On the dynamics of a Solar Sail near L_1

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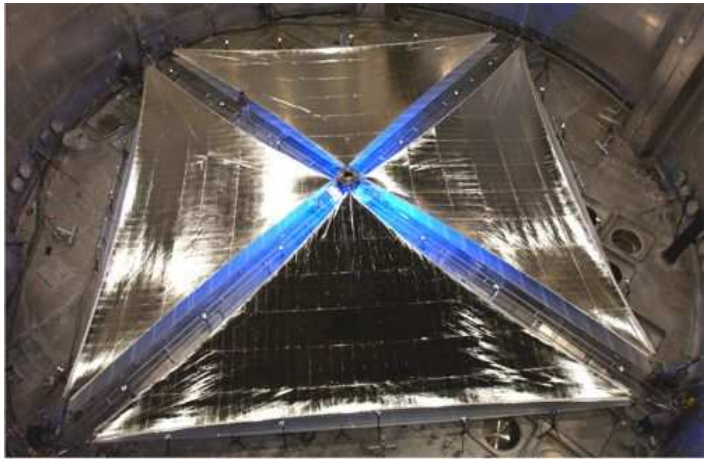
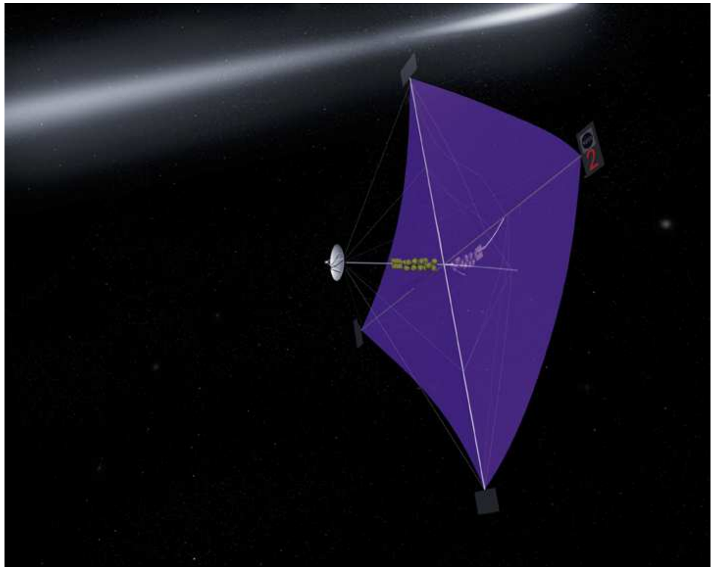
Contents

- Introduction to Solar Sails.
- Families of Equilibria.
- Station Keeping Strategies for a Solar Sail.
- Dynamics around equilibria.

What is a Solar Sail ?

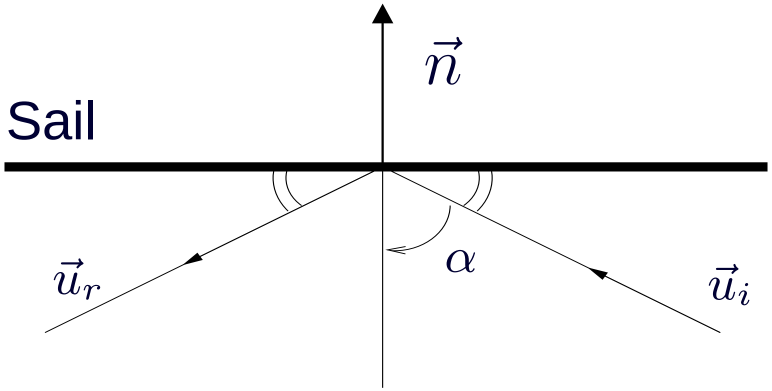
- A Solar Sail is a form of spacecraft propulsion that uses the pressure of light from the Sun to push a satellite.
- The impact of the photons emitted by the Sun onto the surface of the sail and its further reflection produce momentum on it.
- Solar Sails open a new wide range of possible mission that are not accessible for a traditional spacecraft.





The Sail

- We have considered a flat and perfectly reflecting Solar Sail. Hence, the force due to the solar pressure is in the normal direction to the surface of the sail ($\vec{n}$).



- The **sail orientation** is given by the normal vector to the surface of the sail ($\vec{n}$), parametrised by two angles, α and δ.
- The effectiveness of the sail is given by the dimensionless parameter β, the **lightness number**.

Technicalities ...

The force due to the sail given by:

$$\vec{F}_{sail} = \beta \frac{m_s}{r_{ps}^2} \langle \vec{r}_s, \vec{n} \rangle^2 \vec{n}.$$

The parameter β measures the performance of the sail, and relates the ratio between the mass of the spacecraft and the size of the sail.

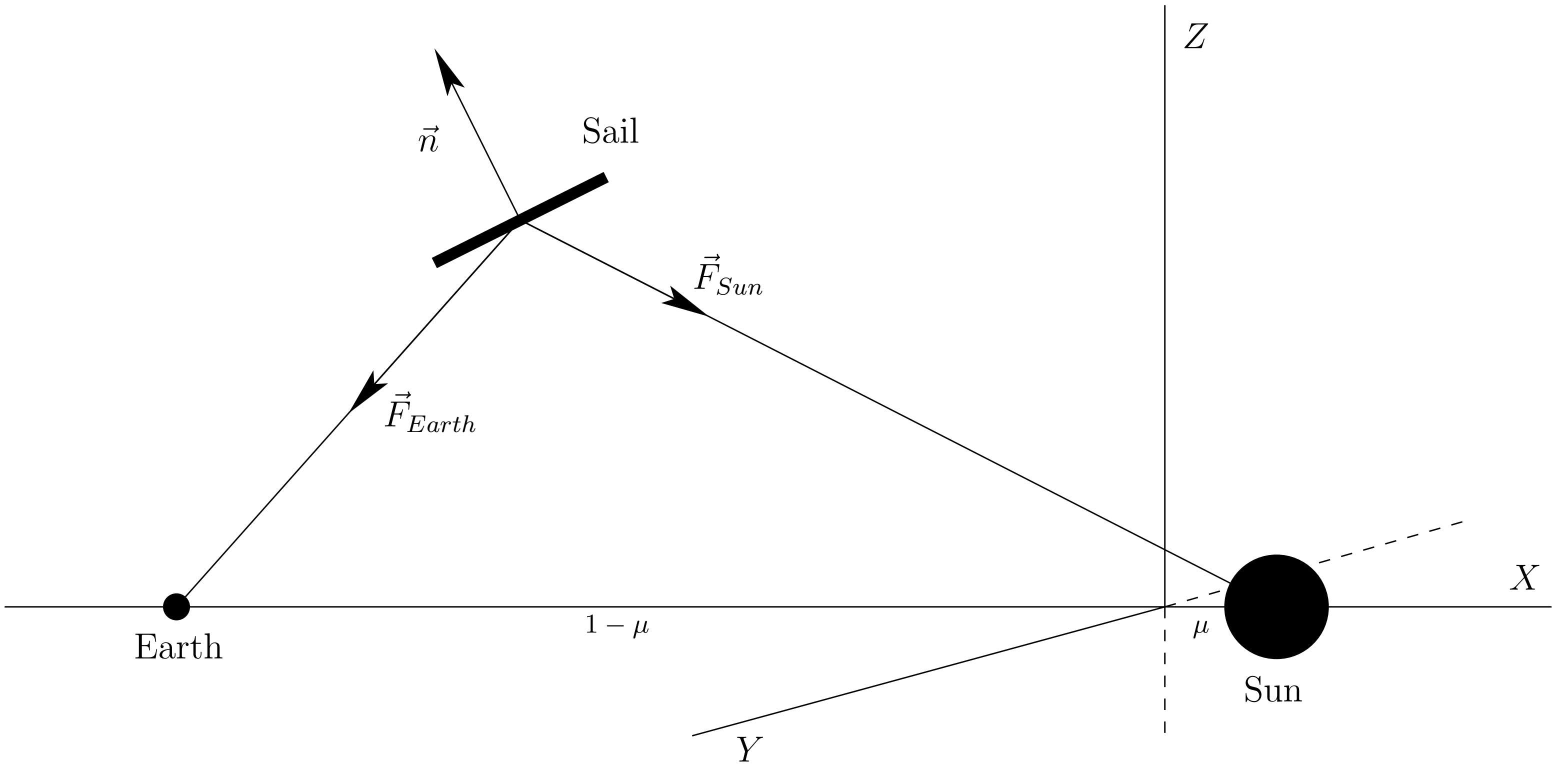
$$\beta = \frac{\sigma^*}{\sigma}, \qquad \sigma^* = \frac{L_s}{2\pi G m_s c} \approx 1.53\, g/m^2.$$

Where, σ is know as the sail's loading. With nowadays technology, it is considered reasonable to take $\sigma \approx 30\, g/m^2$, hence for a spacecraft of $100\, kg$ we need a square sail of $58\, m^2$.

(*Reference*: C. McInnes, "Solar Sail: Technology, Dynamics and Mission Applications.", *Springer-Praxis*, 1999.)

Equations of Motion (I)

We use the Restricted Three Body Problem (RTBP) taking the Sun and Earth as primaries and including the solar radiation pressure to model the motion of the sail.



Equations of Motion (II)

The equations of motion are:

$$\begin{aligned}
\ddot{x} &= 2\dot{y} + x - (1-\mu)\frac{x-\mu}{r_{ps}^3} - \mu\frac{x+1-\mu}{r_{pe}^3} + \beta\frac{1-\mu}{r_{ps}^2}\langle \vec{r}_s, \vec{n}\rangle^2 n_x, \\
\ddot{y} &= -2\dot{x} + y - \left(\frac{1-\mu}{r_{ps}^3} + \frac{\mu}{r_{pe}^3}\right) y + \beta\frac{1-\mu}{r_{ps}^2}\langle \vec{r}_s, \vec{n}\rangle^2 n_y, \\
\ddot{z} &= -\left(\frac{1-\mu}{r_{ps}^3} + \frac{\mu}{r_{pe}^3}\right) z + \beta\frac{1-\mu}{r_{ps}^2}\langle \vec{r}_s, \vec{n}\rangle^2 n_z,
\end{aligned}$$

where,

$$\begin{aligned}
n_x &= \cos(\phi(x,y)+\alpha)\cos(\psi(x,y,z)+\delta), \\
n_y &= \sin(\phi(x,y,z)+\alpha)\cos(\psi(x,y,z)+\delta), \\
n_z &= \sin(\psi(x,y,z)+\delta),
\end{aligned}$$

with $\phi(x,y)$ and $\psi(x,y,z)$ defining the Sun - Sail direction in spherical coordinates ($\vec{r}_s = \vec{r}_{ps}/r_{ps}$).

