An overview on some problems in Celestial Mechanics *

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Abstract

Some topics in Celestial Mechanics are presented. After some general facts on the N–body problem, we devote most of the contents to a simple three degrees of freedom Hamiltonian: The spatial Restricted Three–Body Problem (3D RTBP). It is simple enough to keep the technicalities to a minimum and to illustrate most of the difficulties to be encountered in many Celestial Mechanics problems. Libration points and the local behaviour around them, periodic orbits and invariant tori, stability and practical stability are some of the discussed items. The role of center, stable and unstable manifolds is stressed.

1 Introduction

Celestial Mechanics is a part of Dynamical Systems, mainly Hamiltonian Systems, dealing with the study of the motion of natural and artificial bodies under the action of gravitation. Usually the number of interacting bodies is considered not too large. Other problems, like the dynamics of galaxies and clusters, have usually a different approach. The study of the motion of artificial bodies has some peculiarities (design of missions by means of suitable manoeuvres) and is sometimes denoted as Astrodynamics. We shall not make a distintion here between natural and artificial bodies.

Beyond the gravitational forces we can have relativistic corrections, dissipative forces (as in the case of satellites moving inside the atmosphere) or other nongravitational forces, like in the case of comets approaching the Sun. Usually they can be considered as perturbations of the purely gravitational model. Furthermore, the interacting bodies can be considered as punctual or as finite. When dealing with finite size bodies the mass distribution inside them creates a potential which can be expanded in spherical harmonics, to be considered as a perturbation of the punctual mass attraction. This is the usual approach when studying, for instance, artificial Earth satellites. The perturbations acting on them change slightly the orbit and these changes can, in turn, be used to determine the harmonics. Furthermore, real bodies are non rigid, and this gives rise to tidal effects. All of them will be skipped now and only the case of puntual masses shall be considered.

Continuous symmetries in the equations of motion are associated to first integrals. This allows for a reduction in the number of degrees of freedom. Some elementary first integrals will be used in the formulation, that is, we shall consider, directly, the reduced system. So, we shall be faced to the intrinsic problems of some model of a real problem.

Celestial Mechanics is one of the oldests branches of human knowledge. It starts with the interest in predicting the motion of Sun, Moon, planets and stars. It can be considered as old

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as history itself. This part of Celestial Mechanics is known, actually, as positional astronomy. It has influenced many areas in the physical sciences and most of the areas in mathematics. In present times can be seen as a combination of mechanics (both theoretical and applied) and differential equations. It has a strong degree of multidisciplinarity: It has relations with real and complex function theory, functional analysis (harmonic analysis, operator theory and measure theory), topology, differential geometry, number theory, differential Galois theory, numerical analysis, symbolic computation and computer graphics. This list of topics is not intended to be exhaustive!

Some of the usual techniques to deal with problems in Celestial Mechanics can be considered inside perturbation theory. An approximation to the problem can be integrable, so that the solutions can be expressed in a simple form (at least in theory!) and we can look for solutions of the full problem expanded in power series of some parameter. This requires analyticity of the equations of motion and their dependence with respect to parameters. The solution of the integrable system gives the zero order terms of this expansion. It can be also possible to try to perform changes of variable such that the problem is reduced to a simpler form, hopefully integrable. This is not possible, of course, but averaging methods or normal form techniques often succeed in carrying out such a reduction at the formal level. The lack of convergence of these procedures, due to small divisors problems in most of the cases, asks for bounds on the errors done when several steps of an iterative process are performed. Furthermore, one can derive information about the number of steps to be used so that the estimate of the total error is minimized. Typically this minimum is exponentially small in the small parameter involved in the expansions. More refined information requires the use of time as a complex variable.

From a geometrical point of view a key point consists in trying to understand the structure of the phase space. We can find in it fixed points, periodic orbits and invariant tori. Some of these objects can be (at least partially) of *hyperbolic* character. The related stable and unstable manifolds form a kind of network of connections (the so called *homoclinic tangle*), which together with the previous invariant objects constitute the *skeleton* of the system. The dynamics follows closely the "paths" marked by these connections. The objects which are of purely *elliptic* type, under suitable conditions on the related frequencies, have the nearby phase space "almost foliated" by invariant tori. Here "almost" means that the we have a set of tori depending on parameters (say, amplitudes or "actions") and the parameters range on a Cantorian set of big relative measure. The "gaps" in the Cantor set are due to resonances, but if these resonances are of high order the gaps can be very small. In them there is a mild lack of predictability (or, if you want, some "chaos"). Notice that these gaps can be connected, leading to a *diffusion* of the action variables, but at a very slow rate. Other objects are partially elliptic and partially hyperbolic. Among them we can consider the ones having just an hyperbolic plane, that is, only one positive and one negative eigenvalues (with the same modulus because of the Hamiltonian character). These objects have *center manifolds* of codimension 2, and the stable and unstable manifolds of these center manifolds have codimension 1. They are ideal candidates to act as "barriers" or *confiners* to make difficult the escape of points. In fact these manifolds do not prevent escape: stable and unstable manifolds do not match but, generically, they intersect transversally. The manifolds are highly folded and this, together with small angles at the intersections, produces that the particles of the phase space have to wander for a long time before finding the way out.

A technical difficulty in the sketch above is as follows: For periodic orbits, the elliptic or hyperbolic character is well defined. We know that a time–periodic change of variables reduces the variational flow along the orbit to the one of a constant coefficients matrix (Floquet theory). But this is no longer true when we consider the motion around a torus of dimension two or higher: The variational flow can not be reduced to constant coefficients, in general. Anyway, if the linear system can be put as a constant coefficients one plus a small perturbation, depending on time quasiperiodically, then the reduction is possible for a large (again Cantorian) set of values of the perturbation parameter, provided some non degeneracy conditions hold.

However, most of the problems in Celestial Mechanics are beyond the present limits of the purely analytical techniques. Quite often one has to rely on numerical simulations to understand the behaviour of the solutions of some problem. On the other hand, purely numerical methods are not enough when the number of variables and parameters involved is too large. I consider that, at the present epoch, reliable and complete enough simulations can be done with at most 6 or 7 total variables. A combination of symbolic and numerical techniques, guided by careful analytical studies and the suitable geometric insight, can give very good and useful results.

At the end of these notes there appear some references for most of the topics presented above. These references must be completed with the ones given at the end of the notes written by Àngel Jorba for the course *Mecánica Hamiltoniana y Teoría de Perturbaciones*. For shortness, common references are not duplicated.

2 The N–Body problem

The N–body problem describes the motion of N masses under their mutual garvitational attraction. Assuming the gravitational constant equal to 1, the equations of motion are

$$m_i\ddot{x}_i = \sum_{j=1,\ j\neq i}^{N} \frac{m_i m_j (x_j - x_i)}{||x_j - x_i||^3} = \frac{\partial U}{\partial x_i},$$

where

$$U = \sum_{1\leq i<j\leq N} \frac{m_i m_j}{||x_j - x_i||}.$$

The values m_i and x_i denote the mass and the position of the body number i, and $||\ ||$ the euclidean norm. All the masses are assumed to be positive and the position vectors x_i will be considered in $\mathbb{R}^3$. An important fact is that equations are homogeneous of degree -2 with respect to the positions. Introducing $y_i = m_i x_i$ as the linear momentum of the ith particle, the equations of motion can be written in Hamiltonian form

$$\dot{x}_i = \frac{\partial H}{\partial y_i},\ \dot{y}_i = -\frac{\partial H}{\partial x_i} \text{ with } H = \frac{1}{2}\sum_{i=1}^{N} \frac{||y_i||^2}{m_i} - U.$$

The equations above are several elementary first integrals: The total linear momentum is constant, and hence, the center of mass of the system moves on a line with constant velocity, giving 6 integrals. Furthermore we have the angular momentum (with three components) and the Hamiltonian (or energy) itself. There are not more than these 10 first integrals. The initial system has $3N$ degrees of freedom and, hence, $6N$ first order equations. Using the integrals above and the fact that the we can select the origin of longitudes at will, it is possible to reduce the 3D N–body problem to a Hamiltonian with $3N-5$ degrees of freedom. Beyond the elementary case of the two–body problem, which reduces to one degree of freedom and therefore is integrable, and which is assumed to be well known by the audience of these lectures, the simplest problem, the 3–body problem, has already 4 degrees of freedom (it reduces to 3 degrees of freedom in the planar case).

