periodic matrix A with frequency vector ω. The goal of this section is to put these equations in a suitable geometrical framework.

From now on X will represent both $\mathbb{R}^n$ or $\mathbb{C}^n$, depending on the context, furnished with an Euclidean inner product, $\langle \cdot, \cdot \rangle$, and the corresponding norm $|\cdot|$.

Definition 2.1.1 *A* flow *on Z is a triple $(Z, \mathbb{R}, \Phi)$, being Z a topological space, where*

$$\begin{array}{rccc} \Phi: & \mathbb{R} \times Z & \longrightarrow & Z \\ & (t; z) & \mapsto & \Phi(t; z) \end{array}$$

is a continuous map such that

(i) $\Phi(0; z) = z$, *for all* $z \in Z$.

(ii) $\Phi(t + s; z) = \Phi(s; \Phi(t; z))$, *for all* $z \in Z$, $t, s \in \mathbb{R}$. Z_ϕ *will also be written as* $Z(\phi)$.

Remark 2.1.2 *Many of these definitions can also be applied to* discrete *flows, that is, when the* time *is $\mathbb{Z}$ instead of $\mathbb{R}$ (see [71]). The discrete analogue of the continuous flow (2.2) is given by the following map of $\mathbb{R}^n \times \mathbb{T}^d$ to itself*

$$\bar{x} = \tilde{A}(\theta)x, \qquad \bar{\theta} = \theta + 2\pi\omega.$$

The natural scenario when dealing with linear equations with quasi-periodic coefficients will be the following

Definition 2.1.3 *By a* vector bundle *Z with* base *Ω and* projection *π we mean that*

(i) Z and Ω are Hausdorff spaces and $\pi : Z \to \Omega$ is an exhaustive and continuous mapping.

(ii) For each $\phi \in \Omega$ the set $Z_\phi := \pi^{-1}(\phi)$, which will be called a fiber *(or, more concretely, the fiber over ϕ), is a vector space.*

(iii) For each $\phi \in \Omega$, there is an open neighbourhood $U \subset \Omega$ of ϕ, a finite-dimensional vector space X and a homeomorphism

$$\tau : \pi^{-1}(U) \to X \times U$$

that maps elements in the same fiber Z_η to elements in $X \times U$ with the second component equal to η and for η fixed it acts over Z_η as a linear isomorphism.

Remark 2.1.4 *As a consequence of the definition and by continuity, the function $\phi \mapsto \dim Z_\phi$ is constant on any connected component of Ω.*

Remark 2.1.5 *Definition (2.1.3) is a precise formulation of 'a bundle is locally the product of the base times a vector space, and these vector spaces are locally isomorphic'.*

Remark 2.1.6 *As a vector bundle is locally a product of a finite-dimensional vector space times the base, we shall use the notation (x, ϕ) to refer to elements in Z, where ϕ is a point in the base space Ω and $x \in Z_\phi = \pi^{-1}(\phi)$. Thus x is a vector. This motivates a definition.*

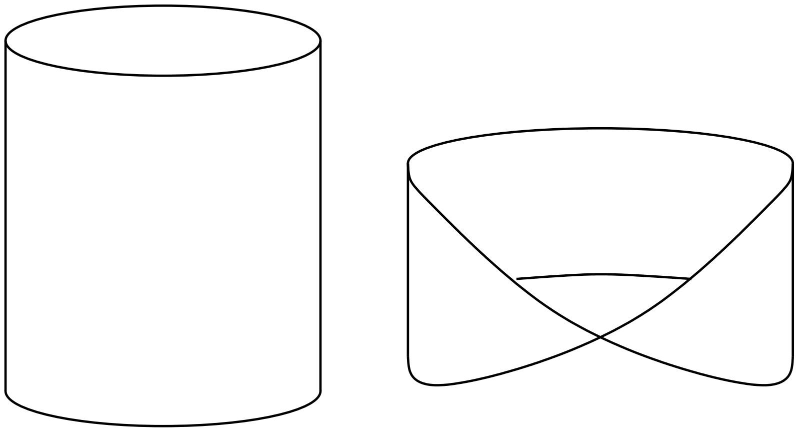

Figure 2.1: Example of two bundles: the cylinder (left) and the Möbius band (right).

Definition 2.1.7 *If $V \subset Z$ is a subset (with no additional structure) of a vector bundle Z and if ϕ is in Ω, we define the* fiber V_ϕ *as*

$$V_\phi = V(\phi) = \{x \in Z_\phi; (x, \phi) \in V\}$$

and more generally, for a certain subset, $N \subset \Omega$,

$$V(N) = \{x \in Z_\phi; (x, \phi) \in V, \phi \in N\}.$$

We now formulate the natural notion of a subbundle of a given bundle.

Definition 2.1.8 *Let (Z, Ω, π) be a vector bundle as above. A subset $V \subset Z$ is said to be a* (continuous) subbundle *of Z if, and only if, V is a closed set in Z with the following two properties:*

(i) For each ϕ in Ω, the fiber $V(\phi)$ is a linear subspace of Z_ϕ.

(ii) The function $\phi \in \Omega \mapsto \dim V(\phi)$ is constant on each connected component of Ω.

The following properties of subbundles are quite natural and can be found in [71],

Proposition 2.1.9 ([71]) *(i) If V is a subbundle of Z, then the linear subspaces $V(\phi)$ vary continuously with $\phi \in \Omega$.*

(ii) A subbundle $V \subset Z$ is itself a vector bundle with base Ω and projection $\pi_{|V}$.

We now come to a characterization of the kind of flows that we will be interested in. As we will see, all flows defined by linear equations with quasi-periodic coefficients can be included in this category.

Definition 2.1.10 *Let Z be a vector bundle with compact Hausdorff base Ω and projection $\pi : Z \to \Omega$. A (real or complex) flow on Z, (Z, Φ) is said to be a* linear skew-product flow, (LSPF), *if it can be represented in the form*

$$\Phi(t; (x, \phi)) = (\varphi(t; (x, \phi), \sigma(t; \phi))$$

such that

(i) $\sigma : \Omega \times \mathbb{R} \to \Omega$ *is a flow on* Ω.

(ii) The map

$$\begin{array}{ccc} Z_\phi & \longrightarrow & Z_{\sigma(t;\phi)} \\ x & \mapsto & \varphi\left(t;(x,\phi)\right) \end{array}$$

is linear, that is, there exists a square matrix of dimension n*, depending only on* t *and* ϕ *such that*

$$\varphi\left(t;(x,\phi)\right) = M(t;\phi)x,\ x \in Z_\phi$$

Proposition 2.1.11 ([71]) *(i)* $M(t;\phi) : Z_\phi \to Z_{\sigma(t;\phi)}$ *is non-singular. Moreover*

$$M\left(t;\phi\right)^{-1} = M\left(-t;\sigma(t;\phi)\right)$$

for all $t \in \mathbb{R}$ *and* $\phi \in \mathbb{T}^d$.

(ii) $\dim Z_\phi = \dim Z_{\sigma(t;\phi)}$ *for all* $t \in \mathbb{R}$ *and* $\phi \in \mathbb{T}^d$.

Example: Linear equations with quasi-periodic coefficients. Before going into more definitions and properties, we want to apply all these concepts to the case which interests us.

To this end recall that we are dealing with homogeneous equations of the type

$$x' = A(t)x, \tag{2.3}$$

where $x \in \mathbb{R}^n$, $A : \mathbb{R} \to L(\mathbb{R}^n)$ is such that there exists a (continuous at least) lifting $\tilde{A} : \mathbb{T}^d \to L(\mathbb{R}^n)$ and a frequency vector $\omega \in \mathbb{R}^d$ satisfying that $A(t) = \tilde{A}(\omega t)$ for all $t \in \mathbb{R}$. Therefore, we can consider equation (2.3) depending on each initial phase $\phi \in \mathbb{T}^d$ as we did in the previous chapter

$$x'(t) = \tilde{A}(\omega t + \phi)x(t). \tag{2.4}$$

Our vector bundle will be $Z = \mathbb{R}^n \times \mathbb{T}^d$, the base space $\Omega = \mathbb{T}^d$ (which is a compact Hausdorff space) with the obvious projection $\pi : \mathbb{R}^n \times \mathbb{T}^d \to \mathbb{T}^d$ on the second component. Now Z is *globally* the product of an Euclidean vector space $\mathbb{R}^n$ times the compact Hausdorff base $\Omega = \mathbb{T}^d$ and hence a vector bundle.

Let us now define our linear skew-periodic flow on Z. Let $\varphi(t;(x,\phi))$ denote the solution of equation (2.4) with initial condition x, evaluated at time t. We now define the map

$$\begin{array}{rccc} \Phi : & \left(\mathbb{R}^n \times \mathbb{T}^d\right) \times \mathbb{R} & \longrightarrow & \mathbb{R}^n \times \mathbb{T}^d \\ & ((x,\phi),t) & \mapsto & \left(\varphi\left(t;(x,\phi)\right), \phi + \omega t\right) \end{array} \tag{2.5}$$

which is a linear skew-product flow because $(\phi, t) \mapsto \phi + \omega t$ is a flow on $\mathbb{T}^d$ and, for fixed t and ϕ, the mapping $x \mapsto \varphi\left(t;(x,\phi)\right)$ is linear. Indeed, as (2.4) is a linear ordinary differential equation, there exists a fundamental matrix, $M(t;\phi)$, depending on t and ϕ, which is non-singular and of dimension n, with $M(0;\cdot) = Id$. This fundamental matrix satisfies that

$$\varphi\left(t;(x,\phi)\right) = M(t;\phi)x.$$

If the mapping $(t;\phi) \mapsto M(t,\phi) \in GL(\mathbb{R}^n)$ takes values on some subgroup of $G \subset GL(\mathbb{R}^n)$ (such as $SL(n,\mathbb{R})$, $SO(3)$, $Sp(n)$, ...) we shall say that it is a linear skew-product flow defined on G.

We now want to study a bit more the objects that we have just introduced and especially their *dynamical* meaning. To do this we first consider a trivial example.

Consider the equation on $\mathbb{R}^2$

$$x' = \Lambda x,$$

where Λ is a 2×2 diagonal matrix having only λ_1 and λ_2 (with $\lambda_1 < 0 < \lambda_2$) as diagonal elements. Then a fundamental matrix is

$$\begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}.$$

From this fundamental matrix we observe that the vector spaces

$$W^s = \{(x, 0); x \in \mathbb{R}\}$$

and

$$W^u = \{(0, y); y \in \mathbb{R}\}$$

are invariant under the above flow and that initial conditions on these vector spaces tend to the origin when $t \to +\infty$ (for W^s) or $t \to -\infty$ (for W^u). Moreover these two subspaces generate all $\mathbb{R}^2$,

$$\mathbb{R}^2 = W^s \oplus W^u.$$

We now want to extend these dynamical notions to general linear skew-product flows.

Definition 2.1.12 *Let Z be a vector bundle and Φ a flow on it. We shall say that $V \subset Z$ is an* invariant subbundle *of Z if it is a subbundle of Z and $\Phi(t; V) = V$ for all $t \in \mathbb{R}$.*

In the case of linear equations with quasi-periodic coefficients, the set $\{0\} \times \mathbb{T}^d$ is always an invariant subbundle.

The following definition will be of importance in the sequel. It generalizes the direct sum of linear subspaces to vector bundles. The definition is quite natural,

Definition 2.1.13 *Let $V_1, \ldots, V_p \subset Z$ be subbundles of Z and let $N \subset \Omega$ be a subset. If, for all $\phi \in N$, $V_1(\phi) \oplus \cdots \oplus V_p(\phi) = Z_\phi$, we shall write this as*

$$Z(N) = V_1(N) \oplus \cdots \oplus V_p(N) \quad \textit{(Whitney sum)}$$

and, if $N = \Omega$, we shall simply write

$$Z = V_1 \oplus \cdots \oplus V_p \quad \textit{(Whitney sum).}$$

For instance, for linear equations with constant coefficients, let $E_1, \ldots, E_p$ be the generalized eigenspaces of the constant matrix A. Then the subsets of Z, defined as $V_i = E_i \times \mathbb{T}^d$ are invariant subbundles, because the solutions of (2.4) are precisely $x(t; \phi) = \exp(tA)x(0)$, and the generalized eigenspaces of A and $\exp(tA)$ for all t are the same. Hence we have a decomposition

$$Z = V_1 \oplus \cdots \oplus V_p \quad \text{(Whitney sum).}$$

Note that in this example, all these invariant subbundles are themselves a direct product. By Floquet theory the same example can be extended to linear equations with periodic coefficients.

Definition 2.1.14 *Let Z be a bundle with base Ω and projection π. A* projector *on Z is defined to be a continuous mapping*

$$P : Z \longrightarrow Z$$

such that

(i) It maps each fiber Z_ϕ ($\phi \in \Omega$) into itself.

(ii) For each $\phi \in \Omega$, $P(\cdot, \phi)$ is a projection on Z_ϕ.

That is, a projector can be written as

$$P(x, \phi) = (\hat{P}(\phi)x, \phi)$$

where P is jointly continuous in x and ϕ and $\hat{P}(\phi)^2 = \hat{P}(\phi)$ for all $\phi \in \Omega$. Given a projector as above, it is natural to define its *range* as the set

$$\mathcal{R} = \{(x, \phi) \in Z;\ P(x, \phi) = (x, \phi)\}$$

and its *null space* as

$$\mathcal{N} = \{(x, \phi) \in Z;\ P(x, \phi) = (0, \phi)\}$$

The following proposition confirms the naturality of such objects:

Proposition 2.1.15 ([71]) *Let $P : Z \to Z$ be a projection on a vector bundle, as above. Then the following properties hold:*

(i) The sets $\mathcal{R}$ and $\mathcal{N}$ are subbundles of Z and they are complementary, that is,

$$\mathcal{R} \oplus \mathcal{N} = Z \text{ (Whitney sum)}$$

(ii) Given any two complementary subbundles V_1 and V_2, there is a unique projector such that its range is V_1 and its null space V_2.

This proposition says that the notions of complementary subbundles and projectors are clearly equivalent. This will prove useful in the following important definition, which refers to certain behaviours that are feasible in the kind of dynamical systems that we are dealing with. It is the generalization of the kind of behaviour that we observed in the previous example of a linear equation with constant coefficients.

Definition 2.1.16 *Let N be a subset of the base space Ω of a vector bundle (Z, Ω, π). We shall say that a LSPF Φ (where $\Phi(t; x, \phi) = (M(t; \phi)x, \sigma(t; \phi))$) defined on Z admits an* exponential dichotomy over *N if there is a projector*

$$P : Z(N) \longrightarrow Z(N)$$

and positive constants K and α such that the following inequalities hold

$$\left|M(t; \phi)P(\phi)M^{-1}(s; \phi)\right| \leq Ke^{-\alpha(t-s)},\ s \leq t$$

$$\left|M(t; \phi)\left(I - P(\phi)\right)M^{-1}(s; \phi)\right| \leq Ke^{-\alpha(s-t)},\ s \geq t$$

for all $\phi \in N$. If N is all Z, we shall simply say that Φ admits an exponential dichotomy.

By the above *equivalence* between projectors and complementary invariant subbundles, there is an equivalent formulation of the concept of exponential dichotomy using complementary invariant subbundles.

Example 2.1.17 *Assume that we have a LSPF* Φ *on* Z*. For each* $\lambda \in \mathbb{R}$*, we can define another flow* Φ_λ *on* Z *by*

$$\Phi_\lambda(t; x, \phi) = \left(e^{-\lambda t} M(t; \phi)x, \sigma(t; \phi)\right)$$

which is also a LSPF on Z*. Furthermore, the sets*

$$\mathcal{B}_\lambda = \left\{(x, \phi) \in Z;\ \sup_{t \in \mathbb{R}} \left|e^{-\lambda t} M(t; \phi)x\right| < \infty\right\}$$

$$\mathcal{S}_\lambda = \left\{(x, \phi) \in Z;\ \lim_{t \to +\infty} \left|e^{-\lambda t} M(t; \phi)x\right| = 0\right\}$$

$$\mathcal{U}_\lambda = \left\{(x, \phi) \in Z;\ \lim_{t \to -\infty} \left|e^{-\lambda t} M(t; \phi)x\right| = 0\right\}$$

are clearly invariant subsets of Z *under both the flows* Φ_λ *and* Φ*. For every fixed* $\phi \in \Omega$*, the fibers* $\mathcal{B}_\lambda(\phi)$*,* $\mathcal{S}_\lambda(\phi)$ *and* $\mathcal{U}_\lambda(\phi)$ *are linear subspaces of* Z_ϕ *and for* $\mu \leq \lambda$ *we have the inclusions* $\mathcal{S}_\mu \subset \mathcal{S}_\lambda$ *and* $\mathcal{U}_\lambda \subset \mathcal{U}_\mu$*.*

The question of whether $\mathcal{S}_\lambda$ and $\mathcal{U}_\lambda$ are complementary invariant subbundles under Φ_λ or not is one of the motivations to introduce the Sacker-Sell spectrum, which will be done in the following section.

2.2 The Sacker-Sell spectrum

We first formulate the main tool for reducibility that we shall use in the present chapter.

Definition 2.2.1 *Let* Φ_λ *be a LSPF on* (Z, Ω, π)*, a vector bundle, and assume that* Ω *is a compact Hausdorff space. Consider, for all* $\lambda \in \mathbb{R}$*, the flow* Φ_λ *(defined in the previous section). For all* $\phi \in \Omega$*, we define the* resolvent of Φ at ϕ*,* $\rho(\phi, \Phi)$*, as*

$$\rho(\phi, \Phi) = \{\lambda \in \mathbb{R};\ \Phi_\lambda \textit{ admits an exponential dichotomy over } \{\phi\}\}$$

and its spectrum at ϕ *as* $\Sigma(\phi, \Phi) = \mathbb{R} - \rho(\phi, \Phi)$*. In general, we call the* resolvent of Φ*,* $\rho(\Phi)$*, by*

$$\rho = \rho(\Phi) = \bigcap_{\phi \in \Omega} \rho(\phi, \Phi)$$

and the Sacker-Sell spectrum of Φ *by its complementary over* $\mathbb{R}$*.*

Example 2.2.2 *According to the definition above,* λ *is in the resolvent set of the equation*

$$x' = \tilde{A}(\omega t + \phi)x$$

if, and only if, the shifted equation

$$x' = \left(\tilde{A}(\omega t + \phi) - \lambda I\right) x$$

admits exponential dichotomy.

The following proposition supplies us with basic properties of the spectrum defined above.

Proposition 2.2.3 ([71]) *(i) If Φ_λ admits an exponential dichotomy over $\phi \in \Omega$, then Φ_λ admits an exponential dichotomy over the* hull *of ϕ in Ω, defined as the following subset of Ω*

$$H(\phi) = \overline{\{\sigma(t;\phi);\ t \in \mathbb{R}\}}$$

(ii) As a consequence,

$$\Sigma(\phi) = \Sigma(H(\phi))$$

for all $\phi \in \Omega$. In particular, if $H(\phi) = \Omega$ for all $\phi \in \Omega$ (that is, if the flow is minimal*), then $\Sigma = \Sigma(\phi)$ does not depend on the chosen ϕ.*

The definition of the hull leads us to another important definition. A subset N of Ω is said to be *minimal* with respect to the flow Φ if $H(N) = \Omega$. In general, the flow is said to be *minimal* whenever $H(\{\phi\}) = \Omega$, for all $\phi \in \Omega$.

Example 2.2.4 *Consider the above example of a flow induced by a linear equation with quasi-periodic coefficients. Let ω be the frequency vector. If there are s components of the vector which are rationally independent, then for any $\phi \in \mathbb{T}^d$, the hull $H(\phi)$ is homeomorphic to a s-dimensional torus in $\mathbb{T}^d$ and the Sacker-Sell spectrum $\Sigma(\phi)$ is constant over this s-dimensional torus. If the frequencies are not commensurable, then the Sacker-Sell spectrum does not depend on the chosen $\phi \in \mathbb{T}^d$.*

Some equivalent conditions to have an exponential dichotomy can be found easily,

Proposition 2.2.5 ([71]) *Let N be a compact invariant set and assume that N is a minimal set in the flow σ on Ω (this means that $H(N) = \Omega$). Then the following statements are equivalent*

(i) λ is in the resolvent set of the Sacker-Sell spectrum, $\rho(N)$.

(ii) The flow Φ_λ has an exponential dichotomy over N.

(iii) $\mathcal{B}_\lambda = \{0\} \times \Omega$, the zero section of $Z(N)$. That is, the only elements of $Z(N)$ whose orbit under the flow Φ_λ is bounded are the trivial ones.

The following theorem describes the spectrum:

Theorem 2.2.6 (Spectral Theorem, [71]) *Let Φ be a LSPF on a vector bundle Z with compact Hausdorff base Ω. Let $n = \dim Z$ and $N \subset \Omega$ an invariant compact connected set. Then the Sacker-Sell spectrum $\Sigma(N)$ is the union*

$$\Sigma(N) = [a_1, b_1] \cup \cdots \cup [a_p, b_p]$$

of p non-overlapping compact intervals with $p \leq n$ and $a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_p \leq b_p$, and we shall call the above expression the spectral decomposition of (Z, Φ). *Furthermore, if $\lambda_0, \ldots, \lambda_k$, with $k \leq p$, are chosen in the resolvent set so that*

$$\lambda_0 < a_1 \leq b_1 < \lambda_1 < a_2 \leq b_2 < \lambda_2 < \ldots$$

then, for $1 \leq i \leq k$, the set

$$V_i(N) = \mathcal{S}_{\lambda_i}(N) \cap \mathcal{U}_{\lambda_{i-1}}(N)$$

is an invariant subbundle of $Z(N)$ called the spectral subbundle *associated to the spectral interval $[a_i, b_i]$. Its dimension is $1 \leq n_i \leq n$ and $n_1 + \ldots n_k = n$. Moreover we have that*

$$Z(N) = V_1(N) \oplus \cdots \oplus V_p(N) \text{ (Whitney sum)}$$

Finally, the spectrum $\Sigma_i(N)$ of $(V_i(N), \Phi_{|V_i(N)})$ is $\Sigma_i(N) = [a_i, b_i]$.

The proof is also found in [71]. It uses the following *uniformization lemma* which is interesting for its own sake

Lemma 2.2.7 ([71]) *Let N be a compact invariant set in Ω and $\lambda \in \mathbb{R}$. Then the following statements are valid:*

(i) If $|\Phi_\lambda(t;z)| \to 0$ as $t \to +\infty$ for each $z \in N$, then $\lambda \in \rho(N)$, $\Sigma(N) \subset (-\infty, \lambda)$ and $\mathcal{S}_\mu$ is the whole bundle $Z(N)$ for all $\mu \geq \lambda$.

(ii) If $|\Phi_\lambda(t;z)| \to 0$ as $t \to -\infty$ for each $z \in N$, then $\lambda \in \rho(N)$, $\Sigma(N) \subset (\lambda, +\infty)$ and $\mathcal{U}_\mu$ is the whole bundle $Z(N)$ for all $\mu \leq \lambda$.

If nothing is said explicitly, we shall consider the spectral decomposition with respect to $N = \Omega$.

Remark 2.2.8 *Let V_i be a spectral subbundle. The unique projector $P_i : Z \to Z$ such that its range is V_i is called the* spectral projector *and its kernel is the Whitney sum $\bigoplus_{i \neq j} V_j$.*

Remark 2.2.9 *If the matrices of the flow $M(t;\phi)$ belong to some subgroup of $GL(n)$ then the properties of this group will imply that for the Sacker-Sell spectrum has some special properties. For instance, if the group is $SL(2,\mathbb{R})$, then if λ belongs to the spectrum, then also $-\lambda$ is in the spectrum.*

Following the example from the previous section of a flow generated by a linear equation with constant coefficients, all the spectral intervals reduce to points which are the real parts of the eigenvalues of the matrix. The corresponding spectral invariant subbundles are also the subbundles corresponding to eigenvalues with equal real parts. The analysis for periodic systems is analogous to case of constant coefficients because of the reducibility given by Floquet theory. In both cases all spectral intervals reduce to points. This concept deserves a definition.

Definition 2.2.10 *We shall say that a LSPF Φ on a vector bundle Z has* pure point spectrum *if all spectral intervals degenerate to points. If, moreover, $n = p$, we shall say that Φ has* full spectrum*. If there is an interval in the spectrum, we shall speak of* absolutely continuous spectrum.

The following proposition clarifies the second hypothesis in the definition.

Proposition 2.2.11 ([71]) *Consider the equation $x' = A(t)x$, with quasi-periodic coefficients. If one spectral subbundle is one-dimensional, then the corresponding spectral interval is a point. Hence, if there are exactly n spectral intervals, then each interval is degenerate.*

We now want to further explore these concepts, focusing mainly on linear equations with quasi-periodic coefficients. In the following section we relate the spectrum of a linear skew-product flow to the existence of Lyapunov exponents. In the other two sections we prove a reducibility theorem.

2.3 Relation with Lyapunov exponents

It turns out that the Sacker-Sell spectrum is strongly connected to the theory of Lyapunov characteristic exponents, which we present in this section. For this purpose consider a LSPF Φ on a vector bundle (Z, Ω) and assume that Ω is compact and invariantly connected. This means that Ω cannot be written as the union of two disjoint nonempty compact invariant sets, and this is true whenever $\Omega = \mathbb{T}^d$ and the flow is linear with rationally independent frequencies.

Let

$$\Sigma = \Sigma(\Omega) = [a_1, b_1] \cup \cdots \cup [a_k, b_k]$$

and

$$Z = V_1 \oplus \cdots \oplus V_k \qquad \text{(Whitney sum)}$$

be the spectral decomposition of (Z, Φ) given by theorem 2.2.6, the intervals $[a_i, b_i]$ being ordered so that $b_i < a_{i+1}$, and let $P_i : Z \to V_i$ be the spectral projectors.

Given a point $(x, \phi) \in Z$, with $x \neq 0$, the usual definition of the four Lyapunov characteristic numbers is

$$\beta^+_{sup}(x, \phi) = \limsup_{t \to +\infty} \frac{1}{t} \log |\Phi(t; (x, \phi))|, \tag{2.6}$$

$$\beta^+_{inf}(x, \phi) = \liminf_{t \to +\infty} \frac{1}{t} \log |\Phi(t; (x, \phi))|, \tag{2.7}$$

$$\beta^-_{sup}(x, \phi) = \limsup_{t \to -\infty} \frac{1}{t} \log |\Phi(t; (x, \phi))|, \tag{2.8}$$

$$\beta^-_{inf}(x, \phi) = \liminf_{t \to -\infty} \frac{1}{t} \log |\Phi(t; (x, \phi))|. \tag{2.9}$$

Using the characterization of the spectral intervals and the associated subbundles we have the following

Theorem 2.3.1 ([71]) *If $(x, \phi) \in V_i$, being V_i the i-th spectral subbundle associated to the spectral interval $[a_i, b_i]$, and $x \neq 0$, then the four Lyapunov characteristic numbers defined above lie in $[a_i, b_i]$. In particular, if $a_i = b_i$, then for all $(x, \phi) \in V_i$ and $x \neq 0$ the four characteristic exponents agree and the limits*

$$\lim_{t \to +\infty} \frac{1}{t} \log |\Phi(t; (x, \phi))| = \lim_{t \to -\infty} \frac{1}{t} \log |\Phi(t; (x, \phi))|$$

exist and equal a_i.

Therefore, considering this result for all the spectral subbundles, together with some work, we obtain the following,

Theorem 2.3.2 ([71]) *For all $(x, \phi) \in Z$, with $x \neq 0$, the four Lyapunov characteristic exponents lie in the spectrum Σ. More precisely, let $(x, \phi) \in Z$, with $x \neq 0$, and define q and q' by*

$$q = \max\{i : P_i(\phi)x \neq 0\},$$

$$q' = \min\{i : P_i(\phi)x \neq 0\}.$$

Then one has the inequalities

$$a_q \leq \beta^+_{inf}(x, \phi) \leq \beta^+_{sup}(x, \phi) \leq b_q,$$

$$a_{q'} \leq \beta^-_{inf}(x, \phi) \leq \beta^-_{sup}(x, \phi) \leq b_{q'}.$$

Finally, we also have some information on the amount of elements of Z having Lyapunov exponent one endpoint of a spectral interval.

Theorem 2.3.3 ([71]) *Assume that $M \subset \Omega$ is a minimal set with respect to the flow σ on Ω. Then the set G of all $\phi \in M$ such that there exist $x_1^-, \dots, x_p^-, x_1^+, \dots, x_p^+$ in X with the property*

$$\beta^+_{sup}(x_i^+, \phi) = b_i \quad \textit{and} \quad \beta^-_{inf}(x_i^-, \phi) = a_i \qquad (i = 1, \dots, k)$$

is a residual G_δ-subset of Ω.

2.4 Smoothness of spectral subbundles

From now on, we restrict ourselves to the case of a linear equation with quasi-periodic coefficients

$$x' = \tilde{A}(\phi + \omega t)x$$

where $\tilde{A} : \mathbb{T}^d \to L(\mathbb{R}^n)$ is a mapping from $\mathbb{T}^d$ to $L(\mathbb{R}^n)$, the space of linear operators, with the supremum norm. We assume that $\tilde{A} \in C^\alpha(\mathbb{T}^d)$, for $\alpha = 0, \dots, \infty, a$.

Theorem 2.4.1 (Smoothness of spectral subbundles, [46]) *In the above situation, let ω be an irrational frequency vector on $\mathbb{R}^d$. Let V_i, for $i = 1, \dots, p$, denote the spectral subbundles of the associated flow and $V_i(\phi)$ denote its spectral subspaces, for $\phi \in \mathbb{T}^d$. Assume that $\tilde{A} \in C^\alpha(\mathbb{T}^d)$, for $\alpha = 0, \dots, \infty, a$. Then we can choose a local basis in $V_i(\phi)$ that it is a C^α function of ϕ. Equivalently the spectral projectors*

$$P_i : \mathbb{T}^d \longrightarrow \{P \in L(\mathbb{R}^n);\ P^2 = P\}$$

are of class C^α.

In order to prove this theorem it suffices to pick any $\phi_0 \in \mathbb{T}^d$ and restrict the values of ϕ that we consider to a suitable small neighbourhood of ϕ_0. Now let $[a, b]$ denote a given spectral interval and let V be the corresponding spectral subbundle. Next let μ, λ be chosen in the resolvent set so that $(\mu, \lambda) \cap \Sigma = [a, b]$. This means that the shifted equations

$$x' = \Big(\tilde{A}(\phi + \omega t) - \mu I\Big)\, x \qquad \text{and} \qquad x' = \Big(\tilde{A}(\phi + \omega t) - \lambda I\Big)\, x$$

admit exponential dichotomies. The stable and unstable subbundles associated to these dichotomies are, for $\nu = \mu, \lambda$,

$$\mathcal{S}_\nu = \{(x, \phi);\ P_\nu(\phi)x = x\} \qquad \text{and} \qquad \mathcal{U}_\nu = \{(x, \phi);\ P_\nu(\phi)x = 0\}$$

and, by the spectral theorem, the spectral subbundle V is given by $V = \mathcal{S}_\lambda \cap \mathcal{U}_\mu$; that is, $V(\phi) = \mathcal{S}_\lambda(\phi) \cap \mathcal{U}_\mu(\phi)$ for all $\phi \in \mathbb{T}^d$. The bundles $\mathcal{S}_\lambda$ and $\mathcal{U}_\mu$ are transversal in a neighbourhood of ϕ. The transversality (i.e., for all ϕ in a neighbourhood of ϕ_0 the vector spaces $\mathcal{S}_\lambda(\phi)$ and $\mathcal{U}_\mu(\phi)$ generate all $Z(\phi)$) comes from the fact that $Z(\phi) = \mathcal{S}_\mu(\phi) \oplus \mathcal{U}_\mu(\phi)$ and we have the inclusion $\mathcal{S}_\mu(\phi) \subset \mathcal{S}_\lambda(\phi)$ (because $\mu < \lambda$).

In order to prove that V depends smoothly on ϕ in a neighbourhood of ϕ_0, it suffices to prove that both $\mathcal{S}_\lambda$ and $\mathcal{U}_\mu$ depend smoothly on ϕ in a neighbourhood of ϕ_0, due to the transversality condition. Without loss of generality (replacing $\tilde{A}$ by $\tilde{A} - \nu I$, if necessary) we can assume that $\nu = 0$.

Below we state a theorem that will imply the previous one

Theorem 2.4.2 ([46]) *Let U be an open set in $\mathbb{R}^k$ and let $A(\cdot,\cdot) : U \times \mathbb{R} \to L(\mathbb{R}^n)$ satisfy the following conditions*

(i) A is continuous and of class C^β in u, where $\beta = 0, 1, 2, \ldots, \infty, a$.

(ii) A and all its derivatives with respect to u (up to and including order β) are bounded on $U \times \mathbb{R}$ and equi-continuous in u.

Assume that for $u_0 \in U$ the differential equation

$$x' = A(u_0, t)x \tag{2.10}$$

admits an exponential dichotomy with projection P_0. Then there is a neighbourhood V of u_0, with $V \subset U$, such that, for $u \in V$, the differential equation

$$x' = A(u, t)x \tag{2.11}$$

admits an exponential dichotomy with projection P_u. Furthermore, one has that $P_{u_0} = P_0$ and P_u is of class C^β on V. Therefore one can choose a C^β-basis for the range $\mathcal{R}(P_u)$ and the null space $\mathcal{N}(P_u)$.

To apply this theorem to the proof of the previous one, we take ϕ in a small neighbourhood around ϕ_0 and we consider the domain for ϕ to be in the parameter space. Then we can apply the above theorem and deduce C^α-smoothness on a narrower domain.

2.5 A Reducibility Theorem

In this last section we apply the definitions and results that we have given to state and prove a theorem on the reducibility of certain classes of linear equations with quasi-periodic coefficients.

The main result in this section is due to R. Johnson and G. Sell, and it can be stated in the following way:

Theorem 2.5.1 (Reducibility under full spectrum assumption, [46]) *Let*

$$x' = \tilde{A}(\phi + \omega t)x, \; x \in \mathbb{R}^n \tag{2.12}$$

be a quasi-periodic differential equation defined on a torus $\Omega = \mathbb{T}^d$ with an irrational flow $(t; \phi) \mapsto \phi + \omega t$. Assume that the three following conditions hold

(i) **(Strong non-resonance)** *There exist constants $K > 0$ and $\tau > 0$ such that*

$$|\langle \mathbf{k}, \omega \rangle| \geq \frac{K}{|\mathbf{k}|^\tau}, \quad \textit{for all } \mathbf{k} \in \mathbb{Z}^d - \{0\}$$

(ii) **(Smoothness)** *$\tilde{A} \in C^\alpha(\mathbb{T}^d)$, for $\alpha \geq d + \tau + 2$.*

(iii) **(Full spectrum)** *Equation (2.12) has full spectrum with $\Sigma = \{\alpha_1, \ldots, \alpha_n\}$.*

Then there exists a quasi-periodic Lyapunov-Perron transformation $x = P(t)y$ such that it takes (2.12) into

$$y' = By$$

where B is the constant matrix diagonal matrix $B = diag(\alpha_1, \ldots, \alpha_n)$. Furthermore, the quasi-periodic matrix has the form

$$P(t) = \tilde{P}(\bar{\omega}t)$$

where $\bar{\omega} = \frac{\omega}{2}$ and $\tilde{P} : \mathbb{T}^d \to L(\mathbb{R}^n)$ is of class $C^\beta(\mathbb{T}^d)$ with $\beta = \alpha - \tau - d - 1$.

Remark 2.5.2 *The full spectrum hypothesis is usually hard to check unless we are dealing with perturbations of hyperbolic systems.*

To see how the full spectrum assumption gives rise to the above properties we first give a definition and a property, both from [46],

Definition 2.5.3 *An equation*

$$x' = A(t)x \tag{2.13}$$

is said to satisfy the Lillo property *if there exist real numbers $\lambda_1 < \lambda_2 < \cdots < \lambda_{n+1}$, a constant $K > 0$ and n solutions $x_1(t), \ldots, x_n(t)$ such that the following inequalities hold*

$$\frac{1}{K} e^{\lambda_i (t-s)} \leq \frac{|x_i(t)|}{|x_i(s)|} \leq K e^{\lambda_{i+1}(t-s)}$$

for $s \leq t$ and $1 \leq i \leq n$.

This means that equation (2.13) has n linearly independent solutions with exponential growth. The following proposition relates the full spectrum property with the Lillo property. It is an easy consequence of the spectral theorem and of the exponential dichotomy in the resolvent set.

Proposition 2.5.4 ([46]) *The following two statements are equivalent:*

(A) Equation (2.13) has full spectrum with $\Sigma = \{\alpha_1, \ldots, \alpha_n\}$.

(B) Equation (2.13) has the Lillo property. In this case one also has the inequalities $\lambda_i < \alpha_i < \lambda_{i+1}$ for $i = 1, \ldots, n$.

There is a weaker concept than Lillo property which we shall also use later. We formulate it now.

Definition 2.5.5 *Equation (2.13) is said to satisfy the* Bylov property *if there exist n solutions $x_1(t), \ldots, x_n(t)$ with the property that*

$$\inf_{t \in \mathbb{R}} \left| \det \tilde{X}(t) \right| > 0$$

where $\tilde{X}(t)$ denotes the matrix with columns $\tilde{x}_1(t), \ldots, \tilde{x}_n(t)$, being, for each $i = 1, \ldots, n$,

$$\tilde{x}_i(t) = \frac{1}{|x_i(t)|} x_i(t)$$

We now turn to the proof of the reducibility theorem 2.5.1. We shall first give a sketch of proof and explain the problem that prevents this argument from being actually a proof. Later on we write down the whole proof. Let us first introduce some notation.

Let Ω denote, as usual, the standard d-dimensional torus, $\mathbb{T}^d = \mathbb{R}^d/(2\pi\mathbb{Z})^d$, and let $p : \mathbb{R}^d \to \mathbb{T}^d$ be its corresponding quotient map. The irrational twist flow $(t;\phi) \mapsto \phi+\omega t$ on $\mathbb{T}^d$ lifts to the flow $(t;y) \mapsto y+\omega t$ on $\mathbb{R}^d$. This means that for all $y \in \mathbb{R}^n$, we have that $p(y)+\omega t = p(y+\omega t)$ on $\mathbb{T}^d$.

The full spectrum assumption implies that there exist n one-dimensional invariant subbundles V_i in $Z = \mathbb{R}^n \times \Omega$ with the property that

$$Z = V_1 \oplus \cdots \oplus V_n \quad \text{(Whitney sum)}.$$

This means that for each $\phi \in \Omega$ the equation

$$x' = \tilde{A}(\phi + \omega t)x \tag{2.14}$$

has the Bylov property (because it satisfies the full spectrum, and thus the Lillo, property). After a normalization, the initial conditions

$$\{x_1(\phi), \ldots, x_n(\phi)\}$$

of the n solutions referred in the Bylov property form a basis of unit vectors in the fiber Z_ϕ which, recall, is a linear space of dimension n.

We now give the heuristic argument. Let $P(\phi)$ the linear transformation that maps the standard basis of $\mathbb{R}^n$ onto $\{x_1(\phi), \ldots, x_n(\phi)\}$. Since the original equation is of class C^α, then, by the smoothness of the spectral subbundles, it turns out that the $x_i(\phi)$ are also of class C^α on Ω. Also the change of variables $x = P(\phi + \omega t)y$ transforms equation (2.14) to a diagonal matrix. If we could define the transformation *globally* and not just locally in each point of Ω, with the required smoothness, then the proof would be finished.

However, the change of basis may not be globally defined. This shouldn't come as a surprise, because it is a phenomenon that also occurs in the periodic case, and it is known as the *period doubling*: it could happen that, after crossing a generator of the fundamental group of Ω, one of the basis vectors $x_i(\phi)$ returned to its negative $-x_i(\phi)$. In the periodic case we can work this out considering transformations with double the period of the original system. The generalization to the case $\Omega = \mathbb{T}^d$, with $d > 1$, is to consider a suitable finite covering of Ω. Let's write down the details.

Let

$$S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\},$$

being $|\cdot|$ the Euclidean norm on $\mathbb{R}^n$. For each $1 \le i \le n$ we denote by W_i a connected component of $V_i \cap (S^{n-1} \times \Omega)$. By theorem 2.4.1, each W_i is a C^α-embedded manifold of $S^{n-1} \times \Omega$. Let $\eta_i : W_i \to \Omega$ the projection on the second component. This map is locally a C^α-diffeomorphism, due to the smoothness of V_i and to its structure, which is locally the product of an n-dimensional vector space times the base Ω. Furthermore, each W_i is invariant under the flow induced on $S^{n-1} \times \Omega$ (this flow is induced by the projection of $\mathbb{R}^n - \{0\}$ onto the sphere S^{n-1}), because each of the subbundles V_i are also invariant by the original flow.

The full spectrum assumption implies that the only way that the frame $\{x_1(\phi), \ldots, x_n(\phi)\}$ can change when crossing a generator of the fundamental group (that is, making one of the d topologically different turns) is for some $x_i(\phi)$ to be replaced by its negative $-x_i(\phi)$. In particular, *turning twice*, all the frame is mapped to itself. This later remark together with the local diffeomorphism given by each of the η_i, implies that every W_i is either a 1-cover or a 2-cover of Ω, depending on whether the corresponding normalized vector $x_i(\phi)$ needs one or two turns to be mapped to itself.

Due to the structure of a finite covering, we can lift the map $\eta_i : W_i \to \Omega$ to a unique map $\tilde{\eta}_i : \mathbb{R}^d \to W_i$ such that the following diagram commutes

$$\begin{array}{ccc} & & W_i \\ & \nearrow & \downarrow \\ \mathbb{R}^d & \to & \Omega \end{array}$$

for each $1 \le i \le n$. In addition, due to the smoothness of η_i and the projection p, the following statements hold:

(i) Each $\tilde{\eta}_i$ is a C^α-mapping because locally one has that $\tilde{\eta}_i = \eta_i^{-1} \circ p$.

(ii) Each $\tilde{\eta}_i$ is a homomorphism of flows (that is, it makes the flows in the range and in the pre-image commute) because η_i and p are flow homomorphisms and the lifting $\tilde{\eta}_i$ is unique.

(iii) Each $\tilde{\eta}_i$ maps the square

$$[0, 4\pi]^d \subset \mathbb{R}^d$$

onto W_i, because the projection $p : \mathbb{R}^d \to \Omega$ takes intervals of length 2π on the coordinate axes of $\mathbb{R}^d$ onto cycles which generate the fundamental group of Ω. Since each W_i is either a 1-cover or a 2-cover of Ω these cycles *unwind* at most twice under η_i^{-1}.

Now consider $(4\pi\mathbb{Z})^d$ and let $\Omega_2 = \mathbb{R}^d/(4\pi\mathbb{Z})^d$, where $\tilde{p} : \mathbb{R}^d \to \Omega_2$ is the quotient mapping. Then Ω_2 is a 2^d-fold covering of Ω (2^d corresponds to all the possible ways in which paths in Ω_2 *turn* around the generators of the fundamental group of Ω). Let now $\sigma : \Omega_2 \to \Omega$ be the following map:

$$\sigma : x + (4\pi\mathbb{Z})^d \mapsto x + (2\pi\mathbb{Z})^d,$$

which is the operation of winding points in Ω depending on their value on the cover. By this map, if we turn around a generator of the fundamental group of Ω_2, the image turns twice around the corresponding generator of the fundamental group of Ω. Then the following diagram commutes

$$\begin{array}{ccc} & & \Omega_2 \\ & \nearrow & \downarrow \\ \mathbb{R}^d & \to & \Omega \end{array}$$

where the maps $\mathbb{R}^d \to \Omega$ and $\mathbb{R}^d \to \Omega_2$ indicate the quotient maps, and item *(iii)* of the previous list implies that Ω_2 is a covering space of each W_i, for $1 \le i \le n$. We now want to lift the flow that we have on Ω to a natural flow on Ω_2. The way to do it is the following: if $\tilde{\phi} = \tilde{p}(x)$, where x is an element of $\mathbb{R}^d$, then we define the flow $(t; \tilde{\phi})$ as

$$(t; \tilde{\phi}) = \tilde{p}(x + \omega t),$$

which is isomorphic to the irrational twist flow on $\Omega = \mathbb{R}^d/\mathbb{Z}^d$ having frequency vector $\bar{\omega} = \frac{\omega}{2}$.