Equilibrium Points (I)

- The RTBP has 5 equilibrium points (L_i). For small β, these 5 points are replaced by 5 continuous families of equilibria, parametrised by α and δ.

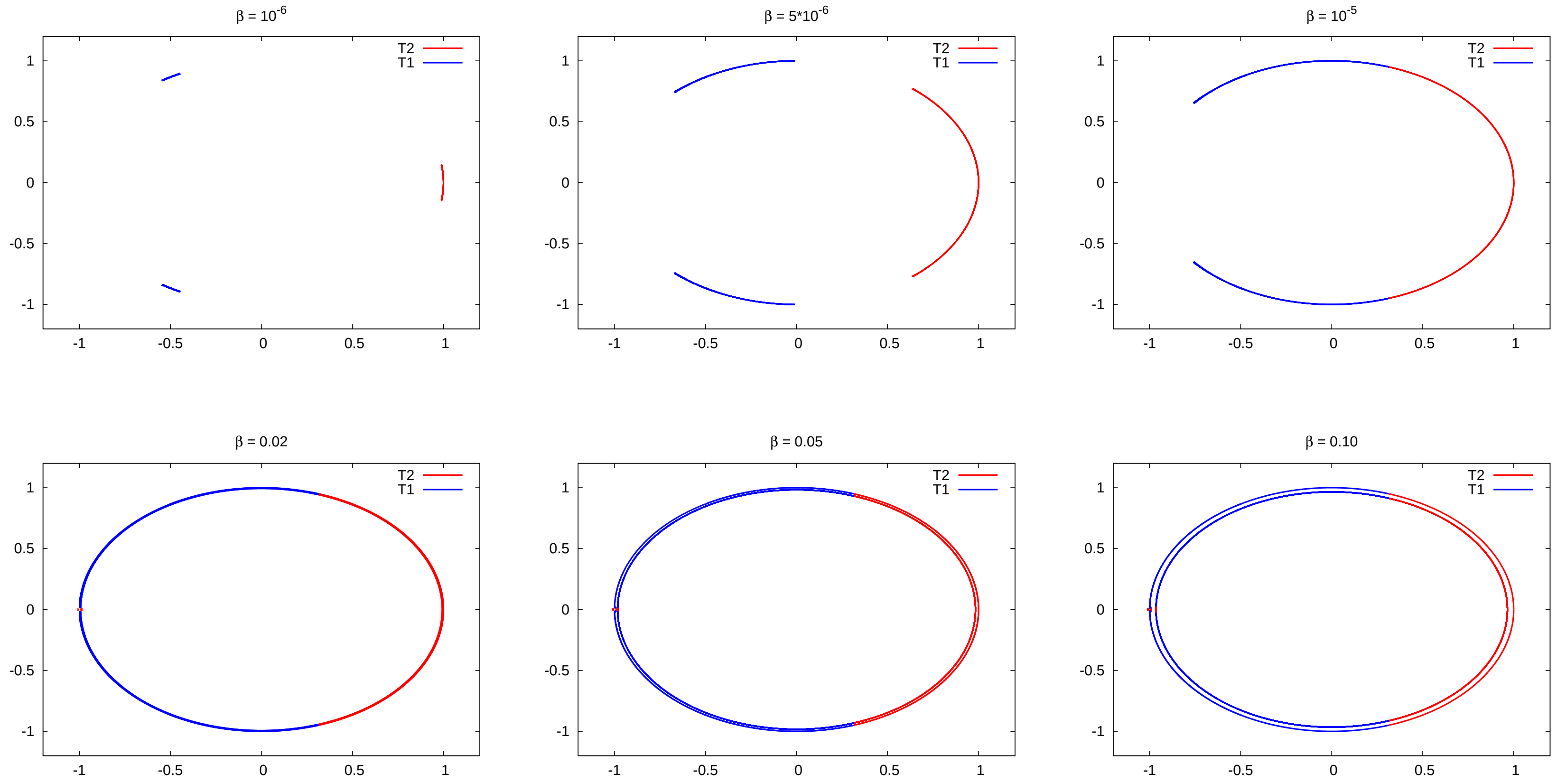
- For a fixed small value of β, we have 5 disconnected family of equilibria around the classical L_i.

- For a fixed and larger β, these families merge into each other. We end up having two disconnected surfaces, S_1 and S_2. Where S_1 is like a sphere and S_2 is like a torus around the Sun.

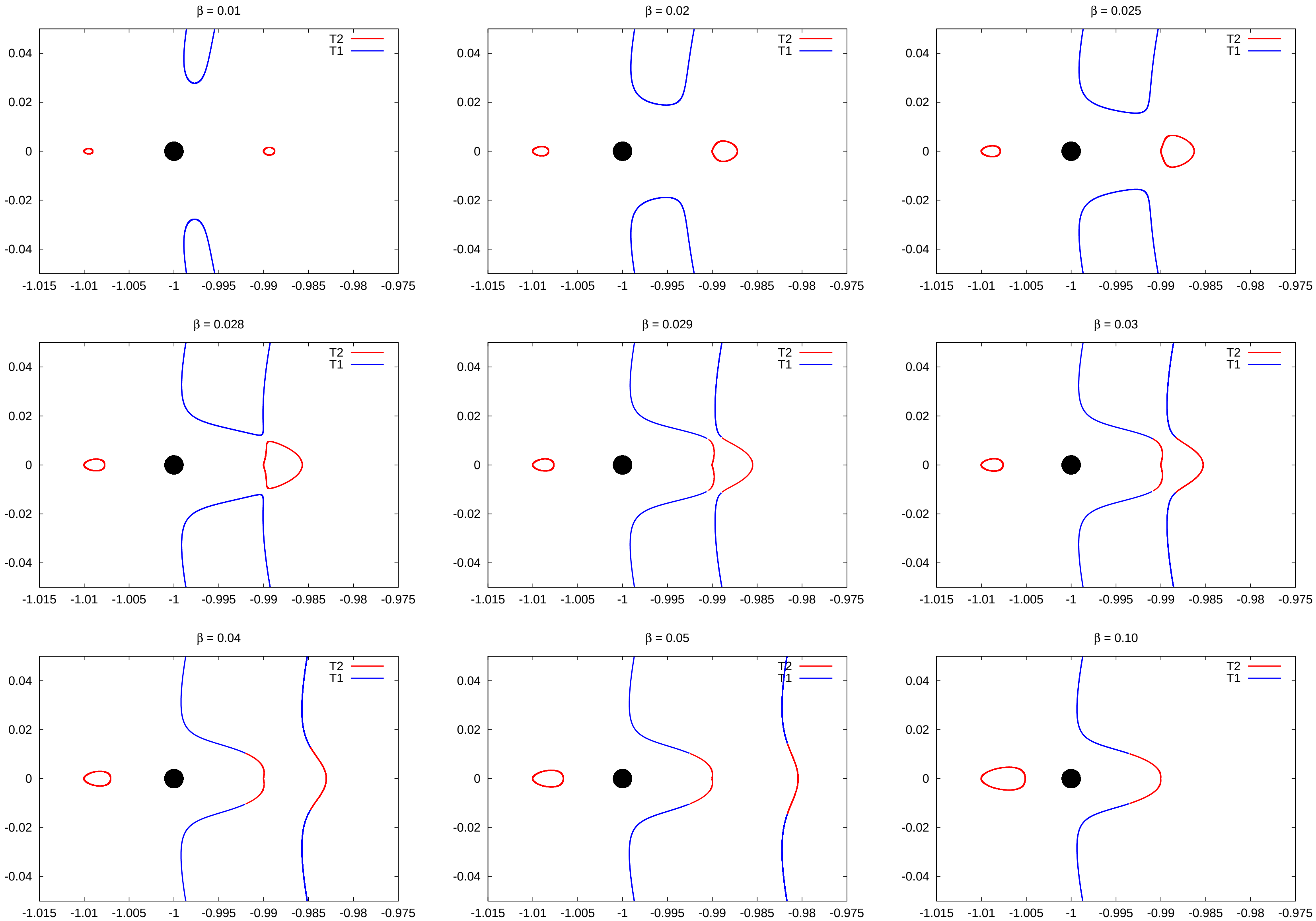
- All these families can be computed numerically by means of a continuation method.

Equilibrium Points (II)

Family of Equilibria on the $\{X, Y\}$ plane.

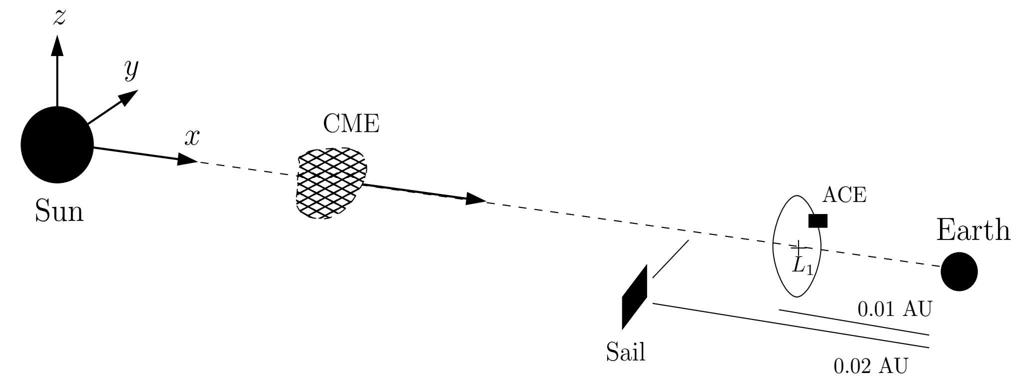


Equilibrium Points (III) [Zoom Close to the Earth]

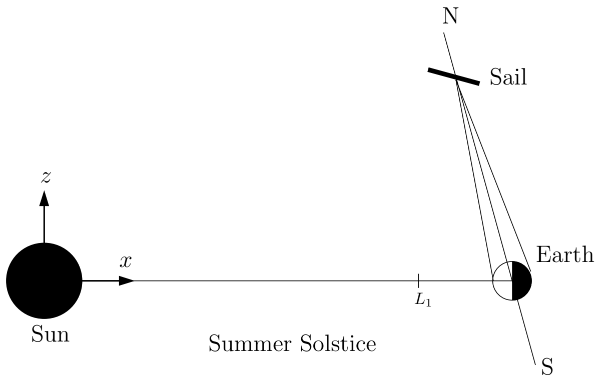


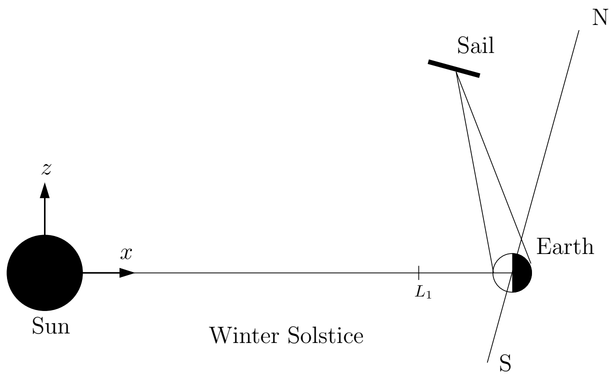
Some Interesting Missions

- Observations of the Sun provide information of the geomagnetic storms, as in the Geostorm Warning Mission.