In the equations above it must be assumed that *no collisions occur*, that is, for all $i \neq j$ one must have $x_i \neq x_j$. Otherwise the equation has singularities. The coincidence of two or more bodies at the same place or *collision*, implies that the acceleration and velocity become unbounded. One can consider different methods to *regularize* the equations of motion. The method which has more physical contents is as follows: Assume that some initial conditions, $x(t_0), y(t_0)$ lead to collision for a time $t_1 > t_0$. Now consider a sequence of initial conditions $x^{(k)}(t_0), y^{(k)}(t_0)$ tending to $x(t_0), y(t_0)$ when k tends to ∞ and not leading to collision for $t \leq t_2$, where $t_2 > t_1$. Such sequences exist in the case of the N–body problem, but this is not true for other more degenerate problems. Let $\tau = t_2 - t_0$ and denote by φ_τ the time τ flow of the problem. If the limit for k tending to ∞ of $\varphi_\tau(x^{(k)}(t_0), y^{(k)}(t_0)$ exists and it is independent of the sequence $x^{(k)}(t_0), y^{(k)}(t_0)$ choosen, let us denote the limit as $\bar{x}(t_2), \bar{y}(t_2)$. Then we say that the *singularity is regularizable*. We are recovering the continuity "after the passage close to collision" and can define $\varphi_\tau(x(t_0), y(t_0) = \bar{x}(t_2), \bar{y}(t_2)$. There is a long literature about the collision of two or more bodies and also about several collisions occurring at the same time. A key tool to study the behaviour close to collision is the so called *collision manifold*. Using some scaling of coordinates and time (a blow up) it is possible to replace the point where the singularity occurs by a smooth manifold, which appears as a boundary of the *physical space*, so that the vector field can be extended analytically to this boundary. The boundary itself is travelled "at infinite physical speed" or "in zero physical time", but the time scaling has a definite limit. Furthermore, the study of this manifold and the limit flow on it, is useful, because it gives relevant information on the behaviour of motion close to exact collision. In particular allows to decide on the regularizability. Important results are that binary collisions (either a single one or several of them at the same time) are always regularizable and, in general (that is, except in some symmetric subproblems of the general N–body problem) no other collision is regularizable in the sense decribed above. Furthermore, when the collision is not regularizable, the particles can escape from collision with unbounded kinetic energy. A combination of consecutive passages near collisions of three or more bodies can lead to funny kinds of motion, escaping to infinity in finite time.

In a kind of "dual" way to the study of the collision of 2 or more bodies, one can also study the types of motion when some bodies escape to infinity. It is also possible to introduce a *manifold at the infinity* and study the motion on it. This is quite useful to understand the sets of initial conditions leading to escape. They are "separated" (at least locally) from the ones going far away but returning to "finite regions", by the stable manifolds of the infinity manifold.

As mentioned before the N–Body problem has 10 elementary (or "classical") first integrals. 6 of them are easy to use in the equations, just by assuming that the center of mass of the system is at rest. We can ask the question if the remaining energy (h) and angular momentum (c) integrals give some bounds on the possible motions. In particular, if some ranges of values of (h, c) imply that the motion of the x variables is bounded (despite, perhaps, binary collisions can occur and the y become unbounded). This idea, with the origin in the RTBP (see next section) was strongly pushed by Smale in 1970. The complete description of the sets $I_{(h,c)}$ of (x, y) with given (h, c) is not even finished for the 3–body problem, despite a lot of information is available in that case. Indeed, there exist ranges for which $I_{(h,c)}$ has some connected component which projects of the x variables on a bounded set. However, for 4 or more bodies there is always only one connected component, and it has unbounded x projection.

When looking for simple solutions of the N–body problem we can proceed as for a general dynamical system, that is, to look for fixed points. There are no fixed points but, if a rotating frame with constant angular velocity is used then there are, indeed, fixed points. We can look for a more general situation by asking that, except by some (vectorial) function, $\rho(t)$, all the bodies are at fixed points. That is, we look for a solution of the form $x_j(t) = \rho(t)a_j$ with constant vectors $a_j,\ j = 1, \ldots, N$. It turns out that then the acceleration acting on each particle is proportional to the position of the particle with respect to the center of mass with the same proportionality constant for all bodies (eventually, this constant depends on time). The function $\rho(t)$ is a solution of Kepler problem $\ddot{\rho} = -\rho/||\rho||^3$. It is a conic, eventually degenerated to a solution on a line. The system to be satisfied by the a_j vectors is an algebraic system. It depends of the masses of the N bodies. The vector $\rho(t)$ must be located on a line (and hence collisions must occur for positive os negative time) unless all the a_j are on a plane. In that case $\rho(t)$ can be a non degenerate conic and in a rotating frame and with a time change of the unit of distance if $||\rho(t)||$ is not constant, the solution of the N–body problem can be seen as a fixed point. It is called a *relative equilibrium*.

Relative equilibria play an important role in the N–body problem. They are related to the changes of topology of the $I_{(h,c)}$ sets and to the equilibria on the collision manifold and on the manifold at infinity. However, only simple results are obtained in the 3–body problem. In that case there are always 5 relative equilibria (also known as *libration points*) independently of the masses of the bodies: Three of them consists on the 3 particles on a line, differing by which one is the mass at the middle. They are denoted as *collinear* configurations and are a particular case of the so called *Moulton configurations*. The last ones appear in number of $N!/2$ in the N–body problem. The other two are the *triangular* configurations and the 3 bodies are at the vertices of an equilateral triangle whatever the masses are. The two triangles differ in the orientation. Furthermore, if we normalize distances so that the moment of inertia of the system $I = \sum_{i=1}^{N} m_i ||x_i||^2$ be equal to 1, then for 3 different masses the values of the potential energy under this normalization (or *normalized potential*) are different at the libration points (of course, they are the same at the two triangular librations) and have a well defined order which is independent on the concrete values of the masses if we assume, for instance, $m_1 < m_2 < m_3$.

For more than 3 bodies the situation is completely different. A systematic study requires the use of the computer. The number of relative equilibria depends on the values of the masses. Furthermore the normalized potentials at the relative equilibria have values which can exchange the order without exchanging the order of the values of the masses.

3 The 3D Restricted Three–Body Problem. Libration points and local stability analysis

A special case of the 3D 3–body problem is obtained when two of the masses are finite and the third mass is infinitesimal, so that the action of this mass on the first two bodies can be neglected. The two massive bodies, known as *primaries* or, better, *primary* and *secondary* if one of the masses is greater than the other, move on an orbit of the two body. For simplicity we shall assume in what follows that they are on circles around the common center of masses. The case of primaries moving on an elliptic orbit is also interesting, but slightly more involved, because it is written as a Hamiltonian with 3 degrees of freedom plus a time periodic perturbation (with the period of the primaries). The 3D RTBP is a useful simple model to study the motion of asteroids and comets and for the design of spacecrafts orbits on the Earth–Sun system (the most important thing we neglect is then the Moon influence if the spacecraft is not too far from the Earth) and on the Earth–Moon system (the most important perturbation being then the one of the Sun). Better models for the design are then restricted 4 body problems, with Sun, Earth and Moon as massive primaries and the spacecraft as infinitesimal body, but they become more involved. Furthermore, the 3D RTBP is a good starting point to develop more realistic models of many problems in Celestial Mechanics.

Let μ be the mass parameter

$$\mu = \frac{\text{mass of the secondary}}{\text{mass of the primary} + \text{mass of the secondary}}.$$

Using the homogeneity of the equations it is possible to scale distances so that the distance between the primaries equal to 1. Then in a frame rotating with angular velocity equals to 1 if the gravitational constant is also equal to 1, the equations of motion of the third particle become autonomous. This system of reference is denoted as *synodical system* in contraposition to the *sidereal system* that we have when considering fixed axes of reference. The primaries are then at rest and we shall assume that the body of mass $1-\mu$ is located at $(\mu, 0, 0)$ while the body with mass μ is at $(\mu - 1, 0, 0)$. The system for the infinitesimal particle can be written in Hamiltonian form with 3 degrees of freedom whose Hamiltonian is

$$H(x,y) = \frac{||y||^2}{2} - x_1 y_2 + x_2 y_1 - \frac{1-\mu}{r_1} - \frac{\mu}{r_2},$$

where $x_j,\ y_j,\ j=1,2,3$ denote the position and momentum of the third particle and $r_1,\ r_2$ denote the distances to the primary and secondary, respectively:

$$r_1^2 = (x_1-\mu)^2 + x_2^2 + x_3^2,\ r_2^2 = (x_1-\mu+1)^2 + x_2^2 + x_3^2.$$

In the Hamiltonian it is clearly seen that the first term denotes the kinetic energy in synodical coordinates, the last two correspond to the potential energy, and the remaining term is due to the fact that the frame is rotating (a Coriolis potential). From the Hamilton equations we have

$$\dot{x}_1 = y_1 + x_2,\ \dot{x}_2 = y_2 - x_1,\ \dot{x}_3 = y_3$$

and then the first integral, H, can be expressed in terms of x and $\dot{x}$ as

$$\frac{1}{2}\left(\dot{x}_1^2+\dot{x}_2^2+\dot{x}_3^2\right) - \frac{1}{2}\left(x_1^2+x_2^2\right) - \frac{1-\mu}{r_1} - \frac{\mu}{r_2}.$$

It is clear that the first term must be non negative. For a fixed value of the H not all the positions are available, but only the ones satisfying

$$H + \frac{1}{2}\left(x_1^2+x_2^2\right) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} \geq 0.$$

This defines regions in the *configuration space*, that is the space of the x variables, where the motion is possible. If H is negative and $|H|$ is large, there are three possible regions for the motion: either r_1 is small (and then r_2 is of the order of 1 and $x_1^2+x_2^2$ is also small) and the main contribution comes from the term $(1-\mu)/r_1$, or, in an analogous way, r_2 is small or $x_1^2+x_2^2$ is large. This means that in that case only motion close to one of the primaries or far away from both is allowed. These regions are known as *Hill regions* and they play, in the RTBP, the role of the projections on the configuration space of the $I_{(h,c)}$ sets in the general N–body problem. The boundaries of these regions are obtained by setting $(\dot{x}_1^2+\dot{x}_2^2+\dot{x}_3^2)=0$ and are known by the obvious name *zero velocity surfaces* (ZVS). In the 2D RTBP they are denoted *zero velocity curves*.