The differential system (2.14), which is defined on the standard d-dimensional torus, now lifts to a differential system

$$x' = A(t; \tilde{\phi})x \tag{2.15}$$

on Ω_2, where the matrix defining the flow can be obtained by means of σ and $\tilde{A}$, $A = \tilde{A} \circ \sigma$. This means that for every $\tilde{\phi} \in \Omega_2$, satisfying that $\tilde{\phi} = \sigma(\phi)$ we have that

$$A(\tilde{\phi}) = \tilde{A}\left(\sigma(\tilde{\phi})\right) = \tilde{A}(\phi).$$

Now the LSPF on the bundle $\mathbb{R}^n \times \Omega_2$ induced by (2.15) has invariant subbundles

$$\tilde{V}_i = (\sigma \times Id)^{-1} (V_i)$$

for each $i = 1, \dots n$ (where, recall, V_i are the invariant subbundles associated to the original flow on $\mathbb{R}^n \times \Omega$). Due to their own definition, the sets

$$(\sigma \times Id)^{-1} (W_i) \subset \mathbb{R}^n \times \Omega_2$$

are also invariant sets for the flow on $\mathbb{R}^n \times \Omega_2$ induced by (2.15). In principle, it could happen that the pre-image of W_i under $\sigma \times Id$ is composed of several connected components. Let $\tilde{W}_i$ denote one of these connected components, for $i = 1, \dots, n$. Since Ω_2 is a covering space for each W_i, it follows that each $\tilde{W}_i$ is a 1-cover of Ω_2. That is, each $\tilde{W}_i$ can be represented as the graph of a function $f_i : \Omega_2 \to S^{n-1}$. The regularity of each f_i is C^α since, locally, we can write $f_i = \eta_i^{-1} \circ \sigma$.

Because of definition, for each $\tilde{\phi} \in \Omega_2$ and $i = 1, \dots, n$, $f_i(\tilde{\phi})$ is a unit vector in $\mathbb{R}^n$ and $\{f_1(\tilde{\phi}), \dots, f_n(\tilde{\phi})\}$ is a vector frame for $\mathbb{R}^n$. We now fix a standard basis $\{e_1, \dots, e_n\}$ for $\mathbb{R}^n$ and consider the projection

$$P : \Omega_2 \to L(\mathbb{R}^n)$$

defined by $P(\tilde{\phi})e_i = f_i(\tilde{\phi})$, for $i = 1, \dots, n$. The change of variables $x = P(t; \tilde{\phi})y$ is obviously a LP transformation (according to the definition in the first chapter), because it is invertible at each point and both P and P^{-1} are bounded on Ω_2. This transformation takes the system (2.15) to the form

$$y' = B(t; \tilde{\phi})y,$$

where the matrix $B(\tilde{\phi}) = \operatorname{diag}\left(\lambda_1(\tilde{\phi}), \dots, \lambda_d(\tilde{\phi})\right)$. Since P is of class C^α, each of the λ_i is also of class C^α. We apply the results on diagonal systems (and the Diophantine assumption on ω!) to deduce the existence of a transformation $Q(t; \tilde{\phi})$ that renders the system to a constant coefficient system with diagonal matrix. Finally, due to the construction of the different flows, the composition of transformations can be written as

$$x = \tilde{P}\left(\phi_1 + \bar{\omega}_1 t, \dots, \phi_d + \bar{\omega}_d t\right) z.$$

□

Chapter 3

Reducibility in Schrödinger equation with quasi-periodic potential

In this chapter we will study the reducibility problem in a special linear equation with quasi-periodic coefficients, namely

$$-\frac{d^2}{dt^2}x(t) + q(t)x(t) = 0$$

where $x(t) \in \mathbb{R}$ and $q(t) = Q(\omega t + \phi)$ is a quasi-periodic function. This equation is usually referred in the literature as the one-dimensional Schrödinger equation with a quasi-periodic potential. The spectral theory developed for second-order operators in Hilbert spaces, together with the tools introduced by ergodic theory, enables us to make a more precise description of the reducibility problem. All these theories have been studied in the last thirty years from a dynamical systems point of view and, especially, KAM techniques have proved to be very useful for this study.

This chapter is the longest in the present survey. This is due to the fact that Schrödinger equation with quasi-periodic potential is probably the most studied of all linear equations with quasi-periodic coefficients and a great number of tools have been developed to study it. Even if our motivation is the dynamical problem of reducibility, it is necessary to expose the functional analysis approach to the problem, because it provides a natural framework in which these equations take place.

3.1 Introduction. Useful transformations

Our aim in this chapter is to study Schrödinger equation with quasi-periodic potential

$$-x'' + q(t)x = 0, \tag{3.1}$$

where $x \in \mathbb{R}^d$, and q is a quasi-periodic function with frequency $\omega \in \mathbb{R}^d$ which, in the context of Schrödinger equation, is called the potential. This means that there exists a continuous function $Q : \mathbb{T}^d \to \mathbb{R}$, being $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$, such that $q(t) = Q(\omega t)$.

Related to equation (3.1), it will be useful to study the following family of equations

$$-x'' + q(t)x = \lambda x, \tag{3.2}$$

being λ a real (or sometimes complex) parameter which we shall denote as the spectral parameter or the energy.

Equation (3.1) appears in many branches of mathematics and physics. For instance, the Schrödinger equation with periodic potential arises in a natural way in the quantum theory of solids, to be more precise, in the quantum theory of crystals, for example metals. The ions forming a crystal lattice actually generate a periodic field and one can examine the motion of a free electron in this field. One can, in a number of cases, by introducing compensating additional terms to the ion potential, disregard the interaction of the free electrons. When more frequencies are added to this problem (coming from considering more general lattices or applying a field to periodic lattices), Schrödinger equation with quasi-periodic potential turns to be a useful model (see [5], [57], [3], [74], [75] and references therein).

A useful object to study the solutions of a linear system is the Wronskian. In our two-dimensional setting, the Wronskian of two differentiable complex functions u and v takes the form

$$W(u,v)(t) = u(t)v'(t) - u'(t)v(t).$$

It follows from Liouville theorem that the Wronskian of two solutions of equation (3.1) is a constant function which is nonzero if, and only if, the two solutions are linearly independent.

More general equations such as

$$y'' + a(t)y' + b(t)y = 0, \tag{3.3}$$

can be transformed to the form of equation (3.1) using the techniques (and therefore the conditions on the potential and the frequency) introduced in the first chapter ([28]).

Schrödinger equation with bounded potential has some interesting properties which will be useful in the sequel. One of the most important is the transformation of equation (3.1) to polar coordinates with the reduction imposed by the symmetries that exist in the problem. This also has to do with the linear character of the flow. As it will be necessary later on, we describe both the situation in the complex case (assume that Q is complex in equation (3.1) or simply that Q is real and λ is complex in equation (3.2)).

For each $\phi \in \mathbb{T}^d$, the equation

$$-x'' + Q(\omega t + \phi)x = 0 \tag{3.4}$$

is linear and the fundamental matrix solution $\Phi(t;\phi)$ (with $\Phi(0;\phi) = I$) maps complex lines (that is, 1-dimensional subspaces of $\mathbb{C}^2$) to complex lines. If l is a complex line in $\mathbb{C}^2$, let $l(t)$ be its evolution by the flow so that $l(t) = \Phi(t;\phi)l$. Letting $\mathbb{P}^1(\mathbb{C})$ be the usual space of all complex lines in $\mathbb{C}^2$, we define a flow Ψ on $\Sigma = \mathbb{P}^1(\mathbb{C}) \times \mathbb{T}^d$ as follows

$$\Psi(t;\phi) = (\omega t + \phi, l(t)), \text{ for } \phi \in \mathbb{T}^d,\ l \in \mathbb{P}^1(\mathbb{C}) \text{ and } t \in \mathbb{R}.$$

This description becomes easier in the case when Q is real. Let then $\mathbb{P}^1(\mathbb{R})$ be the space of real one-dimensional subspaces of $\mathbb{R}^2$, whose elements we shall call lines (opposed to complex lines), and $\Sigma_{\mathbb{R}} = \mathbb{P}^1(\mathbb{R}) \times \mathbb{T}^d$ the real bundle on which the equation (3.4) defines a flow. Then we can view $\mathbb{P}^1(\mathbb{R})$ as a subset of $\mathbb{P}^1(\mathbb{C})$ using the usual identification of the Riemann sphere $\mathbb{S}^2$ with $\mathbb{P}^1(\mathbb{C})$ and the identification of $\mathbb{P}^1(\mathbb{R})$ with $\mathbb{R} \cup \{\infty\} \subset \mathbb{P}^1(\mathbb{C})$. Thus, if we want to use coordinates for the real case, we can use this last remark, parameterizing $\mathbb{P}^1(\mathbb{R})$ with an angle φ between 0 and π. Once an initial condition is fixed, which amounts to an initial line (angle) in $\mathbb{P}^1(\mathbb{R})$, the evolution is given by the following equation

$$\varphi' = \cos^2\varphi - q(t)\sin^2\varphi. \tag{3.5}$$

Assuming the knowledge of the evolution of the line given by $\varphi(t)$, then the radius can be directly integrated as follows

$$r(t) = r(0)\exp\left(\frac{1}{2}\int_0^t (q(s)-1)\sin(2\varphi(s))ds\right). \tag{3.6}$$

If we want to stress the dependence on the special parameter λ in the case of equation (3.2) we will write $\varphi(\cdot;\lambda)$ and $r(\cdot;\lambda)$.

Before ending the introduction let us outline the contents of this chapter. In section 3.2 we sketch the spectral theory for Schrödinger equation with quasi-periodic potential. There are three steps in this approach: we first study the spectral theory for general self-adjoint operators on Hilbert spaces, we then pass to Schrödinger equation with a time-dependent potential and, finally, we make all the previous theory a bit more precise in the case of quasi-periodic potentials. In this case we can characterize the spectrum of Schrödinger equation according to a dynamical criterion, which uses the *dynamical* notion of exponential dichotomy that we saw in the previous chapter.

The close relation of the dynamical behaviour of the solutions of Schrödinger equation with quasi-periodic potential with the spectral analysis approach is made more evident in section 3.3, where some of the links between the properties of the rotation number and the spectrum of the Schrödinger operator are described. A study of the properties of the rotation number is important, because in section 3.4 it is used to control the imaginary part of the eigenvalues of the Floquet matrix, which is one of the important points in the KAM approach to reducibility. We also include some theory on the Lyapunov exponents for Schrödinger equation, showing its relation with the spectral objects defined in section 3.2.

Section 3.4 is the central section in the chapter, because we state a theorem by H. Eliasson, on the almost everywhere reducibility of Schrödinger equation with quasi-periodic potential on the assumption of analyticity, strong non-resonance and closeness to constant coefficients. Some previous reducibility results are also discussed. The steps of the proof of the main reducibility result are sketched, because these techniques have been extended to more general situations as we shall see in the following chapter.

Finally, we pay some attention to the converse results on reducibility for Schrödinger equation with quasi-periodic potential, that is, conditions under which a certain Schrödinger equation is not reducible to constant coefficients. We mention some results on this equation and on its discrete analog, especially on a example, called the *almost Mathieu operator*, for which one can describe the reducible and non-reducible zones.

3.2 Some spectral theory

3.2.1 Statement of the problem

The classical spectral theory of Schrödinger operators with bounded potential provides us with useful tools to study the ultimate behaviour of solutions of equation (3.1). As we are dealing with a linear case, the ultimate behaviour is quite definitive for the classification of these solutions, as not all kinds of behaviour are possible in a linear system.

This theory was first developed by H. Weyl in 1910. The idea is the following. Assume that $\Lambda \subset \mathbb{R}$ is a certain interval, and to fix ideas, take it $\Lambda = (-\infty, \infty)$. Then on the set $C_c^\infty(\Lambda)$ of infinitely differentiable functions with compact support on Λ we can consider the operator

$$Hf = H_0 f + Vf = -f'' + Vf \tag{3.7}$$

for a certain real function V and the related eigenvalue problem

$$Hf = \lambda f \tag{3.8}$$

for $\lambda \in \mathbb{C}$. As $C_c^\infty(\Lambda)$ is dense in $L^2(\Lambda)$ (the space of square integrable functions on Λ) it would be nice to define a natural and self-adjoint extension of the above operator on $L^2(\Lambda)$, because the

Hilbert space structure makes the eigenvalue problem (3.8) much clearer. However it is not clear a priori which conditions should be imposed on the quasi-periodic potential (or more generally bounded) so that such a natural extension exists (and has useful properties) and how to perform this extension. This is the purpose of the following section.

3.2.2 Some spectral theory of self-adjoint operators

In this subsection we present some of the basic objects and results in the theory of self-adjoint operators, which will be used in the sequel. We will use the references [68] (for the general theory) and [13] (more adapted to our purposes), where a proper exposition of this theory can be found (see also [16], [65] and [5]).

Domains, adjoints, resolvents and spectra

Let $\mathcal{H}$ be a separable Hilbert space. An *operator* H on $\mathcal{H}$ will be a linear map defined from a vector subspace $\mathcal{D}(H)$ of $\mathcal{H}$, which will be called the *domain* of H, to $\mathcal{H}$. We will assume that the operator is densely defined on $\mathcal{H}$; that is, we will assume that $\mathcal{D}(H) \subset \mathcal{H}$ is a dense subspace.

We will focus on the Hilbert space $\mathcal{H} = L^2(\Lambda)$ (with the usual Lebesgue measure) for a certain interval $\Lambda \subset \mathbb{R}$, bounded or not, and the operator H defined in (3.7). For this operator, which in principle is not defined on the whole Hilbert space, we will construct an extension to $L^2(\Lambda)$ suitable for our purposes. It is important to note that the initial operator is defined by both the action in (3.7) and the domain $\mathcal{D}(H)$.

The *graph* of an operator H as above will be the following subset of $\mathcal{H} \times \mathcal{H}$ (which is naturally a Hilbert space)

$$gr(H) = \{(f, Hf); f \in \mathcal{D}(H)\},$$

and the operator H is said to be *closed* if its graph $gr(H)$ is a closed subset of $\mathcal{H} \times \mathcal{H}$. Another operator H_1 on $\mathcal{H}$ is said to be an *extension* of H if $\mathcal{D}(H) \subset \mathcal{D}(H_1)$ and both operators coincide on $\mathcal{D}(H)$. An operator will be called *closable* if there exists at least one closed extension of this operator. If this is the case, we call its *closure* ($\bar{H}$ if the original operator is H) to the smallest of these extensions.

A very important object is the *adjoint*. The *adjoint* of an operator H is the operator H^* defined by the domain

$$\mathcal{D}(H^*) = \{f \in \mathcal{H};\ \text{there exists an element } g \in \mathcal{H} \text{ s.t. if } h \in \mathcal{D}(H), \langle Hh, f\rangle = \langle h, g\rangle\}$$

and

$$H^* f = g$$

if f and g are as in the definition of $\mathcal{D}(H^*)$. The adjoint of an operator H^*, assuming as customary that $\mathcal{D}(H)$ is dense, is well defined on $\mathcal{D}(H^*)$.

We will use the notation I for the *identity operator* on $\mathcal{H}$ and $L(\mathcal{H})$ for the Banach space of bounded operators on $\mathcal{H}$ furnished with the uniform norm. If the operator H is one-to-one, its *inverse* is the operator H^{-1} defined by the domain

$$\mathcal{D}(H^{-1}) = \{Hf; f \in \mathcal{D}(H)\}$$

and

$$H^{-1} g = f$$

if $Hf = g$. Note that H^{-1} is closed if H is so (because of the Closed Graph Theorem, [68]), but $\mathcal{D}(H^{-1})$ needs not to be dense.

We can naturally define the operator zH on $\mathcal{H}$ for any operator H on $\mathcal{H}$ and $z \in \mathbb{C}$ by the definitions $\mathcal{D}(zH) = \mathcal{D}(H)$ and $(zH)f = z(Hf)$ for all $f \in \mathcal{D}(H)$. It is also checked that $(zH)^* = \bar{z}H^*$. If we want to define the sum of two operators on $\mathcal{H}$ we must be a bit more careful, because it might happen that $\mathcal{D}(H) \cap \mathcal{D}(K) = 0$ even if $\mathcal{D}(H)$ and $\mathcal{D}(K)$ are the domains of two densely defined operators. Anyway, if we define $\mathcal{D}(H+K) = \mathcal{D}(H) \cap \mathcal{D}(K)$ and $(H+K)f = (Hf)+(Kf)$ we can speak of the operator $H + K$ even if it is not densely defined. We always have that $(H + K)^* = H^* + K^*$ if either H or K is bounded. As an important example we have that given an operator H and a complex number z, we can define the shifted operator $H_z = H - zI$ with domain $\mathcal{D}(H_z) = \mathcal{D}(H)$.

The previous example leads us to the core definitions of the spectral theory that we are interested in. The set of complex numbers z for which the operator $H - zI$ is one-to-one from $\mathcal{D}(H)$ onto $\mathcal{H}$ (in which case the operator $(H - zI)^{-1}$ is bounded by the Closed Graph Theorem, [68]) is denoted by $\rho(H)$ and it is called the *resolvent set* of H. In this case, the operator $R(z, H) = (H - zI)^{-1}$ is called the *resolvent operator* at z. The set $\rho(H) \subset \mathbb{C}$ is open and the map $z \in \rho(H) \mapsto R(z, H) \in L(\mathcal{H})$ is strongly analytic. Moreover, the following identity, called the *resolvent identity*, is true:

$$R(z_1, H) - R(z_2, H) = (z_1 - z_2)R(z_1, H)R(z_2, H), \tag{3.9}$$

for all z_1, z_2 in the resolvent set of H. We will come back to these notions later on, when focusing to our particular examples.

The complement $\mathbb{C} - \rho(H)$ of the resolvent set is called the *spectrum* of H, and will be denoted by $\sigma(H)$. The operator H is said to be *symmetric* if H^* is an extension of H (we shall denote this by $H \subset H^*$). This is equivalent to the following identity

$$\langle Hf, g\rangle = \langle f, Hg\rangle$$

for all f and g in $\mathcal{D}(H)$. The operator H is said to be *self-adjoint* if $H^* = H$, that is, if H is symmetric and $\mathcal{D}(H^*) = \mathcal{D}(H)$. A symmetric operator is always closable since $H \subset H^*$ and H^* is closed. Moreover, since H^* is a closed extension of H, the smallest of these, namely H^{**} has to be smaller than H^*. Hence we have the following properties

- $H \subset H^{**} \subset H^*$ whenever H is symmetric.
- $H = H^{**} \subset H^*$ whenever H is closed and symmetric.
- $H = H^{**} = H^*$ whenever H is self-adjoint.

A symmetric operator is said to be *essentially self-adjoint* if its closure is self-adjoint. Hence, it has a unique self-adjoint extension. In fact, the converse statement is also true. We can say even more, H is essentially self-adjoint if and only if H^* is symmetric in which case one has $\bar{H} = H^*$.

Up to now, the real or complex character of z has been irrelevant. However, the following inequality, easy to check, imposes important restrictions for symmetric operators H

$$\|(H - zI)\, f\| \geq |\mathrm{Im}\, z|\|f\|, \qquad z \in \mathbb{C}, f \in \mathcal{D}(H).$$

Therefore, the range of $H - zI$ is closed whenever H is closed and z is not real.

We end this section with a criterion for a real number to be in the spectrum of a self-adjoint operator.

Proposition 3.2.1 (Weyl's Criterion, [68]) *A number $\lambda \in \mathbb{R}$ is in the spectrum of the self-adjoint operator H whenever there exists a sequence $\{f_n; n \geq 1\}$ of unit vectors in the domain of H satisfying*

$$\lim_{n\to\infty} \|(H - \lambda I)\, f_n\| = 0.$$

Resolutions of the identity and the spectral theorem

In this subsection $\mathcal{H}$ will denote a fixed (separable) complex Hilbert space, $(\Omega, \mathcal{B})$ a fixed measurable space and we will talk about a projection in $\mathcal{H}$ to mean an orthogonal (i.e. self-adjoint) projection in $\mathcal{H}$.

Definition 3.2.2 *A function E on $\mathcal{B}$ with values in the space of projections in H is called a* resolution of the identity *(resp.* subresolution of the identity*) of $\mathcal{H}$ on $(\Omega, \mathcal{B})$ if:*

(i) $E(\Omega) = I$ *(resp. (i') $E(\emptyset) = 0$).*

(ii) $E(\cup_{n\geq 1} A_n) = \sum_{n\geq 1} E(A_n)$*, whenever $\{A_n; n \geq 1\}$ is a sequence in $\mathcal{B}$ whose elements are disjoint.*

The convergence in item *(ii)* has to be understood in the sense of the strong convergence of operators. This means that for each fixed $f \in \mathcal{H}$, the series $\sum_{n\geq 1} E(A_n)f$ converges in $\mathcal{H}$ to $E(\cup_{n\geq 1} A_n)f$. Hence for each fixed $f \in \mathcal{H}$, the function $A \in \mathcal{B} \mapsto E(A)f \in \mathcal{H}$ is countably additive, and this is usually called a *countably additive measure.* Since the norm of a projection is either zero or one, the series in item (*ii*) cannot converge in the operator norm of $L(\mathcal{H})$ unless all but finitely many $E(A_n)$ are zero. Consequently, except for trivial cases, E is not countably additive as a function in $L(\mathcal{H})$, and therefore, E is not a $L(\mathcal{H})$-valued measure. Nevertheless one has the following proposition

Proposition 3.2.3 ([13]) *A projection valued function E on $\mathcal{B}$ is a* resolution of the identity *(resp.* subresolution of the identity*) of $\mathcal{H}$ on $(\Omega, \mathcal{B})$ if, and only if:*

(i) $E(\Omega) = I$ *(resp. (i') $E(\emptyset) = 0$).*

(ii) For all f and g in $\mathcal{H}$ the complex function $E_{f,g}$ defined on $\mathcal{B}$ by $E_{f,g}(A) = \langle E(A)f, g\rangle$ is a complex measure (that is, a complex σ-additive set function on $\mathcal{B}$).

We now go to the main theorem in this subsection. Before we make some remarks in order to introduce it better.

Let $\phi = \sum_{1\leq j\leq n} a_j \mathbf{1}_{A_j}$ a simple function, with $a_j \in \mathbb{C}$ and $A_j \in \mathcal{B}$ for $j = 1, \dots, n$. If $g \in \mathcal{H}$, we set

$$\int_\Omega \phi(w)E(dw)g = \sum_{1\leq j\leq n} a_j E(A_j)g.$$

Note that $\int_\Omega \phi(w)E(dw)g$ is the only element $f \in \mathcal{H}$ satisfying that, for all $h \in \mathcal{H}$,

$$\langle f, h\rangle = \int_\Omega \phi(w)E_{g,h}(dw)$$

where the integral on the right side is an integral with respect to a scalar measure and therefore understood in the usual sense. Moreover, using the definition, we have the following property for the norms

$$\left\| \int_\Omega \phi(w)E(dw)g \right\|^2 = \int_\Omega |\phi(w)|^2 E_{g,g}(dw).$$

So, if $g \in \mathcal{H}$ is fixed, for each measurable function ϕ on Ω which is square integrable with respect to the non-negative measure $E_{g,g}$ we define the integral $\int_\Omega \phi(w)E(dw)g$ in the following way. First we approximate the function ϕ by a sequence $\{\phi_n; n \geq 1\}$ in $L^2(\Omega, E_{g,g}(dw))$. Then we notice that

$$\left\| \int_\Omega \phi_n(w)E(dw)g - \int_\Omega \phi_m(w)E(dw)g \right\|^2 = \int_\Omega |\phi_n(w) - \phi_m(w)|^2 E_{g,g}(dw)$$

and finally we define the integral

$$\int_{\Omega} \phi(w)E(dw)g$$

as the limit in $\mathcal{H}$ of the Cauchy sequence $\{\int_{\Omega} \phi_n(w)E(dw)g; n \geq 1\}$. That this is well-defined and the resulting properties are the contents of the theorem:

Theorem 3.2.4 ([13]) *Let E be a resolution of the identity of $\mathcal{H}$ on $(\Omega, \mathcal{B})$. Then*

(i) For each measurable function $\phi : \Omega \to \mathbb{C}$, the subspace

$$\mathcal{D}_{\phi} = \left\{ f \in H; \int_{\Omega} |\phi(w)|^2 E_{f,f}(dw) < \infty \right\}$$

is dense in $\mathcal{H}$ and there is a unique operator $\phi(E)$ with domain $\mathcal{D}_{\phi}$ such that

$$\langle \phi(E)f, g \rangle = \int_{\Omega} \phi(\omega)E_{f,g}(dw)$$

for all $f \in \mathcal{D}_{\phi}$ and $g \in \mathcal{H}$. Moreover, for each $f \in \mathcal{D}_{\phi}$ we have

$$\|\phi(E)f\|^2 = \int_{\Omega} |\phi(w)|^2 E_{f,f}(dw)$$

(ii) If ϕ and ψ are measurable functions on Ω, we have:

$$\mathcal{D}(\phi(E)\psi(E)) = \mathcal{D}_{\psi} \cap \mathcal{D}_{\phi} \text{ and } \phi(E)\psi(E) \subset (\phi\psi)(E)$$

and consequently $\phi(E)\psi(E) = (\phi\psi)(E)$ if, and only if, $\mathcal{D}_{\phi\psi} = \mathcal{D}_{\psi}$.

(iii) For each measurable function ϕ on Ω we have :

$$\phi(E)^* = \bar{\phi}(E) \text{ and } \phi(E)\phi(E)^* = |\phi|^2(E) = \phi(E)^*\phi(E)$$

We try to make this theorem clearer by means of the following example.

Example 3.2.5 *Let μ be a non-negative measure on $(\Omega, \mathcal{B})$ such that the Hilbert space $L^2(\Omega, \mathcal{B}, \mu)$ is separable and let E be the resolution of the identity given by $E(A)f = \mathbf{1}_A f$ for all $f \in \mathcal{H}$ and $A \in \mathcal{B}$. Let ϕ be a measurable function on $(\Omega, \mathcal{B})$. We want to find $\phi(E)$ in this particular example. We have that*

$$\langle \phi(E)f, g \rangle = \int_{\Omega} \phi(w)E_{f,g}(dw)$$

and that, for all $f, g \in \mathcal{H}$, $E_{f,g}$ is the measure on $(\Omega, \mathcal{B})$ defined by

$$E_{f,g}(A) = \langle E(A)f, g \rangle = \langle \mathbf{1}_A f, g \rangle = \int_A f\bar{g}d\mu$$

thus we have that

$$\phi(E)f = \phi f = M_{\phi} f$$

where M_{ϕ} is the multiplication operator on the domain $\mathcal{D}_{\phi}$ given by

$$\mathcal{D}_{\phi} = \left\{ f \in \mathcal{H}; \int_{\Omega} |\phi(w)|^2 |f(w)|^2 \mu(dw) < \infty \right\}.$$

This example shows that, when Ω is a Borel subset of $\mathbb{C}$ and $\mathcal{B}$ is the set of Borel subsets of Ω, then we can view the operator $\phi(E)$ as the function ϕ of the operator $H = \phi_0(E)$, being the function $\phi_0(w) = w$, for all $w \in \Omega$. In some sense, theorem 3.2.4 provides us with a functional calculus for the self-adjoint operators of the form $H = \phi_0(E)$. We shall later see that every self-adjoint operator on the real line is of this form.

Numerous concepts from classical scalar valued measures remain meaningful for projection valued measures. For example, if E is a resolution of the identity of H, we define the *E-essential range* of a measurable function ϕ as the complement of the largest open subset $U \subset \mathbb{C}$ which satisfies that $E(\phi^{-1}(U)) = 0$. The function ϕ is said to be *essentially bounded* if its E-essential range is bounded, the *E-essential supremum* of ϕ being defined as the supremum of the $|z|$ for z in the E-essential range. It is verified that the spectrum of the operator $\phi(E)$, $\sigma(\phi(E))$, for a measurable function ϕ and a resolution of the identity, is nothing but the E-essential range of ϕ.

Definition 3.2.6 *If μ is a non-negative σ-finite measure on $(\Omega, \mathcal{B})$, and if E is a subresolution of the identity of $\mathcal{H}$ on $(\Omega, \mathcal{B})$, by the μ-essential support of E we will refer to any set $A \in \mathcal{B}$ satisfying the following property: There exists $A' \in \mathcal{B}$ with $\mu(A') = 0$ and $E((A \cup A')^c) = 0$ such that for all $A'' \in \mathcal{B}$, we have that $E(A'' \cap A) = 0$ if, and only if, $\mu(A \cap A'') = 0$.*

Note that all the μ-essential supports differ only by μ-negligible sets so we will talk of *the* μ-essential support, even if the latter is not uniquely defined. Note also that this support does not change when we replace μ by an equivalent measure, finite for example. Finally we will simply talk about the essential support, without even mentioning μ, whenever Ω is the real line and μ is the Lebesgue measure.

We now want to further explore the notion of the resolutions of the identity by using the properties of scalar valued measures. More concretely, we will try to find the right analogy of the Lebesgue decomposition of complex-valued measures in the case of resolutions of the identity. We begin with a useful remark.

Let E_1 and E_1 be two subresolutions of the identity of $\mathcal{H}$ on $(\Omega, \mathcal{B})$, and let us assume that $\mathcal{H}$ is the orthogonal direct sum of $\mathcal{H}_1 = E_1(\Omega)\mathcal{H}$ and $\mathcal{H}_2 = E_2(\Omega)\mathcal{H}$. Then the function E defined by $E(A) = E_1(A) + E_2(A)$ for all $A \in \mathcal{B}$ is a resolution of the identity of $\mathcal{H}$ on $(\Omega, \mathcal{B})$ and by theorem 3.2.4, for each measurable function ϕ on Ω, and for $j = 1, 2$ we have

$$E_j(\Omega)\mathcal{D}(\phi(E)) \subset \mathcal{D}(\phi(E)) \text{ and } \phi(E)\mathcal{H}_j \subset \mathcal{H}_j,$$

which is another way of saying that the decomposition reduces the operator, since $\phi(E)$ commutes with the projections $E_j(\Omega)$ on the spaces $\mathcal{H}_j$ and

$$\mathcal{D}(\phi(E_j)) = \mathcal{D}(\phi(E)) \cap \mathcal{H}_j \text{ and } \phi(E_j)f = \phi(E)f \text{ if } f \in \mathcal{H}_j.$$

Note that this tells us that the components of $\phi(E)$ on $\mathcal{H}_1$ and $\mathcal{H}_2$ are the operators $\phi(E_1)$ and $\phi(E_2)$. This idea of decomposition will be applied in what follows.

Recall that each complex measure μ on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ has a decomposition

$$\mu = \mu_{pp} + \mu_{ac} + \mu_{sc},$$

being

- μ_{pp} a *pure point measure* (i.e. a sum of Dirac point masses).
- μ_{ac} an *absolutely continuous measure* with respect to the Lebesgue measure.

- μ_{sc} a *singular continuous measure* (*continuous* means that it has no point masses and *singular* means that it is carried by a set of zero Lebesgue measure).

This Lebesgue decomposition of the scalar measures μ suggests that, given a resolution of the identity of H on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$, say E, we may set

$$\mathcal{H}_{pp}(E) = \mathcal{H}_{pp} = \{f \in \mathcal{H}; E_{f,f} \text{ is pure point }\}.$$

$$\mathcal{H}_{ac}(E) = \mathcal{H}_{ac} = \{f \in \mathcal{H}; E_{f,f} \text{ is absolutely continuous }\}.$$

$$\mathcal{H}_{sc}(E) = \mathcal{H}_{sc} = \{f \in \mathcal{H}; E_{f,f} \text{ is singular continuous }\}.$$

These three subsets are mutually orthogonal closed subspaces that generate the whole Hilbert space $\mathcal{H}$,

$$\mathcal{H} = \mathcal{H}_{pp} \oplus \mathcal{H}_{ac} \oplus \mathcal{H}_{sc}.$$

In fact we have that if $f \in \mathcal{H}$, being $f = f_1 + f_2 + f_3$ is the decomposition of f in the above direct sum, and if $E_{f,f} = \mu_1 + \mu_2 + \mu_3$ is the Lebesgue decomposition of the measure $E_{f,f}$, then necessarily $E_{f_j,f_j} = \mu_j$, for $j = 1, 2, 3$.

If α stands for either *pp*, *ac*, or *sc*, we denote by π_α the projection of $\mathcal{H}$ onto $\mathcal{H}_\alpha$ and if, for f and g in $\mathcal{H}$ and $A \in \mathcal{B}_{\mathbb{R}}$, we set:

$$\langle E_\alpha(A)f, g\rangle = \left(E_{f,g}\right)_\alpha(A),$$

where the left hand side is defined as the right hand side, then one readily checks that collections $\{E_\alpha(A); A \in \mathcal{B}_{\mathbb{R}}\}$ are resolutions of the identity of $E_\alpha(\mathbb{R})\mathcal{H} = \mathcal{H}_\alpha$, for $\alpha = pp, ac, sc$ respectively and that

$$E = E_{pp} + E_{ac} + E_{sc}.$$

E_{pp} (resp. E_{ac}, E_{sc}) is a pure point (resp. absolutely continuous, singular continuous) subresolution of the identity of $\mathcal{H}$ in the sense that, as f varies in $\mathcal{H}$, all the scalar measures $\langle E_{pp}(\cdot)f, f\rangle$ (resp. $\langle E_{ac}(\cdot)f, f\rangle$ and $\langle E_{sc}(\cdot)f, f\rangle$) are pure point (resp. absolutely continuous, singular continuous). For this reason, the above decomposition is called the *Lebesgue decomposition of the resolution of the identity E*.

Consequently, for each measurable function ϕ, the operator $\phi(E)$ can be decomposed in three parts, $\phi(E_{pp})$, $\phi(E_{ac})$ and $\phi(E_{sc})$. In the particular case when $H = \phi_0(E)$, we have three self-adjoint operators $H_{pp} = \phi_0(E_{pp})$, $H_{ac} = \phi_0(E_{ac})$, and $H_{sc} = \phi_0(E_{sc})$ which are called the *pure point part*, *absolutely continuous part* and *singular continuous part* of the operator H. Their spectra are denoted by $\sigma_{pp}(H)$, $\sigma_{ac}(H)$ and $\sigma_{sc}(H)$, and are called the *pure point spectrum*, *absolutely continuous spectrum* and the *singular continuous spectrum* of H. Notice that

$$\sigma(H) = \sigma_{pp}(H) \cup \sigma_{ac}(H) \cup \sigma_{sc}(H),$$

but these sets need not to be disjoint. Sometimes $\sigma_c(H) = \sigma_{ac}(h) \cup \sigma_{sc}(H)$ will be referred as the *continuous part of the spectrum.*

Remark 3.2.7 *In many texts $\sigma_{pp}(H)$ stands for the set of eigenvalues of H, while for us it actually denotes the closure of this set. The problem with defining σ_{pp} as the set of eigenvalues $eigen(H)$ is that it is not true in general that $eigen(H) \cup \sigma_c(H) = \sigma(H)$.*

Finally, we introduce another distinction between the elements of the spectrum of an operator H. The *essential spectrum* of an operator H, $\sigma_{ess}(H)$ is defined as the (closed) subset of the spectrum whose elements are the real numbers λ such for which the rank of the spectral projection $E((\lambda-\epsilon,\lambda+\epsilon))$ is infinite for all $\epsilon>0$. The complementary of this set in the spectrum, which we shall call the *discrete spectrum*, will be denoted by $\sigma_{disc}(H)$

Up to now we have seen that we could associate a self-adjoint operator $\phi_0(E)$ to each resolution of the identity E of $\mathcal{H}$ on $(\mathbb{R},\mathcal{B}_{\mathbb{R}})$. Now we seek for a converse of this result. It goes under the name of the *spectral theorem for self-adjoint operators.*

Theorem 3.2.8 (Spectral Theorem for self-adjoint operators, [13]) *A densely defined operator H on $\mathcal{H}$ is self-adjoint if, and only if, there exists a resolution of the identity of $\mathcal{H}$ on $(\mathbb{R},\mathcal{B}_{\mathbb{R}})$, say E, such that $H=\phi_0(E)$.*

Proof: We have already proved one implication. The proof for the other one goes as follows. For each $f\in\mathcal{H}$, the function

$$\Phi_f(z)=-\langle (H-zI)^{-1}f,f\rangle$$

is defined and analytic outside the spectrum of H (which is a closed subset of the real line, due to the self-adjointness of H), and in particular it is analytic on Π_+, the upper open complex half-plane. The resolvent identity gives

$$-2\text{Im } z\left\|(H-zI)^{-1}f\right\|^2=2i\text{Im }\langle (H-zI)^{-1}f,f\rangle$$

which shows that the imaginary part of Φ_f is non-negative in the upper half-plane and, consequently, it is a Herglotz function. The representation theorem for these functions gives the existence of a finite non-negative measure μ_f on $\mathbb{R}$ such that

$$\langle (H-zI)^{-1}f,f\rangle=\int_{\mathbb{R}}\frac{1}{\lambda-z}d\mu_f(\lambda),\quad z\in\Pi_+.$$

Now, for f and g in $\mathcal{H}$, we set:

$$\mu_{f,g}=\frac{1}{4}\left(\mu_{f+g}-\mu_{f-g}+i\left(\mu_{f+ig}-\mu_{f-ig}\right)\right)$$

and we get

$$\langle (H-zI)^{-1}f,g\rangle=\int_{\mathbb{R}}\frac{1}{\lambda-z}d\mu_{f,g}(\lambda),\quad z\in\Pi_+.$$

Now the uniqueness part of the representation theorem for Herglotz functions gives that for each Borel set A in $\mathbb{R}$ the function $(f,g)\mapsto\mu_{f,g}(A)$ is linear in f and anti-linear in g, and therefore there is a linear map $E(A)$ from $\mathcal{H}$ into itself such that $\langle E(A)f,g\rangle=\mu_{f,g}(A)$ for all f and g in $\mathcal{H}$. The map E defined in this way satisfies that $E(A)$ is an orthogonal projection and that the collection $\{E(A);A\in\mathcal{B}_{\mathbb{R}}\}$ is a resolution of the identity of $\mathcal{H}$ such that $\phi_0(E)=H$. □

3.2.3 Some Spectral theory of 1D Schrödinger operators

We shall now apply the definition and results of the previous section (and introduce some more specific ones) to the one-dimensional Schrödinger equation, which we recall

$$-x''+V(t)x=0.$$

In our case the potential function is $V(t) = Q(\omega t + \phi)$, but the discussion in this subsection holds for much more general functions. We reserve, thus, the notation q for a quasi-periodic potential and V for a general one-dimensional potential. Then the operator

$$Hf = H_0 f + Vf = -f'' + Vf$$

can be defined on $C_c^\infty(\mathbb{R})$ by means of the usual derivative. This operator is clearly symmetric. Indeed,

$$\langle Hf, g\rangle = \int_{\mathbb{R}} \left(-f''(t) + V(t)f(t)\right) g(t)dt = -\int_{\mathbb{R}} f''(t)g(t)dt + \int_{\mathbb{R}} V(t)f(t)g(t)dt.$$

On the other hand

$$\int_{\mathbb{R}} \left(f''(t)g(t) - f(t)g''(t)\right) = \int_{\mathbb{R}} \left(f'g - fg'\right)' dt = 0,$$

because both f and g are of compact support.

Here we have chosen $\mathcal{H}$ to be $L^2(\mathbb{R}, dt)$, but this choice is by no means irrelevant. Indeed, the existence of self-adjoint extensions and the spectrum depend strongly on the Hilbert space chosen. More precisely it depends on the existence and properties of the L^2-solutions of the eigenvalue equation

$$-f'' + V(t)f = \lambda f \tag{3.10}$$

with $\lambda \in \mathbb{C}$. We devote this section to investigate the existence of these solutions, a knowledge that will supply us with the essential self-adjointness of the Schrödinger operator with bounded potential.

First properties. The limit point and the limit circle case. Essential self-adjointness

The first important result will be the following

Lemma 3.2.9 ([13], [15]) *If Im $\lambda \neq 0$ then at least one solution of the eigenvalue equation (3.10) is square integrable near $+\infty$ and at least one solution is square integrable near $-\infty$. Moreover, if for some $\lambda \in \mathbb{C}$ two linearly independent solutions of (3.10) are square integrable near $+\infty$ (resp. $-\infty$), then for all $\lambda \in \mathbb{C}$, all the solutions of (3.10) are square integrable near $+\infty$ (resp. $-\infty$)*

Proof: The proof of the first statement will be proved below, in the discussion of Weyl's m-functions. Let us now prove the second statement. Let $\lambda_0 \in \mathbb{C}$ and let φ_1 and φ_2 be two linearly independent solutions of the eigenvalue equation (3.10) with $\lambda = \lambda_0$ which are square integrable near $+\infty$, and assume (scaling if necessary) that their Wronskian is 1. If φ is a solution of (3.10) with $\lambda = \lambda_1$, for some $\lambda_1 \in \mathbb{C}$, then we have

$$\left(-\frac{d^2}{dt^2} + V - \lambda_0\right)\varphi = (\lambda_1 - \lambda_0)\varphi$$

and thus,

$$\varphi(t) = c_1\varphi_1(t) + c_2\varphi_2(t) + (\lambda_1 - \lambda_0)\int_c^t \left(\varphi_1(t)\varphi_2(s) - \varphi_1(s)\varphi_2(t)\right)\varphi(s)ds$$

for some constants c_1 and c_2. Using the notation

$$\|f\|_{[a,b]} = \left(\int_a^b |f(s)|^2 ds \right)^{\frac{1}{2}}$$

and the Schwartz inequality one gets

$$|\varphi(t)| \leq |c_1||\varphi_1(t)| + |c_2||\varphi_2(t)| + M|\lambda_1 - \lambda_2| \left(|\varphi_1(t)| + |\varphi_2(t)|\right) \|\varphi\|_{[c,t]},$$

provided that M is chosen so that $\|\varphi_1\|_{[c,+\infty)}, \|\varphi_2\|_{[c,+\infty)} < M$. Hence we have

$$\|\varphi\|_{[c,t]} \leq \left(|c_1| + |c_2|\right) M + 2M^2|\lambda_1 - \lambda_0| \|\varphi\|_{[c,t]},$$

from which

$$\|\varphi\|_{[c,t]} \leq 2\left(|c_1| + |c_2|\right) M,$$

provided that c is chosen large enough so that $4M^2|\lambda_1 - \lambda_0| < 1$. This completes the proof, leaving the first part for later on. □

This result leads us to an important definition:

Definition 3.2.10 *The potential function V is said to be in the* limit circle case at $+\infty$ (resp. $-\infty$) *if for some (and hence for all) $\lambda \in \mathbb{C}$, all the solutions of the eigenvalue equation (3.10) are square integrable near $+\infty$ (resp. $-\infty$). Otherwise it is said to be in the* limit point case*.*

In order to prove the essential self-adjointness we will do the following assumption, which is typical in the context of Schrödinger operators with bounded potential. We will assume that V is a measurable real function such that there exist constants $a > 0$ and $C > 0$ satisfying

$$V(t) \geq -a(t^2 + 1), \quad |t| > C. \tag{3.11}$$

This condition is satisfied when $V(t)$ is bounded, which is the case if $V(t) = Q(\omega t + \phi)$, being Q continuous on $\mathbb{T}^d$. Assuming (3.11) we can prove the following results which will lead us to the essential self-adjointness of Schrödinger operators with potentials satisfying (3.11).