- Observations of the Earth's poles, as in the Polar Observer.





AIM of the WORK

We want to:

- understand the natural dynamics of a Solar Sail around an equilibrium point.
- understand the bounded motion around equilibria, periodic orbits, invariant tori.
- understand the geometry of the phase space and how it varies when the sail orientation is changed to derive strategies to control the motion of a Solar Sail.

Station Keeping for a Solar Sail

- Mainly, in all these applications, the equilibrium points are of the class $\mathcal{T}_2$. We want to find a station keeping algorithm for a Solar Sail using Dynamical System Tools.

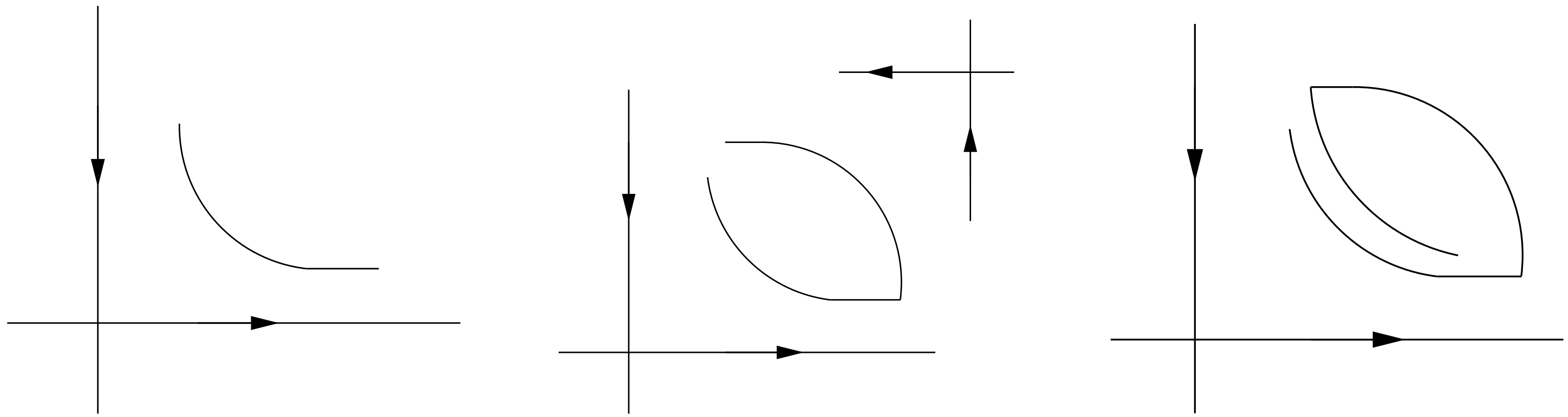
- We need to understand the linear dynamics and how it varies when the sail orientation is changed.

- As a first approach, we can think that the dynamics close to an equilibrium point is centre $\times$ centre $\times$ saddle.

- For small variations of the sail orientation, the fixed points slightly shifts and the eigenvalues and eigendirections also vary slightly.

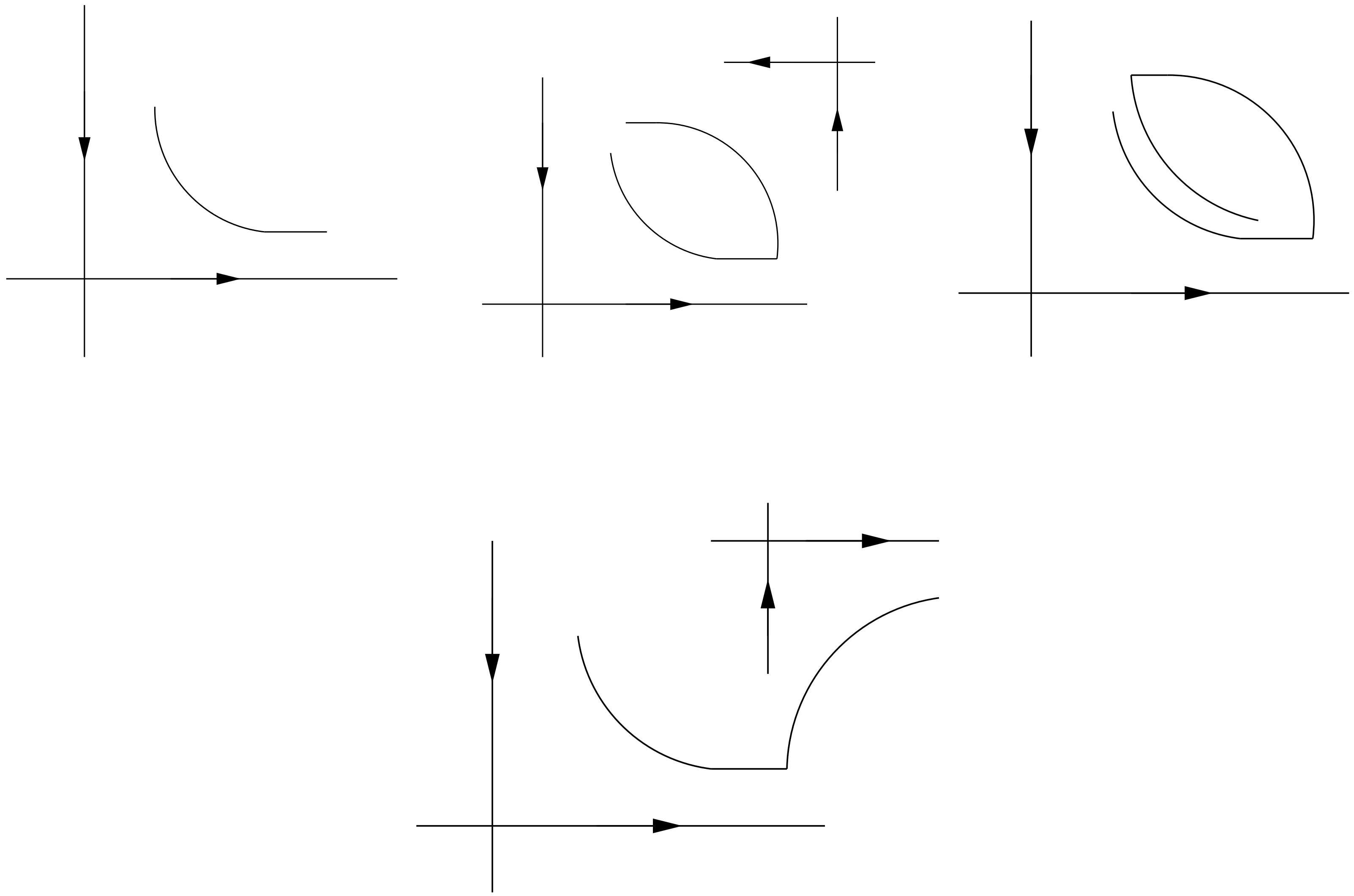
Schematic Idea of the Control Strategy (I)

In the saddle projection:



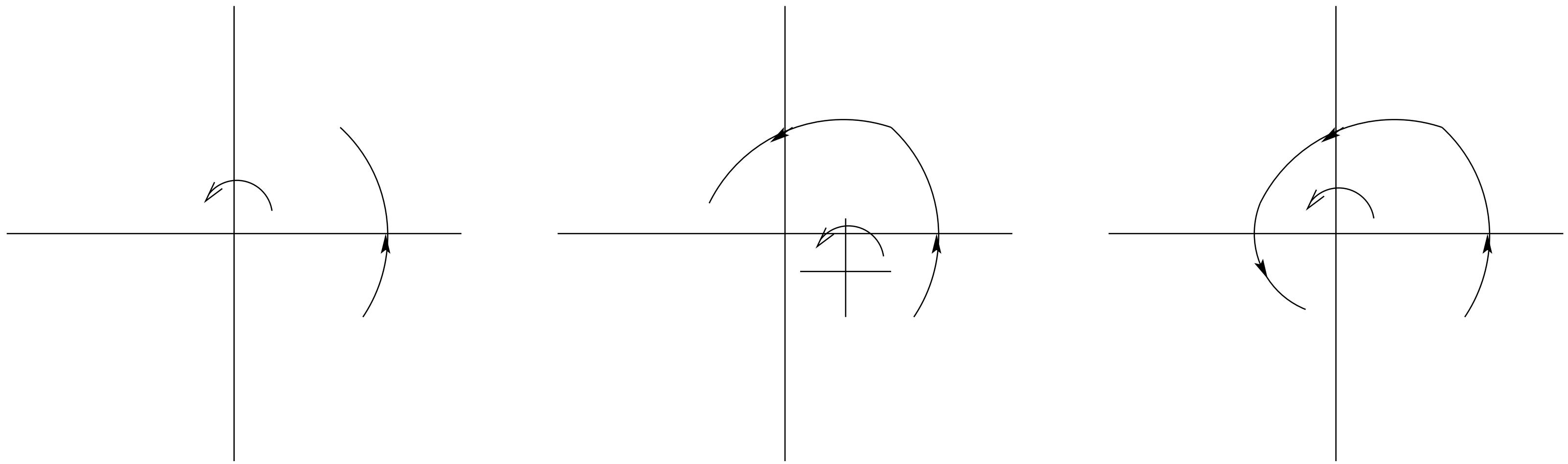
Schematic Idea of the Control Strategy (I)

In the saddle projection:



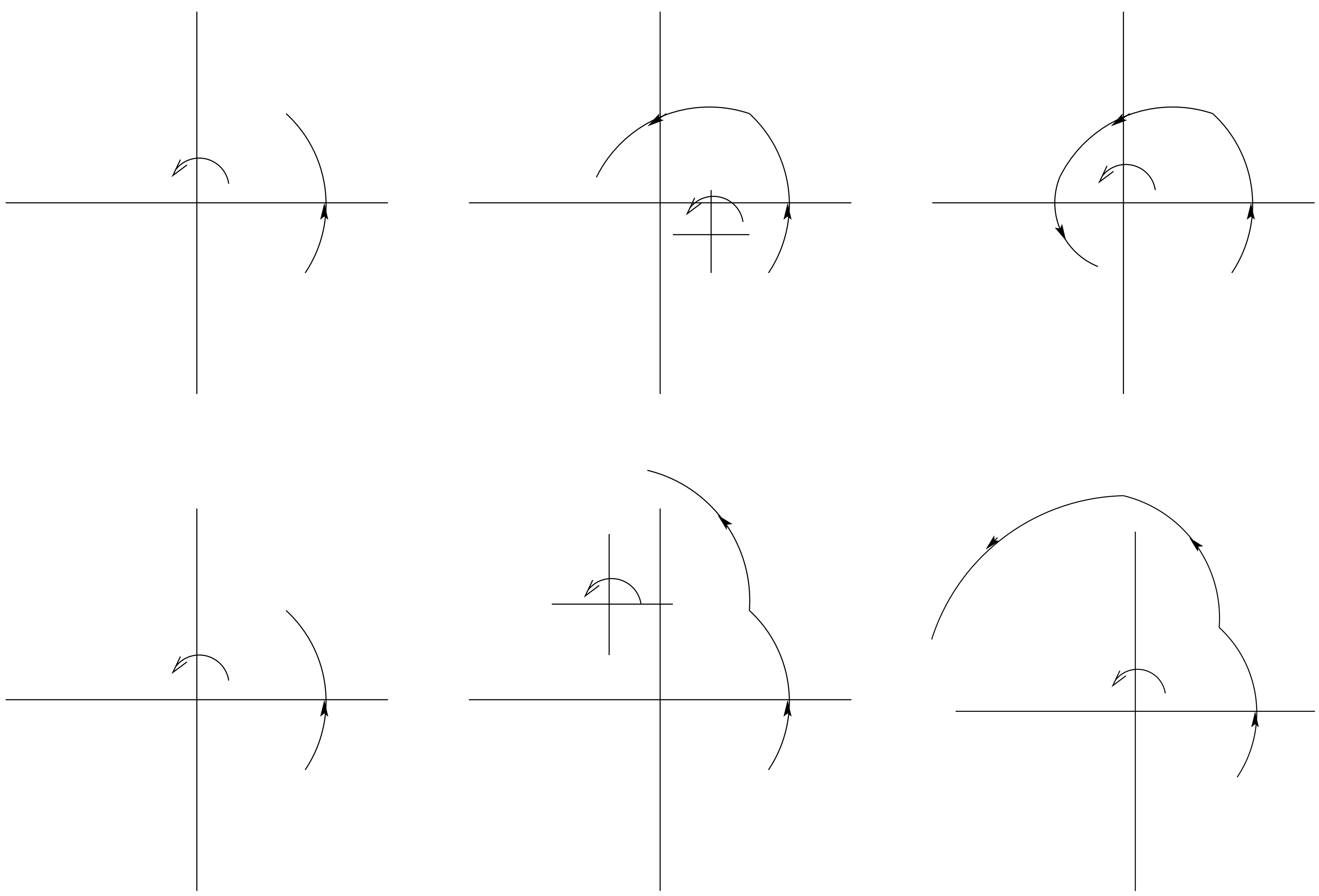
Schematic Idea of the Control Strategy (II)

In each of the elliptic projections:



Schematic Idea of the Control Strategy (II)

In each of the elliptic projections:



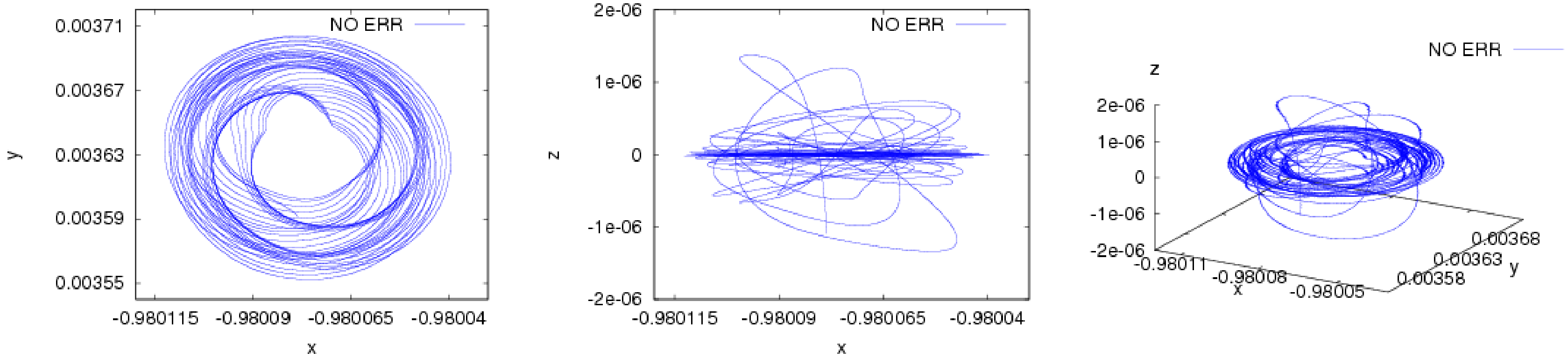
Station Keeping for a Solar Sail

Observations:

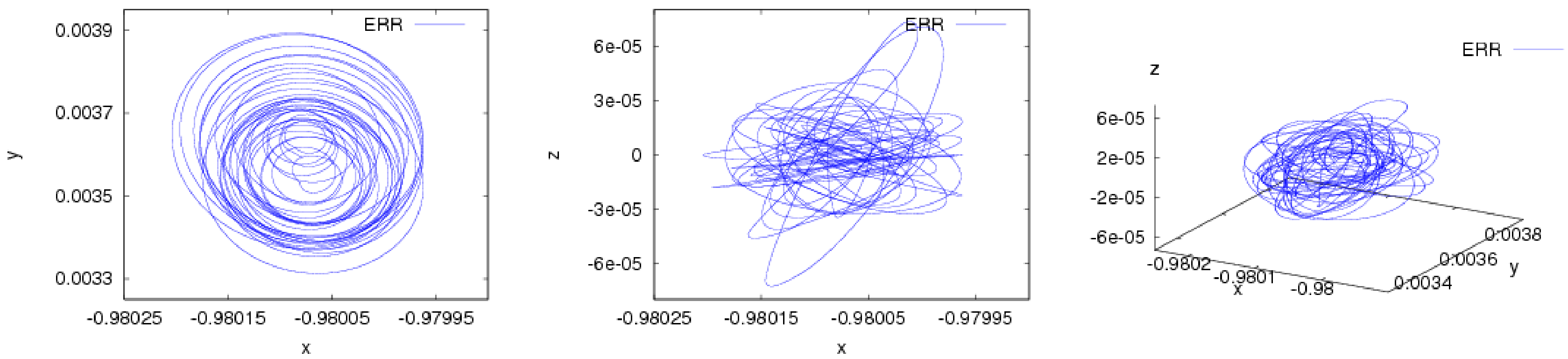
- We need to have expressions for the variation of the fixed point position and eigenvalues and eigevectors with respect to α and δ.
- There are some restrictions of the position of the new equilibria when we change α and δ. We have 2 unknowns and at least 6 conditions.
- We will not always be able to control the solar sail, it all depends on the dynamics close to the equilibrium point.
- All the simulations have been done using the full set of equations, the linear dynamics is just used to decide the change on the sails orientation.

Simulation for the Geostorm Mission (I)

Trajectory with No Errors during manoeuvres

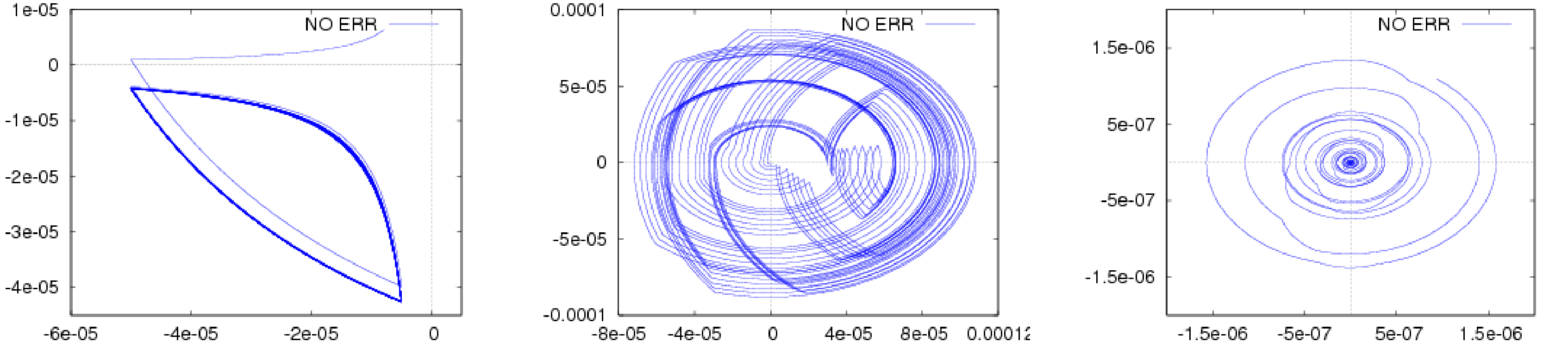


Trajectory with Errors during manoeuvres

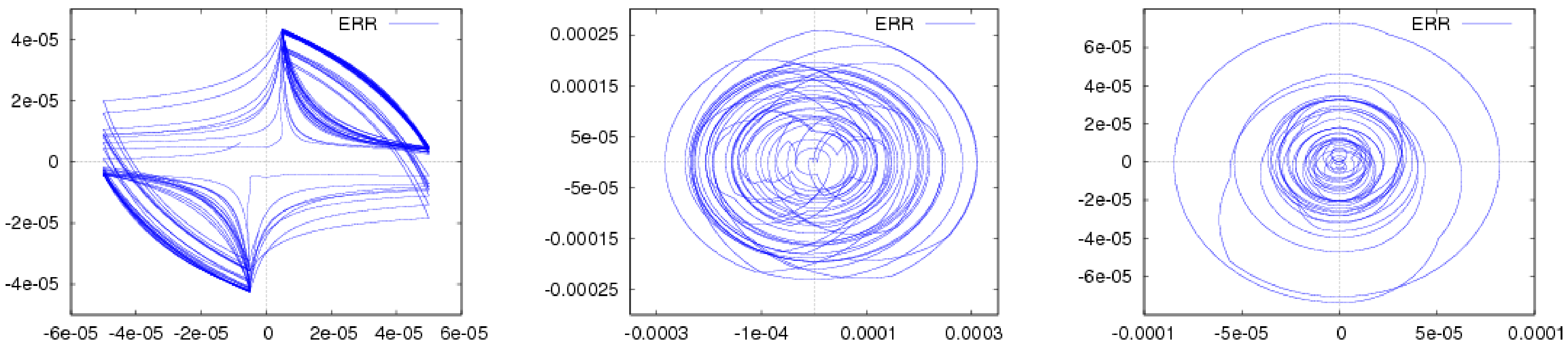


Simulation for the Geostorm Mission (II)

Saddle x Centre x Centre Projection (No Errors)



Saddle x Centre x Centre Projection (Errors)



Some References

- C. McInnes, “Solar Sail: Technology, Dynamics and Mission Applications.”, *Springer-Praxis*, 1999.