When H increases two of these connected components meet at the collinear libration point sitting between both primaries and we have that motions going from "revolutions" around one primary to "revolutions" around the other one are possible. Only two connected Hill regions exist, one bounded and the other unbounded. Further increase of H produces that these two regions meet at the other collinear points if $\mu=1/2$ or at the one which is closer to the secondary if $\mu<1/2$. Only one Hill region exist and it is unbounded. So the motion has no "topological" restrictions to leave to infinity. However, if H is not too large, still a big set of bounded motions exists. Again an increase of H produces that two leaves of the ZVS touch at the third collinear point. For $H=-\frac{3}{2}-\frac{1}{2}(\mu^2-\mu)$ the ZVS leaves the $x_3=0$ plane. This happens at the triangular libration points. From that value on all points on the $x_3=0$ plane are allowable. Finally, further increase of H produces that two leaves of the ZVS move to higher values of $|x_3|$. When $H=0$ all the points of the configuration space are admissible.

As said before there are 5 libration points. The collinear ones are on $x_2=x_3=0$. Let x_{L_j} the values of the x_1 coordinates for $j=1,2,3$. For definiteness we shall assume, for the positions of these points and the primaries the order

$$x_{L_2} < \mu-1 < x_{L_1} < \mu < x_{L_3}.$$

In fact, if μ is small then both $\mu-1-x_{L_2}$ and $\mu-1-x_{L_1}$ are $3^{-1/3}\mu^{1/3}+O(\mu^{2/3})$ and $x_{L_3}=1+O(\mu)$. The triangular points are, of course, located at $(-1/2+\mu,\pm\sqrt{3}/2,0)$. The one with $x_2<0$ is denoted as L_4 and the one with $x_2>0$ as L_5.

We can study the local behaviour around the libration points. This is done by looking first at the linear part of the equations at the L_j points. It is immediate to check that the "planar", that is, the (x_1,x_2,y_1,y_2) variables and the "vertical", that is, the (x_3,y_3) variables are uncoupled at the linear level. On the vertical variables all the libration points have a *center* character. The related eigenvalues are of the form $\pm\omega_v\sqrt{-1}$. On the horizontal variables the collinear points are always of the *center* $\times$ *saddle* type, that is, the eigenvalues are of the form $\pm\lambda$ and $\pm\omega_h$, for all values of the μ parameter. The horizontal behaviour depends on the value of μ. Assume $\mu \in (0,\mu_1)$, where μ_1 is a value to be given below. Then the planar behaviour is of *center* $\times$ *center* type. The two related frequencies are given by

$$\left[\frac{1}{2}(1 \pm (1-27\mu(1-\mu))^{1/2})\right]^{1/2}$$

and, hence $\mu_1 = (1-(23/27)^{1/2})/2 \approx 0.038521$. As the frequencies are related to periodic orbits one denotes them as ω_{short} and ω_{long}, to recall that they are associated to "short" and "long" periodic orbits. They are computed using the formula above with $+$ and $-$ sign, respectively. In general one introduces μ_s such that

$$\frac{\omega_{\text{short}}}{\omega_{\text{long}}} = s, \quad s \in \mathrm{N}$$

as the values of μ at which a s to 1 resonance is produced. The value μ_1 corresponds to the 1 to 1 resonance. For higher values of μ the eigenvalues become complex, of the form $\pm\alpha\pm\beta\sqrt{-1}$, and one says that the planar RTBP at the triangular libration points has a *complex saddle.*

Summarizing, the collinear libration points are always unstable. The triangular ones are linearly stable for $\mu < \mu_1$. In fact it has been proved that, in the planar case they have nonlinear stability in the set

$$\mu \in [0,\mu_1] \setminus \{\mu_2,\mu_3\}.$$

The nonlinear stability of the spatial problem is unknown. Probably there is some amount of Arnol'd diffusion (see the lectures by Àngel Jorba in this course). But there is a domain of "practical stability" which is not too small. See later for details.

Is it possible to have a good picture of the totality of motions in the 3D RTBP? If the mass parameter is not quite small the dynamics becomes extremely complex to be described in simple terms. We can take $\mu \le 10^{-3}$ as good enough mass parameters. Why larger values produce many more difficulties will be described below. The procedures to obtain an answer to the standing question will be sketched in the next sections, but the main ideas are as follows:

- Far away from the primaries the motion is essentially a small ($O(\mu)$) perturbation of the two body problem. The two body problem is degenerate, in the sense that it is *superintegrable*, that is, in canonical variables depends only on one action. Two of the frequencies, associated to the motion of the pericenter and the node, are zero. This can produce a "too neutral" effect when studying some perturbations but they can be avoided, usually, by adding the average of the perturbation to the integrable problem. In most of the cases this breaks the degeneracy. Hence, far away from the primaries we can expect orbits rather close to the ones of the two body problem. The effects of the perturbation turn out to be exponentially small in some negative power of $-H$ for $-H$ large enough. This means that, despite being non integrable, this lack of integrability can only be seen in zones with a width of the order $\exp(-|H|^{-\gamma})$ for some positive γ.

- Close enough to one of the primaries, the effect of the other primary can be considered as a perturbation. So, the motion still is essentially the one of the two body problem. However if μ is small there is an essential difference between both massive bodies. For the primary the domain where it dominates has a radius of the order of 1. For the secondary it is of the order μ^α with some $\alpha < 1/3$. We know that the $L_{1,2}$ points are at a distance of the secondary of the order of $\mu^{1/3}$. At these points the actions of both bodies plus the centrifugal force cancel in the synodical system, and they become rest points in the rotating frame.

- Close to the $L_{1,2}$ points the situation is that of an unstable equilibrium. Furthermore the instability is strong (see below). However, due to the center $\times$ center part there are 4D *center manifolds* around them when we consider all the levels of energy. On a given level this is just a 3D set where the dynamics has a "neutral behaviour". Topologically this set is a 3D spherical surface. On it there are periodic orbits and 2D invariant tori, both of them good candidates for *controlled* spacecrafts orbits. We emphasize on the fact that these orbits require some manouevres (that is, station keeping) to be kept at the right place.

- The L_3 point is unstable. However the instability is quite mild, of the order of $\mu^{1/2}$. A body close to the L_3 point in the Sun–Earth system (assuming no more bodies in the solar system) would remain close to it for centuries. However, the long term effects associated to the unstable/stable manifolds of L_3 or to the ones of the center manifold around L_3 are extremely important.

- The vicinity of the $L_{4,5}$ points has a more subtle dynamics. If we consider a "small enough vicinity" (this means small compared with $\mu^{1/2}$) it looks as stable for all practical purposes. However this "practical stability" can be destroyed by small perturbations. Indeed, as we are assuming μ small, the ω_{short} frequency is close to 1. More concretely, it is $1-\frac{27}{4}\mu+O(\mu^2)$ while the vertical frequency is 1. They are close to resonance and small perturbations can produce a bifurcation to a complex saddle, the so called *Hamiltonian Hopf bifurcation*, leading to instability. Even with the only effect of the nonlinearities of the 3D RTBP this occurs at distances of $L_{4,5}$ which are $O(\mu^{1/2})$ in the *radial direction* (seen from the center of mass), but much larger, of the order of 1, in both angular directions.

Note that the first two items concern motions which occur when H is negative and $|H|$ is large. We are in the case of three connected components of the Hill domain. The motion has some kind of "hierarchy". It can be considered as a pair of two–body problems with a small coupling. This is far from true in the other cases, when the effects of both primaries are comparable.

Now we shall introduce a useful limit problem to study one of the most difficult parts of the 3D RTBP. The equations of motion reduce to the ones of the two body problem if μ goes to zero, except on a vicinity of the small primary. Now let us consider the vicinity of the secondary. Setting the origin of coordinates at the small primary and scaling distances by $\mu^{1/3}$, the RTBP can be written as a limit problem plus an $O(\mu^{1/3})$ perturbation. The limit problem is known as Hill's Problem (HP) and it is described by the following equations

$$\begin{array}{rclcccl} \ddot{x}_1 & = & 2\dot{x}_2 & + & 3x_1 & -x_1 r^{-3}, \\ \ddot{x}_2 & = & -2\dot{x}_1 & & & -x_2 r^{-3}, \\ \ddot{x}_3 & = & & - & x_3 & -x_3 r^{-3}, \end{array}$$

where $r^2 = x_1^2+x_2^2+x_3^2$. The system has a collision singularity at $r=0$ and two libration points, L_1 and L_2, of *center* $\times$ *center* $\times$ *saddle* type located at $(\pm 3^{-1/3}, 0, 0)$. Let

$$h = -\frac{1}{r} + \frac{1}{2}|\dot{x}|^2 - \frac{3x_1^2}{2} + \frac{x_3^2}{2}.$$

Then the corresponding ZVS has a bounded connected component provided $h < -3^{4/3}/2$. In that case there exist the possibility to have bounded motion in the configuration space (but, a priori, it is not possible to exclude the existence of collisions). Otherwise, the infinitesimal particle can escape from the vicinity of the primary.