Lemma 3.2.11 ([13], [15]) *Assumption (3.11) implies that for any $\lambda \in \mathbb{C}$, any solution ψ of the eigenvalue equation (3.10) which is square integrable near $+\infty$ (resp. $-\infty$) necessarily satisfies:*

$$\int^{+\infty} \frac{|\psi'(t)|^2}{t^2} dt < \infty \text{ (resp. } \int_{-\infty} \frac{|\psi'(t)|^2}{t^2} dt < \infty \text{)}$$

Proof: Our assumptions imply that, for any $t_1 < t$, large enough, there exists a positive constant c such that

$$\int_{t_1}^t \frac{\psi''(s)\psi(s)}{s^2} ds = \int_{t_1}^t \frac{(V(s) - \lambda)\psi(s)\psi(s)}{s^2} ds \geq -c \int_{t_1}^{+\infty} |\psi(s)|^2 ds$$

Integrating by parts the left part of the equation we get

$$-\frac{\psi'(t)\psi(t)}{t^2} + \int_{t_1}^t \frac{\psi'(s)}{s^2} ds - 2\int_{t_1}^t \frac{\psi'(s)\psi(s)}{s^3} ds \leq c'$$

for some constant $c' > 0$. Using Schwartz inequality to control the right most term of the previous equation we get

$$-\frac{\psi'(t)\psi(t)}{t^2} + h(t) - c_2\sqrt{h(t)} \leq c_1$$

if we set

$$h(t) = \int_{t_1}^{t} \frac{|\psi'(s)|^2}{s^2} ds$$

and being c_1 and c_2 two positive constants. We now conclude that $h(t)$ cannot converge to $+\infty$ when $t \to +\infty$, because this would imply that $\psi'(t)\psi(t) \geq t^2 h(t)/2$ for t large enough. The latter would mean that $\psi(t)$ and $\psi'(t)$ have the same sign for t large, and this is a contradiction with the square integrability of ψ. $\square$.

As an important corollary we have

Corollary 3.2.12 *Assumption (3.11) implies that for any $\lambda \in \mathbb{C}$ one cannot have two linearly independent solutions which are both square integrable near $+\infty$ (resp. $-\infty$).*

Proof: Let φ and ψ be linearly independent solutions of (3.10). We assume that their Wronskian is 1. Then

$$\frac{1}{t} = \varphi(t)\frac{\psi'(t)}{t} - \psi(t)\frac{\varphi'(t)}{t} \tag{3.12}$$

and if we assume that φ and ψ are linearly independent solutions of (3.10) both square integrable at $+\infty$, then this is a contradiction, because of the lemma, that implies that the right hand side (and therefore also the left one) of equation (3.12) is integrable by Schwartz inequality. This is a contradiction, since $1/t$ is not integrable around $+\infty$. $\square$

This last corollary directly implies the essential self-adjointness of the operator H either in the whole or in the half line (see [13]):

Theorem 3.2.13 ([13]) *Assumption (3.11) implies that the operator H is essentially self-adjoint on $C_c^\infty(\mathbb{R})$.*

It will happen quite often that one solution of the eigenvalue equation decays exponentially at $+\infty$ or $-\infty$. This is typically the case when the energy λ is in the resolvent set of the operator H. In any case one can use the Wronskian to show that every other linearly independent solution has to explode exponentially. This corresponds to the already met notion of exponential dichotomy which we saw in past chapters. It can be useful to have a more general notion

Definition 3.2.14 *A non-zero solution ϕ of the eigenvalue equation $Hf = \lambda f$ is said to be* subordinate at $+\infty$ *if, and only if, any solution ψ which is linearly independent of ϕ satisfies*

$$\lim_{t\to+\infty} \frac{\|\phi\|_{[0,t]}}{\|\psi\|_{[0,t]}} = 0$$

and a notion of subordinacy of a solution at $-\infty$ is similarly defined.

Green's and Weyl's m functions

This subsection is devoted to the introduction of two objects, the so-called Weyl's m functions (also called Weyl-Titchmarsch) and Green's functions for the Schrödinger operator H on a half axis or on the whole real line $\mathbb{R}$.

Recall that we still must prove that if $\operatorname{Im} \lambda \neq 0$, then the equation

$$-x'' + (V(t) - \lambda)x = 0 \tag{3.13}$$

has one solution which is in $L^2(0, +\infty)$ and another solution which is in $L^2(-\infty, 0)$. The arguments here are taken from [15]. Let φ and ψ be two solutions of (3.10) which satisfy that

$$\varphi(0, \lambda) = \sin \alpha, \qquad \varphi'(0, \lambda) = -\cos \alpha \tag{3.14}$$

$$\psi(0, \lambda) = \cos \alpha, \qquad \psi'(0, \lambda) = \sin \alpha, \tag{3.15}$$

where $0 \leq \alpha < \pi$. Then, clearly, ψ and φ are two linearly independent solutions with Wronskian one. Moreover, due to the theory of dependence on initial conditions, they are entire in the parameter λ and continuous in the couple (t, λ). These solutions at zero satisfy

$$\cos \alpha \varphi(0, \lambda) + \sin \alpha \varphi'(0, \lambda) = 0$$

$$\sin \alpha \psi(0, \lambda) - \cos \alpha \psi'(0, \lambda) = 0.$$

We would like to see how other solutions of this system are expressed in terms of these two solutions. This is how Weyl's m-functions appear. Let's be more precise. Let χ be another solution of equation (3.13). Then, unless χ is identically ψ, it is, up to a multiple constant, of the form

$$\chi = \varphi + m\psi,$$

for some complex constant m which obviously depends on λ but not on t. Instead of considering our problem on the whole half-line $(0, +\infty)$ (resp. $(-\infty, 0)$) or even in $\mathbb{R}$, we will try to understand it first in a bounded sub-interval. To this end we fix an element $b \in (0, +\infty)$, we consider the boundary condition

$$\cos \beta \chi(b, \lambda) + \sin \beta \chi'(b, \lambda) = 0 \tag{3.16}$$

and ask what must m be like in order that the previous equation is satisfied, for a fixed $0 \leq \beta < \pi$. Clearly m must satisfy that

$$m = -\frac{\cot \beta \varphi(b, \lambda) + \varphi'(b, \lambda)}{\cot \beta \psi(b, \lambda) + \psi'(b, \lambda)}.$$

As λ, b and β vary, m becomes a function of all these arguments, $m = m(\lambda, b, \beta)$, and since φ, φ', ψ and ψ' are entire on the parameter λ, it follows that m is meromorphic in λ and real for real λ. This is reminiscent of homographic transformations in the complex plane. To see this clearer, we write $z = \cot \beta$ and, if (λ, b) are kept fixed, the previous equation for m becomes

$$m = -\frac{Az + B}{Cz + D} \tag{3.17}$$

where the constants (with obvious definition) are fixed. If z varies over the real line this means (undoing the change for β) that β varies from 0 to π. From the properties of equation (3.17), the real axis of the complex z-plane has as its image a circle C_b in the complex m-plane. This means that a solution χ will satisfy the additional boundary condition if, and only if, m lies on the circle

C_b, a circle which can be expressed by means of the constants A, B, C and D. Indeed, by simple transformations, the center of this circle is

$$\tilde{m}_b = \frac{A\bar{D} - B\bar{C}}{D\bar{C} - C\bar{D}}$$

and the radius is

$$r_b = \frac{|AD - BC|}{|D\bar{C} - C\bar{D}|}.$$

Undoing the changes for the constants A, B, C and D, it is readily seen that the equation of C_b can also be written as

$$W(\chi, \bar{\chi})(b) = 0,$$

and that the center and radius can also be expressed as follows

$$\tilde{m}_b = -\frac{W(\varphi, \bar{\psi})(b)}{W(\psi, \bar{\psi})(b)} \qquad r_b = \frac{1}{|W(\psi, \bar{\psi})(b)|}.$$

Moreover, the interior of the circle C_b in the complex m-plane is given by

$$\frac{W(\chi, \bar{\chi})(b)}{W(\psi, \bar{\psi})(b)} < 0.$$

Now we recall Green's formula, namely,

$$\int_{t_1}^{t_2} \left((Hf)\bar{g} - f(\overline{Hg})\right) dt = (W(f, \bar{g})(t_2) - W(f, \bar{g})(t_1))$$

(this is the reason for H being defined as $-d^2/dt^2 + V(t)$ instead of $d^2/dt^2 + V(t)$) from which we can conclude that

$$W(\psi, \bar{\psi})(b) = 2i(\text{Im } \lambda) \int_0^b |\psi|^2 dt$$

and

$$W(\chi, \bar{\chi})(b) = 2i(\text{Im } \lambda) \int_0^b |\chi|^2 dt + W(\chi, \bar{\chi})(0).$$

Since by definition of m, $W(\chi, \chi)(0) = -2i\text{Im } m$, the last equation reads

$$W(\chi, \bar{\chi})(b) = 2i(\text{Im } \lambda) \int_0^b |\chi|^2 dt + 2i\text{Im } m$$

and the equation for the interior of the circle becomes

$$\int_0^b |\chi|^2 dt < \frac{\text{Im } m}{\text{Im } \lambda},$$

provided $\text{Im } \lambda \neq 0$. Therefore points m are on C_b if, and only if,

$$\int_0^b |\chi|^2 dt = \frac{\text{Im } m}{\text{Im } \lambda},$$

and under the same hypothesis the radius r_b is given by

$$\frac{1}{r_b} = 2\text{Im } \lambda \int_0^b |\psi|^2 dt. \tag{3.18}$$

Now let $0 < a < b < +\infty$. Then if m is inside or on C_b

$$\int_0^a |\chi|^2 dt < \int_0^b |\chi|^2 \leq \frac{\text{Im } m}{\text{Im } \lambda}$$

which implies that m is inside C_a and therefore C_a contains the interior of C_b if $a < b$. This means that keeping λ fixed, with Im $\lambda > 0$, and letting $b \to \infty$, the circles C_b converge either to a circle C_∞ or to a point m_∞. We now discuss these two possibilities.

If the C_b converge to a circle, then its radius $r_\infty = \lim r_b$ is strictly positive and, from the expression of the radius (3.18), we get that $\psi \in L^2(0, +\infty)$. If $\hat{m}_\infty$ is any point on this limit circle C_∞, then, as showed above, $\hat{m}_\infty$ is C_b for any $b > 0$, hence

$$\int_0^b |\varphi + \hat{m}_\infty \psi|^2 \, dt < \frac{\text{Im } \hat{m}_\infty}{\text{Im } \lambda}$$

which, letting $b \to \infty$, it implies that $\phi + \hat{m}_\infty \psi$ belongs to $L^2(0, +\infty)$. The same argument holds if $\hat{m}_\infty$ reduces to the point m_∞ (that is, if we are in the second of the alternatives).

Therefore, if Im $\lambda \neq 0$, there always exists a solution of $Hf = \lambda f$ which is of class $L^2(0, +\infty)$. The difference between these cases is that when $C_b \to C_\infty$ converge to a circle then *all* solutions are of class $L^2(0, +\infty)$ for Im $\lambda \neq 0$, because both ψ and $\varphi + \hat{m}_\infty \psi$ are so, and this is the reason for the name *limit circle case*. On the other hand, when $C_b \to m_\infty$, the result $\lim r_b \to 0$ implies that ψ is not of class $L^2(0, +\infty)$ and, therefore, in this case there is only one linearly independent solution belonging to $L^2(0, +\infty)$ for Im $\lambda \neq 0$ which is precisely $\phi + m_\infty \psi$. This is the justification for the name *limit point case*.

Summing up, we have proven the following

Theorem 3.2.15 ([15]) *If Im $\lambda \neq 0$ and if φ and ψ are the linearly independent solutions of $Hf = \lambda f$ satisfying the conditions (3.14), then the solution $\chi = \varphi + m\psi$ satisfies the real boundary condition (3.16) if, and only if, m lies on a circle C_b in the complex plane whose equation is*

$$W(\chi, \bar{\chi})(b) = 0.$$

As $b \to \infty$ either $C_b \to C_\infty$, a limit circle, or $C_b \to m_\infty$, a limit point. All the solutions of $Hf = \lambda f$ belong to $L^2(0, +\infty)$ in the former case, and if Im $\lambda \neq 0$, exactly one linearly independent solution is $L^2(0, +\infty)$ in the latter case. Moreover, in the limit-circle case, a point is on the limit circle $C_\infty(\lambda)$ if, and only if, $W(\chi, \bar{\chi})(\infty) = 0$. □.

In the limit-point case, if m is any point on C_b, then $m \to m_\infty$, the limit point, and this holds independently of the choice of β in the boundary condition (3.16). In particular, this will hold when $\beta = 0$, and thus the limit point is given by

$$m_\infty(\lambda) = \lim_{b \to +\infty} \frac{\varphi(b, \lambda)}{\psi(b, \lambda)}$$

Definition 3.2.16 *We call Weyl's m-function, $m = m(\lambda)$, for Im $\lambda \neq 0$ associated to the equation $Hf = \lambda f$ the previous limit.*

The following result gives the analyticity of the m-function

Theorem 3.2.17 ([13], [15]) *In the limit point case, the limit point $m(\lambda)$ is an analytic function of λ in the upper half plane Π_+ and in the lower half plane Π_-. Im $m(\lambda) > 0$ for Im $\lambda > 0$ and if $m(\lambda)$ has zeros or poles in the real axis, the latter are simpler.*

Proof: From the expression of the radius and the center in terms of φ and ψ it turns out that these objects for the circle C_1 are continuous functions of λ whenever Im $\lambda > 0$. Thus, since C_b is in the interior of the disk C_1 if $b > 1$, it follows that if λ is restricted to a compact set K in Π_+, then the points $m = m(\lambda, b, \beta)$ on C_b are uniformly bounded when $b \to \infty$. The functions $m_b(\lambda, \beta) = m(\lambda, b, \beta)$, as they are meromorphic and bounded, they are analytic in K. Hence, by Cauchy's theorem they form an equi-continuous set on K, and m_b converges uniformly to m_∞. Being the uniform limit of analytic functions, m_∞ itself is analytic on K, and hence on Π_+.

Since m_∞ is in the interior of C_b, it follows that Im $m_\infty > 0$ on Π_+. This also proves that if m_∞ has zeros or poles on the real axis, then they are simple (in the case of poles) and that the poles have negative residue. $\square$.

We already know that in the cases we are interested in, that is, in quasi-periodic potentials, the imaginary points Im $\lambda \neq 0$ are always in the resolvent set. It may happen (and it fact in our examples it will happen) that there are open subsets in the real line which belong to the resolvent set (quite typically the spectrum will be a Cantor subset of the real line). If this is the case, one would like to know to what extend are Weyl's m-functions extendable through these *holes* or *gaps* in the spectrum. The following theorem shows some light on this problem

Theorem 3.2.18 ([30]) *If $\lambda \in \mathbb{R}$, then the eigenvalue equation $Hf = \lambda f$ has a subordinate solution at $+\infty$ if, and only if, either the boundary limit $m(\lambda + i0)$ exists as a finite real number (in which case χ is subordinate at $+\infty$) or if $\lim_{\epsilon \to 0+} |m(\lambda + i\epsilon)| = +\infty$ (in which case ψ is subordinate at $+\infty$).*

We will come back on this question later on, when referring to the case of Schrödinger equation with quasi-periodic potential.

In the above discussion of the m-function, the dependence of m on the endpoint a and the boundary condition α was not considered. From now on we will restrict the notation $m_\alpha(\lambda)$ to the case $a = 0$ and the notation $m(\lambda)$ to the case $\alpha = \pi/2$. In this case we have that the boundary conditions become

$$\varphi(a; \lambda) = 1, \qquad \varphi'(a; \lambda) = 0$$
$$\psi(a; \lambda) = 0, \qquad \psi'(a; \lambda) = 1,$$

and therefore $\varphi(\cdot, \lambda)$ and $\psi(\cdot, \lambda)$ are the normalized solutions at a. Therefore if, again, we denote by χ a L^2-solution at $+\infty$, one sees that

$$m(a, \lambda) = \frac{\chi'(a)}{\chi(a)}. \tag{3.19}$$

If we want to recover the m-function for the values of α at a we just have to use the following easy relation

$$m_\alpha(a, \lambda) = \frac{\sin\alpha + m(a; \lambda)\cos\alpha}{\cos\alpha - m(a, \lambda)\sin\alpha}$$

Moreover, we can change the potential performing a shift $t \mapsto V(t + s)$, for a fixed s to recover the value of the Weyl's m-functions for other values of the initial boundary condition a. Thus if, for Im $\lambda \neq 0$, we denote by $\chi(t,\lambda)$ the only solution of $H\chi = \lambda\chi$ which is square integrable near $+\infty$ and has value 1 at zero, then we can restate the equation (3.19) by

$$\chi(t,\lambda) = \exp\left(\int_0^s m(s,\lambda)ds\right).$$

This solution implies also that the m-function $m = m(t,\lambda)$ is a solution of the following Ricatti equation

$$\frac{d}{dt}m(t,\lambda) + m(t,\lambda)^2 = V(t) - \lambda. \tag{3.20}$$

This Ricatti equation will be of fundamental importance when speaking about the reducibility on the resolvent set in Schrödinger equation with quasi-periodic potential.

We now introduce a useful object to study the structure of the operator H: Green's function. To this end, recall that, associated to the boundary-value problem

$$Hx = \lambda x$$

$$\sin\alpha x(a) - \cos\alpha x'(a) = 0$$
$$\cos\beta x(b) + \sin\beta x'(b) = 0,$$

and for any λ in the resolvent set (that is for any λ such that the operator $H - \lambda I$ on the space $L^2([a,b])$ has a bounded inverse), there is a Green function, $G_{\alpha,\beta}^{[a,b]}(s,t;\lambda)$, which is defined as the integral kernel of the resolvent of the operator $H_\lambda = H - \lambda I$ in $[a,b]$ with boundary phases α and β. More precisely, the operator $H_{\alpha,\beta}^{[a,b]}$ defined by $Hf = -f'' + Vf$ on the space of functions f on $[a,b]$ satisfying the above boundary conditions is essentially self-adjoint and the resolvent operator of its unique self-adjoint extension has an integral kernel given by the Green's function

$$G_{\alpha,\beta}^{[a,b]}(s,t;\lambda) = \begin{cases} \psi(s,\lambda)\left(\varphi(t,\lambda) + m_{\alpha,\beta}^{[a,b]}(\lambda)\psi(t,\lambda)\right) & (s \le t) \\ \psi(t,\lambda)\left(\varphi(s,\lambda) + m_{\alpha,\beta}^{[a,b]}(\lambda)\psi(s,\lambda)\right) & (s > t) \end{cases},$$

and, if we are in the limit-point case, we may let b go to $+\infty$ so that $m_{\alpha,\beta}^{[a,b]}(\lambda)$ converges to $m_\alpha(a,\lambda)$, irrespectively of the possible values of the angle β. Then $G_{\alpha,\beta}^{[a,b]}$ converges to the function

$$G_{\alpha}^{[a,+\infty)}(s,t;\lambda) = \begin{cases} \psi(s,\lambda)\left(\varphi(t,\lambda) + m_\alpha(a,\lambda)\psi(t,\lambda)\right) & (s \le t) \\ \psi(t,\lambda)\left(\varphi(s,\lambda) + m_\alpha(a,\lambda)\psi(s,\lambda)\right) & (s > t) \end{cases}$$

and this is also an integral kernel for the resolvent operator defined by H on the subspace of $L^2(a,+\infty)$ satisfying the boundary conditions at the left endpoint.

Note that in both cases ($b < +\infty$ and $b = +\infty$), Green's function value at (s,t) with $t > s$ can be computed by multiplying the value of any solution of the eigenvalue equation $Hf = \lambda f$ which satisfies the boundary condition at the left endpoint of the interval by the value of any solution satisfying the boundary condition at the right endpoint and dividing the product by the Wronskian of these two solutions. In this interpretation, the boundary condition at infinity is understood as the square integrability of the function. This last fact is general and this special form of Green's function can also be used in the case of the Schrödinger operator in the whole line $\mathbb{R}$.

We shouldn't get the wrong impression that the spectrum of the Schrödinger operator (even in the case of a quasi-periodic potential) does not depend on whether we are in the half-line problem or in the whole line problem, as this won't usually be the case in our problems. To distinguish between these problems we will use the superscript + (resp. −) to refer to objects in the context of the Schrödinger operator on the positive (resp. negative) half-line, whereas the notation without superscripts will refer to the operator defined on the whole line.

From the following theorems we will deduce the relationship between the spectra σ^+, σ^- and σ, which are subsets of the real line.

Theorem 3.2.19 ([13]) *The essential support of the various parts of the spectral measure σ_α^+ are given by*

(i) ess-supp $\sigma_\alpha^+ = \mathbb{R} - \{\lambda \in \mathbb{R};$ *a subordinate solution exists, not satisfying the b.c. at zero*$\}$

(ii) ess-supp $\sigma_{\alpha,ac}^+ = \{\lambda \in \mathbb{R};$ *no subordinate solution exists*$\}$

(iii) ess-supp $\sigma_{\alpha,s}^+ = \{\lambda \in \mathbb{R};$ *a subordinate solution exists, satisfying the b.c. at zero*$\}$

(iv) ess-supp $\sigma_{\alpha,sc}^+ = \{\lambda \in \mathbb{R};$ *a subordinate solution exists, satisfying the b.c. at zero, but it is not square integrable at* $+\infty\}$

(v) ess-supp $\sigma_{\alpha,pp}^+ = \{\lambda \in \mathbb{R};$ *a subordinate solution exists, satisfying the b.c. at zero, and it is square integrable at* $+\infty\}$

where b.c. states for boundary condition.

For the essential support of the spectrum of the operator on the whole line we have the following

Theorem 3.2.20 ([13]) *An essential support of σ_{ac} is given by $S_+ \cup S_-$, where*

$$S_\pm = \{\lambda \in \mathbb{R};\ \text{the eigenvalue equation has no subordinate solution at } \pm\infty\}$$

and an essential support for the singular component of σ is given by

$$S_s = \{\lambda \in \mathbb{R};\ \text{the eigenvalue equation has a solution subordinate both at } +\infty \text{ and } -\infty\}$$

3.2.4 Some spectral theory of 1D Schrödinger operators with quasi-periodic potential

Now we turn to our case of interest

$$-x'' + q(t)x = \lambda x, \tag{3.21}$$

being q a quasi-periodic function. We will review the concepts already seen for this particular setup. As we will see, the particularities of our equation will result imply stronger properties of the objects already defined. As we will use the properties of quasi-periodicity of the potential $q(t) = Q(\omega t + \phi)$, we will write q instead of V in order to stress this quasi-periodicity. Moreover, in the sequel we will assume that Q is continuous and that the frequency vector $\omega \in \mathbb{R}^d$ is rationally independent, unless otherwise stated.

The eigenvalue problem on the whole line revisited. Characterization of the spectrum

In this subsection, we plan to give a useful characterization of the spectrum of the operator H on the whole line. It should be regarded as a nearly self-contained account of those results on the eigenvalue problem on the whole line which will be heavily used in the sequel. Summing up what we have seen in previous sections we have the following

Theorem 3.2.21 *Consider the Schrödinger operator with quasi-periodic potential q, whose extension Q to $\mathbb{T}^d$ is continuous and whose frequency vector is irrational. Let H be the corresponding self-adjoint extension to $L^2(\mathbb{R})$. Then, for λ in the resolvent set, $\rho(H)$, of this operator (and, in particular, this is true for those λ with Im $\lambda \neq 0$) there exist two linearly independent solutions χ_+ and χ_- which belong, respectively, to $L^2(0,+\infty)$ and $L^2(-\infty,0)$.*

Proof: For Im $\lambda \neq 0$ this has been proved in the previous section, in the discussion of Weyl's m-functions. Now assume that λ is real and belongs to the resolvent set. By definition, the operator $(H-\lambda I)$ defined in some dense subset of $L^2(\mathbb{R})$, and therefore the only solution in $L^2(\mathbb{R})$ of the eigenvalue problem $(H-\lambda I)f = 0$ is zero.

We will use theorem 2.2.5 in a suitable formulation for our purposes.

Theorem 3.2.22 ([71], [42]) *Suppose that the flow on the compact space $\mathbb{T}^d$ is minimal (in particular this is true is the frequency vector ω is rationally independent in the quasi-periodic case) and that the function q is continuous and bounded. Then the system*

$$\begin{pmatrix} x(t) \\ x'(t) \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ q(t)-\lambda & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ x'(t) \end{pmatrix} \tag{3.22}$$

admits an exponential dichotomy if, and only if, all nontrivial solutions of the above system (3.22) are not bounded in $\mathbb{R}$.

As a first consequence, for Im $\lambda \neq 0$, the system satisfies an exponential dichotomy, because we saw that there exist two linearly independent solutions, one in $L^2(0,+\infty)$ and the other in $L^2(-\infty,0)$. Now let Im $\lambda = 0$ in the resolvent set. We want to see that the system has an exponential dichotomy, and to this end we must prove that the only solution which is bounded is the trivial one. Let (ψ,ψ') be a solution of the system (3.22) which is bounded by a constant C in $\mathbb{R}$. If this solution is non-zero, then it cannot belong to $L^2(\mathbb{R})$, because of the resolvent assumption for λ. Thus, the limit

$$\lim_{T\to+\infty} \int_{-T}^{T} |\psi(t)|^2\, dt = +\infty$$

must hold. Now let g_n a function in $C_c^\infty(\mathbb{R})$ with support in the interval $[-n-1, n+1]$ and value 1 on $[-n,n]$. Assume, moreover, that

$$\sup_{t\in\mathbb{R}, n\geq 1} \{|g_n(t)|, |g_n'(t)|, |g_n''(t)|\} < C$$

for a constant C'. Set

$$\psi_n(t) = \frac{1}{\|g_n\psi\|_{L^2(\mathbb{R})}} g_n(t)\psi(t),$$

which defines a sequence of functions in $C_c^2(\mathbb{R})$ with $L^2(\mathbb{R})$-norm one. Now it is satisfied that

$$-\psi_n''(t) + (q(t)-\lambda)\,\psi_n(t) = \frac{-1}{\|g_n\psi\|_{L^2(\mathbb{R})}} \left(g_n''(t)\psi(t) + 2g_n'(t)\psi'(t)\right),$$

and therefore

$$\|-\psi_n'' + (q-\lambda)\,\psi_n\|_{L^2(\mathbb{R})} \le \frac{K}{\|g_n\psi\|_{L^2(\mathbb{R})}}$$

for a suitable constant K, because both g_n' and g_n'' are zero on $(-n, n)$. This would mean that

$$\lim_{n\to+\infty} \|-\psi_n'' + (q-\lambda)\,\psi_n\|_{L^2(\mathbb{R})} = 0$$

for a sequence of elements in $C_c^2(\mathbb{R})$ with L^2-norm one. This is, using Weyl's criterion 3.2.1, we obtain a contradiction with the fact that H is a self-adjoint operator with domain $\mathcal{D}(H) = C_c^\infty(\mathbb{R})$ and $\lambda \in \mathbb{R}$ is in the resolvent set of this operator. This altogether gives the desired conclusion that ψ must necessarily be zero. This implies that the system has an exponential dichotomy and, because of the volume preservation the invariant bundles V_+ and V_- have both dimension one. Therefore the corresponding solutions are linearly independent. These solutions are clearly in $L^2(0,+\infty)$ and $L^2(-\infty,0)$ respectively, because of the exponential bounds around $+\infty$ and $-\infty$. □

Note that we have thus proved the core points of the following important result, which is the characterization of the spectrum for the whole line operator that we will use:

Corollary 3.2.23 *Assume that the potential V is of the type $V(t) = Q(\tau(t;\phi))$, where $Q : Y \to \mathbb{R}$ is continuous, Y is a compact space and $\tau(\cdot;\cdot) : \mathbb{R}\times Y \to Y$ is a minimal flow (in particular this is true for quasi-periodic equations). Then a value λ is in the resolvent set of the Schrödinger operator H on the whole real line if, and only if, the two-dimensional system (3.22) has an exponential dichotomy.*

Proof: We have already seen in the previous proof that if λ was in the resolvent set, either real or complex, then the system had an exponential dichotomy. Now we must see the converse, namely, that if system (3.22) has an exponential dichotomy for $\lambda \in \mathbb{C}$, then λ is in the resolvent set of the operator H on L^2. If Im $\lambda \neq 0$ then there is nothing to prove, as these points are always in the resolvent set. If λ is real and the system has an exponential dichotomy, we are going to use the splitting given by the dichotomy to give an integral kernel (a Green's function) for the inverse of the operator. Let χ_+ and χ_- be linearly independent solutions of the Schrödinger equation which are in $L^2(0,+\infty)$ and $L^2(-\infty,0)$ respectively. For simplicity let us take the Wronskian equal to one. Let

$$G(s,t;\lambda) = \begin{cases} \chi_-(s)\chi_+(t) & \text{for} \quad s \le t \\ \chi_-(t)\chi_+(s) & \text{for} \quad s > t \end{cases}$$

We have that

$$|G(s,t;\lambda)| \le C(\lambda)\exp(\alpha(\lambda)|t-s|),$$

because the exponential dichotomy imposes a certain rate of decrease of χ_+ and χ_-. Hence, we have that the operator

$$(T\psi)(t) = \int_{\mathbb{R}} G(s,t;\lambda)\psi(s)ds$$

is a bounded linear map from $L^2(\mathbb{R})$ to itself (see, for instance, [69]) and satisfies

$$(H-\lambda I)T\psi = \psi$$

for all $\psi \in C_c^\infty(\mathbb{R})$, and thus for all $\psi \in L^2(\mathbb{R})$. □

In the case of quasi-periodic potentials with continuous extensions to $\mathbb{T}^d$ we have the following

Corollary 3.2.24 *The spectrum of the Schrödinger operator on $L^2(\mathbb{R})$ with potential $V(t) = Q(\omega t + \phi)$, where $Q : \mathbb{T}^d \to \mathbb{R}$ is continuous and ω is rationally independent, does not depend on the initial phase ϕ on the torus $\mathbb{T}^d$.*

Proof: If ω is rationally independent, the exponential dichotomy extends all over $\mathbb{T}^d$ by 2.2.5, and by the previous characterization of the spectrum, the result follows. □

Remark 3.2.25 *Here we have used that Q is continuous, although corollaries 3.2.23 and 3.2.24 are also true if the flow defined by equations (3.22) is only continuous on $\mathbb{R}^2 \times \mathbb{T}^d$.*

With these useful properties of the spectrum, we can now proceed to a more detailed description of the special features of Schrödinger equation with quasi-periodic potential.

Weyl's and Green functions in the resolvent set and their extensions to $\mathbb{T}^d$

In the previous subsection we defined Weyl's m-functions for the Schrödinger equation and λ in the set $\operatorname{Im} \lambda \neq 0$. Recall that these functions, taking the customary boundary conditions $a = 0$ and $\alpha = \pi/2$ were defined as

$$m_+(t; \lambda) = \frac{\chi_+'(t; \lambda)}{\chi_+(t; \lambda)}$$

where we write the underscript $+$ for the eigenvalue problem on the positive line, that is, when χ_+ is square integrable at $+\infty$ and satisfies the boundary conditions at zero. The reason for such a distinction is that one may (and should) also consider the problem on the negative half-line, which will be denoted by the underscript $-$, and the corresponding Weyl's m-functions, given by

$$m_-(t; \lambda) = \frac{\chi_-'(t; \lambda)}{\chi_-(t; \lambda)},$$

where χ_- is the solution of (3.21) satisfying the boundary condition at zero and being square integrable at $-\infty$.

Recall that for real λ belonging to the resolvent set there also exist real solutions χ_+ and χ_- square integrable at $+\infty$ and $-\infty$ respectively and satisfying the boundary condition at zero. So it would be nice to define the functions m_+ and m_- for λ in this set. The problem is that, if λ is real, the solutions $\chi_\pm(\cdot; \lambda)$ have zeroes, and therefore we cannot define $m_\pm$ as above.

However, as $\chi_\pm$ are real functions for real λ in the resolvent set, from the following transformation

$$m_\pm \mapsto \tilde{m}_\pm = \frac{m_\pm}{1 + i m_\pm} = \frac{\chi_\pm'}{\chi_\pm + i\chi_\pm'}$$

we get a transformed m-function with no singularities. A more elegant way to express this is to say that the m functions take values in the projective space $\mathbb{P}^1(\mathbb{C})$, for λ in the resolvent set. We will also say that $m_\pm$ is a *projective coordinate* on the resolvent set. When we want to work in $\mathbb{C}$ (and this will we done whenever we have to do some calculus) we will use the following notation. We say that a function m belongs to the class $\mathcal{F}_P$, where $\mathcal{F}$ stands for some class of functions, (for instance continuous on $\mathbb{R}$, or analytic quasi-periodic) if there exist complex constants a, b, c, d, with $ad - bc = 1$ such that

$$\frac{am + b}{cm + d} \in \mathcal{F}.$$

With the above projectivization we have thus a flow on $\mathbb{P}^1(\mathbb{C}) \times \mathbb{T}^d$, for Im $\lambda \neq 0$, given by the following Ricatti equation

$$\frac{d}{dt}m(t;\lambda) + m(t;\lambda)^2 = -\lambda + Q(\theta), \qquad \theta' = \omega. \tag{3.23}$$

By a theorem of Scharf ([72], [61]), in the quasi-periodic case, the functions $m_\pm$, for Im $\lambda \neq 0$, are also quasi-periodic with the same frequency vector ω. That is, there exist unique continuous functions $M_\pm(\cdot,\lambda) : \mathbb{T}^d \to \mathbb{P}^1(\mathbb{C})$ such that

$$m_\pm(t,\lambda,\phi) = M_\pm(\omega t + \phi; \lambda),$$

where ϕ is the initial phase for θ. Moreover, for Im $\lambda \neq 0$, these extensions are given by

$$M_\pm(\phi;\lambda) = \frac{{\chi_\pm}'(0;\lambda,\phi)}{\chi_\pm(0;\lambda,\phi)}$$

where, again, we wrote $\chi(\cdot;\cdot,\phi)$ to stress the dependence on the initial phase on the torus $\mathbb{T}^d$.

The mentioned theorem by Scharf just states the continuity of the functions $M_\pm$ defined on the torus. Nevertheless, it is also true that these functions have always directional derivatives in the direction of ω, so that we can write down the Ricatti equation (3.23) for these extensions

$$D_\omega M_\pm(\theta;\lambda) + M_\pm(\theta;\lambda)^2 = -\lambda + Q(\theta), \qquad \theta' = \omega. \tag{3.24}$$

In the case when the extension Q of the quasi-periodic function q is smooth, this smoothness will imply a greater regularity of the extensions $M_\pm$ as it is stated by the following theorem:

Theorem 3.2.26 ([61]) *Let $\mathcal{F} = \mathcal{D}^\alpha(\omega)$ (where , from now on, $\mathcal{D}^\alpha(\omega)$ will stand for the set of quasi-periodic functions with frequency $\omega \in \mathbb{R}^d$ whose extension to the d-dimensional torus is of class C^α) for $\alpha = r, \infty, a$. If $q \in \mathcal{F}$ and if λ is in the resolvent set $\rho(\lambda)$ then the functions*

$$m = m_\pm \in \mathcal{F}_P.$$

More precisely, $m_\pm \in \mathcal{F}$ and $\tilde{m}_\pm \in \mathcal{F}$ for Im $\lambda \neq 0$ and Im $\lambda = 0$ respectively.

This theorem gives, as corollary, reducibility in the resolvent set, as we shall see in section 3.4.

Proof: We shall indicate the proof for the case Im $\lambda \neq 0$ and the differences between this case and the other.

The result will follow from a series of results. The first step is to state conditions under which we can prove that a continuous function on the d-dimensional torus, which has directional derivatives in the direction ω (a condition written as $M \in C^0_\omega(\mathbb{T}^d)$), and satisfying an equation of the type

$$D_\omega M + P(A,M) = 0, \;\; P(A,M) = A_0 + A_1 M + \frac{1}{2}A_2 M^2, \tag{3.25}$$

where A_i, for $i = 1,2,3$, are complex valued functions on $\mathbb{T}^d$, can be continued in a neighbourhood to other functions of the same type. More precisely we have the following

Lemma 3.2.27 ([61]) *Suppose $M \in C^0_\omega(\mathbb{T}^d)$ and $A = (A_0, A_1, A_2)$ satisfy (3.25) and*

$$Re\ [A_1 + A_2 M]_{\mathbb{T}^d} \neq 0,$$

where $[\cdot]$ stands for the average of a function on the torus. Then there exists a neighbourhood U of A in $C^0 \times C^0 \times C^0$ and a unique analytic map

$$\Phi : U \to C^0_\omega(\mathbb{T}^d), \;\; \Phi(A) = M$$

such that $N = \Phi(B)$ satisfies the equation $D_\omega N + P(B,N) = 0$ for all $B \in U$.

Sketch of proof of the lemma: The Lemma follows from an application of the Implicit Function Theorem in Banach Spaces. Consider the analytic map from $C^0 \times C^0 \times C^0 \times C^0_\omega$ into C^0 given by

$$(B_0, B_1, B_2, N) \mapsto D_\omega N + P(B, N)$$

which vanishes at A. The only point to check is that its first partial derivative with respect to N at $N = M$, that is, the linear map taking $X \in C^0_\omega$ into

$$D_\omega X + (A_1 + A_2 M)X = G$$

on C^0 has a bounded inverse if the conditions of the lemma are satisfied. The bound comes from an application of the theorem on averages for continuous mappings of the torus, and the bound for $|\text{Re}\,[A_1 + A_2 M]|$ gives a bound for the inverse of the above linear mapping, which completes the proof of the lemma. □

The second step is to use the previous result to deduce the regularity of the solutions of equation (3.24).

Lemma 3.2.28 ([61]) *If A and M satisfy the hypothesis of lemma 3.2.27 and*

$$A \in C^\alpha(\mathbb{T}^d), \qquad \text{for } \alpha = r, \infty, a,$$

then also

$$M \in C^\alpha(\mathbb{T}^d).$$

Proof of the lemma: This regularity result follows easily from lemma 3.2.27, since Φ is analytic in a neighbourhood of A. Let's do the case $r = 1$, for instance. Let $A \in C^1(\mathbb{T}^d)$ and let $\hat{\theta}_k$ denote the k-th unit vector. Then

$$M(\theta + t\hat{\theta}_k) = \Phi\left(A(\theta + t\hat{\theta}_k)\right)$$

is well defined for t small because of the lemma and it is continuously differentiable

$$\frac{d}{dt} M(\theta + t\hat{\theta}_k)_{|t=0} = \Phi'(A) \frac{\partial}{\partial \theta_k} A(\theta),$$

showing that $M \in C^1(\mathbb{T}^d)$. To show analyticity, the Cauchy-Riemann equations are checked by the same method. □

Now we can finish the proof of theorem 3.2.26

For Im $\lambda \neq 0$, the extension $M = M_\pm$ of $m = m_\pm$ is a solution of the Ricatti equation (3.24) which corresponds to the choice

$$A_0 = Q - \lambda, \quad A_1 = 0 \quad A_2 = 2$$

in equation (3.25). Assume Im $\lambda > 0$ and the case Im $\lambda < 0$ follows similarly. Then, by the ergodicity of the flow

$$\text{Re}\,[A_1 + A_2 M] = 2\text{Re}\,[M] = \text{Re}\,[m] < 0$$

because $m(t, \lambda)$ is a Herglotz function and therefore has negative real part in the upper plane and hence negative average. This completes the proof when Im $\lambda \neq 0$, as lemma 3.2.28 can be applied.

For Im $\lambda = 0$ and λ in the resolvent set, we consider, as before,

$$\tilde{m} = \frac{\chi'}{\chi + i\chi'}$$

for which one checks that it satisfies the Ricatti equation

$$\tilde{m}' + \left(a_0 + a_1\tilde{m} + \frac{1}{2}a_2\tilde{m}^2\right) = 0$$

with

$$a_0 = \lambda - q, \quad a_1 = 2i(q-\lambda), \quad a_2 = 2(q-\lambda) + 2.$$

Moreover, $\tilde{m} \in \mathcal{D}^0(\omega)$, for the same arguments (due to Scharf) as for Im $\lambda \neq 0$. Hence $\tilde{m}$ extends to a function $\tilde{M} \in C^0_\omega(\mathbb{T}^d)$ which satisfies the equation

$$D_\omega\tilde{M} + \left(A_0 + A_1\tilde{M} + \frac{1}{2}A_2\tilde{M}^2\right) = 0,$$

being A_j the extensions to $\mathbb{T}^d$ of a_j, for $j = 1, 2, 3$ respectively. In order to prove the theorem it suffices to show that $[A_1 + A_2\tilde{M}] = [a_1 + a_2\tilde{m}]$ has non-zero real part. To this end, notice that

$$a_1 + a_2\tilde{m} = 2\frac{(\chi + i\chi')'}{\chi + i\chi'},$$

and therefore, by the ergodicity,

$$[a_1 + a_2\tilde{m}] = \lim_{T\to\infty}\frac{2}{T}\int_0^T \frac{(\chi(s) + i\chi'(s))'}{\chi(s) + i\chi'(s)}ds.$$

Since χ, χ' do not vanish simultaneously (this is an easy consequence of Sturmian theory), we can write

$$\chi(s) + i\chi'(s) = \exp(f(s))$$

for a suitable function $f \in C^\alpha(\mathbb{R})$. Then the above limit can be expressed as

$$\lim_{T\to\infty}\frac{2}{T}\int_0^T f'(s)ds = \lim_{T\to\infty}\frac{2\,(f(T) - f(0))}{T}$$

so that

$$\text{Re}\,[A_1 + A_2\tilde{M}] = \lim_{T\to+\infty} 2\frac{\ln|\chi(T) + \chi'(T)|}{T} < 0$$

because of the exponential dichotomy in the resolvent set. □.

So, for each λ the resolvent set, we have defined a map

$$M : \theta \in \mathbb{T}^d \mapsto M(\theta; \lambda) \in \mathbb{P}^1(\mathbb{C})$$

which is of the same kind of regularity as Q, and such that Weyl's m functions are of the form $m(t; \lambda, \phi) = M(\omega t + \phi; \lambda)$ for all $\phi \in \mathbb{T}^d$ and $t \in \mathbb{R}$. A similar study can be done for the Green's functions in the resolvent set for the eigenvalue problem either in the half or in the whole line. Recall that, in our case, these functions could be written as

$$G(s, t; \lambda) = \frac{\chi_+(t,\lambda)\chi_+(s,\lambda)}{W(\chi_+(\cdot,\lambda), \chi_-(\cdot,\lambda))}, \quad \text{for } s \leq t$$

so

$$G(t, t; \lambda) = \frac{\chi_+(t,\lambda)\chi_+(t,\lambda)}{W(\chi_+(\cdot,\lambda), \chi_-(\cdot,\lambda))} = \frac{1}{m_-(t;\lambda) - m_+(t;\lambda)}$$

and this last expression always makes sense, because $m_- - m_+$ is never zero in the resolvent set. Indeed, if Im $\lambda \neq 0$, this is true, because in this set the imaginary parts of m_- and m_+ have different signs and are not zero. For real λ in the resolvent set we have an exponential dichotomy due to the already mentioned results and we have that, for some t, $m_-(t;\lambda) = m_+(t,\lambda)$ is equivalent to that

$$\chi_+(t;\lambda)\chi'_-(t;\lambda) - \chi'_+(t;\lambda)\chi_-(t;\lambda) = 0$$

which is the Wronskian. This cannot be zero, because χ_+ and χ_- are linearly independent solutions, since they correspond to the two bundles that give the exponential dichotomy.

We finish this subsection with a proposition that relates the exponential dichotomy and the associated invariant subbundles to the projective coordinates $M_\pm$ in the case of Im $\lambda \neq 0$

Proposition 3.2.29 ([42]) *Suppose that $\lambda \in \mathbb{C}$ with Im $\lambda \neq 0$ in equation (3.21) with irrational frequency vector ω. Write as*

$$\mathbb{C}^2 \times \mathbb{T}^d = V^+(\lambda) \oplus V^-(\lambda)$$

the decomposition in invariant subbundles given by the exponential dichotomy in system (3.22) with $\dim V^\pm(\cdot) = 1$. *Let, for all $\phi \in \mathbb{T}^d$, $N_\pm(\phi,\lambda) \in \mathbb{P}^1(\mathbb{C})$ be the projective coordinate of the complex line $V^\pm(\phi;\lambda)$ (see the notation of the previous chapter). Then*

(i) signIm $\lambda \cdot$ Im $N_\pm(\phi;\lambda) = \pm 1$ for all $\phi \in \mathbb{T}^d$.

(ii) $N(\phi;\lambda) = M(\phi;\lambda)$ are the Weyl M-functions in the notation above.