- D. Lawrence and S. Piggott, “Solar Sailing trajectory control for Sub-L1 stationkeeping”, *AIAA 2005-6173*.

- J. Bookless and C. McInnes, “Control of Lagrange point orbits using Solar Sail propulsion.”, *56th Astronautical Conference 2005.*

- A. Farrés and À. Jorba, “Solar Sail surfing along families of equilibrium points.”, *Acta Astronautica* Volume 63, Issues 1-4, July-August 2008, Pages 249-257.

- A. Farrés and À. Jorba, “A dynamical System Approach for the Station Keeping of a Solar Sail.”, *Journal of Astronautical Science*. Volume 63, No. 2, April-July 2008, Pages 199-230.

From now on we fix $\alpha = 0$

Here:

- The system is not Hamiltonian but time reversible by

$$R : (t, X, Y, Z, \dot{X}, \dot{Y}, \dot{Z}) \mapsto (-t, X, -Y, Z, -\dot{X}, \dot{Y}, -\dot{Z}).$$

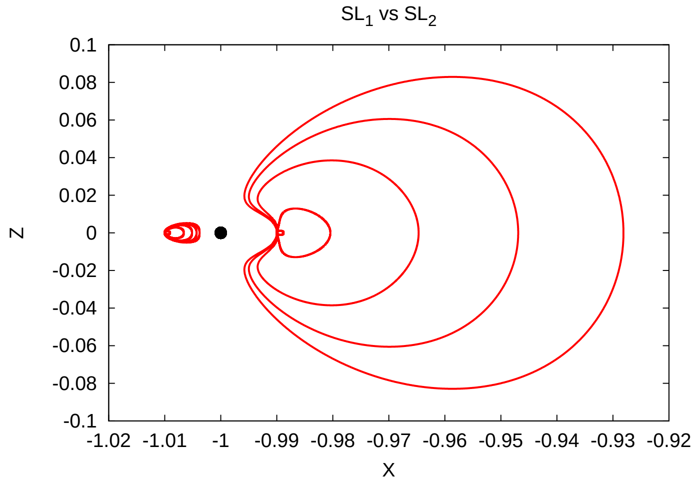
- We have 5 disconnected families of equilibrium points parametrised by δ. We call them SL_i for $i = 1, \ldots, 5$.

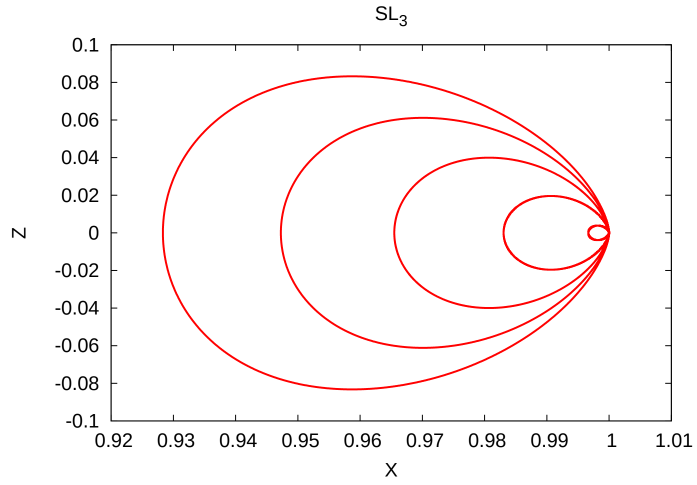
- Three of these families ($SL_{1,2,3}$) are on the $Y = 0$ plane and remain fixed by the reversibility R. The behaviour is of the type *centre* $\times$ *centre* $\times$ *saddle*.

- The other two families ($SL_{4,5}$) are close to $L_{4,5}$ and are not fixed by R. The behaviour is of the type *sink* $\times$ *sink* $\times$ *source* or *sink* $\times$ *source* $\times$ *source*.

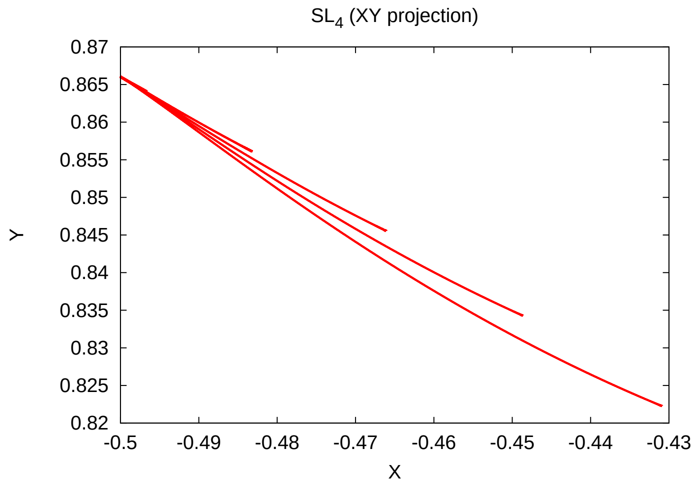
Family of equilibria for $\alpha = 0$

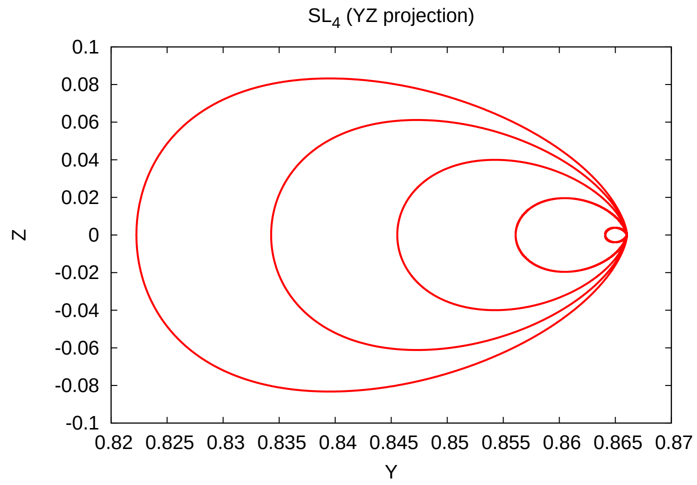
SL_1, SL_2 and SL_3 families of equilibria





Different projections for SL_4 (SL_5 symmetric to SL_4 w.r.t $Y = 0$)





From now on we will focus on the family of equilibrium points around SL_1 for $\beta = 0.051689$.

- First we will describe the periodic motion around an equilibrium point, and how it varies when δ varies.
- Second we will describe the dynamics around an equilibrium point, and how it varies when δ varies.

NOTE:

- This system is time reversible by R and conservative $\forall \delta \in [-\pi/2, \pi/2]$.
- For the particular case $\delta = 0, \delta = \pm\pi/2$ this system is also Hamiltonian.

Periodic Motion

Devaney - Lyapunov Centre Theorem:

Let $\dot{x} = f(x)$, with $f \in \mathcal{C}^2$ and $x \in \mathbb{R}^{2n}$ be an autonomous R-reversible dynamical system, where $\dim(Fix(R)) = n$. Let p_0 be a fixed point such that $R(p_0) = p_0$, and with $\pm \mathrm{i}\omega,\ \pm\lambda_2, \ldots,\ \pm\lambda_n$ as eigenvalues.

Then, if $\forall \lambda_i$ we have that $\mathrm{i}\omega/\lambda_i \notin \mathbb{Z}$, there exists a one-parametric family of periodic orbits emanating from p_0, where the period of these orbits tends to $2\pi/\omega$ when approaching p_0.

Motion around the equilibrium points

If we linearise around a certain equilibrium point for a fixed δ:

$$\begin{aligned}\phi(t) &= A_0[\cos(\omega_1 t+\psi_1)\vec{v}_1+\sin(\omega_1 t+\psi_1)\vec{u}_1] \\ &+ B_0[\cos(\omega_2 t+\psi_2)\vec{v}_2+\sin(\omega_2 t+\psi_2)\vec{u}_2] \\ &+ C_0 e^{\lambda t}\vec{v}_\lambda + D_0 e^{-\lambda t}\vec{v}_{-\lambda}\end{aligned}$$

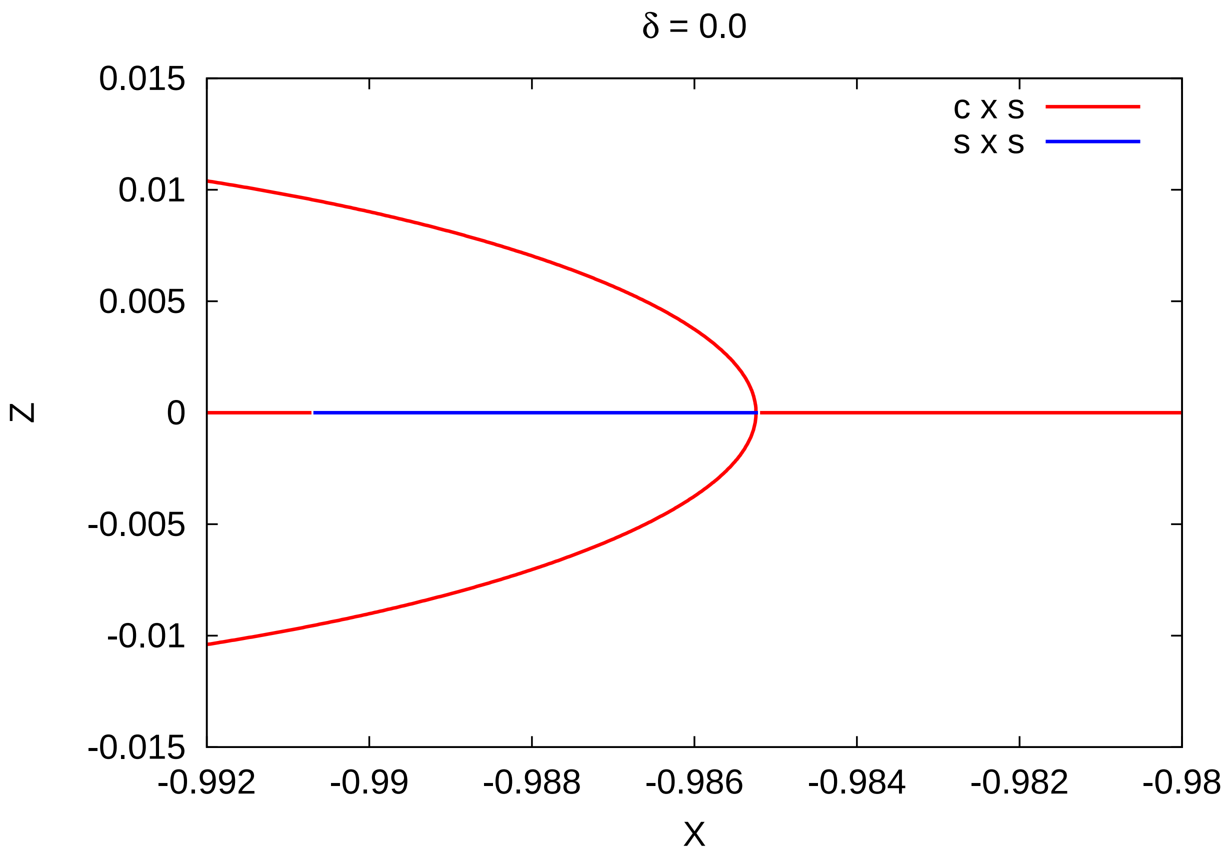
From the Devaney - Lyapunov Centre Theorem, if $\omega_1/\omega_2 \notin \mathbb{Z}$ then we have two families of periodic orbits.