There is a strong numerical evidence that the HP is non integrable. However a rigourous proof of the non integrability is missing. The difficulty of this "limit problem" is that is not close enough (globally) to any integrable problem. There are no parameters at hand to use the methods of perturbation theory. However the HP can be formulated in terms of very nice Hamiltonians giving rise to polynomial equations of motion. One possible way to see this and, at the same time, cope with the difficulties of the collision in the 3D setting is by means of the Kustaanheimo–Stiefel (KS) transformation to regularize the binary collision. This is an extension of a former device introduced by Levi-Civita for the planar two body problem. It is equivalent to a formulation of the equations of motion using quaternions.

Let $u \in \mathbb{R}^4$ and

$$L(u) = \begin{pmatrix} u_1 & -u_2 & -u_3 & u_4 \\ u_2 & u_1 & -u_4 & -u_3 \\ u_3 & u_4 & u_1 & u_2 \\ u_4 & -u_3 & u_2 & -u_1 \end{pmatrix}.$$

We make the transformation $x = L(u)u$ where x is embedded in $\mathbb{R}^4$ taking the fourth component as zero. Then

$$x_1 = u_1^2 - u_2^2 - u_3^2 + u_4^2,\ x_2 = 2(u_1u_2 - u_3u_4),\ x_3 = 2(u_1u_3 + u_2u_4),\ r = ||u||^2,$$

and the time t is scaled to a new time, s, through $t' = ||u||^2$, where $'$ denotes differentiation with respect to s. There is, of course, some freedom in obtaining u from x. The related changes of velocities are $\dot{x} = ||u||^{-2}2L(u)u'$ or $u' = \frac{1}{2}L^{\mathrm{T}}(u)\dot{x}$. The equations of motion become

$$u'' = \frac{h}{2}u + \frac{1}{4}\frac{\partial}{\partial u}\left[||u||^2\left(\frac{3}{2}(u_1^2 - u_2^2 - u_3^2 + u_4^2)^2 - 2(u_1u_3 + u_2u_4)^2\right)\right] + \frac{||u||^2}{2}\begin{pmatrix} u_1p_1 + u_2p_2 \\ -u_2p_1 + u_1p_2 \\ -u_3p_1 - u_4p_2 \\ u_4p_1 - u_3p_2 \end{pmatrix},$$

where h is given by

$$h = \frac{1}{2}||\dot{x}||^2 + V - \frac{1}{r},\ V = -\frac{3}{2}x_1^2 + \frac{1}{2}x_3^2$$

and

$$p_1 = \frac{4}{||u||^2}(u_2u_1' + u_1u_2' - u_4u_3' - u_3u_4'),\ p_2 = \frac{4}{||u||^2}(-u_1u_1' + u_2u_2' + u_3u_3' - u_4u_4').$$

The system has too many equations. This is compensated by the fact that $u_4u_1' - u_3u_2' + u_2u_3' - u_1u_4'$ is a first integral. These equations are suitable for both theoretical studies and numerical integrations. Note that the KS transformation is not 1 to 1. Every $x \in \mathbb{R}^3$ is transformed to the u space as a circle. However any point on the circle under the flow of the u equations, produces the same point when going back to the x space.

Using either the "cartesian" or the KS transformation it is easy to check that the $L_{1,2}$ points have eigenvalues $\pm\omega_p\sqrt{-1} = \pm(28^{1/2} - 1)^{1/2}\sqrt{-1}$, $\pm\omega_v\sqrt{-1} = \pm 2\sqrt{-1}$ and $\pm\lambda = \pm(28^{1/2} + 1)^{1/2}$. In one period of the planar periodic orbits (respectively of the vertical ones, of the primaries) the unstable mode amplifies by a factor $\exp(2\pi\lambda/\omega_p) \sim 2013.60575934$ (respectively $\exp(\pi\lambda) \sim 2643.91315231$, $\exp(2\pi\lambda) \sim 6990276.75696$. Hence, the nearby periodic orbits have very large dominant eigenvalues of the differential of the Poincaré map. For small μ the behaviour is similar. "Fortunately" this instability decreases when the amplitudes of the orbits increase.

4 Periodic orbits and invariant tori

Before starting the discussion about some of the main ingredients of the phase space, that is, periodic orbits and tori, we should be aware of the system of reference. If we consider the synodical frame, as it is rotating with unit angular velocity around the x_3 axis, a periodic orbit in this system will be seen as a periodic orbit in the sidereal system only if the period is conmensurable with 2π. Reciprocally, a periodic orbit in the sidereal system will be seen as periodic in the synodical one also if the period is conmensurable with 2π. This motivates that some questions should be better studied in one system or the other. Otherwise the motion can appear as more complicated when the source of the complexity is the change of reference system.

In the regions where the motion is close to the two–body problem we can consider the infinitesimal particle as moving (on the sidereal frame!) on a slowly changing ellipse (we assume here $H << -1$.) The line of the nodes and the pericenter are no longer fixed, in principle, but moving with small angular velocity. In that case one considers the system as having one fast and two slow frequencies. By using averaging with respect to the fast frequency one can reduce the system (formally!) to a system with only two degrees of freedom. To carry this out assume we use some canonical set of variables to describe the motion of the two–body problem. It is important to select "good" variables for this.

Let a, e, i, M, ω and Ω the classical Keplerian elements, that is, the semimajor axis, eccentricity, inclination, mean anomaly, argument of the pericenter and argument of the node. These variables have a very simple geometrical meaning but they are not canonically conjugate. To consider a more general case, assume also we consider the motion of a body of mass m_1 attracted by a fixed point of mass m_0. Let x the position of m_1 with respect to m_0 (for simplicity we use the same name to denote the body and its mass). Let also $y = m_1\dot{x}$. Then (x, y) form a set of canonically conjugated variables and the Hamiltonian is $F = \frac{1}{2}||y||^2/m_1 - m_0m_1/||x||$. We can introduce the *Delaunay elements* $\ell, g, \theta, L, G, \Theta$ as follows

$$\ell = M,\ \ g = \omega,\ \ \theta = \Omega,\ \ L = \sqrt{m_0a},\ \ G = L\sqrt{1-e^2},\ \ \Theta = G\cos(i).$$

The actions (L, G, Θ) are conjugated to the angles (ℓ, g, θ) and the Hamiltonian reduces to

$$F_0 = -\frac{m_0^2m_1}{2L^2},$$

which shows clearly the degeneracy of the two–body problem. The Delaunay variables are not convenient for small eccentricities and inclinations (for instance, in the lunar or planetary theory). It is better to introduce the *Poincaré variables*. This is done in two steps. First we define

$$\Lambda = m_1L,\ H = m_1L - m_1G = m_1\sqrt{m_0a}(1-\sqrt{1-e^2}),$$

$$Z = m_1G - m_1\Theta = m_1\sqrt{m_0a(1-e^2)}(1-\cos(i)),$$

which is completed to a canonical set of variables by choosing

$$\lambda = \ell + g + \theta,\ h = -g - \theta,\ \zeta = -\theta.$$

Finally we define

$$\xi = \sqrt{2H}\cos(h),\ \eta = \sqrt{2H}\sin(h),\ p = \sqrt{2Z}\cos(\zeta),\ q = \sqrt{2Z}\sin(\zeta).$$

The couples (Λ, λ), (ξ, η), (p, q) are canonically conjugate. If we prefer a more compact form, just define

$$a = \frac{1}{\sqrt{2}}(\xi - \eta\sqrt{-1}),\ b = \frac{1}{\sqrt{2}}(p - q\sqrt{-1}),$$

and the couples $(a, -\bar{a}\sqrt{-1})$, $(b, -\bar{b}\sqrt{-1})$ are also conjugated. Of course, for high inclinations and/or eccentricities it can be convenient to use other sets. Do not exist non redundant variables which allow to describe all the possible Keplerian motions without singularities, because $\mathbb{S}^2$ is non parallelizable.

Then, if the two–body Hamiltonian is perturbed to $F_0 + F_1$, where F_1 stands for the perturbation, it is possible to expand F_1 as power series in the $a, \bar{a}, b, \bar{b}$ variables, negative powers of Λ and Fourier series in λ. If several bodies are close to Keplerian motion, the perturbations will be expanded in the Poincaré variables of all of them. In the case of interest, the RTBP, assume we consider the perturbation of the secondary on the infinitesimal particle expressed in the sidereal system. This is a simpler problem because the position of the secondary only depends on one angle, changing linearly with the time, t. It is useful to introduce an angle φ instead of time modulus 2π having as canonically conjugated variable an action, J, which has to be added to F_0.