The spectrum on the whole line v.s. the spectrum on the half-line

We end the exposition on the spectral properties of equation (3.21) by exposing some of the differences between the spectrum of the operator H on the whole line and the spectra of the operators $H^\pm$ on the positive and negative half-lines.

Theorem 3.2.30 ([72], [40], [42]) *Let the flow on Ω be minimal (this is true for equation (3.21) if the frequency vector is irrational). Then the essential spectra of H^+ and H^- coincide and are equal to*

$$\sigma_{ess}(H^+) = \sigma_{ess}(H^-) = \mathbb{C} - E,$$

where E is the set of λ for which equations (3.22) have an exponential dichotomy. In other words

$$\sigma_{ess}(H^+) = \sigma_{ess}(H^-) = \sigma(H)$$

Theorem 3.2.31 ([40], [42]) *Under the assumptions of the previous theorem, the operator H on the whole line has no isolated eigenvalues, that is*

$$\sigma(H) = \sigma_{ess}(H)$$

We will not give the proofs of the above theorems, which can be found in [13].

3.3 Ergodic invariants. The rotation number and the upper Lyapunov exponent

The existence of a rotation number is one of the most remarkable features of Schrödinger equation with quasi-periodic potential, and it is so because it uses many of the particularities of this equation, such as the Prüfer transformation and the ergodicity of the flow in $\mathbb{T}^d$. We will follow the approach of R. Johnson and J. Moser ([44]), but another introduction to the same object by M. Herman can be found in [36].

3.3.1 The rotation number for real potential and real λ

Here we are going to define the rotation number for real λ and show that it exists and defines a continuous function of λ. To this end we write Schrödinger equation with quasi-periodic potential as a first order system

$$\left(\begin{array}{c} x(t) \\ x'(t) \end{array}\right)' = \left(\begin{array}{cc} 0 & 1 \\ -\lambda + Q(\theta) & 0 \end{array}\right)\left(\begin{array}{c} x(t) \\ x'(t) \end{array}\right), \qquad \theta' = \omega. \tag{3.26}$$

Recall that, by means of the Prüfer transformation, the evolution of a line in $\mathbb{R}^2$ under the above flow is given by the equation

$$\varphi' = \cos^2\varphi - (q(t) - \lambda)\sin^2\varphi \tag{3.27}$$

where, by the arguments with which we began the chapter, the argument φ is taken modulus π. We will denote by $F(\varphi, \phi)$ the left hand side of this equation. This equation gives raise to a flow on the space $B = \mathbb{P}^1(\mathbb{R}) \times \Omega$ (homeomorphic to $\mathbb{T}^{d+1}$ because $\Omega = \mathbb{T}^d$), which we will denote by $\Phi(t; \phi_0, \varphi_0) = (\varphi(t; \varphi_0, \phi_0), \phi_0 + \omega t)$, where $\varphi(t; \varphi_0, \phi_0)$ is the solution of (3.27) with initial conditions $\varphi = \varphi_0$, $\phi = \phi_0$ for $t = 0$.

The rotation number is introduced to study the *average* variation of the phase φ over the time. It is therefore sensible to define it as the following limit

$$\frac{\varphi(t; \varphi_0, \phi_0) - \varphi(0; \varphi_0, \phi_0)}{t} = \frac{1}{t}\int_0^t F\left(\Phi(s; \varphi_0, \phi_0)\right) ds, \tag{3.28}$$

when the time t goes to infinity and φ is considered in the lift. There are many things to prove. We will first show that these time averages converge for *all* $(\varphi_0, \phi_0) \in B$ and that the convergence is uniform in B. Note that this is a remarkable fact, because the convergence is for all the elements of B instead of a set of total measure as it is typical in the context of ergodic objects.

There are some easy remarks on the above limit that can be made. First, note that if the limit (3.28) converges for some $(\varphi_0, \phi_0) \in B$, then it must also converge for another (φ_1, ϕ_0), with $0 < \varphi_1 - \varphi_2 < \pi$ and the limit is independent of φ. Indeed, let $\varphi_j(t) = \varphi(t; \varphi_j, \phi_0)$ for $j = 0, 1$. Then we have that $0 < \varphi_1(t) - \varphi_2(t) < \pi$ for all t because otherwise the theorem on uniqueness of solutions of equation (3.27) would be contradicted. Hence

$$\varphi_1(t) - \varphi_0(t)$$

is bounded and the claim follows.

We now want to state the above problem in terms of classical ergodic theorems. Due to the compactness of the space B, which is topologically a $d+1$-dimensional torus, the flow Φ_t possesses at least one invariant normalized measure, say ν. This means that ν is invariant under the flow Φ and that $\nu(B) = 1$. Moreover, by Birkhoff Ergodic Theorem, we conclude that there exists a function $F^* \in L^1(B, \nu)$, such that

$$\lim_{T\to+\infty} \frac{1}{T}\int_0^T F(\Phi(t; \varphi, \phi))dt = F^*(\varphi, \phi)$$

for ν-almost every $(\varphi, \phi) \in B$, where the integration is taken with respect to the Lebesgue measure. More precisely, there is a set $B_0 \subset B$, with $\nu(B - B_0) = 0$ such that the convergence holds for all $(\varphi, \phi) \in B_0$. Moreover we have that F^* satisfies

$$\int_B F^* d\nu = \int_B F d\nu$$

and F^* is invariant under the flow Φ_t.

We have already shown that the convergence for one φ_0 implies the convergence of (3.28) for *all* $\varphi \in \mathbb{T}^1$. From this we conclude that B_0 can be chosen to be of the form $B_0 = \mathbb{P}^1(\mathbb{R}) \times \Omega_0$, where $\Omega_0 \subset \Omega$, and F^* is independent of φ. Therefore, we can consider F^* as a function on Ω (defined ν-almost everywhere) which is invariant under the irrational flow given by ω. Since this flow is uniquely ergodic and preserves only the usual Haar measure μ on $\mathbb{T}^d$, which is ergodic, we conclude that F^* agrees with a constant, say α, on a set $B_1 = \mathbb{T}^1 \times \Omega_1$, where $\mu(\Omega - \Omega_1) = 0$. Being the measure normalized, it must agree with

$$\alpha = \int_B F d\nu. \tag{3.29}$$

As this argument works for any invariant measure ν of the flow Φ_t on B, we conclude that

$$\int_B (F - \alpha)\, d\nu = 0$$

holds for any invariant measure ν on B. From all these preliminaries, the existence of the limit in (3.28) will follow from the following lemma, which uses the methods of Krylov and Bogoljubov ([6])

Lemma 3.3.1 ([44]) *Let G be a continuous function on B such that*

$$\int_B G d\nu = 0$$

for any invariant measure for the flow F on B. Then

$$\frac{1}{b-a}\int_a^b G(\Phi_t(\beta))dt \to 0, \quad \text{as } b - a \to \infty$$

for all $\beta = (\varphi, \phi)$ and the convergence is uniform.

Proof: Since the space of continuous functions on B, $C(B)$, with the uniform topology, is separable, we can find a dense linear subspace D generated by a countable set of functions. We assume that the statement is false for some function $G \in C(B)$. We may then choose D so that $G \in D$ and select sequences b_j, a_j, β_j such that $b_j - a_j \to \infty$ and

$$\lim_{j\to\infty} \frac{1}{b_j - a_j}\int_{a_j}^{b_j} G\left(\Phi_t(\beta_j)\right) dt \to \delta \neq 0.$$

We may also assume that $\beta_j \to \beta$. Using the Cantor diagonal process, and using that D is countable, we can pick a subsequence, which we call a_j, b_j, β_j again, such that

$$\frac{1}{b_j - a_j}\int_{a_j}^{b_j} H\left(\Phi_t(\beta_j)\right) dt$$

converges for all $H \in D$. This limit defines a linear functional $l = l(H)$, $H \in D$, and since l is bounded (with norm 1) it extends uniquely to a bounded linear functional on $C(B)$. Since $b_j - a_j \to \infty$ one verifies that l is invariant. By the Riesz Representation Theorem (see, for instance, [68]) l defines an invariant measure ν on B. But by our assumption,

$$\int_B G d\nu = l(G) = \delta \neq 0$$

which is a contradiction. □

Now we apply the above lemma to $G = F - \alpha$ for which the hypothesis have already been verified. Hence

$$\frac{1}{T}\int_0^T (F(\Phi_t(\beta)) - \alpha)\, dt$$

converges to zero and the convergence is uniform. Thus the expression (3.28) converges to a limit α which is independent of $\beta = (\varphi_0, \phi_0)$. Moreover, one has that

$$\frac{\varphi(b) - \varphi(a)}{b - a} \to \alpha \text{ for } b - a \to \infty$$

for any solution of (3.27) and the existence of the rotation number has therefore been established. We now want to see more properties of this newly defined object.

Consider the rotation number $\alpha = \alpha(\lambda)$ as a function of $\lambda \in \mathbb{R}$ and let us prove its continuity. Assume for contradiction that α is not continuous at a point $\lambda_0 \in \mathbb{R}$ and let λ_j be a sequence converging to λ with $\alpha(\lambda_j) \to \alpha^* \neq \alpha(\lambda_0)$. Let $F_j = F(\varphi, \phi; \lambda_j) = \cos^2\varphi - (Q(\phi) - \lambda_j)\sin^2\varphi$ and Φ_t^j be the corresponding flow for $\lambda = \lambda_j$. If ν_j is any invariant measure for this flow we have that

$$\alpha(\lambda_j) = \int_B F_j d\nu_j.$$

We may suppose, due to the convergence $F_j \to F$, that $\nu_j \to \nu$ in the weak topology of measures, so that

$$\int_B F_0 d\nu_j \to \int_B F_0 d\nu,$$

and due to all this ν is invariant under the flow induced by F_0, Φ_t^0. Finally, since

$$|F_j - F_0| \leq |\lambda_j - \lambda_0| \to 0,$$

we conclude that

$$\alpha(\lambda_j) = \int_B F_j d\nu_j = O(|\lambda_j - \lambda_0|) + \int_B F_0 d\nu_j \to \int_B F_0 d\nu = \alpha(\lambda_0)$$

which is a contradiction with the assumptions. We therefore obtain

Theorem 3.3.2 ([44]) *For real λ the expression (3.28) or, for short,*

$$\frac{\varphi(t) - \varphi(0)}{t}$$

converges when $t \to +\infty$, uniformly with respect to initial conditions $(\varphi_0, \phi_0) \in B$, to a function $\alpha = \alpha(\lambda)$ which is independent of (φ_0, ϕ_0), but continuously dependent on λ. Moreover, $\alpha(\lambda)$ is monotone increasing, equal to zero for $\lambda \leq \lambda^*$ for some λ^* and $\alpha(\lambda) \to +\infty$ for $\lambda \to \infty$.

Remark 3.3.3 *In some cases the rotation number for the discrete analog of Schrödinger equation with quasi-periodic potential has been shown to have stronger regularity properties on the energy (of Hölder type in the case of one frequency and a certain modulus of continuity in the case of several basic frequencies, see [31] and the references therein).*

There only remain few items of the above theorem to be proved. As already mentioned, a zero of a solution $x(t)$ of equation (3.21) corresponds to a value t_0 for which $\varphi(t_0) = 0$ modulus π and by (3.27) one has that $\varphi'(t_0) = 0$. This shows that φ increases at such a zero and therefore one has one zero per increase of φ by π. Therefore, if $N(T; \lambda, x)$ is the number of zeroes in the time interval $[0, T]$ of a solution $x(t)$, one has that

$$\lim_{T\to\infty} \frac{\pi N(T; \lambda, x)}{T} = \alpha(\lambda).$$

With this identity, the remaining items of the above theorem are an easy consequence of Sturmian theory. Now we prove an important result which reveals the importance of the object already defined. We will call *spectral gap* to any connected component of the resolvent set in the real line.

Theorem 3.3.4 (Gap labelling, [44]) *If q is quasi-periodic with frequency ω and I is an open interval in a spectral gap, then there exist integers $\mathbf{k} \in \mathbb{Z}^d$ such that*

$$\alpha(\lambda) = \frac{\langle \mathbf{k}, \omega \rangle}{2}$$

for all $\lambda \in I$.

Proof: We recall that, in the resolvent set and in the previous notations, $\chi_\pm(t; \lambda)$ (the solutions square integrable in $\pm\infty$) and $G(t, t; \lambda)$ (the Green's function for $s = t$) are well defined and both

$$G(t, t; \lambda) \qquad \text{and} \qquad \frac{d}{dt} G(t, t; \lambda)$$

are quasi-periodic functions with frequency vector ω. The result for G was obtained at the end of section 3.2.4, whereas the result for its derivative uses similar techniques and can be found in [72]. Now we normalize χ_+ and χ_- so that their Wronskian equals one. We then have

$$G(t, t; \lambda) = \chi_+(t; \lambda)\chi_-(t, \lambda)$$

and

$$\frac{d}{dt} G(t, t; \lambda) = \chi_+(t; \lambda)\chi_-'(t; \lambda) + \chi_+'(t; \lambda)\chi_-(t; \lambda),$$

so it is clear that at a zero of $G(t, t; \lambda)$ either χ_+ or χ_- will vanish. Hence, at such zero

$$\frac{d}{dt} G(t, t; \lambda) = \pm \left(\chi_+\chi_-' + \chi_+'\chi_-\right) = \pm 1,$$

the sign depending on whether χ_+ or χ_- is zero. Anyway, $G(t, t; \lambda)$, and similarly its extension $\Gamma(\cdot, \lambda)$ to $\mathbb{T}^d$, has only simple zeroes. Now we use the following result,

Lemma 3.3.5 ([44], [72],[28]) *Let $f(t)$ be a function such that both f and f' are quasi-periodic with the same frequency vector ω. Assume that f has only simple zeroes. Then the number $N(T)$ of zeroes of $f(t)$ in $[0, T]$ satisfies the identity*

$$\lim_{T\to\infty} \frac{\pi N(T)}{T} = \langle \mathbf{k}, \omega \rangle$$

for a suitable $\mathbf{k} \in \mathbb{Z}^d$.

to deduce that the limit

$$\lim_{t\to\infty} \frac{\pi N_2(t;\lambda)}{t} = \langle \mathbf{k}, \omega \rangle,$$

for some $\mathbf{k} \in \mathbb{Z}^d$, if $N_2(T;\lambda)$ is the number of zeroes of $G(\cdot,\cdot;\lambda)$ in the interval $[0,T]$. Let $N_\pm(t;\lambda)$ be the number of zeroes of χ_+ and χ_- in $[0,T]$ then, by Sturmian theory,

$$N_2(t;\lambda) = N_+(t;\lambda) + N_-(t;\lambda)$$

from which we conclude that

$$\frac{\pi N_2(t;\lambda)}{t} \to 2\alpha = \langle \mathbf{k}, \omega \rangle.$$

Since $\lambda \mapsto \alpha(\lambda)$ is continuous it follows that $\alpha(\lambda)$ is constant in I.□

Once a frequency vector ω is fixed, we will refer to the *module of half-resonances* as the set

$$\mathcal{M}(\omega) = \left\{ \frac{\langle \mathbf{k}, \omega \rangle}{2}; \mathbf{k} \in \mathbb{Z}^d \right\}.$$

Remark 3.3.6 (Geometric interpretation of the rotation number) *This theorem gives an interpretation of the rotation number when λ is in the interior of a spectral gap. We have just seen that in this situation the rotation number is in $\mathcal{M}(\omega)$, the module of half-frequencies. That is, there exist integers $\mathbf{k} = (k_1, \ldots, k_d) \in \mathbb{Z}^d$ such that*

$$\alpha(\lambda) = \frac{\langle \mathbf{k}, \omega \rangle}{2}.$$

We are going to give a meaning to these integers. We have already seen, in section 3.2.4, that the projectivization of the solution in $L^2(0,+\infty)$ is quasi-periodic with frequency ω. In particular, the evolution of this angle $\varphi(t)$ is also a quasi-periodic function, but possibly with frequency vector $\omega/2$ instead of ω. This implies that there exists a continuous function

$$\Phi : \mathbb{T}^d \to \mathbb{T}$$

such that

$$\varphi(t) = \Phi\left(\frac{\omega t}{2}\right).$$

Then, for each fundamental generator, given by taking all components in $\mathbb{T}^d$ constant, except from one, say θ_j, which turns around $\mathbb{T}^1$, we can measure its winding by Φ, by means of a suitable lifting, and associate naturally to it an integer i_j which turns to be k_j.

We have seen that the rotation number is a continuous function of λ, which is monotone increasing and is constant at the interior of the spectral gaps. In principle it could happen that there existed other intervals of constancy, apart from the spectral gaps. It will be shown in section 3.3.3 that this is not the case, i.e. that the intervals of constancy of $\alpha(\lambda)$ are exactly the spectral gaps. As we will need some more theory on the extension of the rotation number for complex λ, which will be given in section 3.3.2, we go on with other properties of the rotation number.

Remark 3.3.7 (Hill's equation with quasi-periodic forcing, [12], [11]) *The theorem on gap labelling stated above (together with its* converse, *which will be given in section 3.3.3) can be used to define the analog of resonance tongues for Hill's equation with periodic forcing, namely*

$$x'' + (a + bq(t))\, x = 0, \tag{3.30}$$

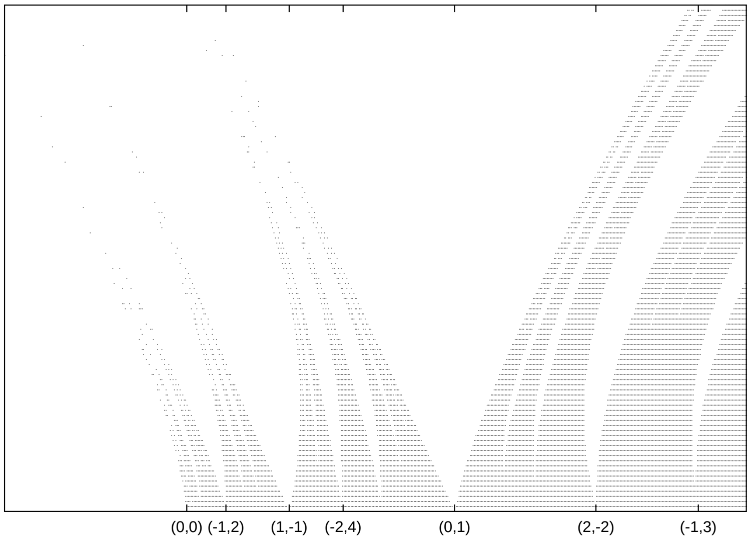

Figure 3.1: Resonance tongues in a Hill's equation with quasi-periodic forcing, (3.30), with two frequencies. Value a is on the horizontal direction and b is on vertical. Marked points correspond to the complementary of tongues (the spectrum). Pairs of the form (k_1, k_2) in the $b = 0$ axis indicate that the resonance tongue which emanates from this point has rotation number $(k_1\omega_1 + k_2\omega_2)/2$

with $(a, b) \in \mathbb{R}^2$ *and* q *a quasi-periodic function with frequency* ω*. Indeed, when* $d = 1$ *in the above equation, the instability zones (called* resonance tongues*, because of their shape and because this instability comes from parametric resonance) are identified by the condition* $|trace(\mathcal{P}_{a,b})| \geq 2$*, where* $\mathcal{P}_{a,b}$ *is the time-period or Poincaré matrix for the first-order system associated to equation (3.30). As all tongues are separated by the spectrum, which in this periodic case consists of non-void intervals, we can easily label tongues from left to right. However, in the quasi-periodic case, the spectrum is quite usually a Cantor set (this will be seen in section 3.3.5), so determining instability zones is not so easy. However, we can define resonance tongues in the* (a, b)*-parameter plane as those points* $(a, b) \in \mathbb{R}^2$ *such that the rotation number is of the form*

$$\frac{\langle \mathbf{k}, \omega \rangle}{2}$$

for a $\mathbf{k} \in \mathbb{Z}^d$*. An easy computation shows that these tongues emanate from the points in the* $b = 0$ *axis of the form*

$$a = \left(\frac{\langle \mathbf{k}, \omega \rangle}{2} \right)^2.$$

This can be seen in figure 3.1. In this context collapsed gaps are called instability pockets*.*

The rotation number has very good continuity properties also on the potential q, which we now state

Proposition 3.3.8 ([44]) *Let* $q_1, q_2 : \mathbb{R} \to \mathbb{R}$ *be quasi-periodic. Then, for any* $\lambda \in \mathbb{R}$*,* $q \mapsto \alpha(\lambda; q)$ *is continuous in the sup-topology. That is, given* $\varepsilon > 0$*, there exists* $\delta = \delta(\varepsilon) > 0$ *such that*

$$|\alpha(\lambda; q_1) - \alpha(\lambda; q_2)| < \varepsilon \qquad \textit{if} \quad \sup_{t \in \mathbb{R}} |q_1(t) - q_2(t)| < \delta.$$

Proof: From classical Sturm's theory we conclude that if $q \leq \tilde{q}$ for all $t \in \mathbb{R}$, then $\alpha(\lambda; q) \leq \alpha(\lambda; \tilde{q})$, using the characterization of the rotation number in terms of the zeroes of the normalized solutions. Therefore, from $|q_1 - q_2| < \delta$ we see that

$$q_2(t) - \delta \leq q_1(t) \leq q_2(t) + \delta$$

for all $t \in \mathbb{R}$ and hence

$$\alpha(\lambda; q_2 - \delta) \leq \alpha(\lambda; q_1) \leq \alpha(\lambda; q_2 + \delta),$$

and since $\alpha(\lambda; q_2 + \delta) = \alpha(\lambda + \delta; q_2)$, we can write this in the form

$$\alpha(\lambda - \delta; q_2) \leq \alpha(\lambda; q_1) \leq \alpha(\lambda + \delta; q_2).$$

Now the result follows from the already proved continuity in λ.□

As we are dealing with quasi-periodic potentials, this result may seem a bit restrictive, as, for instance, it cannot be applied if $q_1(t) = Q(\omega_1 t)$ and $q_2(t) = Q(\omega_2 t)$, where $Q : \mathbb{T}^d \to \mathbb{R}$ is continuous and ω_1, ω_2 are two irrational frequency vectors as close as we want. To work this out we state the following theorem, which gives much better continuity properties for quasi-periodic potentials

Theorem 3.3.9 ([4], [44]) *Let Q_n, for $n \geq 1$, be a sequence of continuous functions on the torus $\mathbb{T}^d$ which converge uniformly to a function Q (hence continuous). Let $\omega_n \in \mathbb{R}^d$ converging to a rationally independent vector $\omega \in \mathbb{R}^d$. Let $\alpha(\lambda; Q, \omega, \phi)$ denote the rotation number for the following Schrödinger equation with quasi-periodic potential*

$$-x''(t) + (-\lambda + Q(\omega t + \phi))\, x(t) = 0,$$

with $\lambda \in \mathbb{R}$ and $\phi \in \mathbb{T}^d$. Then

$$\lim_{n\to\infty} \alpha(\lambda; Q_n, \omega_n, \phi_n) \to \alpha(\lambda; Q, \omega) \tag{3.31}$$

and the convergence is uniform in $(\lambda, \phi) \in J \times \mathbb{T}^d$ for every compact subset $J \subset \mathbb{R}$.

Remark 3.3.10 *Note that in the right hand side of equation (3.31) we haven't written any dependence on the initial phase ϕ. This is because the rotation number with initial phase ϕ is constant over the closure in $\mathbb{T}^d$ of the orbit $\omega t + \phi$ for $t \in \mathbb{R}$, which turns to be the whole space if the components of ω are rationally independent.*

We don't give a proof of this theorem, because it is similar to the one of the continuity in λ. Nevertheless, it is important to stress that all these results can be obtained because the flow for the angles is on a compact space, topologically $\mathbb{T}^{d+1}$. Indeed, the compactness of the space $B = \mathbb{P}^1(\mathbb{R}) \times \mathbb{T}^d$ implies that the set of all normalized Borel measures is compact with the weak topology. The other arguments are similar to the already mentioned proof.

Finally we want to relate the rotation number to a limit, called usually the *integrated density of states* and which is often used in this context. Recall first that we saw that

$$\alpha(\lambda) = \frac{\pi N(a, b; \lambda)}{b - a} \text{ when } b - a \to +\infty,$$

where $N(a, b; \lambda)$ is the number of zeroes of a solution of equation (3.21) in the interval $[a, b]$. Now consider the regular eigenvalue problem on $[a, b]$

$$\begin{cases} -x'' + q(t)x = \lambda x \ \text{ on } \ a \le t \le b \\ x(a; \lambda) = x(b; \lambda) = 0. \end{cases} \tag{3.32}$$

It follows from Sturm's comparison theorem (see, for instance, [15]) that the number $\nu(a, b; \lambda)$ of eigenvalues $\lambda_j \le \lambda$ of the above problem differs from $N(a, b; \lambda)$ by ± 1, so

$$\lim_{b-a\to\infty} \frac{\nu(a, b; \lambda)}{b-a} = \frac{\alpha(\lambda)}{\pi},$$

and the left hand side is often called the *integrated density of states*, used in quantum mechanics (see [4] and references therein) and written as $k(\lambda)$ or $N(\lambda)$. This is the distribution function of a measure dk, which is the *density of states*.

3.3.2 Extension to complex λ and relation with Weyl's m-functions

As we will see the rotation number can be viewed as the imaginary part of a Herglotz function, an analytic function in the upper half-plane. To define such a function it will be very important to use the properties of quasi-periodicity that we proved for Weyl's m functions and their extensions to the torus $\mathbb{T}^d$.

Recall that in theorem 3.2.26 we constructed extensions $M(\cdot; \lambda) : \mathbb{T}^d \to \mathbb{R}$, as regular as Q, for λ in the resolvent set (well, for real λ in this set, we had to slightly modify M if we didn't want to projectivize anything). Let us consider for the moment Im $\lambda > 0$. To any quasi-periodic function (or rather to its continuous extension to $\mathbb{T}^d$) it is naturally associated the average, which always exists. We therefore define

$$w(\lambda) = [M_+(\cdot, \lambda)]_{\mathbb{T}^d} = \int_{\mathbb{T}^d} M_+(\theta; \lambda) d\mu(\theta)$$

where μ stands for the usual Haar measure on the torus. We recall that in section 3.2.4 we established several identities between $M_-(\cdot; \lambda)$, $M_+(\cdot; \lambda)$ and $\Gamma(\cdot; \lambda)$ (the extension of the Green's function $G(t, t; \lambda)$ to the torus) for Im $\lambda > 0$. One of these was

$$\frac{1}{\Gamma(\theta; \lambda)} = M_-(\theta; \lambda) - M_+(\theta; \lambda), \text{ for } \theta \in \mathbb{T}^d,$$

so taking averages it turns out that

$$\left[\frac{1}{\Gamma(\cdot; \lambda)}\right]_{\mathbb{T}^d} = [M_-(\cdot, \lambda)]_{\mathbb{T}^d} - [M_+(\cdot, \lambda)]_{\mathbb{T}^d}.$$

We will now show that the following identity holds

$$[M_-(\cdot, \lambda)]_{\mathbb{T}^d} = -[M_+(\cdot, \lambda)]_{\mathbb{T}^d}.$$

To this end note that

$$m_- + m_+ = \frac{(\chi_-\chi_+)'}{\chi_-\chi_+} = \frac{d}{dt}\log(\chi_-\chi_+) = \frac{d}{dt}\log G(t, t; \lambda).$$

Now, $G(t, t; \lambda)$ is quasi-periodic and its imaginary part is bounded away from zero and, since $\log G(t, t; \lambda)$ is bounded, we conclude that $m_- + m_-$ has zero average and the desired identities follow. Summing up, we have proved the following proposition

Proposition 3.3.11 ([44]) *For Im $\lambda \neq 0$ we have that*

$$-\left[\frac{1}{2\Gamma(\cdot;\lambda)}\right]_{\mathbb{T}^d} = -\left[M_-(\cdot,\lambda)\right]_{\mathbb{T}^d} = \left[M_+(\cdot,\lambda)\right]_{\mathbb{T}^d} = w(\lambda).$$

We have seen that for any $\theta \in \mathbb{T}^d$, the functions $M_\pm(\cdot;\lambda)$ were Herglotz functions. We therefore conclude that the function $w = w(\lambda)$ is a Herglotz function and, in particular, it is holomorphic in the upper half-plane. Moreover, from proposition 3.2.29 we get the following inequalities for Im $\lambda \neq 0$

$$\frac{\text{Im } w(\lambda)}{\text{Im } \lambda} > 0, \qquad \text{Re } w(\lambda) < 0.$$

As a consequence, the function $h(\lambda) = \text{Im } w(\lambda)$ is harmonic in the upper half-plane, hence it has a representation of the form

$$h(\lambda) = \text{Im } \lambda \int_{-\infty}^{+\infty} \frac{d\sigma(s)}{|s-\lambda|^2} \tag{3.33}$$

where $\sigma(s)$ is a monotone increasing function on $\mathbb{R}$ and $d\sigma$ is the associated exterior measure. Moreover, at the points of continuity of σ one has

$$\sigma(s_2) - \sigma(s_1) = \frac{1}{\pi} \lim_{\varepsilon \to 0} \text{Im } w(s + i\varepsilon) ds.$$

Also, for any continuous function $f(s)$ with compact support, we have that

$$\int_{-\infty}^{+\infty} f(s) d\sigma(s) = \lim_{\varepsilon \to 0^+} \frac{1}{\pi} \int_{-\infty}^{+\infty} f(s) h(s + i\varepsilon) ds.$$

We now prove the basic relations of the rotation number and the function w. We start with a useful characterization of the imaginary part of w.

Theorem 3.3.12 ([44]) *If Im $\lambda \geq 0$, let $\phi = \phi(t)$ be a complex solution of Schrödinger equation (3.21) satisfying that*

$$Im\ W(\phi, \bar{\phi}) > 0 \qquad for \quad t = 0, \tag{3.34}$$

where $W(\phi, \bar{\phi}) = \phi\bar{\phi}' - \phi'\bar{\phi}$ is the Wronskian. Then the limit $h(\lambda)$ of

$$h(t;\lambda) = \frac{-1}{t} \arg \frac{\phi(t;\lambda)}{\phi(0;\lambda)} = \frac{-1}{t} \int_0^t Im\ \frac{\phi'(s;\lambda)}{\phi(s;\lambda)} ds$$

for $t \to \infty$ exists and is independent of the solution chosen (as long as (3.34) holds). For real λ we have $h(t;\lambda) \to \alpha(\lambda)$ as $t \to \infty$. Moreover, for Im $\lambda > 0$, h is the imaginary part of w, in the above notation.

From this theorem we can also derive the following.

Theorem 3.3.13 (cf.[44]) *The function $h(\lambda) = Im\ w(\lambda)$ is continuous in the closed upper half-plane Im $\lambda \geq 0$, and*

$$\lim_{\varepsilon \to 0^+} h(\lambda + i\varepsilon) = \alpha(\lambda)$$

for real λ.

The proofs can be found in the same reference. To prove theorem 3.3.13, we define

$$\phi(t;\lambda) = \phi_1(t;\lambda) - i\phi_2(t;\lambda),$$

being ϕ_1 and ϕ_2 two linearly independent solutions with Wronskian one, which are real for real λ. Then (3.34) holds and ϕ is entire with no zeroes if $t \geq 0$ and Im $\lambda \geq 0$. Hence the function $h(t;\lambda)$ is harmonic in the set Im $\lambda \geq 0$ for every $t \geq 0$ and we can therefore represent it in the form of (3.33) for a suitable monotone increasing function $\sigma(t;\lambda)$ which satisfies that

$$\sigma(t;\lambda_2) - \sigma(t;\lambda_1) = \frac{1}{\pi}\int_{\lambda_1}^{\lambda_2} h(t;\lambda)d\lambda$$

for *all* real λ_1, λ_2, because we have already seen that $h(t;\lambda)$ is continuous in Im $\lambda \geq 0$. Now, due to theorem 3.3.12, we have that $h(t;\lambda) \to h(\lambda)$ when $t \to \infty$ in the open upper plane Im $\lambda > 0$.

Therefore, for the σ of the limit function $h(\lambda)$, we have that, at the points of continuity of σ,

$$\sigma(t;\lambda_2) - \sigma(t;\lambda_1) \to \sigma(\lambda_2) - \sigma(\lambda_1)$$

as $t \to +\infty$; i.e.,

$$\sigma(\lambda_2) - \sigma(\lambda_1) = \lim_{t\to+\infty} \frac{1}{\pi}\int_{\lambda_1}^{\lambda_2} h(t;\lambda)d\lambda$$

at such points of continuity. On the other hand, using theorem 3.3.12 it can be shown that the convergence $h(t;\lambda) \to \alpha(\lambda)$ for real λ is uniform on compact intervals $[\lambda_1, \lambda_2]$. Therefore, we can exchange the limit and integral to get

$$\sigma(\lambda_2) - \sigma(\lambda_1) = \frac{1}{\pi}\int_{\lambda_1}^{\lambda_2} \alpha(\lambda)d\lambda$$

at these points of continuity. Hence, we find the representation

$$h(\lambda) = \frac{\text{Im } \lambda}{\pi}\int_{-\infty}^{+\infty} \frac{\alpha(s)}{|s-\lambda|^2}ds, \quad \text{for Im } \lambda > 0. \tag{3.35}$$

From this representation and from the continuity of α it follows that $h(\lambda + i\varepsilon) \to \alpha(\lambda)$ as $\varepsilon \to 0^+$ and for real λ. So h is continuous in the upper half-plane, as we wanted to prove.□

Incidentally, from (3.35) we find the representation

$$\frac{w(\lambda) - w(\lambda_0)}{\lambda - \lambda_0} = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\alpha(s)}{(s-\lambda)(s-\lambda_0)}ds, \quad \text{for Im } \lambda, \text{Im } \lambda_0 > 0$$

which determines $w(\lambda)$ up to an additive constant provided α is known for real values of λ. We shall see that the role of this multiplicative constant is not trivial at all. Note that we may as well define a function $w(\lambda)$ in the lower half plane setting

$$w(\lambda) = \left[\frac{-1}{2G(t,t;\lambda)}\right],$$

for Im $\lambda \neq 0$. It is easily seen that the following is true

$$w(\bar{\lambda}) = \overline{w(\lambda)}, \quad \frac{\text{Im } w(\lambda)}{\text{Im } \lambda} > 0 \quad \text{for Im } \lambda \neq 0. \tag{3.36}$$

Moreover, as we have shown that for $\lambda \leq \lambda^*$ the rotation number is zero, we conclude from theorem 3.3.13 and (3.36) that $w(z)$ is real for real $\lambda \leq \lambda^*$. In other words $w(z)$ extends analytically through $(-\infty, \lambda^*)$. However, w cannot be seen as a one-valued function on the resolvent set. Indeed, if I is an interval in a spectral gap, then

$$\text{Im } w(\lambda + i\varepsilon) \to \alpha(\lambda) = \alpha_I \in \mathcal{M}(\omega), \text{ when } \varepsilon \to 0^+.$$

Moreover, by (3.36) we have that

$$w(\lambda + i\varepsilon) - w(\lambda - i\varepsilon) = 2i\text{Im } w(\lambda + i\varepsilon) \to 2\alpha_I, \quad \text{if } \lambda \in I \text{ and } \varepsilon \to 0^+;$$

i.e., $w(\lambda)$ suffers a jump of $2i\alpha_I$, which is non-zero unless I lies in the lowest gap $(-\infty, \lambda^*)$. Note that the problem for the extension here is the imaginary part of w, as we shall see in more detail in section 3.3.4.

To end this section we make some comments on the possible differentiability of the rotation number and the complex extension w with respect to the potential. We have seen that the rotation number $\lambda \mapsto \alpha(\lambda)$ for real λ may be a Cantor function. Therefore it is not likely that we have always differentiability for $\alpha(\lambda)$ outside the spectral gaps, where we know that the derivative exists and is equal to zero (see [64] for some results and references about this problem). However we will consider differentiation in the following sense.

Consider again Schrödinger equation (3.21) and let p be another quasi-periodic function with frequency module contained in the frequency module of q. Assume moreover that Im $\lambda \neq 0$. Then we may talk about the derivative $\delta w/\delta q$ if it satisfied that

$$\frac{d}{d\varepsilon} w(\lambda; q + \varepsilon p)_{\varepsilon=0} = \left[\frac{\delta w}{\delta q} p\right]_{\mathbb{T}^d}.$$

This will be the content of the following theorem, whose proof can be found in [44],

Theorem 3.3.14 ([44]) *For Im $\lambda \neq 0$, the functional derivative $\frac{\delta w}{\delta q}$ exists and it is given by*

$$\frac{\delta w}{\delta q}(t) = -G(t, t; \lambda)$$

and the λ-derivative, defined in the usual way is

$$\frac{dw}{d\lambda}(\lambda_0) = [G(\cdot, \cdot; \lambda_0)]_{\mathbb{T}^d}.$$

Remark 3.3.15 *If the spectrum of H contains an interval, as it is the case for periodic potentials, then it can be shown ([60], [43]) that such a functional derivative exists and that the functional derivative is positive in this interval. This is of great importance for the methods in section 3.3.5 to prove Cantor spectrum of some generic Schrödinger equations.*

3.3.3 The spectral functions for the half and whole line

We now want to use the representation for harmonic functions to derive an important relation of the rotation number with the spectrum of the Schrödinger operator on the whole line. To this end recall that every positive harmonic function $h(\lambda)$ in Im $\lambda > 0$ can be represented in the form

$$h(\lambda) = \text{Im } \lambda \left(\int_{-\infty}^{+\infty} \frac{d\varrho(s)}{|\lambda - s|^2} + c \right), \tag{3.37}$$

where ϱ is a monotone increasing function and $c \geq 0$ is a constant. Moreover, at the points of continuity of ϱ, we have that

$$\varrho(\lambda_2) - \varrho(\lambda_1) = \lim_{\varepsilon \to 0^+} \frac{1}{\pi} \int_{\lambda_1}^{\lambda_2} h(s + i\varepsilon) ds$$

and

$$c = \lim_{t \to \infty} \frac{h(it)}{t}.$$

We now apply this representation to the functions Im $M_\pm(\theta;\lambda)$, and denote the corresponding densities and constants by $\varrho_\pm$ and $c_\pm$. From standard asymptotic estimates of the solutions (see, for instance, [37]) it is seen that $c_+ = c_- = 0$. Then the following result is true

Proposition 3.3.16 ([44]) *Let μ be the normalized Haar measure on $\mathbb{T}^d$ and λ_1, λ_2 two real numbers. Then, for almost all $\theta \in \mathbb{T}^d$, the function $\varrho_+(\theta, \lambda)$ is continuous at λ_1 and λ_2 and the relation*

$$\varrho_+(\theta;\lambda_2) - \varrho_+(\theta;\lambda_1) = \lim_{\varepsilon \to 0^+} \frac{1}{\pi} \int_{\lambda_1}^{\lambda_2} Im\ M_+(\theta;\lambda + i\varepsilon) d\lambda$$

holds with bounded convergence in θ.

With this proposition in mind we can relate the functions $\varrho(\theta;\lambda)$, which we shall call the *half-line spectral functions*, to the w-function. Recall that, for Im $\lambda > 0$,

$$w(\lambda) = [M_+(\cdot;\lambda)]_{\mathbb{T}^d}.$$

Now, if $f : \mathbb{R} \to \mathbb{R}$ is a continuous function with compact support, we have that, by the above proposition,

$$\lim_{\varepsilon \to 0^+} \frac{1}{\pi} \int_{-\infty}^{+\infty} f(\lambda) \left(\int_{\mathbb{T}^d} \text{Im } M_+(\theta;\lambda + i\varepsilon) d\mu(\theta) \right) d\lambda$$
$$= \lim_{\varepsilon \to 0^+} \frac{1}{\pi} \int_{-\infty}^{+\infty} f(\lambda) \text{Im } w(\lambda + i\varepsilon) d\lambda = \int_{\mathbb{T}^d} \int_{-\infty}^{+\infty} f(\lambda) d\varrho(\theta;\lambda) d\mu(\theta).$$

Since Im $w(\lambda + i\varepsilon) \to \alpha(\lambda)$ uniformly on compact λ-intervals, we get

$$\int_{-\infty}^{+\infty} f(\lambda)\alpha(\lambda) d\lambda = \int_{\mathbb{T}^d} \int_{-\infty}^{+\infty} f(\lambda) d\varrho(\theta;\lambda) d\mu(\theta). \tag{3.38}$$

We will use then the notation

$$\int_{\mathbb{T}^d} (d\varrho(\theta;\lambda))\, d\mu(\theta) = \alpha(\lambda) d\lambda,$$

where $d\lambda$ is the Lebesgue measure on $\mathbb{R}$, to say that equation (3.38) is satisfied for all functions $f : \mathbb{R} \to \mathbb{R}$ which are continuous with compact support.

We can do a similar analysis for the spectral functions on the whole line. To this end we recall that the Green's function $G(t,t;\lambda)$ for Im $\lambda > 0$ is quasi-periodic with frequency ω. That is, we can find a function $\Gamma(\theta;\lambda)$ in the torus, through which G can be expressed. Moreover, for each $\theta \in \mathbb{T}^d$, $\Gamma(\theta;\cdot)$ is holomorphic with positive imaginary part in Im $\lambda > 0$. This means that there exists a *whole line spectral function* $\hat{\varrho}(\theta;\lambda)$ on $\mathbb{R}$ such that

$$\text{Im } \Gamma(\theta;\lambda) = \text{Im } \lambda \int_{-\infty}^{+\infty} \frac{d\hat{\varrho}(\theta;s)}{|s - \lambda|^2} \tag{3.39}$$

when Im $\lambda > 0$. The following lemma shows that we can also *integrate* the measures $d\hat{\varrho}(\theta;\lambda)$ and relate this to the measure on $\mathbb{R}$ defined by α.

Lemma 3.3.17 ([44]) *If Im* $\lambda > 0$ *then*

$$\int_{\mathbb{T}^d} (d\hat{\varrho}(\theta;\lambda))\, d\mu(\theta) = \frac{1}{\pi} d\alpha(\lambda)$$

in the sense that, for all continuous functions $f : \mathbb{R} \to \mathbb{R}$ *with compact support, the following holds*

$$\int_{\mathbb{T}^d} \int_{-\infty}^{+\infty} f(\lambda) d\hat{\varrho}(\theta;\lambda) d\mu(\theta) = \frac{1}{\pi} \int_{-\infty}^{+\infty} f(\lambda) d\alpha(\lambda).$$

Now we can prove the converse of theorem 3.3.4

Theorem 3.3.18 ([44]) *The support of the measure* $d\alpha(\lambda)$ *agrees with the spectrum* $\sigma(H)$ *of equation (3.21) on* $L^2(-\infty, +\infty)$.

Proof: By theorem 3.3.4 it is clear that we only have to prove that if I is a bounded open interval on which the rotation number $\alpha(\lambda)$ is constant, then

$$\sigma(H) \cap I = \emptyset \tag{3.40}$$

From the integration of measures of the previous lemma we obtain that

$$\int_{\mathbb{T}^d} \int_I \hat{\varrho}(\theta;\lambda) d\mu(\lambda) = \frac{1}{\pi} \int_I d\alpha(\lambda),$$

considering the integration with respect to a bounded interval as the multiplication by a characteristic function and approximating by continuous functions with compact support. Therefore we have that, for almost all $\theta \in \mathbb{T}^d$,

$$\int_I d\hat{\varrho}(\theta;\lambda) = 0.$$

Now let us fix a value of θ in this set of measure one. Then it follows from (3.39) that

$$\text{Im } \Gamma(\theta; \lambda + i\varepsilon) \to 0 \quad \text{for} \quad \lambda \in I,\ \varepsilon \to 0^+.$$

We can apply the reflection principle to get the analyticity of $\Gamma(\theta;\cdot)$ through I, and thus,

$$\frac{-1}{\Gamma(\theta;\lambda)} = M_+(\theta;\lambda) - M_-(\theta;\lambda)$$

is meromorphic on I. Let Z denote the set of zeroes of $\Gamma(\theta;\cdot)$ on I. Then it follows from the inequalities

$$\text{Im } M_+(\theta;\lambda) > 0 > \text{Im } M_-(\theta;\lambda) \quad \text{for Im } \lambda > 0$$

that

$$\text{Im } M_+(\theta; \lambda + i\varepsilon) \to 0 \quad \text{for} \quad \lambda \in I - Z,\ \varepsilon \to 0^+.$$

Hence also $M_-(\theta;\lambda)$ is meromorphic on I and

$$\lim_{\varepsilon \to 0^+} \frac{1}{\pi} \int_0^\lambda \text{Im } M_+(\theta; s + i\varepsilon) ds = \varrho(\theta;\lambda)$$

is a piecewise constant function on I, with possible jumps on Z. On the other hand, $\varrho(\theta;\lambda)$ is the spectral function of the operator $H^+ = H^+_\theta$ on the half line, and therefore, $\sigma(H^+_\theta) \cap I$ contains only isolated point eigenvalues, because they correspond to the jumps in $\varrho(\theta; \lambda)$. In particular this means that it has no intersection with the essential spectrum of this half-line operator,

$$\sigma_{ess}(H_\theta^+) \cap I = \emptyset.$$

But we already saw (theorems 3.2.30 and 3.2.31) that the sets $\sigma_{ess}(H_\theta^+)$ and $\sigma(H_\theta)$ agree, and that they are independent of θ (because of the rational incommensurability of ω). This concludes the proof.□

Remark 3.3.19 *The above theorem in combination with (3.3.4) shows that a point* $\lambda \in \mathbb{R}$ *is in the spectrum* $\sigma(H)$ *of equation (3.21) on* $L^2(-\infty, +\infty)$ *if and only if the rotation number is not constant at* λ.