Let us assume that the periodic orbits emanating from ω_2 have a larger Z oscillation that ω_1.

- We call $\mathcal{P}$ - Family, to the family emanating from ω_1.
- We call $\mathcal{V}$ - Family, to the family emanating from ω_2.

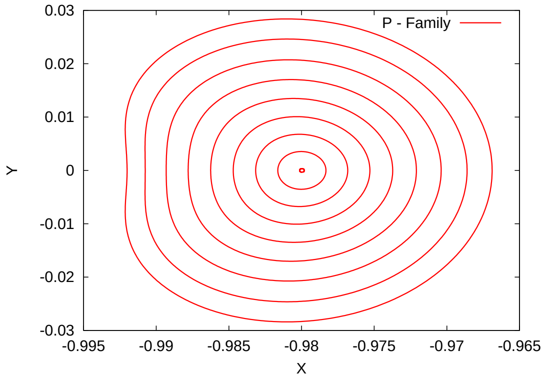
$\mathcal{P}$ - Family of Periodic Orbits (I)

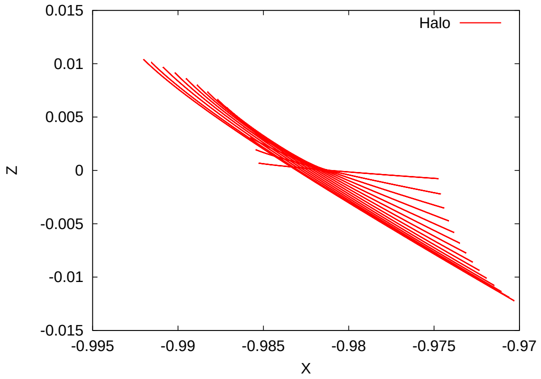
- We have computed the planar family for $\delta = 0$. (i.e. sail perpendicular to Sun - line direction).

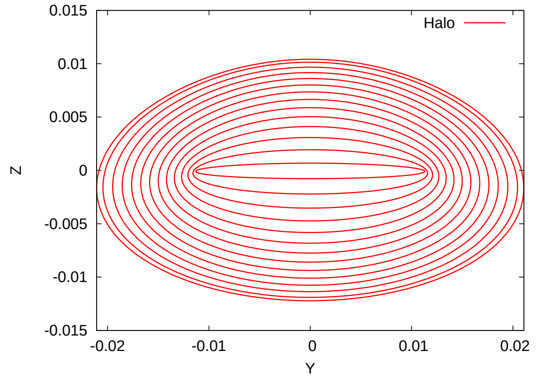


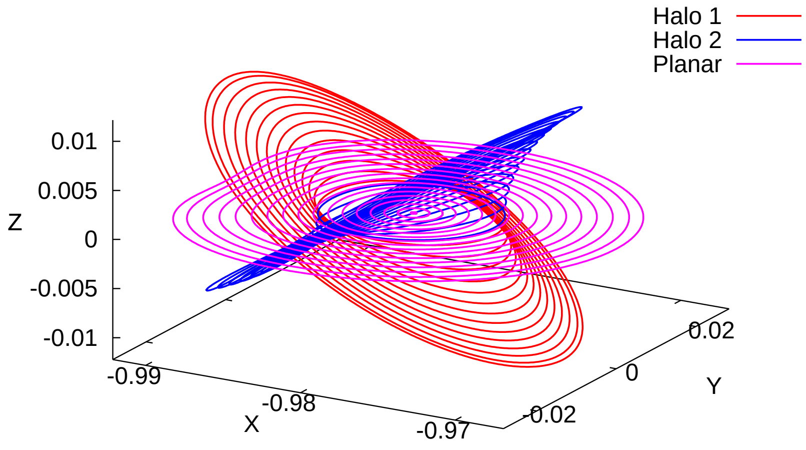
$\mathcal{P}$ - Family of Periodic Orbits (II)

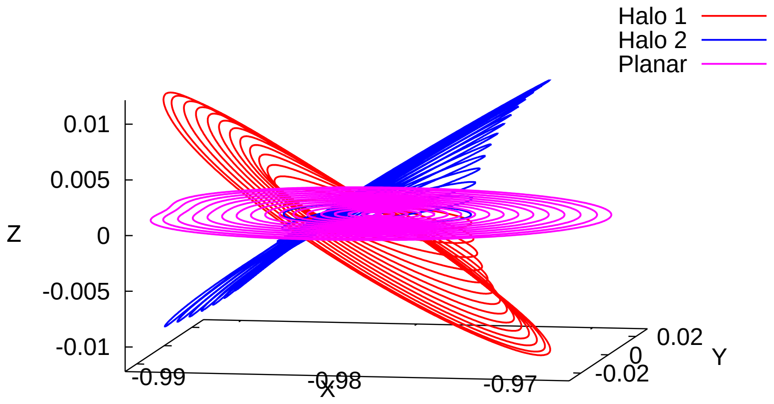
Periodic Orbits for $\delta = 0$.