The next step is to try to cancel the dependence with respect to λ and t. To this end one can use a simple formalism introduced by Giorgilli and Galgani (many other equivalent formulations are also widely used). It is method based on Lie series, that we shall describe now. Compare with the Normal Forms methods described in Jorba's lectures.

Assume we are given a Hamiltonian of the form $F_0 + F_1 + F_2 + \ldots$, where the indices denote the smallness with respect to some perturbation parameter (in our case it can be the mass parameter μ). Let us denote F_j as $F_{j,0}$, the terms of the initial Hamiltonian. We look for a $G = G_1 + G_2 + \ldots$, which is a Taylor–Fourier series, where the subscripts refer also to the order in the small parameter. The formal series G and the transformed Hamiltonian will be found inductively. Let us compute recurrently the functions $F_{j,k}$, $j \geq 0$ and $k \geq 1$ by

$$F_{j,k} = \sum_{\ell=1}^{k} \frac{\ell}{k} [G_\ell, F_{j,k-\ell}],$$

where $[\ ,\ ]$ denotes the Poisson bracket. The Hamiltonian then is transformed to $\overline{F} = \sum_k \overline{F}_k$, where $\overline{F}_k = \sum_{m=0}^{k} F_{m,k-m}$ gathers all the terms of degree k in the small parameter.

At this stage we do not yet know the terms G_ℓ. Suppose then, that $G_1, G_2, \ldots, G_{k-1}$ are already known. Then the expression of all the terms appearing in $F_{m,k-m}$, for m from 0 to k, can be obtained completely, except for the term $[G_k, F_{0,0}]$. We can write

$$M_k + [G_k, F_{0,0}] = \overline{F}_k,$$

where the M_k are known from the previous steps of normalization. To solve this *homological* equation for G_k we just note that, $F_{0,0}$ only contains a term in Λ and J. We have a relation for the partial derivatives with respect to λ and φ. If we assume that no resonances occur (that is, we are not looking at a periodic solution with period conmensurable with 2π) no formal problem occurs in the elimination, step by step, of the dependence of F with respect to these "fast" angles. We shall end with a Hamiltonian which is reduced to 2 degrees of freedom up to some order in the small parameter, the remainder being assumed to be small enough to be neglected. Of course, the transformations leading to this "averaged form" depend on time through two different angles: λ and φ.

Then we can look at the properties of this averaged Hamiltonian. Fixed points will be equivalent to 2D tori in sidereal coordinates and can be equivalent to periodic orbits in the synodical ones. Periodic orbits or 2D tori in the averaged system should have very small frequencies, and going back through the transformations we can obtain tori of dimension 3 or 4. This covers the formal aspects. To obtain rigourous proof one has to estimate the remainders of the averaging procedure. Typically (that is, under "nice" non resonance conditions and assuming all the perturbations smooth enough in the domain of interest) it can be shown that they are exponentially small in the perturbation parameter if one selects the right order to stop the averaging process.

This order is not constant: it depends on the value of the parameter. Smaller values require a higher order of the averaging. The "nice" character of the non resonance is stated as follows: We say that some frequencies $\omega \in \mathbb{R}^n$ satisfy a *diophantine condition* if the following holds

$$|(k,\omega)| > \frac{c}{||k||^\tau},\ \forall k \in \mathbb{Z}^n \setminus \{0\},$$

for some positive values of c and τ. The symbol $||k||$ denotes here any norm of the integer vector k, and $(\ ,\ )$ the scalar product. The set of "bad" ω not satisfying this condition for $\tau > n-1$ and some positive c has a measure which goes to zero when c goes to zero. For the proofs of the corresponding theorems one should require c not too small. Its minimum value is related to the size of the perturbation parameter. However, despite the set of "bad" ω has small measure, it contains open and dense sets. Its complement is almost of full measure but nowhere dense.

In the case of fixed points of the averaged Hamiltonian one has to study also its character. If they are hyperbolic the proof of the existence of 2D tori or periodic orbits in the initial system does not present major difficulties. If they are of center $\times$ center or of center $\times$ saddle types, one should require to be not too close to resonance also when one considers the frequencies of the transformation and the frequencies (or frequency) at the fixed point. As all these frequencies, except the one of the rotation of the frame, depend slightly on the perturbation parameter, some small intervals of the parameter must be excluded, the possible tori being destroyed by the resonance. We find, finally, a Cantor set of tori for a fixed value of the parameter or a Cantor set of values of the parameter for a fixed mean frequency of the λ variable.

Now we proceed to the vicinity of the libration points and we shall only work in the synodical frame. The collinear points behave, linearly, as the product of two centers. Hence, we should expect families of periodic orbits which in the limit have frequencies ω_p and ω_v. This is assured by the Lyapunov theorem unless one of the frequencies is a multiple of the other. We can also expect 2D tori, with basic frequencies which, in the limit when we approach the point, have ω_p and ω_v as basic frequencies. The rigourous existence of these tori is more problematic. First, the basic frequencies at the collinear point can be too close to resonant. Furthermore the frequencies change with the amplitudes (unless some degeneration occurs), and so we are crossing resonances when the amplitudes are changed. Again this leads to a Cantor set of tori. The proof of the existence of tori follows similar lines to the proof of the KAM theorem: One constructs a sequence of transformations converging to the desired torus with prescribed frequencies. This approach has to be modified if the number of "free" parameters is less than the number of frequencies.

To obtain effective approximations to the periodic orbits and tori one can use symbolic manipulation, to implement analytical expansions up to some order, and/or numerical methods. If an integrable normal form is available, that is, in the absence of resonances at the fixed points, both periodic orbits and tori can be directly obtained. Note that, generically, these expansions can not be convergent in any open set, because of the crossing of resonances. However, if the resonances are of high order, the *resonance zones* associated to them can be so small that they can be neglected for all practical purposes. Their effect will be seen only after a very long time.

Another approach consist in the use of the Lindstedt-Poincaré method. It looks for an analytical expression in terms of suitable amplitudes and angles. An essential point is the fact that the frequencies *must* be assumed to depend on the *amplitudes*. Assume the origin is a fixed point of a differential equation, having as eigenvalues

$$\pm\omega_j i,\ j=1,\ldots,k;\ \lambda_j i,\ j=1,\ldots,p_r;\ \gamma_j \pm \delta_j i,\ j=1,\ldots,p_c,$$

where ω_j, λ_j, γ_j and δ_j denote nonzero real values. Furthermore we assume the eigenvalues are different (despite this is not essential concerning the ones with nonzero real part). As we said, it is assumed that the frequencies, ω_j, satisfy some diophantine condition.

The system of equations can be written, without loss of generality, as

$$\begin{array}{lcllcll}
\ddot{x}_j & & & + & \omega_j^2 x_j & = & g_j(x,y,z,u,v) \quad j=1,\ldots,k, \\
 & & \dot{z}_j & - & \lambda_j z_j & = & h_j(x,y,z,u,v) \quad j=1,\ldots,p_r, \\
\ddot{u}_j & - & 2\gamma_j \dot{u}_j & + & (\gamma_j^2+\delta_j^2)u_j & = & r_j(x,y,z,u,v) \quad j=1,\ldots,p_c,
\end{array}$$

where $y_j = \dot{x}_j,\ v_j = \dot{u}_j$, and the right hand side functions are assumed to begin with terms of degree at least two and are analytical. When needed we shall denote the full set of variables as w. It is clear that the linear part has a solution (*maximal torus*) of the form

$$\begin{array}{c}
x_j^{(1)} = \alpha_j \cos(\theta_j),\ \theta_j = \omega_j t + \phi_j,\ j=1,\ldots,k; \\
z_j^{(1)} = 0,\ j=1,\ldots,p_r;\ u_j^{(1)} = 0,\ j=1,\ldots,p_c,
\end{array}$$

where the ϕ_j are arbitrary phases. Here the superscript $^{(1)}$ denotes that these are the terms of first order in the α amplitudes.

For the full system we shall assume

$$x_j = \sum_{n\geq 1} x_j^{(n)},\ z_j = \sum_{n\geq 1} z_j^{(n)},\ u_j = \sum_{n\geq 1} u_j^{(n)},$$

the terms with superscript $^{(n)}$ being of order n in α. Of course, it is hard to believe that the frequencies with be exactly the same as in the linear case. Hence, we shall replace ω_j by $\hat{\omega}_j = \omega_j + O(|\alpha|)$. We shall obtain immediately more information on these (amplitude depending) frequencies. It will be also useful to introduce $\psi_j = \exp(i\theta_j)$.