3.3.4 The real part of w. The Lyapunov exponents for Schrödinger equation with quasi-periodic potential

In the previous sections, the imaginary part of function w has received a great deal of attention, mainly because of its good properties. This section is devoted to a short survey of results on the *real* part of this function, in principle defined on the resolvent set, but that, with certain restrictions, can be extended to the whole real line. It turns out that this object is strongly related to the Lyapunov exponent. The approach to the problem in this section and the results are mostly taken from [41].

For the discussion on reducibility it will be very important to distinguish between the Schrödinger equation with a *fixed* quasi-periodic potential, and the Schrödinger equation with *varying initial phase* and fixed $Q : \mathbb{T}^d \to \mathbb{R}$ and frequency vector ω. That is, on one hand we will consider

$$-x'' + (-\lambda + q(t))x = 0, \tag{3.41}$$

where $\lambda \in \mathbb{C}$ and $q : \mathbb{R} \to \mathbb{R}$ is a quasi-periodic function whose extension to $\mathbb{T}^d$ is assumed to be at least continuous, while on the other we will consider the extended system

$$-x'' + (-\lambda + Q(\theta))x = 0, \qquad \theta' = \omega, \tag{3.42}$$

where Q is the extension of the previous quasi-periodic function (i.e., there exists an initial phase $\phi \in \mathbb{T}^d$ such that $q(t) = Q(\omega t + \phi)$).

We begin with the definition of the main object of interest in this section

Definition 3.3.20 *We define the* upper Lyapunov exponent *for real* λ, $\beta(\lambda)$, *for equation (3.41) as*

$$\beta(\lambda) = \sup \left\{ \limsup_{t\to+\infty} \frac{1}{2t} \log \left(x(t)^2 + x'(t)^2 \right) \right\}$$

where the supremum is taken over all solutions of the system (3.42). This implies considering all initial conditions for $x(0), x'(0) \in \mathbb{R}$ *and* $\phi \in \mathbb{T}^d$.

Remark 3.3.21 *Due to the linear character of the flow one can always take the supremum on the set of initial conditions given by* $x(0)^2 + x'(0)^2 = 1$.

We will see later on that the function β can also be considered for complex λ. Note that if we write equation (3.42) as a first-order system

$$u'(t) = \begin{pmatrix} 0 & 1 \\ -\lambda + Q(\theta) & 0 \end{pmatrix} u(t), \quad \theta' = \omega, \tag{3.43}$$

where $u \in \mathbb{R}^2$, then the upper Lyapunov exponent can equivalently be written as

$$\beta(\lambda) = \sup \left\{ \limsup_{t \to +\infty} \frac{1}{t} \log \|u(t)\|; \; u \text{ is a solution of (3.43)} \right\}. \tag{3.44}$$

Consider now, for complex λ, the flow on the complex bundle $\Sigma = \mathbb{P}^1(\mathbb{C}) \times \mathbb{T}^d$, using the standard projectivization exposed in the beginning of the chapter. Consider the following function, for $\lambda \in \mathbb{C}$,

$$(l, \theta) \in \Sigma \mapsto f_\lambda(l, \theta) = \mathrm{Re} \, \frac{\langle A_\lambda(\theta) u_0, u_0 \rangle}{\langle u_0, u_0 \rangle}$$

where $u_0 \in \mathbb{C}^2$ is any nonzero vector which represents the projective complex line $l \in \mathbb{P}^1(\mathbb{C})$ and A_λ is the matrix

$$A_\lambda(\theta) = \begin{pmatrix} 0 & 1 \\ -\lambda - Q(\theta) & 0 \end{pmatrix}.$$

Then, if $u(t)$ satisfies (3.43) with $u(0) = u_0$ and initial phase ϕ, one has

$$\frac{1}{t} \left(\log \|u(t)\| - \log \|u(0)\| \right) = \frac{1}{t} \int_0^t f_\lambda(l, \omega s + \phi) ds, \tag{3.45}$$

and therefore the exponential growth of $u(t)$ is determined by a time average of f_λ. We will also use the projective flow on $(\Sigma_{\mathbb{R}}, \mathbb{R})$ where $\Sigma_{\mathbb{R}} = \mathbb{P}^1(\mathbb{R}) \times \mathbb{T}^d$, with the notation from the introduction in this chapter. Recall that the function $M_+(\phi; \lambda)$, for $\mathrm{Im}\, \lambda \neq 0$ gives the direction of the projective lines of solutions of (3.42) with initial phase ϕ that belong to $L^2(0, +\infty)$ in the sense that if $\chi_+(\cdot; \phi)$ is a solution belonging to $L^2(0, +\infty)$, then

$$\chi_+(t; \phi) = \chi_+(0; \phi) \exp \left(\int_0^t M_+(\omega s + \phi; \lambda) ds \right),$$

so that the real part of w can be expressed by means of these solutions

$$\mathrm{Re}\, w(\lambda) = \lim_{t \to +\infty} \frac{1}{2t} \log \left(|\chi_+(t; \phi)|^2 + |\chi_+'(t; \phi)|^2 \right).$$

Thus $\mathrm{Re}\, w(\lambda)$ is the exponential rate of decay of the solutions $u(t; \phi) = (\chi_+(t; \phi), \chi_+'(t; \phi))^T$ of (3.42) in $L^2(0, +\infty)$ and this rate is independent of the initial phase. Using the exponential dichotomy of the system for $\mathrm{Im}\, \lambda \neq 0$ it is easily shown that

$$\beta(\lambda) = -\mathrm{Re}\, w(\lambda),$$

where $\beta(\lambda)$ is defined as in (3.44).

Remark 3.3.22 *We can use similar techniques to those in the proof of lemma 3.3.1 to prove the following in the real case:*

If $\beta(\lambda) = 0$ and $\lambda \in \mathbb{R}$ then $\int_\Sigma f_\lambda d\mu = 0$ for every invariant measure μ on Σ, because we have that the time-average (3.45) is zero for all solutions of (3.43). Thus using lemma 3.3.1 we conclude that

$$\lim_{|b-a|\to+\infty} \frac{1}{b-a}\int_a^b f_\lambda(l(s;l_0,\phi);\omega s+\phi))ds = 0$$

for all $\sigma = (l_0,\phi) \in \Sigma = \mathbb{P}^1(\mathbb{C}) \times \mathbb{T}^d$, and the convergence is uniform in a, b and σ.

If $\beta(\lambda) > 0$ and $\lambda \in \mathbb{R}$ then there are exactly two ergodic measures on Σ, μ_+ and μ_- (see [39]); one has that $\mu_\pm(\Sigma_\mathbb{R}) = 1$, and

$$\int_\Sigma f_\lambda d\mu_+ = -\int_\Sigma f_\lambda d\mu_- = \hat{\beta}(\lambda) > 0.$$

Moreover, $\hat{\beta}$ is the right end-point of the Sacker-Sell spectrum of equations (3.43). It is very important to note that this does not imply an exponential dichotomy, because it could happen, and it certainly does, that the points $\pm\hat{\beta}(\lambda)$ are the endpoints of the same Sacker-Sell spectral interval, therefore corresponding to the same invariant subbundle (the whole space). This should not come as a surprise because positive Lyapunov exponents do not imply uniform hyperbolicity.

The following result gives more details in the case of positive upper Lyapunov exponent.

Proposition 3.3.23 ([41]) *If $\lambda \in \mathbb{R}$ and $\beta(\lambda) > 0$, then $\beta(\lambda) = \hat{\beta}(\lambda)$. Moreover, for almost every $\phi \in \mathbb{T}^d$ (where the measure here is the usual Haar measure), there exist unique solutions $u_\pm$ of (3.43) (up to a constant multiple) with*

$$\lim_{t\to\infty} \frac{1}{t}\log\|u_\pm(t)\| = \pm\beta(\lambda)$$

The proof is similar to the construction of the rotation number and can be found in [41]. The *almost everywhere* part comes from an application of Birkhoff ergodic theorem.

Remark 3.3.24 *It is an easy consequence of the discussion previous to the lemma that if $\beta(\lambda) = 0$ then necessarily $\hat{\beta} = 0$ and the Sacker-Sell spectrum reduces to a single point.*

The rotation number (α) and the Lyapunov exponent (β) are closely related by means of the Thouless formula, which we now state

Theorem 3.3.25 (Thouless formula, ([4], also [77] and [3])) *Let $\beta_0(\lambda) = \sqrt{\max(0,-\lambda)}$ and $\alpha_0(\lambda) = \sqrt{\max(0,\lambda)}$. Then for any quasi-periodic function q on $\mathbb{R}$ we have that for a.e. $\lambda \in \mathbb{R}$*

$$\lim_{T\to+\infty} \frac{1}{\pi}\int_{-\infty}^{T} \log(\lambda-\lambda')\, d(\alpha-\alpha_0)(\lambda')$$

exists and for a.e. pair $(\phi,\lambda) \in \mathbb{T}^d \times \mathbb{R}$, the limit

$$\beta(\lambda,\phi) = \lim_{|T|\to\infty} \frac{1}{|T|}\log\|M(T;\lambda,\phi)\|,$$

being $M(T;\lambda,\phi)$ the fundamental matrix of (3.43), exists and it is equal to

$$\beta(\lambda) = \beta_0(\lambda) + \frac{1}{\pi}\int_{-\infty}^{+\infty} \log(\lambda-\lambda')\, d(\alpha-\alpha_0)(\lambda').$$

In previous sections we showed that, for real λ,

$$\lim_{\varepsilon\to 0^+} \text{Im } w(\lambda+i\varepsilon) = \alpha(\lambda),$$

the rotation number. Now we are going to state a similar result, but with some non-trivial restrictions,

Theorem 3.3.26 ([41]) *(i) If $\lambda \in \mathbb{R}$ and $z \to \lambda$ non-tangentially for Im $z > 0$, then $-$Re $w(z)$ tends to $\beta(\lambda)$. If $\beta(\lambda) = 0$, then β is continuous at λ, and $-$Re $w(z) \to \beta(\lambda)$ whenever $z \to \lambda$, non-tangentially or not.*

(ii) $\lambda \in \mathbb{R} \to \beta(\lambda)$ is upper semi-continuous on $\mathbb{R}$.

(iii) On $\mathbb{R}$, β is non-negative, of first Baire class, and has the mean value property.

We can now prove a nice regularity result for the upper Lyapunov exponent in the resolvent set $\rho(H)$ of the Schrödinger operator with quasi-periodic potential on $L^2(-\infty,+\infty)$.

Proposition 3.3.27 ([41]) *The upper Lyapunov exponent, $\beta(\lambda)$, is harmonic on the resolvent set of the whole-line operator H.*

Proof: Consider the function w, for which we already saw that $\beta(\lambda) = -\text{Re } w(\lambda)$, for Im $\lambda \neq 0$. Recall that, from (3.36), $\overline{w(\lambda)} = w(\bar{\lambda})$. Now, if I is an interval in the resolvent set, then

$$-\beta(\lambda) + i\alpha(\lambda) = \lim_{\varepsilon\to 0^+} w(\lambda+i\varepsilon)$$

and $\alpha(\lambda)$ is constant for $\lambda \in I$, say α_I. We also have that, when we tend to the real axis on the other side then

$$\lim_{\varepsilon\to 0^-} w(\lambda+i\varepsilon) = \beta(\lambda) - \alpha_I$$

for $\lambda \in I$. So, if we define

$$w^*(\lambda) = \begin{cases} w(\lambda), & \text{Im } \lambda > 0, \\ \beta(\lambda) + i\alpha_I, & \lambda \in I, \\ w(\lambda) + 2\alpha_I, & \text{Im } \lambda < 0, \end{cases}$$

then w^* is holomorphic on $\{\text{Im } \lambda \neq 0\} \cup I$ by the reflection principle. It follows that β is harmonic on the resolvent set, because it is the real part of an analytic function. $\square$.

As a corollary we get that the spectrum cannot be too small

Corollary 3.3.28 ([41]) *Let $I \subset \mathbb{R}$ an open interval such that $I \cap \sigma(H) \neq \emptyset$. Then the logarithmic capacity of $\sigma(H) \cap I$ is positive.*

Remark 3.3.29 *This doesn't imply that the spectrum cannot have zero measure, as, for instance, the Cantor ternary set has positive logarithmic capacity and zero measure.*

Proposition 3.3.30 ([41]) *Let $\lambda_0 \in \mathbb{R}$ be the endpoint of a spectral gap I. If $\lambda_n \to \lambda_0$, with $\lambda_n \in I$, then $\beta(\lambda_n) \to \beta(\lambda_0)$.*

Proof: The proof will come from a geometric argument in $\mathbb{P}^1(\mathbb{C})$ and $\mathbb{P}^1(\mathbb{R})$. Recall that for all $\theta \in \mathbb{T}^d$ and λ in the resolvent set, the vectors $(1, M_\pm(\theta;\lambda))$ define complex lines $l_\pm(\theta;\lambda)$ in $\mathbb{C}^2$. If $\lambda \in I$ we can consider the objects $M_\pm$ to be in $\mathbb{P}^1(\mathbb{R})$ instead of $\mathbb{P}^1(\mathbb{C})$, and the lines in $\mathbb{R}^2$ instead of $\mathbb{C}^2$. As $\mathbb{P}^1(\mathbb{R})$ is a circle, we parametrize it by the φ variable, with $0 \le \varphi < \pi$, as usual. The orientation of $\mathbb{P}^1(\mathbb{R})$ is taken to agree with an increase in φ.

Now fix $\theta \in \mathbb{T}^d$. It can be seen that if λ increases through I, then $M_+(\theta;\lambda)$ and $M_-(\theta;\lambda)$ move in opposite directions on $\mathbb{P}^1(\mathbb{R})$ (see, for instance, [15]). Moreover, $M_+(\theta;\lambda)$ and $M_-(\theta;\lambda)$ can never coincide if $\lambda \in I$.

Therefore, as $\lambda_n \to \lambda_0$ with the terms of the sequence in I, the limits

$$\lim_{n\to\infty} l_\pm(\theta;\lambda_n)$$

must exist in $\mathbb{P}^1(\mathbb{R})$ because $\mathbb{P}^1(\mathbb{R})$ is compact. Call these limits $l_\pm(\theta)$. The sets

$$S_\pm = \left\{(l_\pm(\theta),\theta);\ \theta \in \mathbb{T}^d\right\} \subset \Sigma_\mathbb{R}$$

are measurable subsections of $\Sigma_\mathbb{R} = \mathbb{P}^1(\mathbb{R})\times\mathbb{T}^d$, because they are the limit of measurable functions. Hence they define ergodic measures $\mu^\pm$ on Σ via the formulae

$$\int_\Sigma g d\mu_\pm = \int_{\mathbb{T}^d} g(\theta; l_\pm(\theta))d\theta$$

for continuous functions $g : \Sigma \to \mathbb{R}$. We have that, with the previous definitions

$$-\beta(\lambda_n) = \int_\mathbb{T}^d f_{\lambda_n}(\theta, M_+(\theta;\lambda_n))d\theta \to \int_{\mathbb{T}^d} f_{\lambda_0}(\theta; l_+(\theta))d\theta = \int_\Sigma f_{\lambda_0} d\mu_+.$$

Using remark 3.3.22,

$$\int_\Sigma f_{\lambda_0} d\mu_+ = -\beta(\lambda_0)$$

which completes the proof of the theorem. □

Remark 3.3.31 *R. Johnson provided in [41] an example of a Schrödinger equation with limit-periodic potential (a function which is uniform limit of periodic functions with increasing period) such that the upper Lyapunov exponent, $\beta(\lambda)$, is not continuous for a certain value of λ. This function is not quasi-periodic, and in some sense, this example is the opposite of those systems that we are interested in.*

Remark 3.3.32 *Very recently, J. Bourgain and S. Jitomirskaya ([9]) have shown the continuity of the Lyapunov exponent with respect to the energy λ for the discrete analogue of the Schrödinger equation when the potential is analytic (with no assumptions on the frequency) and only one frequency is considered. Moreover, they also show continuity with respect to the frequency at every irrational point. When the Lyapunov exponent is positive in a compact interval for the energies then the Lyapunov exponent and the rotation number can be shown to be jointly Hölder continuous in this interval ([31]).*

3.3.5 Application to Cantor spectrum

To conclude this section, we shall speak a little bit about the existence of Cantor spectrum in Schrödinger equation with quasi-periodic potential. Recall that we saw that this spectrum was a closed subset of the real line. It will turn out that quite typically this set is nowhere dense, a situation that will be referred as *Cantor spectrum*.

It is not immediate from the discussion and properties of the rotation number (increasing strictly on the spectrum and constant over the intervals of the resolvent set) that the spectrum is a Cantor set, because this does not imply that the resolvent set is non void. Indeed, it could happen (and it certainly does) that the spectral gap corresponding to a certain resonance (that is the interior of the set of points with rotation number in the resonance module) is void, in which case we will speak of a *collapsed gap*. This phenomenon also takes place in the periodic case, and, though is not persistent under generic (even quasi-periodic) perturbations (see [61] for this *gap opening*), it can be shown (using reducibility) that in some cases the analysis for its existence is analogous to the periodic case ([11]).

One way to prove the existence of Cantor spectrum is based on the approximation by periodic potentials. We shall focus on two papers. The first, due to J. Moser ([60]), constructs a limit periodic potential of the type

$$q(t) = a_0 + \sum_{j,s=1}^{\infty} \left(a_{js} \cos(s2^{-j}t) + b_{js} \sin(s2^{-j}t) \right) \tag{3.46}$$

such that it opens all gaps (corresponding to iterations of the Poincaré map) of a certain periodic potential. More precisely, the main statement in [60] is

Theorem 3.3.33 ([60]) *Given $\eta > 0$ and q_0 a continuous function of period π, there exists a limit-periodic analytic function q with basic frequencies 2^{-j} ($j = 0, 1, \dots$) with $\|q - q_0\|_\infty < \eta$ for which the Schrödinger equation with potential q has all spectral gaps open and, hence, the spectrum is a Cantor set.*

There is a very nice geometrical discussion behind this result, which comes from a detailed analysis of the periodic case. In this case we know that the spectral gaps are labeled with the positive integers, because they correspond to those values of λ for which $|\Delta(\lambda)| > 2$, where Δ is the trace of the Poincaré map. On the other hand, one can also consider q as a function of period $2\pi, 3\pi, \dots, m\pi$ and the corresponding spectral gaps. However, it turns out that all the spectral gaps corresponding to higher periods of the Poincaré map are collapsed, because they are all double roots of the $m\pi$-period time map. The idea of [60] is to recurrently define the perturbation (3.46) so that all gaps corresponding to higher periods are opened. To show that previously opened gaps are not closed again it is used the analog of the functional derivative from theorem 3.3.14 for periodic potentials which is also valid for the spectral intervals (i.e. those open intervals contained in the spectrum of the *periodic* operator).

In [43], Moser techniques are heavily used by R. Johnson, together with the functional derivative (and specially its positiveness) to provide a more general result using again the periodic approximation. The main statement of this paper is the following

Theorem 3.3.34 ([43]) *There is a residual subset (i.e., countable intersection of open dense sets) $\mathcal{F} \subset \mathbb{R}^d$ such that, if $\omega \in \mathcal{F}$, then the following statement holds. There is a residual subset $V = V(\omega) \subset C^\delta(\mathbb{T}^d)$, with $0 \le \delta < 1$, such that, if $Q \in V$ and $\phi \in \mathbb{T}^d$, then the operator*

$$H = H_\phi = -\frac{d^2}{dt^2} + Q(\omega t + \phi) \tag{3.47}$$

has Cantor spectrum.

The core of the proof is the following theorem,

Theorem 3.3.35 ([43]) *Let $A = \mathbb{R}^d \times C^\delta(\mathbb{T}^d)$, where $0 \le \delta < 1$. There is an open dense subset $W \subset A$ such that, if $(\omega, Q) \in W$, then the two-dimensional system associated to (3.47) has an exponential dichotomy for all $\phi \in \mathbb{T}^d$.*

Finally, note that the existence of Cantor spectrum can also be obtained in combination with reducibility results, as it is the case in [27] and the references therein. This will be discussed in section 3.4.3.

3.4 Reducibility in Schrödinger equation with quasi-periodic potential

Now we come to the main section in this chapter. Here we shall make free use of the notations, definitions and results from previous sections. Up to now we have seen that the behaviour of Schrödinger equation with quasi-periodic potential can be quite accurately described in the resolvent set, and the underlying reason is that, there, the system is uniformly hyperbolic. We will therefore distinguish between reducibility in the resolvent set and reducibility in the spectrum. Moreover, in the latter case, the rotation number (which will correspond to the imaginary part eigenvalues of the reduced matrix), together with its arithmetical properties, will play a key role.

3.4.1 Reducibility in the resolvent set. Uniformly hyperbolic reducibility

The reducibility in this set under the typical Diophantine assumptions on the frequency vector ω, has, in fact, already been established, as we have shown that if λ is in the resolvent set of the operator

$$H = -\frac{d^2}{dt^2} + Q(\omega t + \phi)$$

then the corresponding two dimensional system with this value of λ has an exponential dichotomy with two one-dimensional subbundles. Therefore we are under the full spectrum hypothesis and the results on reducibility by R. Johnson and G. Sell (see theorem 2.5.1) can be applied if suitable arithmetic properties on ω and regularity properties on Q are satisfied.

For the sake of self-containedness we now give a proof of this fact using what we have introduced during this chapter. We follow the approach of [44], although a similar result can be found in many other places. The result that we want to prove is the following:

Theorem 3.4.1 ([61]) *Let $\mathcal{F}$ the space of quasi-periodic functions whose extension to $\mathbb{T}^d$ is of class α (with $\alpha = r, \infty, a$, $r \ge 0$) and with frequency $\omega \in \mathbb{R}^d$. Then, there exists a transformation $T = T(t;\lambda)$ such that the elements of T, T^{-1} belong to $\mathcal{F}$ and the transformation*

$$\begin{pmatrix} x \\ x' \end{pmatrix} = T(t;\lambda) z$$

takes the system (3.43) into

$$z' = D(t;\lambda) z,$$

where $D(t;\lambda)$ is a diagonal matrix with elements in $\mathcal{F}$. Moreover

$$\left[\frac{d}{dt}\log\det T(\cdot;\lambda)\right] = 0.$$

Proof: For Im $\lambda \neq 0$ we choose

$$T(t;\lambda) = \begin{pmatrix} 1 & 1 \\ m_+(t;\lambda) & m_-(t;\lambda) \end{pmatrix}$$

so that the transformed system $z' = D(t;\lambda)z$ possesses

$$\begin{pmatrix} \psi_+(t;\lambda) \\ 0 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 \\ \psi_+(t;\lambda) \end{pmatrix}$$

as solutions. From

$$D = \operatorname{diag}(m_+, m_-)$$

we find that its average is

$$[D] = \operatorname{diag}(w, -w).$$

For Im $\lambda = 0$ and λ in the resolvent set (that is, λ is in a spectral gap I, with rotation number $\alpha(\lambda) = \frac{1}{2}\langle \mathbf{k}, \omega\rangle$) we set

$$T(t;\lambda) = \begin{pmatrix} 1 - i\tilde{m}_+ & 1 - i\tilde{m}_- \\ \tilde{m}_+ & \tilde{m}_- \end{pmatrix} \begin{pmatrix} e^{i\langle \mathbf{k},\omega\rangle} & 0 \\ 0 & 1 \end{pmatrix}$$

so that the transformed matrix is

$$D = \operatorname{diag}(p_+ + i\langle \mathbf{k}, \omega\rangle, p_-),$$

with

$$p_\pm = \tilde{m}_\pm + (q-\lambda)(\tilde{m}_\pm + i) = \frac{(\psi_\pm + i\psi_\pm{}')'}{(\psi_\pm + i\psi_\pm{}')}.$$

From the above definitions and the properties of $\psi_\pm, \tilde{m}_\pm$ it follows that the functions $p_\pm$ are quasi-periodic with frequency ω and that their averages are

$$[p_+] = \bar{w}, \qquad [p_-] = -w.$$

Hence also in this case the average of the transformed matrix is

$$[D] = \operatorname{diag}(\bar{w} + i\langle \mathbf{k}, \omega\rangle, -w) = \operatorname{diag}(w, -w)$$

and $\operatorname{tr}([D]) = 0$, since Im $w = \alpha(\lambda)$, the rotation number.

For both Im $\lambda \neq 0$, and Im $\lambda = 0$ and in the resolvent set, we have that the elements of the transformation T belong to $\mathcal{F}$ due to theorem 3.2.26. Moreover the determinant of this transformation is, respectively

$$\det T = m_- - m_+, \qquad (\tilde{m}_- - \tilde{m}_+)e^{-i\langle \mathbf{k},\omega\rangle t},$$

which is bounded away from zero. Since

$$D = T^{-1}AT - T^{-1}T',$$

where A is the original matrix of the system, we conclude from $\operatorname{tr}[D] = 0$ and $\operatorname{tr} A = 0$ that

$$0 = \left[\ \operatorname{tr} T^{-1}T'\right] = \lim_{T\to\infty}\int_0^T \frac{(\det T)'}{\det T}dt,$$

proving the theorem □

Remark 3.4.2 *Using the results from the first chapter on diagonal systems we can prove reducibility assuming regularity on the potential and certain arithmetic conditions on frequency vector ω.*

Remark 3.4.3 *Note that the reducing transformation of the above result is quasi-periodic with frequency ω and not $\frac{\omega}{2}$ as usual. The price that we must pay for such property is that the transformed matrix is complex, just as it happens in the periodic case.*

3.4.2 Reducibility in the spectrum. KAM techniques

In the previous section we have shown reducibility in the resolvent set. The arguments have been based on the uniform hyperbolicity of the flow. In the spectrum the situation is much more different and the reducibility will be obtained near constant coefficients and by means of KAM (Kolmogorov-Arnol'd-Moser) theory. If we seek for the reducibility at a point of the spectrum with a certain rotation number, then the imaginary part of the eigenvalues of the reduced matrix will be this rotation number and its negative. The rotation number plays thus the role of the normal frequency of the system. The approach in this section is taken from the [24], [27] and [25].

When using a perturbative argument we need to have some information about what we are perturbing. In our case it is obvious that close to constant coefficients (i.e. assuming the potential small) we can perform such a perturbative analysis. However, perturbative arguments can also be used for arbitrarily big (but fixed) potentials if we assume that the energy λ of the trivial solution is large enough. This is due to the following transformation.

Consider the first-order system associated to Schrödinger equation with quasi-periodic potential

$$\begin{pmatrix} x \\ x' \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ Q(\theta) - \lambda & 0 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}, \qquad \theta' = \omega. \tag{3.48}$$

After performing the following passage to complex coordinates

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ i\sqrt(\lambda) & -i\sqrt(\lambda) \end{pmatrix} u,$$

the above system becomes

$$u' = \begin{pmatrix} i\sqrt{\lambda} & 0 \\ 0 & -i\sqrt{\lambda} \end{pmatrix} u + \frac{Q(\theta)}{2\sqrt{\lambda}} \begin{pmatrix} -i & -i \\ i & i \end{pmatrix} u, \qquad \theta' = \omega. \tag{3.49}$$

For large λ this can be viewed as a perturbation of a family of rotations with angular velocity $\sqrt{\lambda}$. Therefore, a perturbative analysis in this case is also possible.

As it is customary in KAM theory, we will assume the real analyticity of the function $Q : \mathbb{T}^d \to \mathbb{R}$ in a complex strip and some strong non-resonance conditions. These conditions will refer both to the external frequencies of the forcing (the frequency vector ω) and the internal frequencies of the system (the rotation number). We shall use the following notation. Assume that $\omega \in \mathbb{R}^d$ is the frequency vector of the forcing, that will be taken fixed for the rest of the section. We say that it is *Diophantine* whenever there exist positive constants $N, \tau > 0$ such that

$$|\langle \mathbf{k} \rangle| \geq N^{-1} |\mathbf{k}|^{-\tau}, \qquad \mathbf{k} \in \mathbb{Z}^d - \{0\}$$

being $\langle \mathbf{k} \rangle = \langle \mathbf{k}, \omega \rangle$ with the usual scalar product in $\mathbb{R}^d$. We say that a real number ρ is *Diophantine with respect to* ω (in fact with respect to its half-frequency module $\mathcal{M}(\omega)$) whenever there exist positive constants $K, \sigma > 0$ such that

$$\left| \rho - \frac{\langle \mathbf{k} \rangle}{2} \right| \geq K^{-1} |\mathbf{k}|^{-\sigma}, \qquad \mathbf{k} \in \mathbb{Z}^d - \{0\},$$

and that it is *rational with respect to* ω if $\rho \in \mathcal{M}$.

The hypothesis on the real analyticity and the smallness of the potential will be stated in the following form. Assume that $Q : \mathbb{T}^d \to \mathbb{R}$ is the lift to $\mathbb{T}^d$ of the potential q, which we consider analytic in a complex neighbourhood $|\text{Im } \theta| < r$ of $\mathbb{T}^d$. To control the size of this function we use the norm

$$|Q|_r = \sup_{|\text{Im } \theta| < r} |Q(\theta)|$$

on the space of analytic functions in this complex strip of the d-dimensional torus.

A natural way to show reducibility is to try to find a Floquet representation. Recall that, if $X(t; \phi)$ is the fundamental solution for the system (3.48) with initial phase $\phi \in \mathbb{T}^d$ for the angular variables, a Floquet representation of X is a representation of X of the form

$$X(t; \phi) = Y \left(\frac{\omega t}{2} + \phi \right)^{-1} e^{Bt} Y(\phi), \tag{3.50}$$

where B is a constant matrix (belonging to $sl(2, \mathbb{R})$ because we are dealing with Schrödinger equation) and $Y : \mathbb{T}^d \to Gl(2, \mathbb{R})$ is an analytic matrix-valued function. Note the usual phenomenon of frequency-halving in the above expression.

Finding a Floquet representation implies solving the non-linear differential equation

$$\langle \nabla_\theta Y(\theta), \frac{\omega}{2} \rangle = -Y(\theta) A(\theta) + B Y(\theta), \tag{3.51}$$

with

$$A(\theta) = A_0 + F(\theta) = \begin{pmatrix} 0 & 1 \\ -\lambda & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ Q(\theta) & 0 \end{pmatrix}.$$

We will use the notation $\partial_\omega \cdot = \frac{1}{2} \langle \nabla \cdot, \omega \rangle$, so that the equation (3.51) becomes

$$\partial_\omega Y(\theta) = -Y(\theta) A(\theta) + B Y(\theta).$$

This is a transformation problem with small divisors, coming both from the external frequencies of the forcing and the internal frequencies of the system, which one may try to solve by a Newton iteration of KAM type as long as F, the perturbation, is small.

In this case, it is natural to look for a transformation Y which is close to the identity and hence a B which is close to A_0. That is, we can consider the splitting

$$Y(\theta) = I + \hat{Y}(\theta), \qquad B = A_0 + \hat{B},$$

being $\hat{Y}$ and $\hat{B}$ the new unknowns. If we write equation (3.51) for these new functions, we get the equation

$$\partial_\omega \hat{Y}(\theta) = \hat{B} + \left[A_0, \hat{Y} \right] - F(\theta) + \left(\hat{B} \hat{Y}(\theta) - \hat{Y}(\theta) F(\theta) \right), \tag{3.52}$$

where $[\cdot,\cdot]$ is the Lie bracket. Linearizing this equation in the perturbation (that is forgetting about terms with square perturbations) we obtain

$$\partial_\omega \hat{Y}_1(\theta) = \hat{B} + [A_0, \hat{Y}_1] - F(\theta) + \left(\hat{B}\hat{Y}_1(\theta) - \hat{Y}_1(\theta)F(\theta)\right). \tag{3.53}$$

In order to get a solution (and a small one!) of this equation we need to use the Diophantine condition not only on the frequency vector, but also on the eigenvalues $\pm i\alpha_0$ of the unperturbed part A_0. For example, if we want to solve (3.53) up to an approximation of order $\varepsilon^{\frac{3}{2}}$ (being ε a small parameter controlling the size of the perturbation F) we need to impose the Diophantine condition

$$|\pm 2\alpha_0 - \langle \mathbf{k}\rangle| \geq \frac{\varepsilon^{\frac{1}{2}}}{|\mathbf{k}|^\tau}, \qquad 0 < |\mathbf{k}| < \frac{1}{r_0}\log(\frac{1}{\varepsilon}) \tag{3.54}$$

for a positive constant $\tau > 0$. These conditions are not fulfilled in general, but since α_0 depends on the energy λ, we may force the above condition to hold simply by excluding those λ for which $\alpha(\lambda)$ violates it. Hence, for the good λ's that remain, we can solve (3.53), and transforming the equation (3.48) by $Y_1(\theta) = I + \hat{Y}_1(\theta)$ gives the system

$$u' = (A_1 + F_1(\theta))\, u, \qquad \theta' = \omega,$$

where $A_1 = A_0 + \hat{B}$ and F_1 is of the size $\varepsilon^{\frac{3}{2}}$, that is, much smaller than F. Then we repeat this procedure with A_1 (whose eigenvalues $\pm i\alpha_1$ also depend on λ) instead of A_0. In this way one constructs a sequence of transformations $Y_j \cdot \dots \cdot Y_1$, for $j = 1, \dots, \infty$, that transforms the system (3.48) more and more closely to a constant coefficient system. Since the Y_j are closer and closer to the identity, it is seen (after some work) that the sequence converges to a transformation $Y(\theta)$ that solves the non-linear equation (3.51).

This approach, perhaps the most natural one, was used by Dinauburg and Sinai ([20], see also [14]) to prove the reducibility in a big set (in a measure sense) close to constant coefficients. However, this procedure requires that one restricts the parameters λ, and an essential point is to control the dependence on λ in order to ensure that we do not exclude too many or even all λ.

The restriction of parameters turns out to be imposed only by the use of transformations close to the identity, and the way to overcome this restriction (as it is done in [61] and [27]) is to enlarge the class of transformations. An *exponential over the torus* is a matrix-valued mapping $Z : \mathbb{T}^d \to SL(2,\mathbb{R})$ of the form

$$Z(\theta) = C^{-1}e^{z(\theta)}C, \qquad z(\theta) = \frac{\langle \mathbf{k}, \theta\rangle}{2}\begin{pmatrix} +i & 0 \\ 0 & -i \end{pmatrix},$$

where C is a non-singular matrix and $\mathbf{k} \in \mathbb{Z}^d$ is fixed. The effect of transforming the constant matrix system

$$u' = A_0 u, \qquad \theta' = \omega$$

by an exponential $u \mapsto Z(\theta)u$ that commutes with A_0 is simply to replace A_0 by

$$\tilde{A}_0 = \left(\alpha_0 + \frac{\langle \mathbf{k}, \omega\rangle}{2}\right)\frac{1}{\alpha_0}A_0.$$

In particular, if $2\alpha_0 + \langle \mathbf{k}, \omega\rangle \approx 0$ so that condition (3.54) is violated for A_0, then this condition will hold for $\tilde{A}_0$.

Hence, using transformations $Y_j(\theta) = Z_j(\theta) + \hat{Y}_j(\theta)$ that are close to exponentials, one can overcome the restriction imposed by (3.54). The perturbation theory of course becomes more complicated since the estimates are less good, but this can be handled. The approach works up to any order for all λ without exception, but when we want to go to the limit we run into problems. The reason is that an infinite product of exponentials may not converge, and it is only for almost every rotation number $\rho(\lambda)$ that one can prove the convergence of the sequence of transformations

$$Y_j \cdot \dots \cdot Y_1 \to Y.$$

In order to show that the exceptional rotation numbers $\rho(\lambda)$ (i.e. those for which we cannot guarantee convergence of the process) are avoided by almost every λ, one must control the dependence of the rotation number on λ. In a general linear quasi-periodic equation this would be difficult, because we have neither an independent characterization of the imaginary parts of the eigenvalues nor an easy way to control their variation in terms of the parameters of the system. However this turns out to be feasible in $SL(2,\mathbb{R})$ since one can use the rotation number together with some information on the function $\lambda \mapsto \rho(\lambda)$.

The exponential transformations were originally introduced by Moser and Pöschel in [61] and they were used by Eliasson ([27]) in a systematic way to prove the following theorem

Theorem 3.4.4 ([27]) *Assume that $\omega \in DC(N,\tau)$ and that $Q : \mathbb{T}^d \to \mathbb{R}$ is real analytic in a strip of width r. Then there exists a constant $C = C(\tau, r, N)$ such that if we define*

$$\lambda_0(s) = \begin{cases} \left(\frac{s}{C}\right)^2 & s \geq C \\ -\infty & s < C \end{cases}$$

then the following holds for $\lambda > \lambda_0(|Q|_r)$.

(i) If the rotation number $\rho(\lambda)$ is Diophantine or rational, then there exists a matrix $B = B(\lambda)$ in $sl(2,\mathbb{R})$ and an analytic matrix-valued function $Y : \mathbb{T}^d \to GL(2,\mathbb{R})$, also depending on λ, such that

$$X(t;\phi) = Y\left(\frac{\omega t}{2} + \phi\right) e^{Bt} Y(\phi)^{-1}.$$

(ii) If $\rho(\lambda)$ is neither Diophantine nor rational, then

$$\liminf_{|t|\to\infty} |X(t;\phi_0) - X(0;\phi_0)| < \frac{1}{2}|X(0;\phi_0)| \qquad \text{and} \qquad \lim_{|t|\to\infty} \frac{|X(t;\phi_0)|}{t} = 0,$$

for all $\phi_0 \in \mathbb{T}^d$.

Remark 3.4.5 *The result gives no information on the dependence of the matrix $B(\lambda)$ with respect to λ when λ is in a spectral gap. However, an iterative scheme can be done (see [61]) to obtain a family of matrices which are real analytic in the interior of the gap and have C^∞ extensions to the boundaries of the gap. On the other hand, such results can be obtained from the above result* a posteriori *([11]).*

Remark 3.4.6 *In the second item of theorem (3.4.4) it is essential to ask only for point convergence in ϕ_0, because the result comes from a perturbation method which is not absolutely convergent.*

The proof of this theorem as already sketched consists of three main steps. This will be a little bit more developed in the following subsections.

The small divisor lemma

Let $\mathcal{B}_r$ be the space of all analytic functions $F : \mathbb{T}^d \to gl(2, \mathbb{C})$ in a complex strip of width r for which

$$|F|_r = \sup_{|\mathrm{Im}\ \theta|<r} |F(\theta)| < \infty.$$

Then $|\cdot|_r$ is a norm making $\mathcal{B}_r$ a Banach space. Let

$$\mathcal{B} = \bigcup_{r>0} \mathcal{B}_r.$$

If F depends on a parameter $\lambda \in \Delta \subset \mathbb{R}$, we say that it is C^2 in λ whenever $\lambda \mapsto F_\lambda \in \mathcal{B}_r$ is C^2. As a notation, and assuming that the parameter is known, say λ, we will write $\partial^\nu F$ to denote the derivative with respect to this variable. We say that it is *piecewise* C^2 (pw.-C^2) in λ on some set $\Delta \subset \mathbb{R}$ if there exists a finite set $\{\lambda_i\}$ in Δ such that F is C^2 in $\Delta - \{\lambda_i\}$ and such that the right and left limits of $\delta^\nu F$, for $\nu = 0, 1, 2$, exist at all points λ_i, whenever such a limit makes sense.

For each $F \in \mathcal{B}_r$ we define

$$\hat{\sigma}(F) = \left\{ \mathbf{k} \in \mathbb{Z}^d; \hat{F}(\mathbf{k}) \neq 0 \right\},$$

where $\hat{F}(\mathbf{k})$ is the $\mathbf{k}$-th Fourier coefficient of F.

Lemma 3.4.7 (Small divisor lemma, [27]) *Let $A = A(\lambda) \in sl(2, \mathbb{C})$ have eigenvalues $\pm e(\lambda)$, and assume that $|A(\lambda) - \tilde{\lambda} J| < 3$ for some $\tilde{\lambda}(\lambda)$. Let $F \in \mathcal{B}_r$ and assume*

$$|i\langle \mathbf{k} \rangle \pm 2e(\lambda)| \geq K^{-1} |\mathbf{k}|^{-1}, \qquad \mathbf{k} \in \hat{\sigma}(F).$$

Then there exists a unique $Y \in \mathcal{B}$ such that

$$\partial_\omega Y = [A, Y] + F \qquad \text{and} \qquad \hat{\sigma}(Y) \subset \hat{\sigma}(F).$$

The matrix Y satisfies the estimate

$$|Y|_s \leq c \frac{K^2}{(r-s)^{3\tau}} |F|_r, \qquad \text{for} \qquad s < r,$$

where the constant c depends only on τ. Moreover, if $F, A \in \mathcal{B}_r$ are C^2 or pw.-C^2 in λ, then so is $Y \in \mathcal{B}_s$, and the following estimates hold for $s < r$

$$|\partial Y|_s \leq c \left[\frac{K^2}{(r-s)^{3\tau}} |\partial F|_r + \left(\frac{K^2}{(r-s)^{3\tau}} \right)^2 |\partial A| |F|_r \right],$$

$$\left|\partial^2 Y\right|_s \leq c \left[\frac{K^2}{(r-s)^{3\tau}} |\partial^2 F|_r + \left(\frac{K^2}{(r-s)^{3\tau}} \right)^2 (|\partial^2 A| |F|_r + |\partial A| |\partial F|_r) + \left(\frac{K^2}{(r-s)^{3\tau}} \right)^3 |\partial A|^2 |F|_r \right].$$

The proof uses standard estimates for the convergence on a complex strip of the torus of a formal solution of a system of linear homological equations of the type

$$(\partial_\omega u_{2l})(\theta) + u_{1l}(\theta) + u_{0l} = 0$$

for $1 \leq l \leq l_0$ and for all $\theta \in \mathbb{T}^d$ using the formal solution in terms of the Fourier series as we did in the first chapter.

The inductive lemma

Let $A \in sl(2,\mathbb{R})$. In particular, this means that the two eigenvalues of A coincide up to a sign and, since A is real, they are both real or purely imaginary. Hence the imaginary parts of the eigenvalues are $\pm i\alpha$ and we can fix the sign of α inducing an orientation. More precisely we shall require that

$$A = \alpha M J M^{-1},$$

being M a matrix with $\det M > 0$ and J the matrix

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

We shall call α the *rotation number* of A, because it is consistent with previous definitions. If $A = A(\lambda)$ is continuous on λ, then also $\alpha(\lambda)$ is continuous in λ (we already saw this in much more general situations), and if $A(\lambda)$ is pw-C^2 for $\lambda \in \Delta \subset \mathbb{R}$, then $\alpha(\lambda)$ is pw.-continuous on Δ (there is the problem of a collapse of the eigenvalues) and pw-C^2 on $\Delta - \alpha^{-1}(0)$.