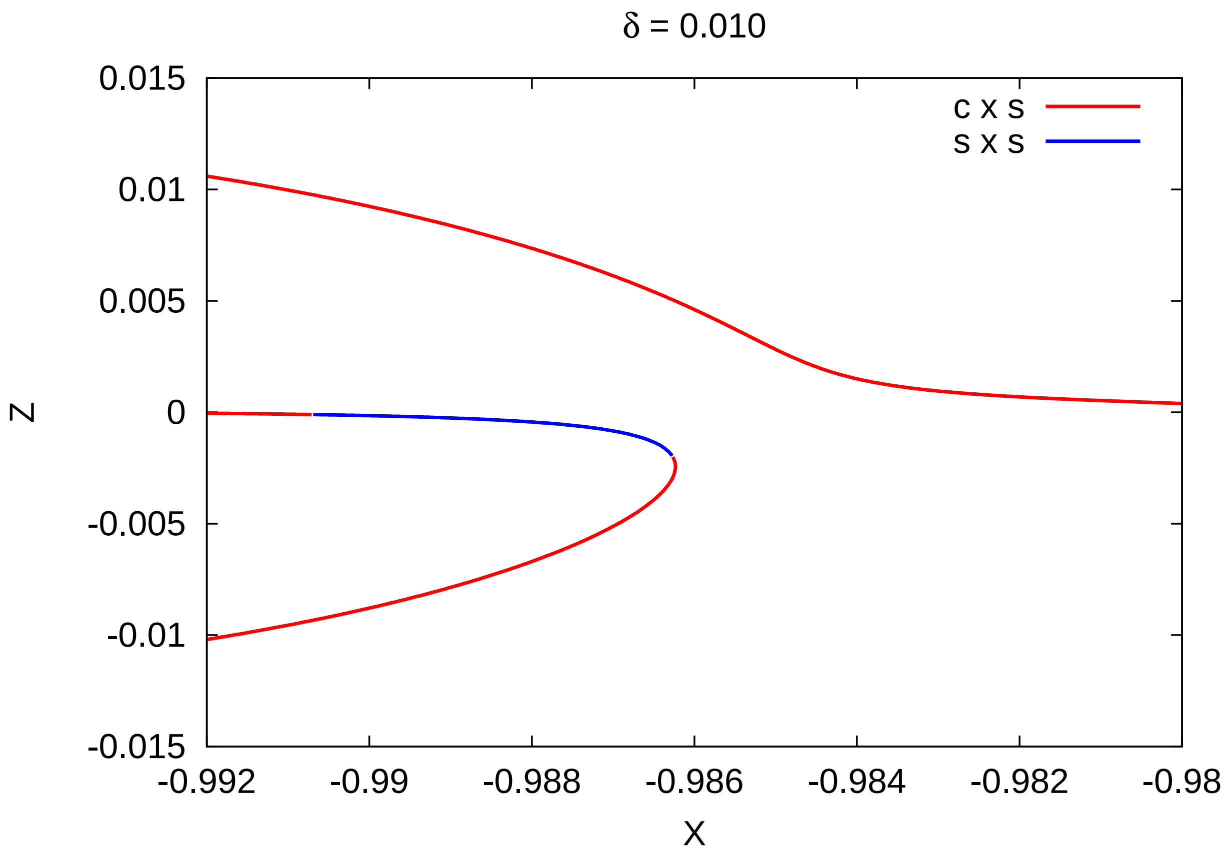






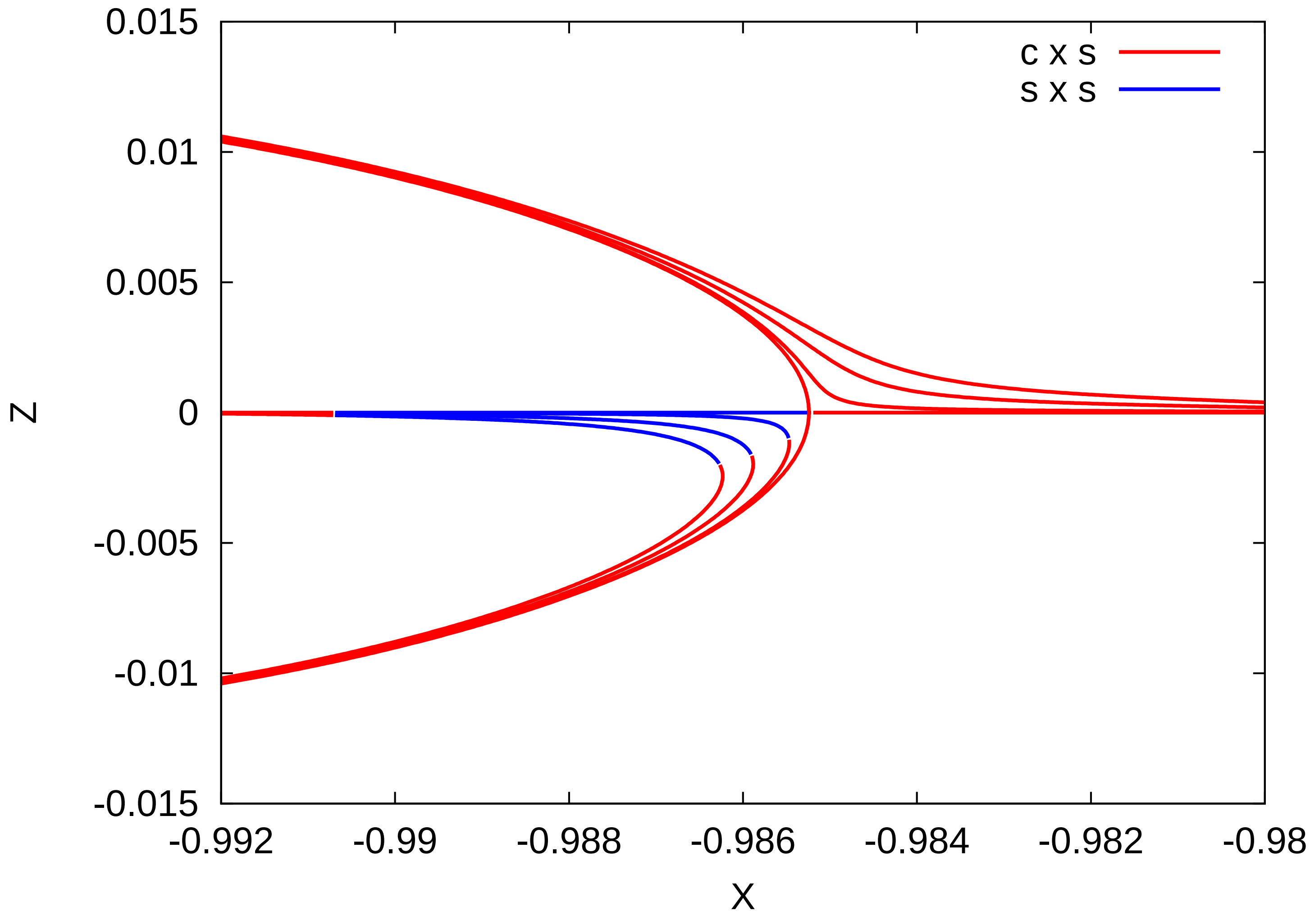
$\mathcal{P}$ - Family of Periodic Orbits (III)

- We have computed the planar family for $\delta = 0.01$.



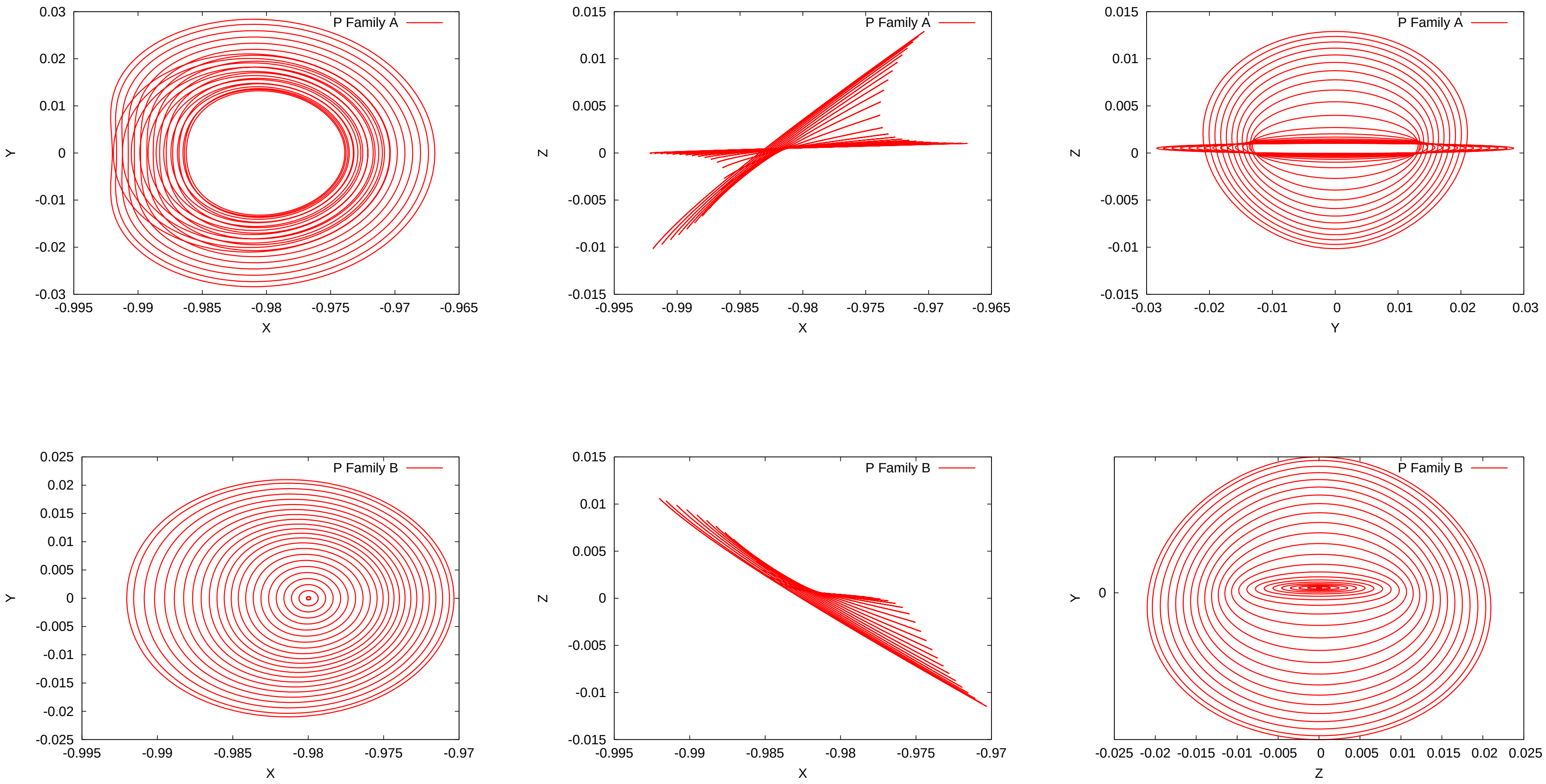
$\mathcal{P}$ - Family of Periodic Orbits (IV)

From $\delta = 0$ to $\delta = 0.01$

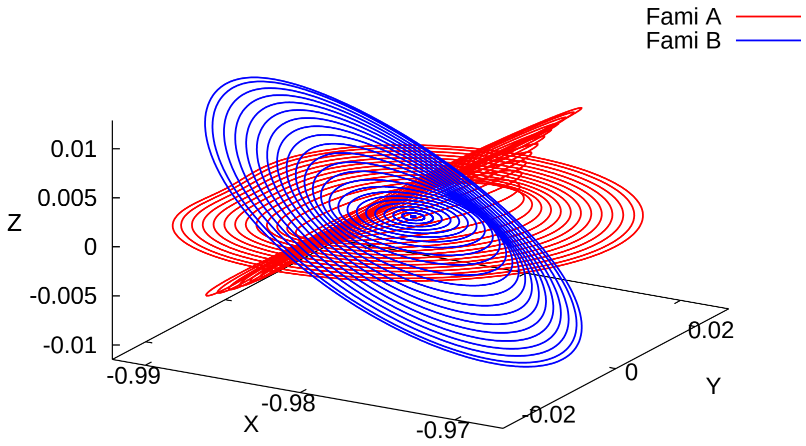


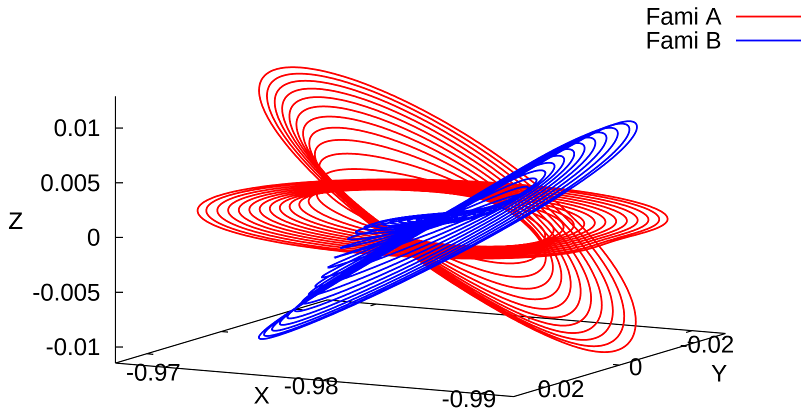
$\mathcal{P}$ - Family of Periodic Orbits (V)

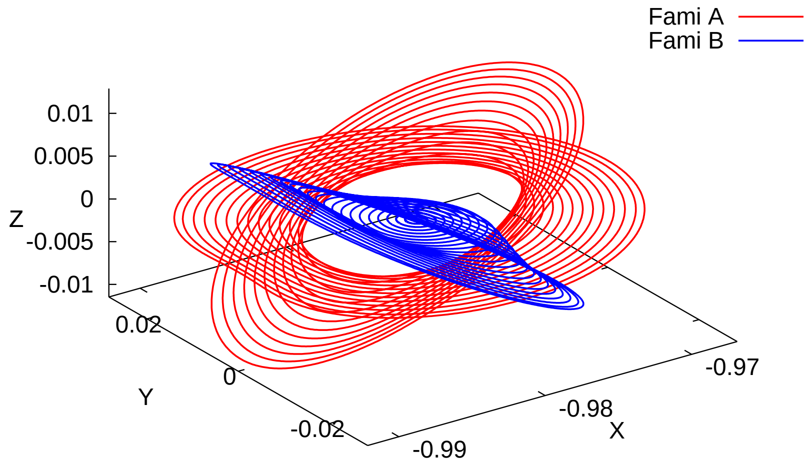
Periodic Orbits for $\delta = 0.01$.