Then we can try to proceed as follows: Imagine that we insert $w^{(1)}$ in the right hand side of the equations, but collecting only the terms $O(|\alpha|^2)$. Then, in principle, we solve for $w^{(2)}$ and we proceed to insert $w^{(1)}+w^{(2)}$ in the right hand side, collecting now the terms $O(|\alpha|^3)$. Proceeding in this way we can imagine that the expansion can be carried out to any order in $|\alpha|$.

Let us look in detail to the computation of $w^{(n)}$. The right hand side terms will be of the form

$$\mathrm{coef}_{m,q}\alpha^m \psi^q,$$

where $\mathrm{coef}_{m,q}$ is a numerical coefficient, $m \in \mathbb{Z}^k$ is a multiindex with $m_j \geq 0$ for all its components, $q \in \mathbb{Z}^k$ is also a multiindex satisfying the following constraints:

$$|q| \leq m,\ |q_j| \leq \alpha_j,\ q_j - m_j \text{ even } j=1,\ldots,k.$$

Furthermore the coefficients $\mathrm{coef}_{m,q}$ satisfy additional conditions ensuring that the solution is real. Assuming a similar expression for the components of $w^{(n)}$ we have

$$\begin{array}{rcllr}
x_j^{(n)} & = & \sum_{m,q} & & x_{n,m,q}\alpha^m\psi^q, \\
\dot{x}_j^{(n)} & = & \sum_{m,q} & i(q,\hat{\omega}) & x_{n,m,q}\alpha^m\psi^q, \\
\ddot{x}_j^{(n)} & = & \sum_{m,q} & (-(q,\hat{\omega})^2) & x_{n,m,q}\alpha^m\psi^q,
\end{array}$$

and similar expressions for the z and u components.

By substitution in the equations we obtain that the unknown coefficients can be determined provided the divisors which appear are different from zero. Before looking at these divisors we note that we are considering terms of degree exactly n in α. As $\hat{\omega}$ contains terms of different degrees in α, we should only take into account the terms of degree 0 in α. Then, in the x equations the divisors are

$$-(q,\omega)^2 + \omega_j^2,$$

while in the z and u equations they are, respectively,

$$i(q,\omega) - \lambda_j \text{ and } -(q,\omega)^2 - 2\gamma_j i(q,\omega) + (\gamma_j^2+\delta_j^2).$$

The last two expressions are never zero, the second one because the real part is different from zero, and the third one because the imaginary part is never zero if $q \neq 0$, and in this case the real part is nonzero. We turn to the first condition. It is clear that it will be zero if and only if $q = e_j$, where e_j denotes the vector having 1 at the j-th place and 0 elsewhere. This term arises, of course, from a term containing ψ_j^1 as only part in ψ.

To solve this difficulty we remind that in the left hand side we can find other terms of degree n in α coming from the α-depending part of $\hat{\omega}$. For instance, the previous problem can not appear if $n = 2$. Indeed, no terms with $|q| = 1$ can appear if $n = 2$ because of the parity conditions. The first time it can appear is for $n = 3$. Then, coming from the $x^{(1)}$ terms, there will be contributions of the form $-(e_j, \hat{\omega})^2 + \omega_j^2$, giving rise to terms in ψ_j^1 of the form

$$\alpha_j \times (-2\omega_j \hat{\omega}_j^{(2)}),$$

where $\hat{\omega}_j^{(2)}$ denote the terms of degree 2 in $\hat{\omega}_j$. These terms should cancel the terms coming from the previously obtained components of x_j. They are of order 3 in α and contain, indeed, the factor α_j because of the parity conditions. Hence, at this point all the components $\hat{\omega}_j^{(2)}$ can be determined. In a similar way, when computing $w^{(n)}$ for n odd, the terms of $\hat{\omega}_j^{(n-1)}$ can be obtained.

Of course, depending on the problem we can find that there is no solution at all when we try to determine the terms which should cancel the resonant terms.

Note also that there are terms which remain free. These are, at order n odd, the coefficient in the x_j variable of ψ_j, because it cancels directly on the left hand side. But this indeterminacy can be skipped by the following consideration. Assume we select for this coefficient some expression (of order n, of course) in α. Then, in the final expression, the coefficient of ψ_j in x_j will be $\alpha_j + O_2(\alpha)$. Denote this last expression by $\overline{\alpha}_j$. It follows immediately that if we choose $\overline{\alpha}_j$ as coefficient of ψ_j in $x_j^{(1)}$, then all the "free" terms become zero. Hence, we can take them directly as zero. This is equivalent to define α_j as the coefficient of ψ_j in the complete expression of x_j.

The Lindstedt-Poincaré method works quite well in the case of the libration points to obtain periodic orbits and to obtain invariant tori if ω_p and ω_v are non resonant.

Direct numerical methods can be used easily to obtain periodic orbits. As the RTBP is autonomous we can not look for a point w_0 in the phase space such that after some time we return to the same point: $G(w_0) = \phi_T(w_0) - w_0 = 0$, where ϕ_T denotes the time T flow and try to solve $G(w_0) = 0$ by using, for instance, Newton's method. Indeed, as one of the eigenvalues of the monodromy matrix of any periodic orbit of an autonomous system is equal to 1, the differential of G at w_0 is not invertible. This reflects the fact that any point $\phi_t(w_0)$ is also a solution for all t. Furthermore, as the system is Hamiltonian, then 1 must be, at least, a double eigenvalue. To solve this difficulty one should select a fixed level of energy and a Poincaré section, to skip both unity eigenvalues. Alternative procedures do not fix the Poincaré section but modify Newton's method to look for the closest to the initial guess point (e.g. under the euclidean norm) of the corresponding periodic orbit. Unless we are at some bifurcation (additional unity eigenvalues) no difficulties appear.

More problematic is the direct numerical computation of invariant tori. The case of the $L_{1,2}$ points is specially critical due to the strong instability. However, the instability can be skipped with a *reduction to the center manifold.* We describe how to carry out the computations. After perfoming a linear change of variables we can assume that the Hamiltonian is written as

$$H = \lambda x_1 y_1 + \frac{1}{2}\omega_p(x_2^2 + y_2^2) + \frac{1}{2}\omega_v(x_3^2 + y_3^2) + \sum_{j \geq 3} H_j(x_1, x_2, x_3, y_1, y_2, y_3),$$

where H_j denotes a homogeneous polynomial of degree j.

Then we pass to an intermediate normal form just trying to cancel in H all the terms such that the total degree in the (x_1, y_1) variables is 1. This is accomplished by making a canonical transformation with a suitable generating function (to be determined). This is possible because there are not problems of small divisors. The denominators appearing in the determination of the generating function (using some complexification to reduce the amount of computation) are of the form

$$(k_1 - l_1)\lambda + \sqrt{-1}(k_2 - l_2)\omega_p + (k_3 - l_3)\omega_v,$$

with modulus bounded from below by $|\lambda|$. Let $\widetilde{H}$ be this intermediate normal form Hamiltonian. By rearranging terms it can be written as

$$\widetilde{H} = H^0(x_2, x_3, y_2, y_3) + \sum_{j_1+j_2>1} (x_1^{j_1} y_1^{j_2}) H^{j_1,j_2}(x_2, x_3, y_2, y_3).$$

One should have in mind that the new variables are not the sames as the previous ones, despite we keep the same name. The transformation is close to the identity, the closer as the closer is the orbit to $L_{1,2}$.

The fact that small divisors problems do not show up do not means at all that the procedure is convergent. In general one obtains only a $\mathcal{C}^\infty$ manifold. The norm of the coefficients of order n can increase as $n!$ or even as a power of $n!$.

It is immediately seen that $I_1 = x_1 y_1$ is a first integral. The reduction to the center manifold is obtained by setting $I_1 = 0$. So it remains a two degrees of freedom Hamiltonian, H^0, and furthermore

$$H^0 = \frac{1}{2}\omega_p(x_2^2 + y_2^2) + \frac{1}{2}\omega_v(x_3^2 + y_3^2) + \sum_{j\geq 3} H_j^0(x_2, x_3, y_2, y_3).$$

Now we are ready to use a direct numerical method without any instability problem. This method can also work with unstable problems (but one should be careful and combine with a parallel shooting technique). It is not required to assume that the system is Hamiltonian. The main idea is the *synthesis* of a kind of *return map* to the initial point. The problem is reduced to a *fixed point* problem for a suitable operator. Assume, for simplicity, that we want to compute a 2D torus. Using a suitable Poincaré section we shall have, at least locally, that the torus intersects the section on an invariant curve under a symplectic diffeomorphism. Assume that we know the rotation number of this invariant curve (or that we look for an invariant curve with a prescribed rotation number). Assuming the curve smooth enough, it must be conjugated to a rotation on $\mathbb{S}^1$ with the same rotation number, $\rho \in \mathbb{R} \setminus \mathbb{Q}$. If we consider on $\mathbb{S}^1$ the map $z \mapsto z + \rho$, some iterates, $n_1, n_2, \ldots, n_k$, give points of the form $n_j\rho$ which are rather close to 0 modulus 1. Assume that we know some function, f, at the points $(n_j\rho)$, where ( ) denotes that we have substracted the closest integer. Then it is possible to obtain an approximation to the value of f at 0 by, say, interpolation. For instance, $f(0) \sim \sum_{j=1}^k \gamma_j f((n_j\rho))$, where γ_j are suitable weights. Let us synthesize the *return map* by using this interpolation procedure. We have

$$\mathcal{R}(p) = \sum_{j=1}^k \gamma_j T^{n_j}(p),$$

where $\mathcal{R}(p)$ denotes the approximate return map and T the Poincaré map. We are now faced to a problem similar to the one found before for the periodic orbits: all the points of the curve shall be approximate solutions of this equation! This is solved by taking again a "section" inside the initial Poincaré section, that is, by fixing a coordinate which changes (locally) along the invariant curve.