We now come to some of the typical steps in KAM setup. Let $(r_j)_j$ be a decreasing sequence of positive numbers such that

$$r_j - r_{j+1} \geq 2^{-j}\frac{r_1}{2} \qquad \text{for} \qquad j \geq 1.$$

Let

$$\varepsilon_{j+1} = \varepsilon_j^{1+\sigma},$$

where $0 < \sigma < 1$, and let

$$N_j = \frac{2\sigma}{r_j - r_{j+1}} \log\left(\frac{1}{\varepsilon_j}\right)$$

for each j. From all these definitions we obtain the estimates

$$\varepsilon_j^{\sigma} \leq \left(\frac{4\sigma}{r_1(1+\sigma)} \log\left(\frac{1}{\varepsilon_1}\right)(2+2\sigma)^j\right)^{-4\tau} \leq N_j^{-4\tau} \tag{3.55}$$

for all j, assuming that ε_1 is small enough. For example, it suffices to consider

$$\varepsilon_1 \leq c r_1^{\left(\frac{4\tau}{\sigma}+1\right)},$$

for some small constant c that only depends on τ and σ. From now on all constants will be denoted by c and they will depend only on σ and τ, where σ is a fixed small number ($\sigma \leq 1/33$ will do).

Let now $A_1(\lambda) \in sl(2,\mathbb{R})$ and $F_1(\cdot,\lambda) \in \mathcal{B}_r$ be real and pw.-C^2 in $\lambda \in \Delta$. We assume that $\operatorname{tr}\ [F_1(\cdot,\lambda)]_{\mathbb{T}^d} = 0$, and

$$\begin{cases} \left|A_1(\lambda) - \tilde\lambda_1(\lambda)J\right| < 2, & \text{for some } \tilde\lambda_1 = \tilde\lambda_1(\lambda), \\ \left|\partial^\nu\left(A_1(\lambda)\right)\right| < \varepsilon_1^{-\nu\sigma} & \nu = 1,2, \\ \left|\partial^\nu F_1\right|_{r_1} < \varepsilon_1, & \nu = 0,1,2. \end{cases} \tag{3.56}$$

The sequences $(\varepsilon_j)_j$, $(r_j)_j$, $(N_j)_j$ (which are strongly interdependent) will be of fundamental importance in the sequel. The sequence $(N_j)_j$ will measure number of resonances that will have to be taken into account at each step of the inductive process. The sequence $(r_j)_j$ will measure the width of the analyticity strip of the mappings that we will recursively define on the torus. Finally the sequence $(\varepsilon_j)_j$ will determine the sets of $\lambda \in \Delta$ for which we can apply a new transformation.

With these preliminaries we can state the main lemma that will be used to prove the theorem.

Lemma 3.4.8 (Inductive lemma, [27]) *There exists a constant $C = C(\tau, \sigma)$ such that, if $\varepsilon_1 < C r_1^{((4\tau/\sigma)+1)}$, then for all $j \geq 1$ there exists $F_{j+1} \in \mathcal{B}$, Y_{j+1} and A_{j+1}, the latter in $sl(2,\mathbb{R})$ and both real and pw.-C^2 in λ and with tr $[F_{j+1}]_{\mathbb{T}^d} = 0$, verifying the equation*

$$\left\langle Y'_{j+1}\left(\frac{\theta}{2}\right), \frac{\omega}{2}\right\rangle = \left(A_j + F_j(\theta)\right) Y_{j+1}\left(\frac{\theta}{2}\right) - Y_{j+1}\left(\frac{\theta}{2}\right)\left(A_{j+1} + F_{j+1}(\theta)\right). \tag{3.57}$$

Let α_j be the rotation number of A_j and let $\Lambda_j(\mathbf{k})$ be a finite union of intervals such that

$$\left\{\lambda : |2\alpha_j(\lambda) - \langle \mathbf{k}\rangle| < \varepsilon_j^{\sigma}\right\} \subset \Lambda_j(\mathbf{k}) \subset \left\{\lambda : |2\alpha_j(\lambda) - \langle \mathbf{k}\rangle| < 2\varepsilon_j^{\sigma}\right\}$$

for $0 < |\mathbf{k}| \leq N_j$. Let also

$$\Lambda_j(0) = \Delta - \bigcup_{0<|\mathbf{k}|\leq N_j} \Lambda_j(\mathbf{k}),$$

and $r_j - r_{j+1} = r_j/2$, if $\lambda \notin \Lambda_j(0)$, and $r_j - r_{j+1} = 2^{-j} r_1/2$ if $\lambda \in \Lambda_j(0)$. Then we have the following estimates:

$$\left|\partial^{\nu}\left(Y_{j+1}(\cdot,\lambda) - \exp\left\{\frac{\langle \mathbf{k}\rangle}{\alpha_j(\lambda)} A_j(\lambda)\right\}\right)\right|_{r_{j+1}} < \varepsilon_j^{1/2}, \qquad \nu = 0,1,2, \qquad \lambda \in \Lambda_j(\mathbf{k}); \tag{3.58}$$

$$\left|\partial^{\nu} F_{j+1}\right|_{r_{j+1}} < \varepsilon_{j+1}, \qquad \nu = 0,1,2; \tag{3.59}$$

$$\left|\partial^{\nu}\left(A_{j+1}(\lambda) - \left(1 - \frac{\langle \mathbf{k}\rangle}{2\alpha_j(\lambda)}\right) A_j(\lambda)\right)\right| < \varepsilon_j^{2/3}, \qquad \nu = 0,1,2, \qquad \lambda \in \Lambda_j(\mathbf{k}); \tag{3.60}$$

$$\left\{\begin{array}{ll} |\partial^{\nu} A_{j+1}| < \varepsilon_{j+1}^{-\nu\sigma}, & \nu = 1,2 \\ |A_{j+1}(\lambda)| < 32\,|\alpha_{j+1}(\lambda)|\, N_{j+1}^{\tau} & \text{if } |\alpha_{j+1}(\lambda)| \geq \frac{1}{4} N_{j+1}^{-\tau}. \end{array}\right. \tag{3.61}$$

The idea of the lemma is the following. We assume that A_j and F_j satisfy the inequalities (3.58)-(3.61) for j and then we try to find Y_{j+1}, A_{j+1} and F_{j+1} satisfying (3.57) with the required properties. In this inductive scheme the first step is to check that all this is true for the first iteration.

Clearly, the first estimate of (3.61) is fulfilled by assumption, and the second one is trivial unless $|A_1| \geq 8$. In this case, the initial assumptions just imply that $|\tilde{\lambda}_1(\lambda)| > 6$, because of (3.56), and hence $|\alpha_1| > |\tilde{\lambda}_1| - 2$. Now this gives immediately $|A_1| < 2|\alpha_1|$ which implies the second estimate of (3.61) for $j = 1$.

The core of the proof is to control the behaviour of the sets $\Lambda_j(\mathbf{k})$ for $0 < |\mathbf{k}| < N_j$. Note that the convergence of the sequence $(\alpha_j(\lambda))_j$ to the rotation number of λ will imply the convergence of the sets $\Lambda_j(\mathbf{k})$ to the set of λ's with rational rotation number $\langle \mathbf{k}, \omega\rangle/2$ (for which we want to prove reducibility). On the other hand the set $\Lambda_j(0)$ will converge to a subset of the spectrum and of Δ, which, as we will see, will be the set of those λ's with Diophantine rotation number. All this makes it necessary to distinguish between the cases $\lambda \in \Lambda_j(0)$ and $\lambda \notin \Lambda_j(0)$.

Case 1. Suppose that $\lambda \in \Lambda_j(0)$. Then, from the definition of this set, we have that for all $0 < |\mathbf{k}| < N_j$,

$$|2\alpha_j(\lambda) - \langle \mathbf{k}\rangle| \geq \varepsilon_j^{\sigma}.$$

Moreover, by the conditions (3.61) for previous $k \le j$, we have that $|A_j(\lambda) - \tilde{\lambda}(\lambda)J| < 3$ for $\tilde{\lambda}_j = 0$ or $\tilde{\lambda}_1$. Indeed, if

$$\lambda \in \bigcap_{1}^{j-1} \Lambda_k(0),$$

then we must have that

$$\left| A_j(\lambda) - \tilde{\lambda}_1 J \right| < 2 + \varepsilon_1^{2/3} + \cdots + \varepsilon_{j-1}^{2/3} < 3.$$

On the other hand, if

$$\lambda \in \bigcap_{k+1}^{j-1} \Lambda_l(0) \cap \Lambda_k(\mathbf{k}),$$

with $\mathbf{k} \neq 0$, then

$$|A_j(\lambda)| < \varepsilon_k^{2/3} + \cdots + \varepsilon_{j-1}^{2/3} + \left| \left(1 - \frac{\langle \mathbf{k} \rangle}{2\alpha_k}\right) A_k \right| < 2\varepsilon_k^{2/3} + 2\varepsilon_k^{\sigma} \left| \frac{A_k}{2\alpha_k} \right| < 2\varepsilon_k^{2/3} + \varepsilon_k^{\sigma} 32 N_k^{\tau} < 34 N_k^{\tau} \varepsilon_k^{\sigma}, \tag{3.62}$$

by the assumption (3.61) for k, because $\lambda \in \Lambda_k(\mathbf{k})$ implies that, if $\mathbf{k} \neq 0$, then

$$|2\alpha_k| \ge N_k^{-\tau} - 2\varepsilon_k^{\sigma} \ge \frac{1}{2} N_k^{-\tau}.$$

Let now G be the truncated Fourier series

$$G(\theta) = \sum_{0 < |\mathbf{n}| \le N_j} \hat{F}_j(\mathbf{n}) e^{i\langle \mathbf{n}, \theta \rangle},$$

and define Y as the solution of

$$\partial_\omega Y - [A_j, Y] = G, \qquad \hat{\sigma}(Y) \subset \hat{\sigma}(G),$$

which exists uniquely by lemma 3.4.7. Then we define

$$\begin{cases} Y_{j+1}(\theta) = I + Y(2\theta) \\ A_{j+1} = A_j + \hat{F}_j(0) \\ F_{j+1} = (I+Y)^{-1} \left[F_j Y - Y \hat{F}_j(0) + \left(F_j - G - \hat{F}_j(0) \right) \right]. \end{cases}$$

These matrices are pw.-C^2 in $\lambda \in \Lambda_j(0)$ and tr $A_{j+1} = 0$. Moreover, they satisfy the equation (3.57) and, therefore

$$\text{tr } [F_{j+1}]_{\mathbb{T}^d} = [\text{ tr } F_{j+1}]_{\mathbb{T}^d} = \left[\text{ tr } Y_{j+1}^{-1}(A_j + F_j) Y_{j+1} \right]_{\mathbb{T}^d} - \left[\text{ tr } Y_{j+1}^{-1} \partial_\omega Y \right]_{\mathbb{T}^d}$$

$$= -\left[\text{ tr } \left(\sum (-1)^k Y^k \right) \partial_\omega Y \right]_{\mathbb{T}^d} = -\sum (-1)^k \frac{1}{k+1} \text{ tr } \left[\partial_\omega (Y^{k+1}) \right]_{\mathbb{T}^d} = 0,$$

so we only need to consider the estimates involved in the lemma.

In order to do so, we note that, reducing the analyticity band

$$|\partial^\nu G|_s < c\varepsilon_j N_j^{\tau+1}, \qquad s = r_{j+1} + \frac{r_j - r_{j+1}}{2}.$$

Then by lemma 3.4.7 and (3.61) for j we get the estimate

$$|\partial^{\nu} Y|_{r_j+1} < c\varepsilon_j^{1-(2+3\nu)\sigma} N_j^{\tau(1+3(\nu+1))+1}, \qquad \nu = 0, 1, 2$$

and this gives (3.58) for $j+1$. To prove (3.60) for $j+1$ one must observe that

$$\left|\partial^{\nu}(F_j - G - \hat{F}_j(0))\right|_{r_{j+1}} < c\varepsilon_j^{1+2\sigma} N_j^{\tau+1}$$

and that

$$\left|(I+Y)^{-1}\right|_{r_{j+1}} < 2,$$

which, differentiating F_{j+1} and after some manipulation gives the desired bound.

The estimates (3.61) and the first one of (3.61) for $j+1$ are an easy consequence of the definition of A_{j+1} and (3.60), already shown. The second estimate of (3.61) must only be considered whenever $|A_{j+1}(\lambda)| \geq 8$, similarly to the first step of the iteration. In this case we have that $\lambda \in \cap_{k=1}^{j} \Lambda_k(0)$, i.e.

$$|A_{j+1}(\lambda) - \tilde{\lambda}_1 J| < 2 + 2\varepsilon_1^{2/3} < \frac{5}{2}.$$

Hence $|\tilde{\lambda}_1| > 5$. But then $|\alpha_{j+1}(\lambda)| \geq |\tilde{\lambda}_1| - \frac{5}{2} > \frac{|\tilde{\lambda}_1|}{2}$. Hence

$$\left|\frac{A_{j+1}(\lambda)}{\alpha_{j+1}(\lambda)}\right| \leq 2 + \frac{5}{|\tilde{\lambda}_1|} \leq 3,$$

which proves the second estimate of (3.61) for $j+1$ and case 1.

Case 2: Suppose now that $\lambda \in \Lambda_j(\mathbf{k})$, with $\mathbf{k} \neq 0$, and let

$$Z(\theta) = \exp\left\{\frac{\langle \mathbf{k}, \theta \rangle}{2\alpha_j(\lambda)} A_j(\lambda)\right\},$$

that is, Z is an exponential in the sense defined before. Then the homological equation, as already stated, becomes

$$\partial_{\omega} Z = (A_j + F_j)Z - Z(B_j + G_j),$$

where

$$B_j = \left(1 - \frac{\langle \mathbf{k} \rangle}{2\alpha_j}\right) A_j, \qquad G_j = Z^{-1} F_j Z.$$

Note that G_j can be defined on $\mathbb{T}^d$ instead of $(2\mathbb{T})^d$, because it can be expressed in terms of F_j (which is defined on $\mathbb{T}^d$) and of products of two elements of Z_j (which are defined on $(2\mathbb{T})^d$ and thus products of two elements are defined on $\mathbb{T}^d$). The rotation number of B_j is $\beta_j = \alpha_j - \frac{\langle \mathbf{k} \rangle}{2}$, and satisfies

$$|2\beta_j - \langle \mathbf{n} \rangle| \geq (5N_j)^{-\tau} - 2\varepsilon_j^{\sigma} \geq 2\varepsilon_j^{\sigma}, \qquad 0 < |\mathbf{n}| \leq 5N_j,$$

because of the definition of β_j and the condition $\lambda \in \Lambda_j(\mathbf{k})$. Moreover, by (3.61) for j, we have that

$$|B_j| < 2\varepsilon_j^{\sigma} \left|\frac{A_j}{2\alpha_j}\right| < 1.$$

Let now G be the following truncated Fourier series (which is different from the first one)

$$G(\theta) = \sum_{0 < |\mathbf{n}| \leq 5N_j} \hat{G}_j(\mathbf{n}) e^{i\langle \mathbf{n}, \theta \rangle}.$$

Then we let Y be the unique solution of

$$\partial_\omega Y - [B_j, Y] = G, \qquad \hat{\sigma}(Y) \subset \hat{\sigma}(G),$$

and we define

$$\begin{cases} Y_{j+1}(\theta) = Z(2\theta)\,(I + Y(2\theta)) \\ A_{j+1} = B_j + \hat{G}_j(0) \\ F_{j+1}(\theta) = (I + Y(2\theta))^{-1}\left[G_j(\theta)Y(2\theta) - Y(2\theta)\hat{G}_j(0) + (G_j(\theta) - G(\theta) - \hat{G}_j(0))\right]. \end{cases}$$

As in the first case, all the requirements are fulfilled. The estimates, however, become a bit more tedious, because of the transformation to an exponential.

Sketch of proof

We now sketch the proof of the theorem 3.4.4. Each λ belongs to a unique set $\cap\Lambda_j(\mathbf{k}_j)$, for a sequence $0 \leq |\mathbf{k}_j| \leq N_j$ (because of the estimates in the inductive lemma) which may be void. Moreover, starting the iteration with our initial setting, it is clear that

$$r_j \to r_0 \geq 0, \qquad |F_j|_{r_j} \to 0 \qquad \text{and } A_j \to A \text{ pointwise}$$

as $j \to \infty$.

Suppose now that λ is such that $\mathbf{k}_j = 0$ for all j sufficiently large. For such λ we can conclude that the product

$$\prod_{j=2}^{n} Y_j \to Y$$

converges in $|\cdot|_{r_0}$ with $r_0 > 0$. If λ is not of this type, the convergence is unclear. However we have that

$$|Y_j(0) - I| < \varepsilon_j^{1/2}$$

for all j, so the product above is convergent at zero. But more is true. For each λ, the product

$$\prod Y_j\left(\frac{\omega}{2}t\right)$$

converges uniformly on compact intervals of $\mathbb{R}$. In order to see this we only need to note that

$$\left|Y_{j+1}\left(\frac{\omega}{2}t\right) - I\right| \leq \varepsilon_j^{1/2} + |Z_j(\omega t) - I|,$$

where

$$Z_j(\omega t) = \exp\left(\frac{\langle \mathbf{k}_j \rangle}{2\alpha_j} A_j t\right) = \cos\left(\frac{\langle \mathbf{k}_j \rangle}{2} t\right) I + \sin\left(\frac{\langle \mathbf{k}_j \rangle}{2} t\right)\frac{A_j}{\alpha_j},$$

by the first estimate of (3.58).

Now, if $\mathbf{k}_j = 0$ then the bound will be for all t, and if $\mathbf{k}_j \neq 0$, then, also by the lemma, $|2\alpha_j - \langle \mathbf{k}_j \rangle| < 2\varepsilon_j^\sigma$, which implies that $|A_j| < 32|\alpha_j| N_j^\tau$ by (3.61). Let now $j_1 < j_2 < \ldots$ be all the indices j for which $\mathbf{k}_j \neq 0$. Then, after some estimates (see [27]) it is seen that for t with $tN_{j_k}^\tau \varepsilon_{j_k}^\sigma$ small we get

$$|Z_{j_{k+1}}(\omega t) - I| < 70 t N_{j_k}^\tau \varepsilon_{j_k}^\sigma, \tag{3.63}$$

which shows that the product converges uniformly for t bounded.

Now from the items of theorem 3.4.4 it only remains to show two things. The first one is that up to now, the conditions on the rotation number ρ have only been written in terms of the sequence α_j and more specifically in terms of the sequence $\mathbf{k}_j$ for which $\lambda \in \Lambda_j(\mathbf{k}_k)$. Recall that we were only able to show uniform convergence on a strip of the torus whenever $\mathbf{k}_j = 0$ for all j large enough. We must now relate this condition with the arithmetical properties of the rotation number ρ. The second item which remains to do is to perform the initial transformations to render the original system, either with large energy or close to constant coefficients, to a suitable form before applying the iterative process.

Let now X_1 be a solution of the system

$$X_1'(t) = (A_1 + F_1(\omega t))\, X_1(t).$$

Then, by lemma 3.4.8 we have a representation

$$X_1(t) = Y\left(\frac{\omega}{2}t\right) e^{At},$$

disregarding the kind of convergence of the product of functions Y_j on the torus $\mathbb{T}^d$, which will be decided in a moment in terms of the rotation numbers involved. Moreover, if X_1 has rotation number $\tilde{\rho}(\lambda)$, then we conclude from this representation that

$$\tilde{\rho}(\lambda) = \frac{1}{2}\sum_{j=1}^{\infty}\langle \mathbf{k}_j\rangle + \alpha(\lambda),$$

for $\lambda \in \cap\Lambda_j(\mathbf{k}_j)$ and $\alpha(\lambda) = \lim \alpha_j(\lambda)$. Let

$$\rho_{j+1} = \alpha_{j+1} + \frac{1}{2}\sum_{k=1}^{j}\langle \mathbf{k}_k\rangle, \qquad \lambda \in \bigcap_{k=1}^{j} \Lambda_k(\mathbf{k}_k).$$

That the sequence $(\rho_j)_j$ converges uniformly in λ is the content of the following lemma:

Lemma 3.4.9 ([27]) *(i)* $|\rho_{j+1} - \rho_j| < c\varepsilon_j^{1/4}$ *for all* j. *In particular, the sequence* $\{\rho_j\}_j$ *converges uniformly to the limit* $\tilde{\rho}$.

(ii) If $\lambda \in \bigcap_1^\infty \Lambda_j(\mathbf{k}_j)$ *and* $\tilde{\rho}(\lambda)$ *is Diophantine or rational, then* $\mathbf{k}_j = 0$ *for all* j *sufficiently large.*

Proof: For the first item (i) it suffices to show that

$$\left|\alpha_{j+1}(\lambda) - \left(\alpha_j(\lambda) - \frac{\langle \mathbf{k}\rangle}{2}\right)\right| < c\varepsilon_j^{1/4}$$

for $\lambda \in \Lambda_j(\mathbf{k})$, which follows from the condition (3.61) for $j+1$.

For (ii) we proceed by contradiction. Suppose that

$$|2\tilde{\rho}(\lambda) - \langle \mathbf{k}\rangle| \geq K^{-1}|\mathbf{k}|^{-s}, \qquad |\mathbf{k}| \geq N,$$

for some N, K, as $s > 0$. Suppose also that there exists an increasing sequence $(j_k)_k$, such that $\mathbf{k}_{j_k} \neq 0$. Hence, by the definition of the sets $\Lambda_j(\mathbf{k})$,

$$|2\alpha_{j_k}(\lambda) - \langle \mathbf{k}_{j_k}\rangle| < 2\varepsilon_{j_k}^{\sigma}.$$

Then

$$\left|2\tilde{\rho}(\lambda) - \sum_{1}^{j_k}\langle \mathbf{k}_l\rangle\right| \leq 2|\tilde{\rho}(\lambda) - \rho_{j_k}(\lambda)| + |2\alpha_{j_k}(\lambda) - \langle \mathbf{k}_{j_k}\rangle| < 4\varepsilon_{j_k}^{\sigma}.$$

On the other hand, due to the Diophantine assumptions,

$$\left|\tilde{\rho}(\lambda) - \sum_{1}^{j_k}\langle \mathbf{k}_l\rangle\right| \geq K^{-1}(N_{j_1} + \cdots + N_{j_k})^{-s} \geq cK^{-1}N_{j_k}^{-2s},$$

because it can be shown that $N_{j_l}^3 < cN_{j_{l+1}}$. Now this implies that

$$\varepsilon_{j_k}^{-\sigma} \leq \text{const.}\left(\log\left(\frac{1}{\varepsilon_1}\right)(2+2\sigma)^{j_k}\right)^{2s}$$

for infinitely many k. This is incompatible with $\tilde{\rho}(\lambda)$ being either rational of Diophantine, due to condition (3.55) in the setup. □

To end the proof of theorem 3.4.4 we must distinguish between the case of high energies and the case of a small potential in order to perform the preliminary transformations. We sketch the case of a small potential.

Let $\lambda_0(s)$ as in theorem 3.4.4, and define

$$\mathbf{C}(\tau, r) = C(\tau, \sigma)r^{((4\tau/\sigma)+1)}$$

for $\sigma = 1/33$ say, where C is the constant of lemma 3.4.8. Suppose that $|Q|_r < \mathbf{C}(r, \tau)$, and let $\lambda \in (-1, 1)$. Now equation (3.48) can be expressed as

$$X_1'(t) = (A_1(\lambda) + F_1(\omega t, \lambda))\, X_1(t),$$

with $A_1 = B$ and F_1 as defined in the preliminaries and X_1 having rotation number $\rho(\lambda) = \tilde{\rho}(\lambda)$. Moreover

$$\left\{\begin{array}{ll} |A_1(\lambda)| < 2 & \\ \left|\left(\frac{\partial}{\partial\lambda}\right)^{\nu} A_1(\lambda)\right| < \varepsilon_1^{-\nu\sigma}, & \nu = 1, 2 \\ \left|\left(\frac{\partial}{\partial\lambda}\right)^{\nu} F_1(\cdot, \lambda)\right|_r < \varepsilon_1 = C(\tau, \sigma)r^{((4\tau/\sigma)+1)} & \nu = 0, 1, 2 \end{array}\right.$$

and now the result follows from an application of lemmas 3.4.8 and 3.4.9. If $\lambda \leq -1$, then we are in the resolvent set if $|Q|_r < 1$ and the result is also true. If $\lambda \geq 1$, then this case is covered by the large potential, which is treated performing the transformations defined in the preliminaries.

It only remains to prove the *non-absolutely convergent* part of the theorem, which gives a bound for the rate of escape of points in the spectrum with rotation number $\tilde{\rho}(\lambda)$ being neither Diophantine nor rational. In this case, we let $\lambda \in \Lambda_j(\mathbf{k}_j)$. If $\mathbf{k}_j = 0$ for j large enough, then X, the fundamental matrix, has a Floquet representation, so the result trivially follows. Assume, therefore, that there exists an increasing sequence of integers $1 \leq j_1 < j_2 < \dots$ such that $\mathbf{k}_{j_k} \neq 0$. Then we can apply any number of steps of the iterative process (because the frequencies of the forcing are Diophantine and we have some control on the set of bad rotation numbers) to deduce that $\lim A_j(\lambda) = 0$. Therefore, we only need to worry about the limit of the product $\prod Y_j(\frac{\omega}{2}t)$.

Let now $t = t_1 + \cdots + t_k$, for $k \geq 2$ with $|t_l| \leq 4\pi N_{j_l}^{\tau}$ for all l. Choose

$$t_k = \frac{4\pi}{\langle \mathbf{k}_{j_k}\rangle}$$

and t_{k-1} so that

$$t_{k-1} + t_k = n\frac{4\pi}{\langle \mathbf{k}_{j_{k-1}} \rangle}, \qquad \text{for some } n,$$

and so on. Then we get that, by construction, $Z_{j_1}(\omega t) = I$ and, for all j_l, with $l \geq 1$,

$$|Z_{j_{l+1}}(\omega t) - I| = |Z_{j_{l+1}}(\omega T_l) - I| < 70 T_l N_{j_l}^{\tau} \varepsilon_{j_l}^{\sigma}$$

as in (3.63), where

$$T_l = \begin{cases} t_1 + \cdots + t_l & l \leq k-1 \\ t_1 + \cdots + t_k & l \geq k. \end{cases}$$

Since

$$|T_l| \leq 4\pi (N_{j_1}^{\tau} + \ldots N_{j_l}^{\tau}) \leq 4\pi N_{j_l}^{2\tau},$$

and

$$\left| Y_{j+1}\left(\frac{\omega}{2} t\right) - I \right| < \varepsilon_j^{1/2}$$

for all $j \neq j_l$ it follows that

$$\left| \prod Y_j \left(\frac{\omega}{2} t\right) - I \right| < c\varepsilon_1^{\sigma/4},$$

and this gives the first part of the theorem 3.4.4(ii). We now go to the second part.

Let t be such that

$$\frac{1}{|\langle \mathbf{k}_{j_k} \rangle|} \leq t < \frac{1}{|\langle \mathbf{k}_{j_{k+1}} \rangle|}, \qquad k \geq 2.$$

Then, by (3.60) and (3.63) we have the following possibilities

$$\left| Y_{j+1}\left(\frac{\omega}{2} t\right) - I \right| < c \begin{cases} \varepsilon_j^{1/2} & j \neq j_l & \\ N_{j_{l-1}}^{2\tau} \varepsilon_{j_{l-1}}^{\sigma} & j = j_l, & l \geq k+2 \\ (1 + |A_j| |\langle \mathbf{k}_j \rangle|^{-1}) & j = j_l, & l \leq k \\ (1 + |tA_j|) & j = j_{k+1}, & \end{cases}$$

Therefore,

$$\left| \prod_2^{k+1} Y_{j_{l+1}} \left(\frac{\omega}{2} t\right) \right| \leq c^k \prod_2^k \left(\left| \frac{\langle \mathbf{k}_{j_l} \rangle}{\langle \mathbf{k}_{j_{l-1}} \rangle} \right| + \left| \frac{A_{j_l}}{\langle \mathbf{k}_{j_{l-1}} \rangle} \right| \right) \left| \frac{\langle \mathbf{k}_{j_l} \rangle}{\langle \mathbf{k}_{j_k} \rangle} \right| (1 + |tA_{j_{k+1}}|), \tag{3.64}$$

and since

$$\left| \frac{\langle \mathbf{k}_{j_l} \rangle}{\langle \mathbf{k}_{j_{l-1}} \rangle} \right| \leq \frac{2\varepsilon_{j_l}^{\sigma} + 2|A_{j_l}|}{|\langle \mathbf{k}_{j_{l-1}} \rangle|} \leq 70 N_{j_{l-1}}^{2\tau} \varepsilon_{j_{l-1}}^{\sigma}$$

by (3.62), the above expression converges to 0 as $k \to \infty$. This together with the bound $2|\langle \mathbf{k}_{j_1} \rangle||t|$ for the last term of (3.64) proves the desired limit

$$\lim_{t \to \infty} \frac{|X(t) - X(0)|}{t} = 0.$$

□

3.4.3 Some more results and applications

Eliasson's paper [27] contains many other interesting results. The first one that we state is an immediate corollary of theorem 3.4.4

Corollary 3.4.10 *For almost every $\lambda \in \sigma(H) \cap (\lambda_0, +\infty)$, the following inequality holds*

$$2\rho(\lambda)\rho'(\lambda) \geq 1.$$

In particular, the set of all $\lambda > \lambda_0$ for which $\rho(\lambda)$ is neither Diophantine nor rational is of measure 0.

The reason for this is the following. It is shown in [18] that $2\rho(\lambda)\rho'(\lambda) \geq 1$ for almost all λ belonging to the set

$$\{\lambda \in \sigma(H); \beta(\lambda) = 0\},$$

where $\beta(\lambda)$ is the upper Lyapunov exponent of (3.1). If now $\rho(\lambda)$ is Diophantine, then $\beta(\lambda) = 0$ by the first item of theorem 3.4.4 and if $\rho(\lambda)$ is neither Diophantine nor rational, then by the second item of the same theorem we also conclude that $\beta(\lambda) = 0$. Hence $\beta(\lambda) = 0$ for almost all λ in the spectrum and in the domain of application of the theorem (that is, either in the upper part of the spectrum or close to constant coefficients).

The following theorem relies on the techniques of 3.4.4 but requires additional work. It deals with the description of the endpoints (both collapsed and not) of spectral gaps and the existence of non-reducible situations in the domain $\lambda > \lambda_0(|Q|_r)$.

Theorem 3.4.11 ([27]) *For $\lambda > \lambda_0(|Q|_r)$ the following holds:*

(i) The matrix $A = A(\lambda) = 0$ if $\{\lambda\}$ is a collapsed gap and it is nilpotent different from zero if $\{\lambda\}$ is the endpoint of a non-collapsed gap.

(ii) For a generic set of Q's in the $|\cdot|_r$-topology, there exist values of $\lambda > \lambda_0(|Q|_r)$ for which the fundamental matrix is unbounded and $\rho(\lambda)$ is neither Diophantine nor rational.

We will come back on the second item in section 3.5, because it is a result on non-reducibility.

Finally, in [27], the following result on the nature of the spectrum of Schrödinger equation with quasi-periodic forcing is also shown

Theorem 3.4.12 ([27]) *For $\lambda > \lambda_0(|Q|_r)$ the following holds*

(i) For a generic potential Q in the $|\cdot|_r$-topology, $\sigma(H) \cap (\lambda_0(|Q|_r), +\infty)$ is a Cantor set.

(ii) $\sigma(H) \cap (\lambda_0(|Q|_r), +\infty)$ is purely absolutely continuous. In particular, there are no point eigenvalues in $(\lambda_0(|Q|_r), +\infty)$.

The first item follows from theorem 3.4.4. From this theorem it also follows that if $\lambda > \lambda_0(|Q|_r)$ no point spectrum can occur. The second item requires some effort, because of the possible existence of some singular continuous spectrum in $\sigma(H) \cap (\lambda_0(|Q|_r), +\infty)$. To rule this out it is shown that all spectral measures are absolutely continuous with respect to the Lebesgue measure on the set $(\lambda_0(|Q|_r), +\infty)$.

3.5 Some remarks on non-reducibility in Schrödinger equation with quasi-periodic potential

Up to now we have discussed the reducibility of Schrödinger equation with quasi-periodic potential, and the results, in the case of analytic potential and Diophantine frequencies are quite definitive either close to constant coefficients or with high energies (they can both be written as perturbations of systems with constant coefficients). It is therefore natural to look for *converse* results. This means looking for conditions under which there is no reducibility, and to try to understand the transition between reducibility and non-reducibility. The results in this direction are not yet complete, especially when trying to understand the transition from reducibility to non-reducibility. Therefore, and as a thorough discussion of these results would take us too far, we will give only some results, together with some speculation.

We begin this section with an example of a non-reducible Schrödinger equation.

An example with point eigenvalues

In this subsection we will give an example due to R. Johnson and J. Moser ([44]) exhibiting point eigenvalues. Recall that a point λ in the spectrum $\sigma(H)$ of a Schrödinger equation is said to be a point eigenvalue if for this value of λ there is a solution belonging to $L^2(-\infty,+\infty)$. It is clear that point eigenvalues imply non-reducibility, because solutions of a reducible system cannot tend to zero backward and forward in time.

Examples of quasi-periodic Schrödinger equations exhibiting this behaviour were given by Gordon ([33]), but there the frequency vector was not Diophantine. The next example will be less *exceptional.*

The construction of this example depends on finding an odd quasi-periodic function

$$f(t) = F(\omega_1 t, \omega_2 t)$$

with

$$F(\theta_1, \theta_2) = \sum_{n=1}^{\infty} a_n \sin(j_n\theta_1 - k_n\theta_2) \tag{3.65}$$

for which

$$g(t) = \int_0^t f(s)ds \geq c|t|^{1-\delta} \qquad \text{for} \qquad |t| \geq 1, c > 0 \tag{3.66}$$

holds for some constant $\delta \in (0,1)$. Then the function

$$\phi(t) = e^{-g(t)}$$

belongs to $L^2(-\infty,\infty)$ and is a solution of $\phi'' - q\phi = 0$ with

$$q = f' + f^2.$$

Hence ϕ is an eigenfunction for this q with eigenvalue $\lambda = 0$. To construct such a function (3.66) we set

$$\omega_1 = \frac{1}{e} = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!}, \qquad \omega_2 = 1,$$

and one checks that e^{-1} satisfies the following Diophantine condition

$$|\langle \mathbf{k}, \omega \rangle| \geq \frac{c}{|\mathbf{k}|^\sigma} \tag{3.67}$$

for any $\sigma > 2$ and a suitable c. We set

$$j_n = (n-1)!, \qquad k_n = (n-1)! \sum_{m=0}^{n-1} \frac{(-1)^m}{m!}$$

$$\varepsilon_n = |j_n \omega_1 - k_n \omega_2| = (n-1)! \left| \sum_{m=0}^{n-1} \frac{(-1)^m}{m!} \right|$$

and with $\delta \in (0,1)$,

$$a_n = (j_n \omega_1 - k_n \omega_2) \varepsilon_n^\delta.$$

We can easily estimate ε_n by

$$\frac{1}{n+1} < \varepsilon_n < \frac{1}{n}.$$

Therefore ε_n is a monotone decreasing sequence and

$$|a_n| = \varepsilon_n^{1+\delta} < \frac{1}{n^{1+\delta}}.$$

Thus the sum in (3.66) is absolutely convergent and f is defined. We can say even more. It is a real analytic function of t since for complex t the sum

$$\sum_{n=1}^{\infty} |a_n| \sinh(\varepsilon_n |\mathrm{Im}\ t|) < \infty$$

is convergent. Therefore q is also real analytic and it only remains to prove the estimate (3.66). This comes from noting that

$$1 - \cos \varepsilon_n t \geq 1 \qquad \text{for} \qquad \frac{\pi}{2\varepsilon_n} \leq t \leq \frac{\pi}{\varepsilon_n}.$$

It should be pointed out that the function F on the torus is only continuous and not smooth, even though f is real analytic. Hence also the extension Q of q is not smooth.

Some speculation

The previous example illustrates a common mechanism for the loss of reducibility: there are solutions of Schrödinger equation, with fixed energy and potential, which belong to $L^2(\mathbb{R})$. There is a stronger situation which is *Anderson localization.* To state this properly, consider the Schrödinger equation for a wave function $\psi(t,x)$:

$$-i\frac{\partial \psi}{\partial t} + H_\phi \psi = 0, \qquad \psi(0,x) = \psi_0(x). \tag{3.68}$$

We can relate some properties of the Spectrum of H to the spread of the wave function. To measure the latter we introduce, for instance, the quantity

$$r(t) = \int_{\mathbb{R}} |x\psi(t,x)|^2 dx.$$

We shall say that there is *Anderson localization* whenever

$$r^2(t) \leq \text{constant};$$

it follows from the RAGE theorem (see [13] and references therein) that in this case the operator H_ϕ has pure-point spectrum. Pure-point spectrum for almost every $\phi \in \mathbb{T}^d$ implies that the dynamical behaviour of the solutions of the lifted system

$$z' = \begin{pmatrix} 0 & 1 \\ -\lambda + Q(\theta) & 0 \end{pmatrix} z, \qquad \theta' = \omega$$

is *non-uniformly hyperbolic* in $\mathbb{R}^2 \times \mathbb{T}^d$: the system has non-zero Lyapunov exponents but no (continuous) hyperbolic structure. This is a famous result of Kotani (see [50] and also the references in [13]). For more general potentials, such as random potentials, Anderson localization seems to be the dominant situation.

There are many results on non-uniformly hyperbolic behaviour of the solutions of Schrödinger equation, both for the continuous case (ours) and the discrete case. This latter case comes when we approximate the continuous operator H by its finite-difference approximation

$$(H^d\psi)(n) = -\left(\psi(n+1) + \psi(n-1)\right) + Q(\phi + n\omega)\psi(n),$$

and the Hilbert space is now $l^2(\mathbb{Z})$. An analogous theory holds (see [13] or [65] and references therein). For instance, for the *almost Mathieu equation*,

$$-\left(\psi(n+1) + \psi(n-1)\right) + b\cos(\phi + n\omega)\psi(n) = \lambda\psi(n), \tag{3.69}$$

where now $d = 1$ so $\phi \in \mathbb{T}$, Herman ([36]) proved using an elegant argument based on the subharmonicity of the Lyapunov exponent that if $b > 2$, then the Lyapunov exponent is positive and, therefore, the behaviour of the solutions at the spectrum is non-uniformly hyperbolic. For $b < 2$ the spectrum is absolutely continuous and no localization occurs (conjectured in [3], and proved in [56] and [19]). For $\lambda = 2$, it is showed in [34] that, if ω is Diophantine, then the spectrum is singular continuous and of measure zero. Note that having absolutely continuous spectrum does not imply reducibility everywhere at the spectrum. In fact for Liouville-type rotation numbers, there is generically no reducibility under the hypothesis of theorem 3.4.11 for continuous operators. Again in the almost Mathieu equation for $b > 2$ the distinction between singular continuous and localized eigenstates, depends on the arithmetic properties of both ω (the frequency) and the initial phase $\phi \in \mathbb{T}$. See [38] for the proofs and references of the above description of the behaviour of the almost Mathieu equation. To end the description of this model, we remark that the measure of the spectrum of the almost Mathieu equation, $|\sigma(\omega, b)|$ has been shown to be

$$|4 - 2|b||$$

for a subset of frequencies ω which has full Lebesgue measure, but excludes noble numbers such as the golden mean (see [57]).

The results on the almost Mathieu equation are quite definitive. One may think that the situation for more general analytic (or Gevrey) finite difference Schrödinger operators is similar. There is a remarkable feature of the almost Mathieu equation which makes the discussion much more difficult in the general case, which is the high symmetry of the spectrum. The spectrum is, indeed, pure point, pure singular continuous or pure absolutely continuous (depending on whether $b > 2$, $b = 2$ or $b < 2$). In more general situations, the line between localization and absolutely

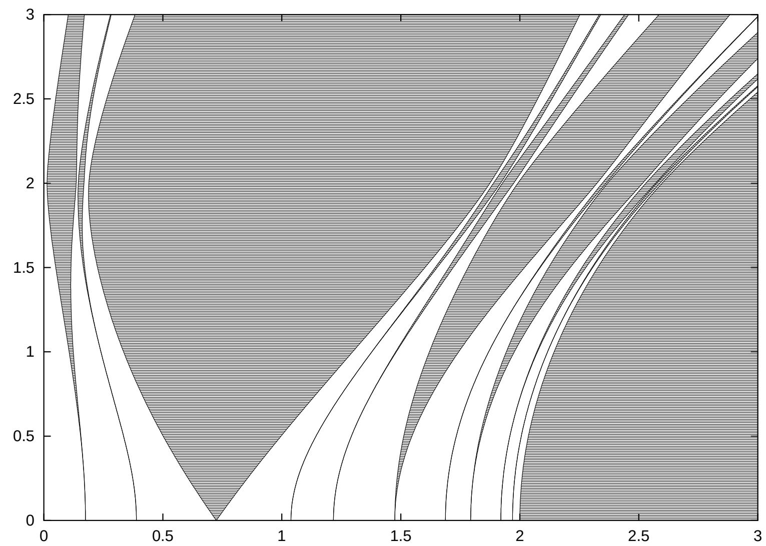

Figure 3.2: The first eleven resonance tongues for the Almost Mathieu Equation (3.69). Parameter b is in the vertical direction and λ is in the horizontal one. Only positive values of λ are plotted because the system is symmetric with respect to $\lambda = 0$.

continuous spectrum does not correspond to a fixed value of b, but it is a *line* in the (λ, b)-parameter lane. This makes it necessary to study Schrödinger equation with two parameters, namely Hill's equation with quasi-periodic forcing ([12], [11]). Moreover, the possible appearance of collapsed gaps ([11]) makes it more difficult to obtain results for *general* Schrödinger equations. We now state a result on localization.

Assume that $Q : \mathbb{T}^d \to \mathbb{R}$ is C^∞ and satisfies the Gevrey-type estimates

$$|Q|_{C^r} \leq C(r!)^2 K^r, \qquad \text{for all } r \geq 0, \tag{3.70}$$

and that the following transversality conditions hold:

$$\max_{0\leq r\leq s} |\partial_\theta^r \left(Q(\theta+\phi) - Q(\phi)\right)| \geq \xi > 0 \tag{3.71}$$

and

$$\max_{0\leq r\leq s} |\partial_\theta^r \left(Q(\theta+\phi) - Q(\phi)\right)| \geq \xi \inf_{k\in\mathbb{Z}} |\phi - 2\pi k| \tag{3.72}$$

for all $(\phi, \theta) \in \mathbb{T}$ and some s. Then the following theorem holds

Theorem 3.5.1 ([23]) *Assume that ω satisfies a Diophantine condition $DC(\kappa, \tau)$, and that the conditions (3.70), (3.71) and (3.72) hold. Then there exists a constant $\varepsilon_0 = \varepsilon_0(C, K, \xi, s, \kappa, \tau)$ such that, if $0 < |\varepsilon| < \varepsilon_0$, then the operator*

$$H_\phi^d = -\left(\psi(n+1) + \psi(n-1)\right) + \frac{1}{\varepsilon} Q(\phi + n\omega)\psi(n)$$

has a pure point spectrum for almost every $\phi \in \mathbb{T}^d$. Moreover the Lebesgue measure of the resolvent set in the interval

$$\left[\inf_{\theta\in\mathbb{T}} \frac{Q(\theta)}{\varepsilon}, \sup_{\theta\in\mathbb{T}} \frac{Q(\theta)}{\varepsilon}\right]$$

is $o(\varepsilon)/\varepsilon^2$ as $\varepsilon \to 0$.

Remark 3.5.2 *Note that the transversality conditions (3.71) and (3.72) are fulfilled, with suitable values of ξ and s, whenever Q is an analytic function defined on $\mathbb{T}$ with no shorter period than 2π.*

The proof proceeds by representing the operator as an infinite, tridiagonal matrix and iteratively constructing a sequence of conjugating matrices which converge to an orthogonal matrix, made up of a complete set of orthogonal eigenvectors, that conjugates the tridiagonal to a diagonal matrix. The conditions on Q control the order of almost multiplicities of eigenvalues (for a discussion of this kind of control of the multiplicites see next chapter), which allows the method (of KAM-type) to succeed. Results of this kind, but with more restrictive hypothesis where first obtained in [76], and [29] (where the continuous case of Mathieu's equation with quasi-periodic forcing is treated).