Then one can apply a Newton method to solve the equation $\mathcal{R}(p) = p$, with the differential of $\mathcal{R}$ obtained also by interpolation. This method is suitable to be used in combination with a continuation curve, to be able to follow a family of invariant curves. Of course, at least in theory one should ask for ρ such that they satisfy some diophantine condition, but in practice the method works efficiently provided ρ is not too close to a strong resonance (i.e., a resonance p/q with q small). As said before in the case of periodic orbits, there is no need to fix any coordinate in the invariant curve if Newton's method is combined with a minimization procedure.

An extension of the method to tori of any dimension and any local character, as well as sufficient conditions for the applicability, have been obtained.

A satisfactory description of the vicinity of the collinear points of the 3D RTBP is achieved by combining the methods described above.

5 Stability and practical stability

We turn to the more subtle question of the stability of the triangular libration points. As said before, for $\mu < \mu_0$, the linear behaviour is the one of a product of 3 center, that is, 3 harmonic oscillators. But, of course, is seems that there is no way to prove nonlinear stability because Arnold diffusion can take points away at a very slow rate. For this it is necessary to have a chain of tori of dimension less than 3, which have a partial hyperbolic character and such that the unstable manifold of one of the torus intersects transversaly the stable manifold of the next one.

So we must be satisfied with some *practical stability.* We describe how a practical stability result is obtained.

Let $H = H_2+H_3+\ldots$ be the power expansion of the Hamiltonian around $L_{4,5}$, where H_k contains the homogeneous terms of degree k in positions and momenta. By a linear transformation, H_2 can be reduced to

$$-\frac{1}{2}\omega_s(q_1^2+p_1^2)+\frac{1}{2}\omega_l(q_2^2+p_2^2)+\frac{1}{2}\omega_v(q_3^2+p_3^2),$$

where, as before, ω_s, ω_l stand for ω_{short}, ω_{long}, and ω_v is the frequency in the vertical direction ($\omega_v = 1$).

A canonical transformation $(q,p) \longrightarrow (Q,P)$, obtained by means of a function $G = G_3 + G_4 + \cdots + G_n$ and a Lie series method, is applied to put H in normal form up to order n. We have the new Hamiltonian

$$\mathcal{H} = N_2 + N_3 + \cdots + N_n + \mathcal{R}_{n+1},$$

where N_k are terms of degree k in the normal form and $N_2 = H_2$. If we only keep $N_2 + N_3 + \cdots + N_n$, the system is integrable. Let

$$I_j = \frac{1}{2}(Q_j^2+P_j^2),$$

the new momenta. Then, the diffusion of the momenta is due to

$$\dot{I}_j = [I_j, R_{n+1}],$$

$[\,,]$ being the Poisson bracket. As the remainder $\mathcal{R}_{n+1}$ is of the form $R_{n+1} + R_{n+2} + \cdots$ we introduce some norm $||R_k||$ (e.g., the sum of the modulus of the coefficients, eventually with some weight). Then one obtains, successively, bounds for $||H_k||$, $k \geq 2$, $||G_k||$, $2 < k \leq n$ (only a finite number of small divisors appears) and for $||R_k||$, $k > n$. The bounds of $||R_k||$ are given by means of a recurrence which depends only on the norm of the homogeneous parts of the initial Hamiltonian, $||H_k||$, and on the current small divisors which appear till order n.

In a ball of radius ρ in the (Q,P) variables one has

$$|\mathcal{R}_{n+1}| < \sum_{k>n} \rho^k ||R_k||,$$

where $|\ |$ denotes the sup norm. In a similar way we can bound $|\dot{I}_j|$, the speed of diffusion.

Given T, δ, there exists an initial radius, ρ_0, such that if $(Q,P)_{t=0} \in B_{\rho_0}$ then $(Q,P)_t \in B_{\rho_0(1+\delta)}$ for all $|t| < T$, where B_ρ denotes the ball of radius ρ centered at the origin. We remark that one obtains better results if H_k, N_k, G_k are computed explicitely up to some order for the desired value of μ by means of a symbolic manipulator.

As a result one obtains, in general, Nekhorosev type estimates (i.e., for δ fixed, one has $T \approx \exp(c/\rho^d)$, for some positive constants c, d).

Numerical simulations show a "stable domain" much larger than the one obtained by applying the practical stability criterion above.

6 Center manifolds and the related stable/unstable manifold. Practical confiners

Before is was mentioned the convenience, for the description presented here of the 3D RTBP to consider $\mu < 10^{-3}$. We shall explain first the reason of that choice. Consider first the planar problem. We recall that the triangular point L_5 is the limit of two families of periodic orbits, the *short periodic* family and the *long periodic* family. It was conjectured by Brown that the family of long periodic orbits has a *termination* as a *homoclinic* orbit to the collinear point L_3. That is, Brown guess was that when the orbits of the family go away from L_5 the period increases and tends to infinity when tending to the homoclinic connection.

The conjecture, of course, is false. This would imply a kind of "integrability" of the RTBP which is certainly non integrable. In fact, if the conjecture is true, it implies that the stable and unstable manifolds of L_3 should coincide. What in fact happens is that the conjecture is *formally true*. More concretely: the lack of coincidence of both manifolds can be measured as follows. Assume we compute the unstable manifold $W^u_{L_3}$. It has two branches, one leaving the fixed point to positive values of x_2 and the other to negative values. For definiteness consider the positive one. At the level of energy of L_3 the zero velocity curve has a double point at L_3 and the two branches of the ZVC deviate slightly (for small μ) from the circle with center at the origin and radius equal to x_{L_3}, one in the upper part and the other in the lower part until they meet at an angle with vertex at the origin that, counted from L_3 tends to $\cos^{-1}(\frac{1}{2} - \sqrt{2})$ (Brown's limit) when μ tends to 0. This closed curve leaves the L_5 point inside and, for any angle counted from L_3 between 0 and the limit value, the two intersections of the ZVC with the radius drawn with this angle have a distance which is $O(\sqrt{\mu})$. The manifold $W^u_{L_3}$ follows closely the ZVC (see later), but when it returns to the $x_2 = 0$ line the distance to L_3 is exponentially small in $\sqrt{\mu}$, that is, of the form $O(\exp(-c/\sqrt{\mu})$ for some positive constant c. Analytical estimates, with good agreement with numerical experiments show that the value of c is close to 0.17559. This behaviour is less than any power of μ and, hence, any expansion to any order in μ can not detect the difference. Furthermore, there are values of μ for which the value of $\dot{x}_1$ when $W^u_{L_3}$ returns to $x_2 = 0$ is zero. Then, by the reversibility of the problem, the branch of $W^s_{L_3}$ which begins by going down, has a symmetric behaviour, and both manifolds coincide. But this homoclinic connection only occurs for a countable set $\mu_k,\ k \in \mathbb{N},\ k > k_0$ for some k_0 of values of μ, tending to 0 as C/k for some constant $C \sim 0.0345$, and the projection on the (x_1, x_2) plane leaves both $L_{4,5}$ inside. These are not the connections expected by Brown.

Despite the "first return" of $W^u_{L_3}$ to $x_2 = 0$ does not produce homoclinic orbit, this is true for periodic orbits of the family which has limit period ω_p (the planar Lyapunov family of periodic orbits around L_3), provided the amplitude of the orbit is not too small. A complete description of the evolution of the long periodic family and the relation to the homoclinic orbits to periodic Lyapunov orbits is in progress.

When μ is increased the exponential character of the distance has the effect that small changes in μ are extremely amplified. For instance, the function $\exp(-c/\sqrt{\mu})$ with $c \sim 0.17559$

and $\mu = 10^{-4},\ 10^{-3},\ 10^{-2}$ gives the approximate values $2.4 \times 10^{-8},\ 3.9 \times 10^{-3},\ 1.7 \times 10^{-1}$, respectively. Certainly there are factors in front of these figures, but they give the correct order of magnitude. While 2.4×10^{-8} can be considered small, definitely 1.7×10^{-1} can not be seen as small. This produces that the $W^u_{L_3}$ does not have a first return close to L_3. This fact is clearly seen in the case of the Earth-Moon system.