Finally note that, due to Eliasson's result 3.4.11, in the KAM zone there is generically a dense set of non-reducible systems (from those with rotation number neither Diophantine nor rational).

Chapter 4

Reducibility in higher dimensions

In this chapter we discuss several results concerning the reducibility of linear equations with quasi-periodic coefficients in dimension greater than two.

First of all we present some reducibility theorems that hold in arbitrary groups and which are proved by means of KAM techniques, together with their connection with previous theory, especially the reducibility theorem under the full spectrum assumption that we saw in the second chapter. We present several significant improvements of these results.

In the second part of the chapter we study the problem of reducibility in the context of compact groups (eg. $SO(3)$ or $SU(2)$). For this case many of the techniques of the previous section and the previous chapter are used. It is also necessary to consider some geometrical definitions in order to be able to properly formulate the results for compact groups. We will consider with more detail the case of $SO(3,\mathbb{R})$, for which the main steps of the proof will be outlined, but without getting into any details. Finally some converse results for the compact case are presented.

4.1 Reducibility for general linear differential equations with quasi-periodic coefficients

Consider a linear system of differential equations with quasi-periodic coefficients:

$$x'(t) = A(t)x(t) \tag{4.1}$$

where $x \in \mathbb{R}^n$ and $t \mapsto A \in g \subset gl(n,\mathbb{R})$ (or in $gl(n,\mathbb{C})$) is quasi-periodic with frequency $\omega \in \mathbb{R}^d$ and g is a Lie sub-algebra of $gl(n,\mathbb{R})$. Therefore, and as customary, we can lift the above system to a system in $\mathbb{R}^n \times \mathbb{T}^d$ (resp. $\mathbb{C}^n \times \mathbb{T}^d$) as follows

$$x' = \tilde{A}(\theta)x, \qquad \theta' = \omega, \tag{4.2}$$

where $\theta \in \mathbb{T}^d$ are the angular variables. If we want the group structure given by matrices A to become clearer, we can also consider the corresponding matrix equation

$$X' = \tilde{A}(\theta)X, \qquad \theta' = \omega, \tag{4.3}$$

where $X \in G$, the group generated by the infinitesimal algebra g. This defines a linear skew-product flow on $G \times \mathbb{T}^d$ (for the definition see the second chapter).

We have already met with some conditions under which a general equation such as (4.1) is reducible to constants coefficients by means of a quasi-periodic transformation with frequency

$\omega/2$. Essentially, these conditions were the *full spectrum* (see chapter 2), which amounts to the normal hyperbolicity of the trivial invariant torus given by the zero section $\{0\} \times \mathbb{T}^d$. Moreover the corresponding invariant manifolds need to be one-dimensional and the external frequencies need to satisfy Diophantine conditions in order to guarantee that the system can be reduced to constant coefficients. If the latter conditions are not satisfied, then we can split the system into two boxes which correspond to the stable and the unstable projections.

Historically, the first results on reducibility are covered by the above theory, and they deal with perturbations of hyperbolic systems. Bibliography for these results can be found in the monograph by Bogoljubov, Mitropoliskii and Samolienko ([7]) and the references therein (mainly from Soviet literature). The following theorem, for instance, can be readily interpreted in terms of Sacker-Sell spectral theory and the corresponding reducibility theorems.

Theorem 4.1.1 ([7]) *Consider the linear quasi-periodic equation*

$$x' = (A + \varepsilon Q(\theta))\, x, \qquad \theta' = \omega, \tag{4.4}$$

satisfying the following conditions

(i) The matrix function $Q : \mathbb{T}^d \to gl(n, \mathbb{R})$ is analytic in the complex domain $|Im\ \theta| < \rho$, with $\rho > 0$.

(ii) The frequency vector ω satisfies that there exist constants $K, \sigma > 0$ such that for every $\mathbf{k} \in \mathbb{Z}^d$,

$$|\langle \mathbf{k}, \omega \rangle| \geq \frac{K}{|\mathbf{k}|^{\sigma}}, \qquad \textit{for} \qquad \mathbf{k} \neq 0.$$

(iii) The eigenvalues of the unperturbed matrix A have distinct real parts.

Then, there exists an $\varepsilon_0 > 0$ such that for $|\varepsilon| < \varepsilon_0$ the above system reduces to the form

$$y' = A_0 y, \qquad \theta' = \omega,$$

with A_0 a constant matrix, by means of the non-singular change of variable

$$x = P(\theta) y$$

with $P : \mathbb{T}^d \to GL(n, \mathbb{R})$ real analytic and analytically invertible in the domain $|Im\ \theta| \leq \rho/2$.

Note that this theorem is almost identical to (2.5.1) from chapter 2 if we take into account that the full spectrum is persistent under small perturbations of the system. As already said all these results heavily rely on the uniform hyperbolicity to guarantee reducibility in open domains of the space of matrices.

In practice it turns out that in many cases, and certainly in Hamiltonian systems, we have to decide on the reducibility character of perturbations of elliptic matrices. Here is when the KAM machinery enters with all its power. One possible way to overcome the small divisor problems coming from both the external frequencies ω and, more especially, the internal frequencies which we do not know fore-hand (there is no independent characterization of the imaginary parts of the eigenvalues, although a rotation number can be defined for Hamiltonian linear equations which still keeps some properties, see [45] and [62]), is to assume that we have enough parameters to control these problems. This is a classical idea (see, for instance [58]) and the idea is the following. Assume that we split the perturbed system (4.4) in the following way

$$x' = A_0 x + (\varepsilon Q(\theta, \xi) + \xi))\, x, \qquad \theta' = \omega. \tag{4.5}$$

Then may try to find those ξ close to zero for which there is reducibility with reduced matrix A_0. In this case we can enlarge the class of unperturbed matrices by imposing a Diophantine condition on the imaginary parts of its eigenvalues. This condition is the following. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be the eigenvalues of the unperturbed matrix A_0. Then we will require that the *new* frequency vector $(\lambda, i\omega)$ satisfies a strong non-resonance condition. More precisely, we will assume the existence of positive constants $\kappa, \tau > 0$ such that, for all $\mathbf{k} \in \mathbb{Z}^d$ and $1 \leq j, l \leq n$ the following is true

$$|\lambda_j - \lambda_l + i\langle \mathbf{k}, \omega \rangle| \geq \frac{\kappa}{|\mathbf{k}|^\tau}, \tag{4.6}$$

if $\mathbf{k} \neq 0$. Now we can state a reducibility theorem which guarantees the existence of the desired ξ:

Theorem 4.1.2 ([7]) *For the system of equations (4.5) with the Diophantine conditions (4.6) on both the eigenvalues of the unperturbed matrix and the external frequencies, assume that Q is analytic in θ and ξ on the domain $|Im\ \theta| < \rho$, $|\xi| < \sigma$. Then, it is possible to find a small ε_0 and a real constant matrix ξ_0, with $|\xi_0| \leq 2\varepsilon_0$, such that there exists a change of variable*

$$x = P_0(\theta) y$$

with $P_0 : \mathbb{T}^d \to GL(n, \mathbb{R})$ real analytic on the domain $|Im\ \theta| \leq \rho/2$ which reduces the system (4.5) to the constant coefficient system

$$y' = A_0 y, \qquad \theta' = \omega.$$

The result is achieved by a combination of KAM's quadratic method with perturbation techniques which use the many parameters of the perturbed system (4.5) to control the eigenvalues of the matrices that appear in the iterative process.

In the above result it is imposed that the reduced matrix is exactly A_0. This is a strong condition, because it is natural that, even if the perturbed system is reducible, the reduced matrix is not exactly A_0. Then, it is interesting to know how many systems around the unperturbed one are reducible. An application of the above proposition together with some measure results leads to the following theorem,

Theorem 4.1.3 ([7]) *Consider the system*

$$x' = (A + \varepsilon Q(\theta, \varepsilon))\, x, \qquad \theta' = \omega, \tag{4.7}$$

where

(i) Q is real analytic in (θ, ε) in a complex neighborhood of $\mathbb{T}^d \times \{0\}$.

(ii) The frequency vector ω is Diophantine.

Define now as $\mathcal{R}(\varepsilon; Q)$ the set of matrices A with $|A| \leq 1$ for which the above system is reducible, and on this set take the normalized Lebesgue measure. Then, the following bound holds

$$meas\,(\mathcal{R}(\varepsilon, Q)) \geq \frac{1}{1 + \varepsilon^\alpha},$$

for a suitable constant α depending on Q and the Diophantine condition on the frequency vector ω.

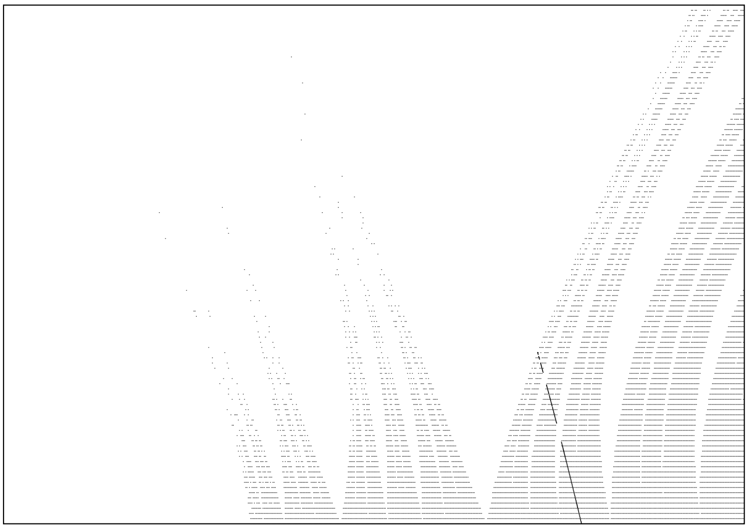

Figure 4.1: Example of a *generic* path in Hill's equation with quasi-periodic forcing. Parameter b is in the vertical direction whereas a is in the horizontal. Marked points are in the spectrum.

In practical problems, both the matrix A and the perturbing matrix Q, depending on one parameter, say ε, are fixed, and one is interested in the set of ε for which the system is reducible. Here we meet with the problem of the lack of parameters in KAM theory. Indeed, note that the previous theorem requires to change the unperturbed part of the system, which is something that we cannot do for applications. That the control of the resonances and small divisors could be performed by making use of only one parameter (under suitable hypothesis) was made evident in [21] (see also [73] and the references therein) and reducibility results using only one parameter were developed in [48], [49], [27] (see also references therein).

Moreover, it turns out that the bounds for the measure of reducible systems close to constant coefficients given by theorem (4.1.3) are not as good as one could expect. As an example think of Hill's equation with quasi-periodic forcing, which we discussed in the previous chapter, depending on a parameter ε. Fix a value of the energy λ such that $\sqrt{\lambda}$ is Diophantine with respect to the external frequency vector ω. Then we ask for the set of ε's (restricted to some neighbourhood of the origin) for which the system

$$x'' + (\lambda + a(\varepsilon) + b(\varepsilon)Q(\theta))\, x = 0, \qquad \theta' = \omega,$$

being $a(\varepsilon)$ and $b(\varepsilon)$ analytic functions of ε, is reducible.

In a classical KAM setup (i.e. looking for reducing transformations which are close to the identity), we do not expect to remove all resonances, but to simply skip them. Therefore, a method of this kind will prove reducibility (assuming that the kind of perturbation is *generical* in a sense to be discussed below) in a subset of those values of ε for which $(a(\varepsilon), b(\varepsilon))$ is not in a resonance tongue in the (a, b)-parameter plane corresponding Hill's equation (see figure 4.1).

The structure of these tongues is in principle unknown, but there are methods to control the measure of their intersections with paths as long as we have some knowledge of the derivative of the path $(a(\varepsilon), b(\varepsilon))$ for $\varepsilon = 0$ (which in general system amounts to control the instantaneous velocity of the eigenvalues of the perturbed matrix for $\varepsilon = 0$) and the asymptotics of the nearby resonance tongues at $b = 0$ (which in general system can be achieved by the smoothness of the potential). In this context and for generic paths, the measure of the forbidden zones in the ε variable is exponentially small.

The following theorem, due to Jorba and Simó, and found in [49] (see also [48] and [63]), provides these kind of estimates and it can be applied in practical situations

Theorem 4.1.4 ([49]) *Consider the system*

$$x' = (A + \varepsilon Q(\theta, \varepsilon))\, x, \qquad \theta' = \omega, \tag{4.8}$$

where $Q : \mathbb{T}^d \times (-\varepsilon_0, \varepsilon_0) \to GL(n, \mathbb{R})$ and A is a non-singular constant matrix of dimension n with n different eigenvalues λ_i. Moreover, assume the following:

(i) Q is analytic with respect to θ in the complex strip $|Im\ \theta| < \rho$ (with $\rho > 0$) and depends on ε in a Lipschitz way.

(ii) The vector $(\lambda_1, \ldots, \lambda_n, i\omega_1, \ldots, i\omega_d)$ satisfies the strong non-resonance condition

$$|\lambda_j - \lambda_l - i\langle \mathbf{k}, \omega\rangle| > \frac{2c}{|\mathbf{k}|^{\gamma_0}}$$

for all $1 \le j, l \le n$ and $\mathbf{k} \in \mathbb{Z}^d - \{0\}$, being $c \ge 0$ and $\gamma_0 \ge d - 1$ fixed fore-hand.

(iii) Let us define

$$\underline{A}(\varepsilon) = A + \varepsilon\, [Q(\cdot, \varepsilon)]\,,$$

where square brackets denote the average on the torus. Let $\lambda_j^0(\varepsilon)$, for $j = 1, \ldots, n$ be the eigenvalues of $\underline{A}(\varepsilon)$. We require that there exist constants $\delta_1, \delta_2 > 0$ such that

$$\frac{\delta_1}{2}|\varepsilon_1 - \varepsilon_2| > \left|\left(\lambda_j^0(\varepsilon_1) - \lambda_l^0(\varepsilon_1)\right) - \left(\lambda_j^0(\varepsilon_2) - \lambda_l^0(\varepsilon_2)\right)\right| > 2\delta_2|\varepsilon_1 - \varepsilon_2| > 0,$$

for all $1 \le j, l, \le n$ and provided that $|\varepsilon_1|$ and $|\varepsilon_2|$ (with $\varepsilon_1 \ne \varepsilon_2$) are less than some positive value ε_0.

Then, there exists a Cantorian set $\mathcal{E} \subset (-\varepsilon_0, \varepsilon_0)$ with positive Lebesgue measure for which system (4.8) can be reduced to a system with constant coefficients

$$y' = B(\varepsilon)y$$

by means of a change of variables ε-close to the identity

$$x = (I + \varepsilon P(\theta, \varepsilon))y,$$

where P is analytic in θ.

If ε_0 is small enough, then

$$meas\, ((0, \varepsilon_0) - \mathcal{E}) \le \exp \frac{-c_1}{\varepsilon_0^{c_2}},$$

for $c_1, c_2 > 0$ (independent of ε_0) and c_2 is any number such that $c_2 < 1/\gamma_0$.

Remark 4.1.5 *This result can be generalized to quasi-periodic perturbations of elliptic equilibrium points, as it is done in [49].*

Remark 4.1.6 *Theorem 4.1.4 is nearly optimal in the sense that, restricting to transformations of the form $I + \varepsilon P(\theta, \varepsilon)$, we cannot handle resonant eigenvalues. Again in the example of Hill's equation with quasi-periodic forcing, asking for transformations close to the identity implies restricting ourselves to the exterior of the tongues. To overcome this problem (see also the discussion preceding to theorem 3.4.4), the set of allowed transformed can be enlarged. This technique has been successfully applied to Schrödinger equation with quasi-periodic potential and skew-product flows in compact groups, as we shall see in section 4.2.*

4.1.1 Idea of proof

In this subsection we will follow, again, [49], together with [48]. The idea of the proof is similar to that of KAM theorem (see [1]) and it consists in the successive application of a quadratic method to overcome the small divisors problems that arise in trying to prove the convergence of the formal series that *solve* the conjugating equation. At each step of the method we must solve a certain linear homological equation.

For a function $F : \mathbb{T}^d \to \mathbb{R}$, $F_{\mathbf{k}}$ shall denote its Fourier coefficients, that is

$$F_{\mathbf{k}} = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} F(\theta) e^{-i\langle \mathbf{k},\theta\rangle} d\theta,$$

for all $\mathbf{k} \in \mathbb{Z}^d$. These, assuming that the function F is analytic in the complex strip $|\mathrm{Im}\ \theta| < \rho$ of the d-dimensional torus, satisfy the following estimate:

$$|F_{\mathbf{k}}| \leq \|F\|_\rho \, e^{-\rho|\mathbf{k}|},$$

where $\|\cdot\|_\rho$ is the supremum norm in the space of real analytic functions defined in a complex strip of width ρ of the torus $\mathbb{T}^d$.

The inductive scheme

Let's recall the system which we want to render to constant coefficients

$$x' = (A + \varepsilon Q_1(\theta, \varepsilon))\, x, \qquad \theta' = \omega. \tag{4.9}$$

Here we have written the subscript 1 to stress that this will be the initial equation of an inductive process. We now want to perform a change of variables close to the identity in order to put the ε-dependence in the above expression to ε^2. This is achieved by means of an averaging method. Let

$$\bar{Q}_1(\varepsilon) = [Q_1(\cdot, \varepsilon)]_{\mathbb{T}^d}$$

denote the average on the torus. If we split equation (4.9) in its zero and non-zero average parts, then it becomes

$$x' = \left(\bar{A}(\varepsilon) + \varepsilon\tilde{Q}_1(\theta, \varepsilon)\right) x, \qquad \theta' = \omega,$$

where $\tilde{Q}_1(\theta, \varepsilon) = Q_1(\theta, \varepsilon) - \bar{Q}_1(\varepsilon)$ has average zero and $\bar{A}(\varepsilon) = A + \varepsilon\bar{Q}_1(\varepsilon)$ is the new time-free part. As we discussed before theorem 3.4.4, the homological equation for the perturbation P of the identity giving rise to the first transformation in the inductive process can be linearized, so that it becomes

$$\partial_\omega P = \bar{A}P - P\bar{A} + \tilde{Q}_1, \qquad \theta' = \omega, \tag{4.10}$$

assuming that we want the transformation to have the frequency vector ω and letting $\partial_\omega = \langle \nabla_\theta, \cdot\rangle$ as usual. In order to be able to obtain a quasi-periodic P from the above expression we must impose a Diophantine condition:

$$|\bar{\lambda}_j - \bar{\lambda}_l - i\langle \mathbf{k}, \omega\rangle| > \frac{c}{|\mathbf{k}|^{\gamma_0}}, \tag{4.11}$$

being $\bar{\lambda}$ the eigenvalues of the matrix $\bar{A}$. Then, making the change of variables

$$x = (I + \varepsilon P(\theta, \varepsilon))\, x_1,$$

the new system becomes

$$x_1{}' = \left(\bar{A}(\varepsilon) + \varepsilon^2 Q_2(\theta,\varepsilon)\right) x_1, \qquad \theta' = \omega, \tag{4.12}$$

where the new perturbation is $Q_2 = (I+\varepsilon P)^{-1}\tilde{Q}_1 P$.

If we apply this procedure r times (postponing the discussion of the resonance condition (4.11) and its control) we get an equation which looks like the following

$$x_r{}' = \left(A_r(\varepsilon) + \varepsilon^{2^r} Q_r(\theta,\varepsilon)\right) x_r, \qquad \theta' = \omega. \tag{4.13}$$

If the norms of A_n and Q_n do not grow too fast, the scheme will be convergent to an equation like

$$y' = B(\varepsilon)y$$

which is already in Floquet form. Therefore, the core of the proof is the convergence of the above scheme.

The resonances

Recall that at each step of the iterative process, we need a Diophantine condition like

$$|\bar{\lambda}_j - \bar{\lambda}_l - i\langle \mathbf{k},\omega\rangle| > \frac{c}{|\mathbf{k}|^{\gamma_0}}$$

in order to be able to solve the linear homological equation (4.10). Note that the eigenvalues λ are changed at each step of the process (because its matrix is changed). Therefore we do not know in advance whether this condition will be fulfilled or not.

To deal with this problem a control for the variation of the eigenvalues at each step is needed, and this control is achieved by the third condition in theorem 4.1.4. Indeed, assume that condition (4.11) is verified at the first step. Then we can solve the linearized homological equation (4.10) and apply the transformation that it defines as an ε-perturbation of the identity. The new time-free matrix, as already stated, is $\bar{A}(\varepsilon) = A + \varepsilon\bar{Q}_1(\varepsilon)$, and of its eigenvalues we only know that the condition (4.10) is satisfied for $\varepsilon = 0$ (when the eigenvalues are precisely those of A). To see how to control the fulfillment of this condition for $\varepsilon > 0$, let $\bar{\lambda}_j(\varepsilon)$ be the eigenvalues of $\bar{A}(\varepsilon)$ for $i = 1,\ldots,n$. As we are assuming the eigenvalues to be different, we can write their Taylor expansion around $\varepsilon = 0$

$$\bar{\lambda}_j(\varepsilon) = \lambda_j + \lambda_j^{(1)}\varepsilon + \lambda_j^{(2)}\varepsilon^2 + \ldots. \tag{4.14}$$

Looking at $\bar{\lambda}_j(\varepsilon)$ as a function of ε, we can avoid the resonant values of $\lambda_j(\varepsilon)$ by avoiding the corresponding values of ε. This implies taking out a (Cantor-like) set of resonant values of $\bar{\lambda}_j(\varepsilon)$ at each step. The key question is whether this can be done without excluding too many ε's or not. To bound the measure of the *resonant* values of ε, we will require relation (4.14) to be Lipschitz from below with respect to ε (although some Hölder condition would also work). Obviously, we need to take out values of ε at each step of the process. It would seem sensible that the condition on the eigenvalues (the third item in the theorem) is defined in terms of the derivatives of the eigenvalues $\lambda_i^{(1)}$, but as at each step we take out a Cantor-like set the dependence on ε cannot be made differentiable. This is why the Lipschitz condition replaces the condition on the derivatives of the eigenvalues. Moreover, as it will the show in a moment, this assumption allows us to control the size (in measure) of the set of ε's that we take out.

The measure of the resonant set

Assuming that ε belongs to an interval $[0, \varepsilon_0]$, with ε_0 small enough, we want to bound the measure of the set of values of ε too close to a resonance. It will be shown that the exponentially small bounds on the measure of this set which appear in the theorem can be achieved from a careful analysis of the Diophantine condition (4.11).

On one hand, we can use the Diophantine condition (4.11) to describe the set of resonant eigenvalues $\bar{\lambda}$. Indeed, the set of resonant $\lambda = (\lambda_1, \ldots, \lambda_n)$ is given by the union

$$\bigcup_{|\mathbf{k}|\neq 0} \left\{ \lambda \in \mathbb{C}^n;\ |\lambda_j - \lambda_l - i\langle \mathbf{k}, \omega\rangle| < \frac{c}{|\mathbf{k}|^{\gamma_0}} \right\},$$

and we want the measure of this set to be as small as possible. This can be done relaxing the exponents of the Diophantine condition. However, doing so causes the convergence of the Fourier series involved in the proof uncertain. Therefore there must be some balance in the choice of the constants of the Diophantine condition at each step. The way used in [49] to overcome this problem is to use, at the r-th step, the Diophantine condition

$$|\lambda_j - \lambda_l - i\langle \mathbf{k}, \omega\rangle| > \frac{c}{|\mathbf{k}|^{\gamma_r}} e^{-\nu_r |\mathbf{k}|} = R(\mathbf{k}, r)$$

where $(\gamma_r)_r$ is a geometrically increasing sequence (taken $\gamma_r = \gamma_0 z^r$, with $1 < z < 2$) and ν_r is precisely

$$\nu_r = \frac{\nu_0}{(r+1)^2}, \qquad \nu_0 = c.$$

At each step of the inductive process the measure of the resonant set for the eigenvalues λ is given by the sum

$$\sum_{\substack{\mathbf{k}\in\mathbb{Z}^d \\ \mathbf{k}\neq 0}} 2R(\mathbf{k}, r),$$

and the total measure is

$$\sum_{r\geq 0} \sum_{\substack{\mathbf{k}\in\mathbb{Z}^d \\ \mathbf{k}\neq 0}} 2\frac{c}{|\mathbf{k}|^{\gamma_r}} e^{-\nu_r |\mathbf{k}|} \tag{4.15}$$

which implies that the convergence of the above series must be examined.

Let us now explain where the exponentially small character of set of resonant values of ε comes from. As already stated, the eigenvalues (and hence its differences) of the matrix A move at each step of the inductive process by an amount of order ε (this comes from the fact that the averaging procedure does not change previously computed terms, but only modifies those of order greater than the step). Let us call $I_{jl}(\varepsilon)$ the interval (with diameter $\mathcal{O}(\varepsilon)$) where the difference between the eigenvalues λ_j and λ_l moves. This implies that if the difference satisfies a condition like (4.11) then the values $\langle \mathbf{k}, \omega\rangle$ are outside $I_{jl}(\varepsilon)$ if $|\mathbf{k}| < N(\varepsilon)$, for a suitable value $N(\varepsilon)$. Therefore, in the sum (4.15) it is enough to start with $|\mathbf{k}| \geq N(\varepsilon)$. This in turn implies that if we are able to prove the exponential decay of $R(\mathbf{k}, r)$ with $\mathbf{k}$, then we will obtain something which is exponentially small with ε_0.

To be able to prove the latter, it is necessary to resort to varying ν and γ at each step in the constants of the Diophantine condition (4.11). Indeed, since the value $\exp(-\nu|\mathbf{k}|)$ will appear in the denominators of the Fourier expansions for solving the homological equation (4.10), the value $\exp(\nu|\mathbf{k}|)$ will be multiplying the coefficients of these series, therefore reducing the width of the analyticity strip. If the value of ν was kept fixed, the functions would be no longer analytic after some steps and the process would have to be stopped. As it is classical in KAM setup, a varying ν_r at each step is chosen, in such a way that the reduction on the analyticity strip becomes bounded.

Which is the reason for changing also at each step the exponent γ in condition (4.11) ? Note that with the choice of ν_r, the exponential $\exp(-\nu_r|\mathbf{k}|)$ goes to one as $r \to \infty$. Thus the addition of some factor in front of the exponential is needed to ensure that the sum with respect to r is still exponentially small. In [49] the value $\gamma_r = \gamma_0 z^r$ is used (although it is not necessary to take precisely this sequence). The value of z must be taken between 1 and 2 to ensure convergence.

The proof of theorem 4.1.4 that we have just sketched is found in the paper [49].

Remark 4.1.7 *In this section we have always assumed that the dependence on the angles θ was analytic in a complex strip. There are, however, some results assuming only some degree of smoothness. This kind of results are usual in KAM theory, see for instance [10]. For finitely differentiable results on reducibility see [8] and references therein.*

Remark 4.1.8 *In some cases one does not want to reduce completely a linear quasi-periodic system, but only up to a small remainder. Then we can speak of effective reducibility or almost reducibility (see [47] and [26]).*

4.2 Reducibility in compact groups

This section is devoted to the results obtained by R. Krikorian in [52], [53] and [51] on the reducibility of linear equations with quasi-periodic coefficients which take values on compact groups. Our attention will focus on the case of $SO(3)$ for which, together with $SU(2)$, there is a fairly complete picture of the *amount* of reducible and non-reducible systems close to constant coefficients.

The techniques used in this section are very wide. KAM results on reducibility like in the previous section proving reducibility for most values of one-parameter families of linear equations with quasi-periodic coefficients, together with the exponential estimates on the measure of the reducible systems are needed. Moreover Eliasson's techniques for handling resonances in Schrödinger equation with quasi-periodic forcing (the use of exponential transformations instead of transformations close to the identity), as well as other covering transformations are used in a systematic way to eliminate resonances. As it happened in the Schrödinger equation (see section 3.4.2), to enlarge the set of transformations we must pay a price, which is the loss of regularity of the dependence of the eigenvalues with respect to the parameter. In Krikorian's work, the control is done using the notion of *Pyartli transversality*, which will be discussed below.

These results on almost everywhere reducibility of quasi-periodic equations of compact groups must be contrasted with the density, in $SO(3,\mathbb{R})$ and close to constant coefficients, of skew-product systems which are uniquely ergodic (it has *only* one ergodic invariant measure), as was proved by Eliasson in [22]. In particular, the system cannot be reducible, because reducibility would imply a certain foliation of the phase space which is incompatible with unique ergodicity. In the non-compact example of Schrödinger equation with quasi-periodic potential, we already saw that, in the analytic setting and close to constant coefficients, there was a dense set of non-reducible systems and a set with total measure of reducible ones. This situation seems to be quite general (see [24], [25], [26]).

4.2.1 Set up

In this section we will focus on the following linear skew-product flow defined on $G \times \mathbb{T}^d$,

$$X'(t) = A(\theta)X(t), \qquad \theta' = \omega, \tag{4.16}$$

where G is a (finite-dimensional) Lie group with Lie algebra g (later on we shall specify more conditions on this group). The matrix A defining equation (4.16) will lie in the Lie algebra g, whereas the fundamental matrix $X(t;\phi)$ (which will be assumed to satisfy $X(0;\phi) = Id$ with initial phase ϕ) will lie in the Lie group G. The dependence on θ will be assumed to be analytic in a certain complex strip of the torus $\mathbb{T}^d$ (some results for C^∞-systems can be found in [52] but we will not treat this case).

We first review some facts on compact Lie groups that will be necessary to state the results on almost everywhere reducibility.

Some theory of compact Lie groups

Let now G be a real compact Lie group, which will be assumed connected, and let us denote by g its Lie algebra. The Lie bracket will be denoted by $[\cdot,\cdot]$, and it is defined for pairs of elements in g. Through this bracket, and for every $X \in g$ we can define a map

$$\mathrm{ad}(X)\cdot Y = [X,Y], \qquad Y \in g,$$

which is a differentiation on the Lie algebra g. Now, using the fact that G is infinitesimally generated by g, we can define the corresponding operation for the Lie group G. Indeed, for all $u \in G$ we can define the operation

$$\mathrm{Ad}(u)\cdot X,$$

for all $X \in g$, as the tangent vector at the identity of the following path

$$u\gamma(t)u^{-1},$$

where γ is any path in G with the conditions

$$\gamma(0) = Id \qquad \text{and} \qquad \gamma'(0) = X.$$

We denote, for all $u, v \in G$,

$$\mathrm{Ad}(u)\cdot v = uvu^{-1}.$$

We can now define the *Cartan-Killing* form κ as the following bilinear form on the Lie algebra g,

$$\kappa(X,Y) = \mathrm{tr}(\mathrm{ad}(X)\circ \mathrm{ad}(Y)),$$

where tr is the trace of a linear operator.

Example 4.2.1 ([66]) *If $g = gl(n,\mathbb{R})$ (for which the Lie group $G = GL(n,\mathbb{R})$ is not compact), the Lie algebra of all square matrices of dimension n, then the explicit formula for the Cartan-Killing form is*

$$\kappa(X,Y) = 2n\,tr(XY) - 2tr(X)\cdot tr(Y), \qquad X,Y \in gl(n,\mathbb{R}).$$

Moreover, we also obtain expressions for $sl(n,\mathbb{R})$,

$$\kappa(X,Y) = 2n\,tr(XY), \qquad X,Y \in sl(n,\mathbb{R}),$$

and for $so(n,\mathbb{R})$

$$\kappa(X,Y) = (n-1)tr(XY), \qquad X,Y \in so(n,\mathbb{R}).$$

Depending on the properties of the bilinear form κ we can give the following definitions

Definition 4.2.2 *A Lie algebra g is said to be* semi-simple *if the Cartan-Killing form κ is non-degenerate, and it is said to be* compact *if κ is negative.*

Remark 4.2.3 *The Lie group G of a semi-simple Lie algebra g is compact if, and only if, the algebra g is compact. In this case, the exponential map $\exp : g \to G$ is surjective (this does not happen, for instance, if $G = SL(2,\mathbb{R})$).*

From now on we shall assume that G is a compact semi-simple group. We now give a fundamental definition in what will follow

Definition 4.2.4 *We shall say that $T \subset G$ is a* maximal torus *whenever it is a subgroup of G which is compact, Abelian, connected and maximal with respect to the inclusion in G.*

Maximal tori are isomorphic to a torus $\mathbb{T}^d$, for a suitable $d \in \mathbb{N}$ depending on the maximal torus that we are considering. The Lie algebra $\mathcal{T} \subset g$ of a maximal torus $T \subset G$ is an Abelian Lie sub-algebra which is maximal for the inclusion in g. Conversely, every Abelian Lie sub-algebra which is maximal, $\mathcal{T} \subset g$, is the Lie algebra of a maximal torus $T \subset G$. We shall call *maximal toric sub-algebras* these sub-algebras.

There are many properties which can help to describe compact Lie algebras and which will be of interest in the sequel. We have already said that for these algebras, the exponential map from g to G is surjective. Moreover, we have also Ascoli's-Arzelà theorem: *If G is a compact group, every equi-continuous family of $Diff^\alpha(\mathbb{T}^d, G)$ is relatively compact in $Diff^\alpha(\mathbb{T}^d, G)$.*

The following theorem will be of importance in the description of maximal tori in compact semi-simple groups

Theorem 4.2.5 ([52]) *Let G be a compact semi-simple group, and let g be its Lie algebra. Then the following is true*

(i) At every point of $u \in G$ (resp. $X \in g$) there is at least a maximal torus (resp. a maximal toric sub-algebra).

(ii) If T and T' (resp. $\mathcal{T}$ and $\mathcal{T}'$) are two maximal tori of G (resp. two maximal toric sub-algebras of g), there exists an element $x \in G$ which conjugates T and T' (resp. $\mathcal{T}$ and $\mathcal{T}'$). That is,

$$T' = x \cdot T \cdot x^{-1}, \qquad (resp. \qquad \mathcal{T}' = Ad(x) \cdot \mathcal{T}).$$

(iii) The center *of a maximal torus T, i.e.*

$$C(T) = \{x \in G;\ xy = yx \text{ for all } y \in T\}$$

is T.

(iv) If $N(T)$ is the normalizator of a maximal torus T,

$$N(T) = \{x \in G;\ Ad(x) \cdot T \subset T\},$$

then T is distinguished in $N(T)$ and the group $W_T = N(T)/T$ has finite cardinal. All groups W_T of this kind are isomorphic, and this group is called the Weyl's group of *G.*

Let T be a maximal torus of a compact semi-simple connected group G and $\mathcal{T}$ its corresponding maximal toric Lie sub-algebra. For every element $H \in \mathcal{T}$, $\mathrm{ad}(H) \in gl(g)$ is an antisymmetric endomorphism of $gl(g)$. On the other hand, as $\mathcal{T}$ is a maximal toric sub-algebra, it is Abelian and hence, all elements $\mathrm{ad}(H)$, with $H \in \mathcal{T}$, commute. This altogether implies that these $\mathrm{ad}(H)$ can be simultaneously diagonalized on $\mathbb{C}$ (see [52] and [66]). We then have the following

Theorem 4.2.6 ([52]) *In the above situation, there exists a set Δ of non-zero real linear forms α on $\mathcal{T}$ which can be written as $\Delta = \tilde{\Delta} \cup (-\tilde{\Delta})$ (hence they are invariant by the operation $\alpha \mapsto -\alpha$) and elements $X_\alpha \in g$ such that*

(i) $g = \oplus_{\alpha \in \tilde{\Delta}} (\mathbb{R}X_\alpha \oplus \mathbb{R}X_{-\alpha})$.

(ii) For all $H \in \mathcal{T}$ and $\alpha \in \Delta$

$$ad(H) \cdot X_\alpha = i\alpha(H) X_{-\alpha}.$$

The elements α are called the *roots* of g with respect to $\mathcal{T}$. An element $X \in g$ it is said to be *regular* if for all $\alpha \in \Delta$, $\alpha(X) \neq 0$ (this means that e^X is contained in only one maximal torus), and it is said to be *generic* if the adherence of the orbit

$$\left(e^{tX}\right)_{t\in\mathbb{R}}$$

is equal to the exponential of the maximal toric algebra, $e^{\mathcal{T}}$. The cardinal of Δ will be denoted by $\tilde{q}$.

Reducibility in the Lie group setting

Following the ideas from the previous sections and chapters, it is natural to look for a Floquet representation of X

$$X(t) = Z(\omega_1 t + \phi_1, \ldots, \omega_d + \phi_d) e^{Bt}, \tag{4.17}$$

where $B \in g$ is the *Floquet matrix*, and $Z \in C^a(\mathbb{T}^d; G)$ is a non-singular analytic transformation. As usual, when such a representation exists we shall say that the system (4.16) is reducible. The existence of a Floquet representation is equivalent to the following *homological equation*

$$A(\theta) = (\partial_\omega Z)(\theta) Z(\theta)^{-1} + \mathrm{Ad}(Z(\theta))B, \qquad \theta \in \mathbb{T}^d, \tag{4.18}$$

where, again,

$$\partial\omega\cdot = \langle D_\theta \cdot, \omega\rangle$$

and, according to the definitions in the previous section

$$\mathrm{Ad}(Z)B = Z \cdot B \cdot Z^{-1}.$$

We will focus on linear equations on compact semi-simple Lie groups which are of the form a time-free matrix plus a (small) perturbation. More precisely and according to the notation in (4.16) we will assume that

$$A(\theta) = \lambda A_0 + F(\theta),$$

where A_0 is independent of θ and $F : \mathbb{T}^d \to g$ is analytic on a strip of the torus.

Now we can state the main theorem for compact semi-simple groups, due to R. Krikorian:

Theorem 4.2.7 ([53]) *Let $\gamma, \sigma, \rho > 0$ and $\Lambda \subset \mathbb{R}$ an interval such that the following conditions hold:*

(i) The frequency vector ω satisfies the Diophantine condition $DC(\gamma, \sigma)$: for all $\mathbf{k} \in \mathbb{Z}^d$, $\mathbf{k} \neq 0$,

$$|\langle \mathbf{k}, \omega \rangle| \geq \frac{\gamma^{-1}}{|\mathbf{k}|^{\sigma}}.$$

(ii) The function $F : \mathbb{T}^d \to g$ is analytic in the complex strip $|Im\ \theta| < \rho$.

(iii) $A_0 \in g$ is a generic element (see the definition above).

Then, there exists $\varepsilon_0 = \varepsilon_0(\Lambda, A_0, \omega, \rho)$ such that, if $|F|_{\rho} \leq \varepsilon_0$, then for almost every $\lambda \in \Lambda$, the skew-product flow

$$X'(t) = (\lambda A_0 + F(\theta))\, X(t), \qquad \theta' = \omega$$

is reducible modulus a χ_G-finite covering, where χ_G is an integer depending only on the group G. That is, for almost every $\lambda \in \Lambda$ there exist $Z_\lambda \in C^a\left((\mathbb{R}/(2\pi\chi_G\mathbb{Z}))^d, G\right)$ and $B_\lambda \in g$ such that

$$\partial_\omega Z_\lambda(\theta) \cdot Z_\lambda(\theta)^{-1} + Ad(Z_\lambda(\theta)) \cdot (\lambda A_0 + F(\theta)) = B_\lambda, \tag{4.19}$$

for all $\theta \in \mathbb{T}^d$, and B_λ does not depend on time. Moreover, if G is a unitary special group $SU(w+1)$, then $\chi_G = 1$.

Remark 4.2.8 *Compare with theorems 3.4.4 and 4.1.4.*

What does all this mean in $SO(3, \mathbb{R})$?

When sketching the proof of theorem 4.2.7, we will focus on the case $G = SO(3, \mathbb{R})$. Here we plan to particularize the definitions and results that we have just given in this particular case. This will (hopefully) make the later discussion clearer. We follow [52].

The group $SO(3, \mathbb{R})$ is the group of all orthogonal transformations of $\mathbb{R}^3$ which have determinant one. Its corresponding Lie algebra, $so(3, \mathbb{R})$ is the algebra of all real antisymmetric matrices (hence with trace zero). If we denote by J_1, J_2, J_3 the following matrices

$$J_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad J_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \qquad J_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

then (J_1, J_2, J_3) is a basis of $so(3, \mathbb{R})$, and for the Lie bracket the following relation holds

$$[J_1, J_2] = J_3, \tag{4.20}$$

together with the other two circular permutations of indices $1, 2, 3$. Using this basis, we can easily express the image of $X \in so(3, \mathbb{R})$ under the linear map $\mathrm{ad}(A)$ for any $A \in so(3, \mathbb{R})$. Indeed, assume that $\mathbf{e}, \mathbf{f} \in \mathbb{R}^3$ are such that

$$A = e_1 J_1 + e_2 J_2 + e_3 J_3 \qquad \text{and} \qquad X = f_1 J_1 + f_2 J_2 + f_3 J_3.$$

Then using property (4.20) it follows that

$$\mathrm{ad}(A) X = [A, X] = \begin{vmatrix} e_2 & f_2 \\ e_3 & f_3 \end{vmatrix} J_1 - \begin{vmatrix} e_1 & f_1 \\ e_3 & f_3 \end{vmatrix} J_2 + \begin{vmatrix} e_1 & f_1 \\ e_2 & f_2 \end{vmatrix} J_3$$

so $\mathrm{ad}(A)X$ is expressed by means of the components of $\mathbf{e} \wedge \mathbf{f}$ in terms of the basis $\mathbf{J}$. We can make this fact more precise by giving an identification of $(so(3,\mathbb{R}), [\cdot,\cdot])$ with $\mathbb{R}^3$, together with the exterior product $\wedge$. Indeed, consider the map

$$\rho : (so(3,\mathbb{R}), [\cdot,\cdot]) \to (\mathbb{R}^3, \wedge)$$

which sends the basis (J_1, J_2, J_3) to the canonical basis of $\mathbb{R}^3$. The computation above shows that

$$\rho([X,Y]) = \rho(X) \wedge \rho(Y)$$

and that

$$\rho \circ \mathrm{Ad}(e^X) \circ \rho^{-1} = \mathrm{Rot}(\rho(X)),$$

where $\mathrm{Rot}(u)$ is the rotation in $\mathbb{R}^3$ given by the axis $u/|u|$ and the angle $|u|$ modulus 2π if $u \neq 0$ and the identity if $u = 0$.

From this we conclude that the eigenvalues of $ad(A)$ as a linear map in $so(3,\mathbb{R})$, for an element A of $so(3,\mathbb{R})$, are 0 and $\pm i\alpha$, where α is precisely $|\rho(A)|$ (the norm being the usual Euclidean norm on $\mathbb{R}^3$). Note that if the non-zero roots of A (the non-zero eigenvalues of $ad(A)$) are $\pm i\alpha$, then the non-zero eigenvalues of A are precisely $\pm i\alpha/2$ (check this for the basis).

Recall that a maximal torus on $SO(3,\mathbb{R})$ is a maximal Abelian subgroup of $SO(3,\mathbb{R})$. From the computation of some Lie brackets, it follows, that the maximal tori are of the form $e^{\mathbb{R}A}$, with $A \in so(3,\mathbb{R})$ and hence maximal toric algebras are of the form $\mathbb{R}A$, with $A \in so(3,\mathbb{R})$. The condition of genericity for an element $A \in so(3,\mathbb{R})$ is simply $A \neq 0$.

Moreover, and by the above description, it is clear that any two tori are conjugated by an element of $SO(3,\mathbb{R})$, so we can skip the reference to a particular torus when referring to the roots of an element of $so(3,\mathbb{R})$. Recall that we saw, by means of a computation, that the Cartan-Killing form on $so(3,\mathbb{R})$ was

$$\kappa(A,B) = \mathrm{tr}(\mathrm{ad}(A)\mathrm{ad}(B)) = 2\mathrm{tr}(AB),$$

for all $A, B \in so(3,\mathbb{R})$. Note that this form is negative and non-degenerate, and it can be used to define the following norm on $so(3,\mathbb{R})$:

$$|A| = \frac{1}{\sqrt{2}} \left(-\kappa(A,A)\right)^{1/2},$$

which is equal to $|\alpha|$, whenever $\pm i\alpha$ are the non-zero eigenvalues of $\mathrm{ad}(A)$ (which in this case are the *roots* of A). If we think in the representation of $so(3,\mathbb{R})$ as $\mathbb{R}^3$, then all the elements A of $so(3,\mathbb{R})$ with $\alpha(A)$ =constant are the spheres of $\mathbb{R}^3$ with radius $\alpha(A)$. See figure 4.2 for a picture of these facts.

We have now given the necessary elements to state theorem 4.2.7 for the particular setting of $SO(3,\mathbb{R})$

Theorem 4.2.9 ([52]) *Let $\gamma, \sigma, \rho > 0$ and $\Lambda \subset \mathbb{R}$ a non-trivial and bounded interval such that the following conditions hold:*

(i) The frequency vector ω satisfies the Diophantine condition $DC(\gamma,\sigma)$: for all $\mathbf{k} \in \mathbb{Z}^d$, $\mathbf{k} \neq 0$,

$$|\langle \mathbf{k}, \omega \rangle| \geq \frac{\gamma^{-1}}{|\mathbf{k}|^\sigma}.$$

(ii) The function $F : \mathbb{T}^d \to so(3,\mathbb{R})$ is analytic in the complex strip $|Im\ \theta| < \rho$.

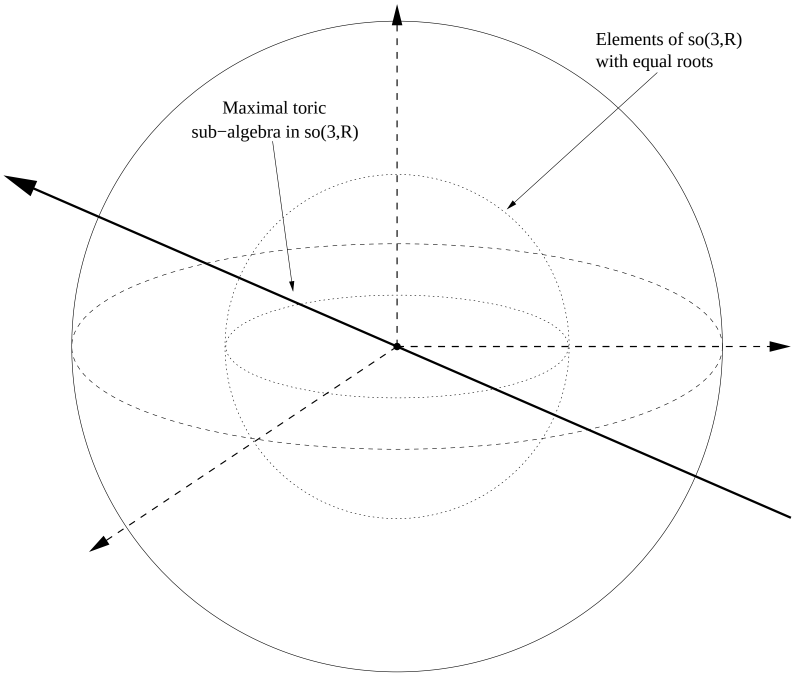

Figure 4.2: Sketch of some elements in $so(3,\mathbb{R})$.

(iii) $A \in so(3,\mathbb{R})$ *is different from zero.*

Then, there exists $\varepsilon_0 = \varepsilon_0(\Lambda, A, \omega, \rho)$ *such that if* $|F|_\rho \le \varepsilon_0$*, then for almost every* $\lambda \in \Lambda$*, the skew-product flow*

$$X'(t) = (\lambda A + F(\theta))\, X(t), \qquad \theta' = \omega$$

is reducible. That is, for almost every $\lambda \in \Lambda$ *there exist* $Z_\lambda \in C^a(\mathbb{T}^d, SO(3,\mathbb{R}))$ *and* $B_\lambda \in so(3,\mathbb{R})$ *such that*

$$\partial_\omega Z_\lambda(\theta) \cdot Z_\lambda(\theta)^{-1} + Ad(Z_\lambda(\theta)) \cdot (\lambda A + F(\theta)) = B_\lambda, \tag{4.21}$$

for all $\theta \in \mathbb{T}^d$*, and* B_λ *does not depend on time.*

Remark 4.2.10 *The perturbation* F *may depend also on the parameter* λ *in a real analytic way and the result is still true (see [52]). In fact, the proof is based on an infinite sequence of transformations in which, after the first step, the perturbation term will also depend on* λ*.*

4.2.2 Sketch of proof in the case of $SO(3,R)$

As it is stated in the theorem, we must *solve* the homological equation (4.21) in order to render equation (4.16) to constant coefficients. The proof consists in the following steps:

- First of all, a theorem similar to (4.1.4) is used to prove reducibility by means of transformations close to the identity for a set of λ of positive measure. Moreover, as λ is closer to zero (say less than a certain constant λ_0), the measure of the *reducible* λ's is exponentially small in λ_0.

- To prove the above result some resonance regions have been let aside. In the second step methodology is given to skip these resonances by enlarging the class of transformations (not close to the identity). The technique has the problem that the dependence of the roots of the unperturbed matrices on the parameter λ (which give rise to small divisor problems in solving homological equations) becomes very bad.

- To control the dependence on the parameter λ it is used the notion of *Pyartli transversality* together with some useful results for the present framework.
- Finally, using Pyartli transversality, it is proved that the inductive procedure described in the second item is convergent.

We will give more details on the above *program* in the following sections.

Results of positive measure

Here we will only present the setting necessary for the sequel, because results of this type have already been discussed in section 4.1.

Assume that $\omega \in DC(\gamma, \sigma)$ and $h_1 = h > 0$ are fixed and that $A_1 \in so(3, \mathbb{R})$ and $F_1 \in C^a_{h_1}(\mathbb{T}^d, so(3, \mathbb{R}))$ are such that

$$|F_1|_{h_1} < \varepsilon_1,$$

with ε_1 small enough.

If N, K are real positive numbers (think of them as *large* numbers), and $P \geq 1$ is an integer, we define the set

$$DS(N, K, P) = \left\{ \alpha \in \mathbb{R},\ |\alpha - \frac{\langle \mathbf{k}, \omega \rangle}{P}| \geq K^{-1},\ \text{ for all } 0 < |\mathbf{k}| \leq N \right\},$$

and we also define the set $RS(N, K, P)$ as its complementary in $\mathbb{R}$. When $P = 1$, we will denote these sets by $DS(N, K)$ and $RS(N, K)$ respectively. Note that, if $\tau > \sigma$ is fixed, then we can define the classical set of points satisfying the Diophantine condition

$$DC_\tau(N, K) = \left\{ \alpha \in \mathbb{R},\ |\alpha - \langle \mathbf{k}, \omega \rangle| \geq \frac{K^{-1}}{|\mathbf{k}|^\tau},\ \text{ for all } 0 < |\mathbf{k}| \leq N \right\},$$

and the inclusions

$$DS(N, K) \subset DC_\tau(N, K) \subset DS(N, KN^\tau)$$

hold. The sets $DS(N, K, P)$ and $RS(N, K, P)$ are, respectively, the sets of Diophantine and nearly resonant real numbers *up to order* N. Moreover, if K is chosen large enough (of the order of a power of N for a fixed N), and if $\alpha \in RS(N, K)$, there exists a unique $0 < |\mathbf{k}_0| \leq N$ such that

$$|\alpha - \langle \mathbf{k}_0, \omega \rangle| \leq K^{-1} |\mathbf{k}_0|^{-\tau} \leq K^{-1}.$$

The reason why these Diophantine and nearly resonant sets actually describe the resonances for our small divisors problem will become clear in a moment.

For the KAM result of reducibility in positive measure we will introduce sequences N_r and K_r going to infinity (we will choose $N_r = \text{constant} \cdot 2^{2r}$ and $K_r = \text{constant} \cdot N_r^\nu$, with ν positive and fixed).

To see where the resonance problem comes from let's proceed a bit further with the proof. We now want to conjugate $A_1 + F_1 = A + F$ to $A_2 + F_2$ so that F_2 is much smaller than F_1 by means of a transformation $Z = e^Y \in SO(3, \mathbb{R})$ which is close to the identity. The homological equation is, therefore,

$$\partial_\omega e^Y e^{-Y} + \mathrm{Ad}(e^Y)(A_1 + F_1) = A_2 + F_2, \tag{4.22}$$

but, as customary, we do not want to solve the whole homological equation, but rather its linear part with respect to Y (i.e. we will apply Newton's quadratic method). The linear homological equation is thus

$$-\partial_\omega Y + \mathrm{ad}(A_1)Y = T_{N_1}F_1 - \hat{F}_1(0), \tag{4.23}$$

where $T_N F$ denotes the truncation of the Fourier series of F up to order N and $\hat{F}(\mathbf{k})$ the Fourier coefficients of F, for $\mathbf{k} \in \mathbb{Z}^d$. Therefore, solving the linear homological equation implies that

$$A_2 = A_1 + \hat{F}_1(0)$$

as it is customary in averaging methods. It is clear now how the resonances of the roots of A (the eigenvalues of the adjoint operator $ad(A)$) can give rise to small divisor problems in solving the linear equation (4.23).

Finally, to prove the result on reducibility in positive measure it is necessary to impose a generic condition like

$$|\partial_\lambda \alpha(A_1(\lambda))| \geq \chi_1 > 0, \qquad \text{for all} \qquad \alpha \in \Delta,$$

(for a similar condition in this context, see [48]). In our case and since $A_1(\lambda) = \lambda A$, this condition is satisfied as long as $A \neq 0$. Now, it is clear that, to obtain a reducibility result for almost every value of λ we must know how to treat the case when, at the r-th step, A_r is in the resonant set $RS(N_r, K_r)$. This is the content of the following section.

Resonances and how to avoid them

Assume that we find a resonance at the r-th step; that is, let $\alpha_r = |A_r|$ be the root of A_r (in this case, we can consider only one root), and assume that it is close to a certain resonance $\langle \mathbf{k}_0, \omega \rangle$ (in the above notations, $\alpha_r \in RS(N_r, K_r)$). Then we can conjugate (far from the identity) $A_r + F_r$ to $\tilde{A}_r + \tilde{F}_r$ through the transformation

$$\tilde{Z}_r(\theta) = \exp\left(-\langle \mathbf{k}_0, \omega \rangle \frac{A_r}{|A_r|} \right).$$

By means of this transformation $\tilde{F}_r$ is of the same order as F_r and the new time-free matrix is

$$\tilde{A}_r = A_r - \langle \mathbf{k}_0, \theta \rangle \frac{A_r}{|A_r|},$$

for which now

$$\alpha(\tilde{A}_r) = \alpha(A_r) - \langle \mathbf{k}_0, \omega \rangle,$$

similar to the exponential transformations that we saw in the section devoted to Eliasson's results on reducibility 3.4.2. Therefore, after the transformation, $\tilde{A}_r \notin RS(N_r, K_r)$ ($\mathbf{k}_0$ was the *only* resonant value of order less than N_r).

Therefore, whenever we find a resonance we apply the previous transformation and we get a conjugation given by

$$G_r(\theta) = e^{\tilde{Y}_{r-1}(\theta)} \tilde{Z}_{r-1}(\theta) \cdot \ldots \cdot e^{\tilde{Y}_1(\theta)} \tilde{Z}_1(\theta).$$

As the transformations $\tilde{Z}$ are far from the identity, the convergence is unsure unless we can show that we only have to remove a finite number of resonances. In [27], Eliasson was able to prove the convergence of the above scheme showing that, whenever the rotation number is either Diophantine or rational with respect to the half frequency module (see lemma 3.4.9 from the previous chapter),

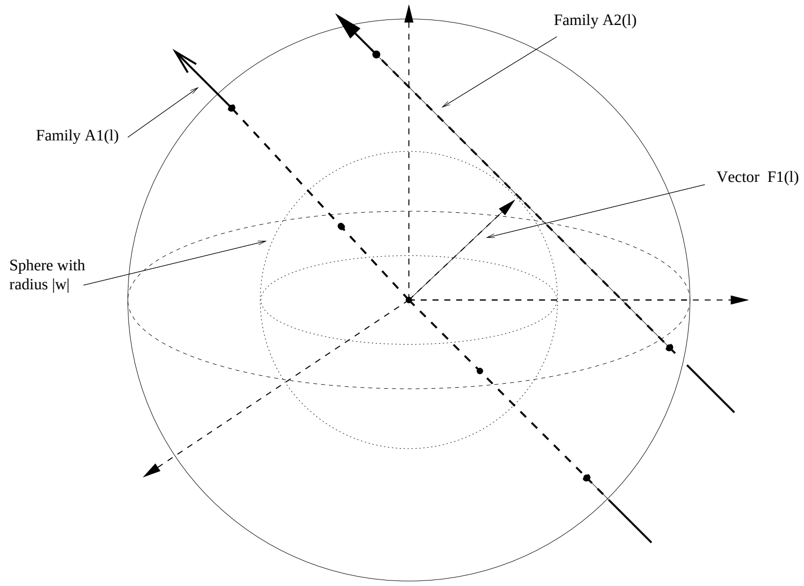

Figure 4.3: Illustration of the loss of smooth dependence of the roots on the parameter λ after removing a resonance in the $so(3,\mathbb{R})$ case. See text for details.

and always in the close-to-constant-coefficients case, we only have to remove a finite number of resonances in the case of Schrödinger's equation with quasi-periodic potential. That the rotation number was defined, together with its relation with the spectral properties of the operator was essential, because, the *role* of our parameter could be played by the rotation number.

We are going to see that it is not possible to keep a good dependence on the parameter λ after the passage close to a resonance. To make the discussion easier, we are going to assume that, for $\lambda \in (-\delta_0, \delta_0)$, $A_1(\lambda)$ is resonant and that after removing the resonance, $\tilde{A}_1(\lambda)$ is of the form $\tilde{A}_1(\lambda) = \lambda v$, with $v \in so(3,\mathbb{R}) \simeq \mathbb{R}^3$ and norm one. Assume also that $\tilde{F}_1(\lambda)$ is a constant $w \in so(3,\mathbb{R}) \simeq \mathbb{R}^3$, and that it is chosen to be orthogonal to v and with small norm, say η. Then we have that $|\tilde{A}_1(\lambda)| = \lambda$, that is, that the dependence of $|\tilde{A}_1(\lambda)|$ with respect to λ is good. However, note that, performing the subsequent averaging step, we have

$$|A_2(\lambda)| = |\tilde{A}_1(\lambda) + \hat{\tilde{F}}_1(0,\lambda)| = |\lambda v + w| = \sqrt{\lambda^2 + \eta^2} = \eta + \frac{\lambda^2}{2\eta} + o\left(\frac{\lambda^2}{2\eta}\right).$$

That is, the derivative of $|A_2(\lambda)|$ with respect to λ vanishes at first order. It can be shown in the same way that if, after the construction of A_{r+1}, we have removed r resonances, then the derivative of $|A_{r+1}(\lambda)|$ with respect to λ might vanish at order $2^r - 1$. The dependence of the roots with respect to λ might therefore become very bad. Geometrically, this means that $A_{r+1}(\lambda)$ might have a contact of order $2^r - 1$ with the spheres of constant radius in $so(3,\mathbb{R}) = \mathbb{R}^3$, that is, the hyper-surfaces with α =constant. See figure 4.3 for an illustration of this phenomenon.

Therefore, we must use a more general notion of transversality so that this kind of situations can be handled. This will be done by means of *Pyartli transversality*, which, was used by R. Krikorian to treat, first the $SO(3,\mathbb{R})$ case ([52]) and later on the general compact semi-simple case ([53]). This will be the content of the following section.

Root dependence. Pyartli transversality

The idea of Pyartli transversality ([67]) is that the information lost on the derivatives of order $j \leq s$ can be recovered looking carefully at the derivatives up to order $2s$. We give the definition and we refer to [52] and [67] for properties:

Definition 4.2.11 *Let $f : (a,b) \subset \mathbb{R} \to \mathbb{R}$ be a function. We shall say that f is (C,c,s)-Pyartli whenever it is of class C^{s+1} and it satisfies that*

$$\max_{1\le j\le s+1} |\partial_t^j f(t)| \le C, \qquad \max_{1\le j\le s} |\partial_t^j f(t)| > c.$$

As we will be working with orders of transversality which tend to infinity, we will need to make hypothesis on the analyticity of f (Gevrey estimates would also work). These will be our s-transversal functions ([52])

Definition 4.2.12 *Let f be a real analytic function defined in an interval (a,b). We shall say that it is (M,δ,c,s)-transversal if*

(i) f can be extended in a complex strip of width δ, $B_\delta = (a-\delta, b+\delta) + i(-\delta,\delta)$ and satisfies that

$$\sup_{z\in B_\delta} |f(z)| \le M,$$

(ii) for all $t \in (a-\delta/2, b+\delta/2) \subset \mathbb{R}$:

$$\max_{1\le j\le s} |\partial_t^j f(t)| > c > 0.$$

Remark 4.2.13 *Using Cauchy inequalities for an analytic function it follows that a (M,δ,c,s)-transversal function on (a,b) is $(M(\delta/2)^{-(s+1)}, c, s)$-Pyartli on $(a-\delta/2, b+\delta/2)$.*

The following result is essential in the control of the roots in the inductive scheme, and it deals with the change of the transversality after small perturbations

Proposition 4.2.14 ([52]) *Let $\mathcal{T}$ be a maximal torus of $so(3,\mathbb{R})$ (i.e., a straight line passing through the origin) and let $\gamma : (a,b) \to \mathcal{T}$ be a (M,δ,c,s)-transversal function on (a,b). Let also $\eta(\cdot) \in C^a_{\delta_\eta}([a,b], so(3,\mathbb{R}))$, (with $0 < \delta_\eta \le \delta$) such that*

$$\max_{z\in W_{\delta_\eta}} |\eta(z)| \le m,$$

being W_{δ_η} the complex fattening of $[a,b]$. Let also

$$\mu(k) = \min\left(1, \max_{0\le j\le k} \sup_{w\in W_{\delta_\eta/2}} |\partial_w^j \eta(w)|\right),$$

(note that $\mu(k) \le m(\delta/2)^{-k}$), and let $1 > \zeta > 0$. Assume that $s \ge 1$, $M \ge 1$, $0 < c < 1$ and $0 < \delta < 1$. Then:

(i) If I is an interval on which $|\gamma(t) + \eta(t)| > \zeta > 0$ for some $\zeta > 0$, then $|\gamma(s) + \eta(s)|$ is a $(M', \delta', c', 2s)$-transversal function on I, with

$$M' = M + 3m,$$

$$\delta' = \frac{\zeta\delta_\eta}{16(M+3m)^2} \qquad \text{and}$$

$$c' = 2^{-s}\sqrt{|(vM^{-1}\delta)^{5s^3} c^{4s} - (6M\delta^{-1})^{2s}\mu(2s)|}.$$

(ii) If $m < (1/11)\zeta$ and if I is an interval on which $\gamma(t) > \zeta > 0$, then $|\gamma + \eta|$ is a (M', δ', c', s)-transversal function on I with

$$M' = M + 3m,$$
$$\delta' = \min\left(\frac{\zeta\delta}{8M}, \delta_\eta\right) \qquad \textit{and}$$
$$c' = \left|c - (2M\zeta^{-1}\delta_\eta^{-1})^{5s} m\right|.$$

The above proposition makes the assertion *the passage through a resonance doubles the order of transversality* rigorous; we shall call this phenomenon the *loss of transversality.* Note that this loss of transversality does not happen when the roots $\alpha(A_r)$ lie in the corresponding Diophantine sets $DS(N_r, K_r)$ at every step r.

If the estimates on transversality are to be useful, and as they depend strongly on the order of transversality s, then it is necessary to show that s_r (the order of transversality of $\lambda \mapsto |A_r(\lambda)|$ at the r-th step) grows very slowly. That is, that the order of transversality doubles only very seldom. The estimates on the occurrence of such transversality order doubling will be given in the following section, together with the sketch of how these can be used in combination with the reducibility on positive measure to prove almost everywhere reducibility.

From positive measure to total measure reducibility

The end of the proof goes as follows. At each step r of the recursion we will define a partition Π_r of the parameter interval Λ into q_r sub-intervals $(\Lambda_{r,j})_{1\leq j\leq r}$. In the interior of each of these sub-intervals, the original flow is conjugated to a new flow defined by $A_r(\lambda) + F_r(\lambda, \cdot)$, being F_r very small (it is of the form constant $\cdot \exp(-(1+\beta)^r)$, with $\beta > 0$).

The technique for handling resonances, together with the notion of transversality leads to a classification of the sub-intervals of the partition Π_r into two classes, Π_r^t (the t-ransversal sub-partition) and Π_r^0 (the *stand-by* sub-partition, the reason for the name will become clearer in a moment).

If an interval $\Lambda_{r,j}$ belongs to the transversal partition Π_r^t (we shall make this evident by writing $\Lambda_{r,j}^t$)) then the map

$$\begin{array}{ccc} \Lambda_{r,j}^t & \to & \mathbb{R} \\ \lambda & \mapsto & |A_r(\lambda)| \end{array}$$

is $(M_r, \delta_r, c_r, s_r)$-transversal, where $M_r = 4^{r-1}M_1$, M_1 fixed, δ_r (the width of the analyticity strip) tends with r to a positive constant, c_r is always greater than the norm of F_r on the strip of width δ_r and s_r, the order of the transversality, satisfies the estimate

$$s_r \leq \text{constant}_k \log_{(k)} r, \tag{4.24}$$

where $k \geq 1$ is a fixed integer, constant$_k$ is a constant depending only on k and $\log_{(k)}$ denotes the k-th iterate of log as long as it makes sense. Therefore, we have that the order of transversality grows very slowly and that in the intervals $\Lambda_{r,j}$ we have useful information on the transversality properties.

The *stand-by* intervals $\Lambda_{r,j}$ belonging to the partition Π_r^0, correspond to those intervals on which we must remove a resonance at the r-th step. Therefore, the transversality condition (recall that we are referring to the generalization of this notion given in the previous section), becomes worse, but, it can be used, not to prove convergence of the method, but to just control the constants involved in the method. Of course, as we want to prove convergence of the method, we can only

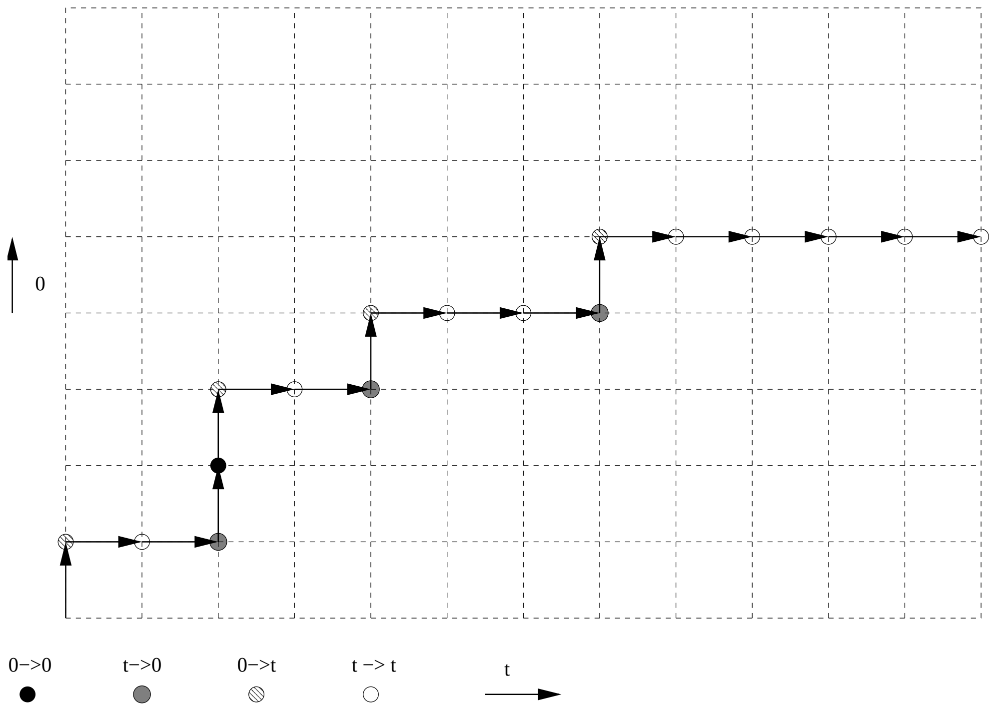

Figure 4.4: Sketch of a possible *path* of a value of λ showing when it belongs to the partition Π_r^t or Π_r^0. The order r is the number of arrows. In this case, the value of λ is always in Π_r^t for $r \geq 13$. A behaviour of this type (paths which are *bounded above*) happens for almost every value of Λ.

be in this sub-partition for a finite number of r's; that is, it is showed that *for almost every $\lambda \in \Lambda$, with respect to the Lebesgue measure, there is an order $r_0 = r_0(\lambda)$, such that if $r \geq r_0(\lambda)$ then $\lambda \in \Lambda^t_{r,j_\lambda}$.* The almost everywhere reducibility comes from this last argument. For the proof, one must consider at each step the four possible bifurcations, that may happen

(t→t): $\lambda \in \Lambda_r^t \cap \Lambda_{r+1}^t$

(0→t): $\lambda \in \Lambda_r^0 \cap \Lambda_{r+1}^t$

(t→0): $\lambda \in \Lambda_r^t \cap \Lambda_{r+1}^0$

(0→0): $\lambda \in \Lambda_r^0 \cap \Lambda_{r+1}^0$

as it is shown in figure 4.4.

This is, obviously, just a short sketch of the proof, which is quite technical. For the complete arguments and a proper exposition we refer, again, to Krikorian's monographs [52] and [53].

4.2.3 Remarks on the general compact case

The proof of the reducibility result for the general compact semi-simple case (theorem 4.2.7) can be found in Krikorian's work [53] and requires some adaption from the proof in the $SO(3,\mathbb{R})$ setting.

One of the problems that are found comparing to the former case is that, apart from resonances of the roots of A with the external frequencies, we have resonances between the different roots of A, which are difficult to control. In general, given a certain maximal torus $\mathcal{T}_A$ passing through

A for which we assume that there are resonances between its roots, it is not always possible to find a $\tilde{Z} \in C^a(\mathbb{T}^d, \mathcal{T}_A)$ which conjugates A_1 to a Diophantine $\tilde{A}_1$. However, one can define a transformation $\tilde{Z} \in C^a(\mathbb{R}^d, \mathcal{T}_A)$ with the desired properties but $(2\pi P\mathbb{Z})^d$-periodic instead of $(2\pi\mathbb{Z})^d$-periodic (being P an integer greater than one).

As there are many roots of the elements $A(\lambda) \in G$, it is necessary to introduce a notion for the transversality of set of roots $\Sigma(\lambda)$, and to study the separation of these roots depending on the parameter $\lambda \in \Lambda$. According to this separation criterion ([53]), the roots of $A(\lambda)$ are split into different *piles* $\Sigma_j(\lambda)$ (from the French *amas*). If the piles are separated at all steps of the process, the transversality of these sets can be controlled and used to prove reducibility (as in the case of $SO(3,\mathbb{R})$). In contrast to the $SO(3,\mathbb{R})$ case, it can happen that two or more piles agglutinate at a certain step (we shall call this *root growth*). It is however possible to show that this last phenomenon happens only a finite number of times for almost every $\lambda \in \Lambda$, and that it is possible to keep the control of transversality between the piles of roots in the meanwhile.

4.2.4 Two results on non-reducibility in compact groups

To end this chapter we give two results on non-reducibility for compact groups. They give a glimpse of the fine structure of reducible and non-reducible zones close to constant coefficients in the case of $SO(3,\mathbb{R})$ and $SU(2)$ (the group of special unitary transformations of $\mathbb{C}^2$). In the light of results from the previous sections, these results on the *density* of non-reducibility apply in those zones (of Lebesgue measure zero in generic one-parameter families) which were excluded in the process for proving reducibility. Compare also with theorem 3.4.11 in the (non-compact) case of $SL(2,\mathbb{R})$.

The first result on non-reducibility is due to H: Eliasson, and refers to the case $SO(3,\mathbb{R})$

Theorem 4.2.15 ([22]) *Consider the following skew-product flow on $SO(3,\mathbb{R})$*

$$X' = (A + F(\theta))\,X, \qquad \theta' = \omega, \tag{4.25}$$

where $\omega \in DC(\gamma,\sigma)$ and that $F \in C^a_h(\mathbb{T}^d, so(3,\mathbb{R}))$. Then there exists a $\varepsilon_0 = \varepsilon_0(d,\sigma,A,\gamma,h)$ such that there is a residual G_δ-set in

$$|F|_h \leq \varepsilon_0$$

for which the skew-product flow (4.25) is uniquely ergodic in $SO(3,\mathbb{R}) \times \mathbb{T}^d$.

In particular there is no reducibility, because for a reducible system in $SO(3,\mathbb{R})$ there must exist a foliation of $SO(3,\mathbb{R}) \times \mathbb{T}^d$ in tori (typically of dimension $d+1$) and hence *many* invariant measures. The proof uses the same technique for removing resonances that we used to prove reducibility. Note that this kind of non-reducibility close to constant coefficients (3.4.11) was also obtained as a *by-product* of reducibility theorems in the case of $SL(2,\mathbb{R})$. This seems to be a quite general situation.

The second result on non-reducibility refers to $SU(2)$ and it is due to R. Krikorian. It focuses on the time-discrete skew-product and proves a conjecture by M. Rychlik ([70]) on the skew-product

$$(X,\theta) \mapsto (A(\theta)X, \theta + \omega), \tag{4.26}$$

where $(X,\theta) \in SU(2) \times \mathbb{T}$ (therefore it covers the case of linear equations with quasi-periodic coefficients in $su(2)$ and *two* frequencies), $\omega \in \mathbb{T}$ and $A \in C^k(\mathbb{T}, SU(2))$. A skew-product like (4.26) will be denoted by (A,ω), and the set of all skew-products of this kind, will be denoted by $SW^k(\mathbb{T}, SU(2))$ and furnished with the C^k-topology. The result deals with the density of non-reducible systems :

Theorem 4.2.16 ([51]) *The set of systems $(A,\omega) \in SW^0(\mathbb{T}^1, SU(2))$ which are not reducible is dense (in the C^0-topology).*

The proof follows from the two following theorems, the first one due to R. Krikorian

Theorem 4.2.17 ([51]) *For all $(A,\omega) \in SW^0(\mathbb{T}, SU(2))$ and $\varepsilon > 0$, there exists an element $(\tilde{A},\tilde{\omega}) \in SW^\infty(\mathbb{T}, SU(2))$, C^∞-reducible and such that*

$$|\omega - \tilde{\omega}|,\ |A - \tilde{A}|_0 \leq \varepsilon.$$

and the second one is due to M. Herman

Theorem 4.2.18 ([35]) *If $\omega \notin \mathbb{Q}$ the adherence in C^∞-topology of the set V_ω consisting of those $A \in C^\infty(\mathbb{T}, SU(2))$ such that (A,ω) is not reducible contains $SU(2)$.*

Bibliography

[1] V. I. Arnol'd. Proof of a theorem of A. N. Kolmogorov on the invariance of quasi-periodic motions under small perturbations of the hamiltonian. *Russian Math. Surveys*, 18:9–36, 1963.

[2] V. I. Arnol'd. *Mathematical methods of classical mechanics.* Springer-Verlag, New York, 1996.

[3] S. Aubry and G. André. Analyticity breaking and Anderson localization in incommensurate lattices. In *Group theoretical methods in physics (Proc. Eighth Internat. Colloq., Kiryat Anavim, 1979)*, pages 133–164. Hilger, Bristol, 1980.

[4] J. Avron and B. Simon. Almost periodic schrödinger operators II.the integrated density of states. *Duke Math. J.*, 50:369–391, 1983.

[5] F. A. Berezin and M. A. Shubin. *The Schrödinger equation.* Kluwer Academic Publishers Group, Dordrecht, 1991.

[6] N. N. Bogoljubov and N. Krylov. La théorie générale de la mesure et son application a l'étude des systèmes dynamiquess de la méchanique non linéaire. *Ann. Math.*, 38:65–113, 1938.

[7] N. N. Bogoljubov, J. A. Mitropoliskii, and A. M. Samoĭlenko. *Methods of accelerated convergence in nonlinear mechanics.* Hindustan Publishing Corp., Delhi, 1976.

[8] F. Bonetto, G. Gallavotti, G. Gentile, and V. Mastropietro. Quasi-linear flows on tori: regularity of their linearization. *Comm. Math. Phys.*, 192(3):707–736, 1998.

[9] J. Bourgain and Jitomirskaya S. Continuity of the Lyapunov exponent for quasiperiodic operators with analytic potential. *Preprint*, 2001. Accessible at `http://www.maia.ub.es/mp_arc`.

[10] H. W. Broer, G. B. Huitema, and M. B. Sevryuk. *Quasi-periodic motions in families of dynamical systems.* Springer-Verlag, Berlin, 1996. Order amidst chaos.

[11] H.W. Broer, J. Puig, and C. Simó. On resonance tongues and instability pockets in Hill's equation with quasi-periodic forcing. 2001. In preparation.

[12] H.W. Broer and C. Simó. Hill's equation with quasi-periodic forcing: resonance tongues, instability pockets and global phenomena. *Bol. Soc. Brasil. Mat. (N.S.)*, 29(2):253–293, 1998.

[13] R. Carmona and J. Lacroix. *Spectral theory of random Schrödinger operators.* Probability and its Applications. Birkhäuser, 1990.

[14] L. Chierchia. Absolutely continuous spectra of quasiperiodic Schrödinger operators. *J. Math. Phys.*, 28(12):2891–2898, 1987.

[15] E.A. Coddington and N. Levinson. *Theory of ordinary differential equations.* McGraw-Hill Book Company, Inc., New York-Toronto-London, 1955.

[16] H. L. Cycon, R. G. Froese, W. Kirsch, and B. Simon. *Schrödinger operators with application to quantum mechanics and global geometry.* Springer-Verlag, Berlin, study edition, 1987.

[17] R. de la Llave. A tutorial on KAM theory. In *Smooth ergodic theory and its applications (Seattle, WA, 1999)*, pages 175–292. Amer. Math. Soc., Providence, RI, 2001.

[18] P. Deift and B. Simon. Almost periodic Schrödinger operators III. The absolute continuous spectrum. *Comm. Math. Phys.*, 90:389–341, 1983.

[19] F. Delyon. Absence of localisation in the almost Mathieu equation. *J. Phys. A*, 20(1):L21–L23, 1987.

[20] E.I. Dinaburg and Y.G. Sinai. The one-dimensional schrödinger equation with quasi-periodic potential. *Funkt. Anal. i. Priloz.*, 9:8–21, 1975.

[21] L. H. Eliasson. Perturbations of stable invariant tori for Hamiltonian systems. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 15(1):115–147 (1989), 1988.

[22] L. H. Eliasson. Ergodic skew systems on $so(3,\mathbb{R})$. *preprint ETH-Zürich*, 1991.

[23] L. H. Eliasson. Discrete one-dimensional quasi-periodic Schrödinger operators with pure point spectrum. *Acta Math.*, 179(2):153–196, 1997.

[24] L. H. Eliasson. One-dimensional quasi-periodic Schrödinger operators—dynamical systems and spectral theory. In *European Congress of Mathematics, Vol. I (Budapest, 1996)*, pages 178–190. Birkhäuser, Basel, 1998.

[25] L. H. Eliasson. Reducibility and point spectrum for linear quasi-periodic skew-products. In *Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998)*, number Extra Vol. II, pages 779–787 (electronic), 1998.

[26] L. H. Eliasson. Almost reducibility of linear quasi-periodic systems. In *Smooth ergodic theory and its applications (Seattle, WA, 1999)*, pages 679–705. Amer. Math. Soc., Providence, RI, 2001.

[27] L.H. Eliasson. Floquet solutions for the one-dimensional quasi-periodic schrödinger equation. *Comm. Math. Phys.*, 146:447–482, 1992.

[28] A Fink. *Almost periodic differential equations.* Number 377 in Lecture Notes in Mathematics. Springer, 1974.

[29] J. Fröhlich, T. Spencer, and P. Wittwer. Localization for a class of one-dimensional quasi-periodic Schrödinger operators. *Comm. Math. Phys.*, 132(1):5–25, 1990.

[30] D.J. Gilbert and D.B. Pearson. On subordinacy and analysis of the spectrum of one-dimensional schrödinger operators. *J. Math. Anal. Appl.*, 128:30–56, 1987.

[31] M. Goldstein and W. Schlag. Hölder continuity of the integrated density of states for quasi-periodic Schrödinger equations and averages of shifts of subharmonic functions. *Ann. of Math. (2)*, 154(1):155–203, 2001.

[32] G. Gómez, J.M. Mondelo, and Simó C. Refined fourier analysis: procedures, error estimates and applications. *Preprint*, 2001. Accessible at `http://www.maia.ub.es/dsg/2001`.

[33] A. Gordon. On the point spectrum of the one-dimensional Schrödinger operator. *Russ. Math. Surv.*, 31:257–258, 1976.

[34] A. Y. Gordon, S. Jitomirskaya, Y. Last, and B. Simon. Duality and singular continuous spectrum in the almost Mathieu equation. *Acta Math.*, 178(2):169–183, 1997.

[35] M.R. Herman. Non topological conjugacy of skew-products in $su(2)$. manuscript.

[36] M.R. Herman. Une méthode pour minorer les exposants de lyapunov et quelques exemples montrant le caractère local d'un théorème d'arnold et de moser sur le tore de dimension 2. *Comment. Math. Helvetici*, 58:262–288, 1981.

[37] E. Hille. *Lectures on ordinary differential equations.* Addison-Wesley, Reading, Mass., 1969.

[38] S. Y. Jitomirskaya. Metal-insulator transition for the almost Mathieu operator. *Ann. of Math. (2)*, 150(3):1159–1175, 1999.

[39] R. Johnson. Ergodic theory and linear differential equations. *J. Diff. Eq.*, 28:165–194, 1978.

[40] R. Johnson. The recurrent Hill's equation. *J. Diff. Eq.*, 46:165–193, 1982.

[41] R. Johnson. Lyapunov numbers for the almost periodic Schrödinger equation. *Illinois J. Math.*, 28(3):397–419, 1984.

[42] R. Johnson. Exponential dichotomy, rotation number and linear differential equations with bounded coefficients. *J. Diff. Eq.*, 61:54–78, 1986.

[43] R. Johnson. Cantor spectrum for the quasi-periodic Schrödinger equation. *J. Diff. Eq.*, 91:88–110, 1991.

[44] R. Johnson and J. Moser. The rotation number for almost periodic potentials. *Comm. Math. Phys.*, 84:403–438, 1982.

[45] R. Johnson and M. Nerurkar. Exponential dichotomy and rotation number for linear Hamiltonian systems. *J. Differential Equations*, 108(1):201–216, 1994.

[46] R. Johnson and G.R. Sell. Smoothness of spectral subbundles and reducibility of quasi-periodic linear differential systems. *J. Diff. Eq.*, 41:262–288, 1981.

[47] À. Jorba, Ramírez-Ros R., and J. Villanueva. Effective reducibility of quasi-periodic linear equations close to constant coefficients. *SIAM J. Math. Anal.*, 28(1):178–188, 1997.

[48] À. Jorba and C. Simó. On the reducibility of linear differential equations with quasiperiodic coefficients. *J. Differential Equations*, 98(1):111–124, 1992.

[49] À. Jorba and C. Simó. On quasi-periodic perturbations of elliptic equilibrium points. *SIAM J. Math. Anal.*, 27(6):1704–1737, 1996.

[50] S. Kotani. Ljapunov indices determine absolutely continuous spectra of stationary random one-dimensional Schrödinger operators. In *Stochastic analysis (Katata/Kyoto, 1982)*, pages 225–247. North-Holland, Amsterdam, 1984.

[51] R. Krikorian. C^0-densité globale des systèmes produits-croisés sur le cercle réductibles. *Ergodic Theory Dynam. Systems*, 19(1):61–100, 1999.

[52] R. Krikorian. Réductibilité des systèmes produits-croisés à valeurs dans des groupes compacts. *Astérisque*, 259, 1999.

[53] R. Krikorian. Réductibilité presque partout des flots fibrés quasi-périodiques à valeurs dans les groupes compacts. *Ann. scient. Éc. Norm. Sup.*, 32:187–240, 1999.

[54] J. Laskar. Introduction to frequency map analysis. In *Hamiltonian systems with three or more degrees of freedom (S'Agaró, 1995)*, pages 134–150. Kluwer Acad. Publ., Dordrecht, 1999.

[55] J. Laskar, C. Froeschlé, and A. Celletti. The measure of chaos by the numerical analysis of the fundamental frequencies. Application to the standard mapping. *Phys. D*, 56(2-3):253–269, 1992.

[56] Y. Last. A relation between a.c. spectrum of ergodic Jacobi matrices and the spectra of periodic approximants. *Comm. Math. Phys.*, 151(1):183–192, 1993.

[57] Y. Last. Almost everything about the almost Mathieu operator. I. In *XIth International Congress of Mathematical Physics (Paris, 1994)*, pages 366–372. Internat. Press, Cambridge, MA, 1995.

[58] J. Moser. Convergent series expansions for quasi-periodic motions. *Math. Ann.*, 169:136–176, 1967.

[59] J. Moser. *Stable and random motions in dynamical systems.* Princeton University Press, Princeton, N. J., 1973. With special emphasis on celestial mechanics, Hermann Weyl Lectures, the Institute for Advanced Study, Princeton, N. J, Annals of Mathematics Studies, No. 77.

[60] J. Moser. An example of schrödinger equation with almost periodic potential and nowhere dense spectrum. *Comment. Math. Helvetici*, 56:198–224, 1981.

[61] J. Moser and J. Pöschel. An extension of a result by dinaburg and sinai on quasi-periodic potentials. *Comment. Math. Helvetici*, 59:39–85, 1984.

[62] S. Novo, C. Núñez, and R. Obaya. Ergodic properties and rotation number for linear Hamiltonian systems. *J. Differential Equations*, 148(1):148–185, 1998.

[63] A. Nunes and X. Yuan. A note on the reducibility of linear differential equations with quasiperiodic coefficients. *Preprint*, 2001.

[64] R. Obaya and M. Paramio. Directional differentiability of the rotation number for the almost periodic Schrödinger equation. *Duke Math. J.*, 66:521–552, 1992.

[65] L. Pastur and A. Figotin. *Spectra of random and almost-periodic operators.* Springer-Verlag, Berlin, 1992.

[66] M. Postnikov. *Lie groups and Lie algebras.* "Mir", Moscow, 1986. Lectures in geometry. Semester V.

[67] A.S. Pyartli. Diophantine approximations on submanifolds of euclidean space. *Funkt. Anal. i. Priloz.*, 3:303–306, 1969.

[68] M. Reed and B. Simon. *Functional Analysis*, volume 1. Academic Press, 1972.

[69] W. Rudin. *Real and Complex Analysis*. McGraw-Hill Book Co., New York, 1987.

[70] M. Rychlik. Renormalization of cocycles and linear ode with almost periodic coefficients. *Invent. Math.*, 110:173–206, 1992.

[71] R.J. Sacker and G. Sell. A spectral theory for linear differential systems. *J. Diff. Eq.*, 27:320–358, 1978.

[72] G. Scharf. Fastperiodische potentiale. *Helv. Phys. Acta*, 24:573–605, 1965.

[73] M. B. Sevryuk. The lack-of-parameters problem in the KAM theory revisited. In *Hamiltonian systems with three or more degrees of freedom (S'Agaró, 1995)*, pages 568–572. Kluwer Acad. Publ., Dordrecht, 1999.

[74] B. Simon. Almost periodic Schrödinger operators: a review. *Adv. in Appl. Math.*, 3(4):463–490, 1982.

[75] B. Simon. Schrödinger operators in the twentieth century. *J. Math. Phys.*, 41(6):3523–3555, 2000.

[76] Ya. G. Sinaĭ. Anderson localization for one-dimensional difference Schrödinger operator with quasiperiodic potential. *J. Statist. Phys.*, 46(5-6):861–909, 1987.

[77] D.J. Thouless. A relation between the density of states and range of localization for one-dimensional random system. *J. Phys. C*, 5:77–81, 1972.