Near the triangular points the system can be considered as three harmonic oscillators, with frequencies, at that point, equal to ω_s, ω_l and ω_v, the last one being equal to 1. For μ small the values of the ω_s, ω_l are, respectively, $1 - 27\mu/8 + O(\mu^2)$ and $(27\mu/4)^{1/2} + O(\mu^{3/2})$. Hence, one can also consider as basic, in a neighborhood of L_5, the frequencies ω_v, ω_l and $\omega_v - \omega_s$, whose values are of the order of 1, $\mu^{1/2}$ and μ, respectively.

Provided the frequencies satisfy some conditions (not too close to a strong resonance) there is a "practical stability" region. The previous consideration on the orders of magnitude shows that no strong resonance occurs if μ is small, except the 1 to 1 resonance between the short period and the vertical modes. Hence, it takes an extremely large time span to increase the actions by a significant amount. We are interested in the global shape of these regions, because the theoretical bounds of the practical stability domain give rather pessimistic estimates.

To this end the following experiment has been done. Given values of ρ, α and x_3, we consider an initial point with synodical coordinates

$$x_1 = \mu + (1+\rho)\cos(2\pi\alpha), \qquad x_2 = (1+\rho)\sin(2\pi\alpha), \qquad x_3,$$

and with zero initial velocity: $\dot{x}_1 = 0,\ \dot{x}_2 = 0,\ \dot{x}_3 = 0$. The L_5 point corresponds to $\rho = 0$, $\alpha = 1/3$ and $x_3 = 0$. Then a numerical integration is started up to a given final time. Points close to L_5 just remain moving around it (typically on a 3D torus). But if the initial conditions are sufficiently far from the libration point, the projection of the orbit on the (x_1, x_2) plane crosses the x_1 axis and then it can be considered as escaped from a "large" vicinity of L_5. Due to the shape of some large tori which seem to exist even for large values of x_3, it is convenient to take as "escape" criterion from the vicinity of L_5 the condition $x_2(t) < x_2^*$ for some negative value of x_2^*. As a practical value one has taken $x_2^* = -0.5$. For the current value of $\mu = 0.0002$ we have found orbits lying on tori "around" L_5 such that $x_2(t)$ reaches values up to -0.233. The time span used for the computations is 10,000 times the period of the massive bodies. This can seem a short time interval, but see the comments below.

Some remarks must be made concerning the results.

- Most of the points close to L_5 give motion on 3D tori. As we take a 3D set of initial conditions and each one of the tori contains, typically, just one point with zero velocity, we are scanning an open set on the phase space. There is no need to use other initial conditions.

- For small values of μ, as in the present case, the transition from the "stable" zone to instability is quite sharp. This means that increasing the 10,000 revolutions to 100,000, for instance, produces very few losses. On the other hand, most of the points which escape go away from the vicinity of L_5 in a few revolutions. Most of the points which escape after a longer time go away after a time interval of the order of $1/\mu$. This is related to the smallest of the basic frequencies, $\omega_v - \omega_s$, as said before.

- For the planar case it is seen that the nonescaping points fill an area that we want to identify. To this end take the ZVC passing through L_3. A description of the behaviour of this curve has been given before. Consider the curve with $\rho = 0$ (that is, at unit distance from the larger body). Then the ZVC is located at a distance $O(\mu^{1/2})$ from it. More concretely, there is a function $K(\alpha) = 1 - \frac{2}{3}\cos(2\pi\alpha) - \frac{1}{3\cos(\pi\alpha)}$ having a double zero at $\alpha = 0$ and a simple one at $\alpha = \alpha_{lim} = \cos^{-1}(\frac{1}{2} - \sqrt{2})/\pi$ such that the ZVC is described by

$\rho = \pm(K(\alpha)\mu)^{1/2} + O(\mu)$. The "boundary" of the stability region on $x_3 = 0$ seems to be rather close to $\rho = \pm\frac{1}{2}(K(\alpha)\mu)^{1/2} + O(\mu)$. This estimate is obtained analytically, at least for α close to zero, and agrees very well with the numerical results.

Just for completeness we mention that in the "integrable" approximation of $W^u_{L_3}$ (coinciding with $W^s_{L_3}$ in the integrable model) is described by $\rho = \pm\frac{\sqrt{7}}{2}(K(\alpha)\mu)^{1/2} + O(\mu)$.

- There is numerical evidence that, in the planar case, the boundary of the "stability" region is related to the stable and unstable manifolds of the center–stable manifold at L_3. This is a codimension 1 manifold. In particular it contains the stable and unstable manifolds of the family of planar Lyapunov orbits around L_3.

- The full set of points in the "stability" zone with zero synodical velocity, in the spatial RTBP, seems to be on a thin shell. The shape of the shell is essentially circular in the horizontal direction and parabolic in the vertical one. Cutting the "stable" zone by $\alpha = 1/3$, the central point, as a function of x_3, is close to $\rho = -0.245x_3^2$, which agrees with an elementary analysis. Close to $x_3 = 0$ the shell has an amplitude in α close to $\frac{5}{12}$ of revolution. The thickness is $O(\mu^{1/2})$ up to high values of x_3. The maximum value of x_3 is slightly larger than 0.86.

- There are families of unstable 2D tori which play also a role in the boundary of the "stability" region. If the initial value of x_3 is small these tori reach a vicinity of the L_3 point. Their sections through $x_3 = 0$ have positive x_1 values. For large initial values of x_3, the sections through $x_3 = 0$ of the 2D tori which seem to be at the "boundary" of the stable region have negative x_1 values.

- The preceding tori have been continuated numerically using a direct numerical method as described above. They appear to form families which emanate from unstable periodic orbits. These periodic orbits are of center $\times$ hyperbolic type and belong to a family. Hence the points of the family form a 2D set which has a 4D center manifold. This center manifold contains unstable 2D tori of the previous item. But it seems that there are several families of periodic orbits which play that role, and not only one. Furthermore, these families seem to depend on the mass parameter. The unstable/stable manifolds of the tori act as "confiners" of the 3D tori emanating from L_5.

Summarizing, according to the numerical experiments in the planar case, most of the stable motions take place in tori 2D and they are "bounded" by planar periodic orbits of the Lyapunov family around L_3. In the spatial case, most of the stable motions take place in 3D tori and they are "bounded" by 2D tori associated to the 1 to 1 resonance between the short period and the vertical frequencies. These 2D tori are located at different places depending, mainly, on the initial value of x_3. Of course, high order resonances play a role and give rise also to chains of heteroclinic connections taking points away, probably from any neighborhood of any stable point, but this requires a very large amount of time and this mechanism can only be seen for rather large values of μ (say, of the order of 10^{-2}) .

Collectively, the objects which act as *practical confiners* of the 3D tori around L_5 seem to be several unstable/stable manifolds of center manifolds of dimension 4. This is reasonable, because these unstable/stable manifolds have then dimension 5, that is, codimension 1. Note that we are not fixing the level of the energy. Really these manifolds are not "boundaries" (see the comments about Brown's conjecture). There is some *splitting of separatrices* for the center manifolds which has as a consequence that they "do not close the box". Points of the phase space "inside the quasibox" can escape but the rate of escape is very slow. As soon as they are "outside" the box the escape takes place in a time which is, roughly, $O(\mu^{-1})$. The objects at the "core" of the center manifolds are the L_3 point and several families of elliptic–hyperbolic periodic orbits.

A final comment, concerning the fact that the triangular points have a couple of frequencies differing only by $O(\mu)$, is that these points are *not robust* in front of external perturbations. These ones can move the eigenvalues at the points and produce an instabilization. Furthermore the variation of the frequencies when we go away from $L_{4,5}$ in the 3D RTBP is also weak. The derivative of the frequencies with respect to the actions (the so called *torsion*) is non degenerate but it is not so far from degenerate. It is also a symmetric non definite matrix, a fact which is known to be able to produce difficulties. This happens, for instance, when we consider the Earth–Moon system and the external perturbation is the effect of the Sun. In simple models like the so called *bicircular model* or some variant of it the triangular points are replaced by periodic solutions, with a period equal to the period of the Sun in the synodical Earth–Moon frame. But the periodic orbit turns out to be unstable. Only at some moderate distance of $L_{4,5}$ stability is recovered. This distance, in the x_3 coordinate, is of the order of 0.25 times the Earth–Moon distance. Numerical simulations using the JPL ephemeris show that the behaviour is essentially the same with this quite realistic model.

7 Conclusions

For small values of the mass parameter μ, the 3D RTBP has several regions where it behaves in a close to integrable way. Not only very close to the primaries or far away from them, but also in a big region around the triangular libration points. If we consider the section through zero synodical velocity, this region is big in all directions but one, which is $O(\mu^{1/2})$. The more important sources of instability are the collinear libration points. The points $L_{1,2}$ produce a fast inestabilization, while the one due to L_3 is milder. Several time scales play a role: 1, $\mu^{-1/2}$ and μ^{-1}.

The methods used to understand the 3D RTBP combine different tools. They seem useful to understand other problems in Celestial Mechanics and in general Hamiltonian Systems.

In fact, many of the results presented above have been checked in the elliptic RTBP, bicircular and quasi–bicircular models and in reallistic models of the solar system (JPL Ephemeris).

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