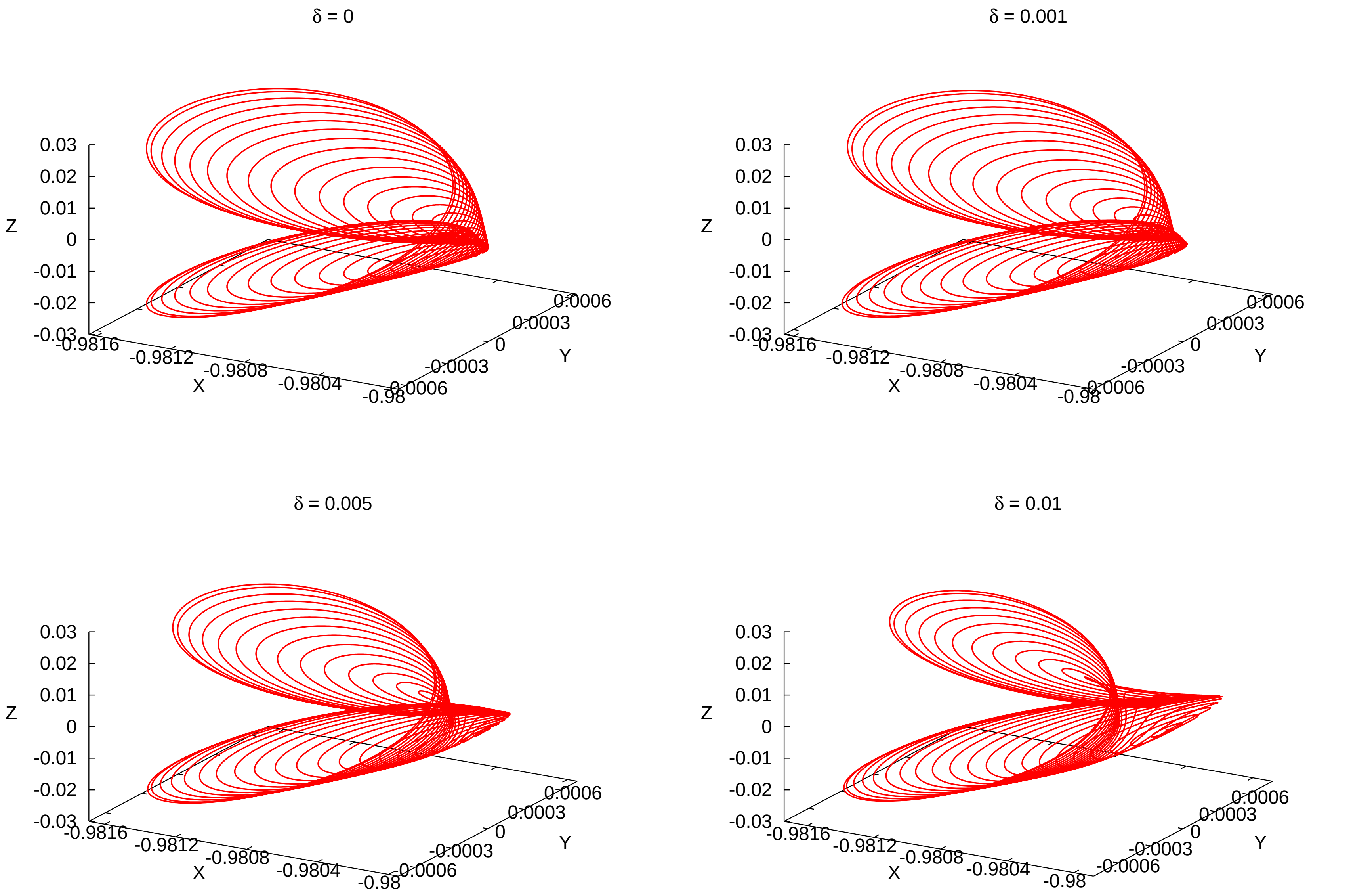
$\mathcal{P}$ - Family of Periodic Orbits (VI)







$\mathcal{V}$ - Family of Periodic Orbits



Quasi-Periodic Motion

- We want to understand the dynamics in an extended neighbourhood of an equilibrium point. We are interested in the trajectories that remain close to the equilibrium point.
- Due to the instability of the fixed point, we cannot take arbitrary initial conditions and integrate them numerically, as they will quickly escape from the vicinity of the fixed point.
- We will decouple up to high order the elliptic from the hyperbolic behaviour, and use this high order approximation of the centre manifold to understand the dynamics.
- As the system is not always Hamiltonian, we will compute formally the power expansion of the graph of the centre manifold.

Reduction to the Centre Manifold

Using an appropriate linear transformation, the equations around the fixed point can be written as,

$$\begin{aligned} \dot{x} &= Ax + f(x,y), \quad x \in \mathbb{R}^4, \\ \dot{y} &= By + g(x,y), \quad y \in \mathbb{R}^2, \end{aligned}$$

where A is an elliptic matrix and B an hyperbolic one, and $f(0,0) = g(0,0) = 0$ and $Df(0,0) = Dg(0,0) = 0$.

- We want to obtain $y = v(x)$, with $v(0) = 0$, $Dv(0) = 0$, the local expression of the centre manifold.
- The flow restricted to the invariant manifold is

$$\dot{x} = Ax + f(x, v(x)).$$

Approximating the Centre Manifold (I)

To find $y = v(x)$ we substitute this expression on the differential equations. Hence, $v(x)$ must satisfy,

$$Dv(x)Ax - Bv(x) = g(x, v(x)) - Dv(x)f(x, v(x)). \tag{1}$$

We take,

$$v(x) = \sum_{|k| \geq 2} v_k x^k, \qquad k \in (\mathbb{N} \cup \{0\})^4,$$

its expansion as power series. Then we solve equation (1) to find the coefficients v_k up to high degree ($|k| = N$).

- $\widehat{v}(x) = \sum_{|k|=2}^{N} v_k x^k$ is a high order approximation of the centre manifold.
- $\dot{x} = Ax + f(x, \widehat{v}(x))$ gives a high order approximation of the motion on the centre manifold.

Approximating the Centre Manifold (II)

- The left hand side of equation (1),

$$L(x) = Dv(x)Ax - Bv(x),$$

is a linear operator w.r.t $v(x)$ that diagonalizes if A and B are diagonal.

- The right hand side of equation (1),

$$h(x) = g(x, v(x)) - Dv(x)f(x, v(x)),$$

can be expressed as, $h(x) = \sum_{|k|\geq 2} h_k x^k$, where h_k depend on v_j in a known way.

- It can be seen that for a fixed degree $|k| = n$, the h_k depend only on the v_j such that $|j| < n$.

Approximating the Centre Manifold (III)

We can solve equation (1) in an iterative way, equalising the left and the right hand side degree by degree.

Notice:

- It is important to have a fast way to find the h_k to go up to high degrees.
- We do not recommend to expand $f(x, y)$ y $g(x, y)$, and then compose with $y = v(x)$. One should find other alternative ways, faster in terms of computational time.
- For instance, compute the terms in a recurrent way and use the recurrence to define an efficient algorithm.
- The matrices A and B don't have to be diagonal, but then one must solve a larger linear system at each degree.

Equations of Motion

For $\alpha = 0$ the equation of motion can be written as:

$$\begin{aligned}
\ddot{X} - 2\dot{Y} &= \frac{\partial \Omega}{\partial X} - \beta \frac{(1-\mu)}{r_{PS}^3} \frac{(X-\mu)Z}{r_2} \cos^2\delta \sin\delta, \\
\ddot{Y} + 2\dot{X} &= \frac{\partial \Omega}{\partial Y} - \beta \frac{(1-\mu)}{r_{PS}^3} \frac{YZ}{r_2} \cos^2\delta \sin\delta, \\
\ddot{Z} &= \frac{\partial \Omega}{\partial Z} + \beta \frac{(1-\mu)}{r_{PS}^3} r_2 \cos^2\delta \sin\delta,
\end{aligned}$$

where,

$$\Omega(X,Y,Z) = \frac{1}{2}\left(X^2 + Y^2\right) + \frac{(1-\mu)(1-\beta\cos^3\delta)}{r_{PS}} + \frac{\mu}{r_{PE}},$$

and,

$$r_{PS} = \sqrt{(X-\mu)^2 + Y^2 + Z^2}, \qquad r_{PE} = \sqrt{(X-\mu+1)^2 + Y^2 + Z^2},$$

$$r_2 = \sqrt{(X-\mu)^2 + Y^2}.$$

Equations of Motion

After translating the fixed point $(X^*, 0, Z^*)$ to the origin, and expanding the equations of motions, we have,

$$
\begin{aligned}
\ddot{x} &= 2\dot{y} + x - \frac{X^*}{\xi} - \mathcal{K}_S \left[\sum_{n\geq 0} c_n T S_n\right]^3 \left(x - \frac{X^*-\mu}{\xi}\right) - \mathcal{K}_E \left[\sum_{n\geq 0} T E_n\right]^3 \left(x - \frac{X^*-\mu+1}{\xi}\right) \\
&\quad -\mathcal{K}_{ss} \left[\sum_{n\geq 0} c_n T S_n\right]^3 \left[\sum_{n\geq 0} d_n T b_n\right] \left(x - \frac{X^*-\mu}{\xi}\right) \left(z + \frac{Z^*}{\xi}\right),
\end{aligned}
$$

$$
\begin{aligned}
\ddot{y} &= -2\dot{x} + \Bigg(1 - \mathcal{K}_S \left[\sum_{n\geq 0} c_n T S_n\right]^3 - \mathcal{K}_E \left[\sum_{n\geq 0} T E_n\right]^3 \\
&\quad -\mathcal{K}_{ss} \left[\sum_{n\geq 0} c_n T S_n\right]^3 \left[\sum_{n\geq 0} d_n T b_n\right] \left(z + \frac{Z^*}{\xi}\right)\Bigg) y,
\end{aligned}
$$

$$
\begin{aligned}
\ddot{z} &= -\left(\mathcal{K}_S \left[\sum_{n\geq 0} c_n T S_n\right]^3 + \mathcal{K}_E \left[\sum_{n\geq 0} T E_n\right]^3\right) \left(z + \frac{Z^*}{\xi}\right) \\
&\quad +\mathcal{K}_{ss} \left[\sum_{n\geq 0} c_n T S_n\right]^3 \left[\sum_{n\geq 0} d_n T b_n\right]^{-1}.
\end{aligned}
$$

where $\mathcal{K}_S = (1-\mu)(1-\beta\cos^3\delta)/\xi^3$, $\mathcal{K}_E = \mu/\xi^3$, $\mathcal{K}_{ss} = \beta(1-\mu)\cos^2\delta\sin\delta/\xi^3$, and ξ is the distance between the fixed point and the Earth.

On the Centre Manifold

We have computed the centre manifold around Sub-L_1 up to degree 16.

- After this reduction we are in a four dimensional phase space (x_1, x_2, x_3, x_4).
- We need to perform suitable Poincaré sections to help us visualise the phase space.
- For $\delta = 0$, we have a first integral, and we can take advantage of this. For the other sail orientations we will use similar ideas.

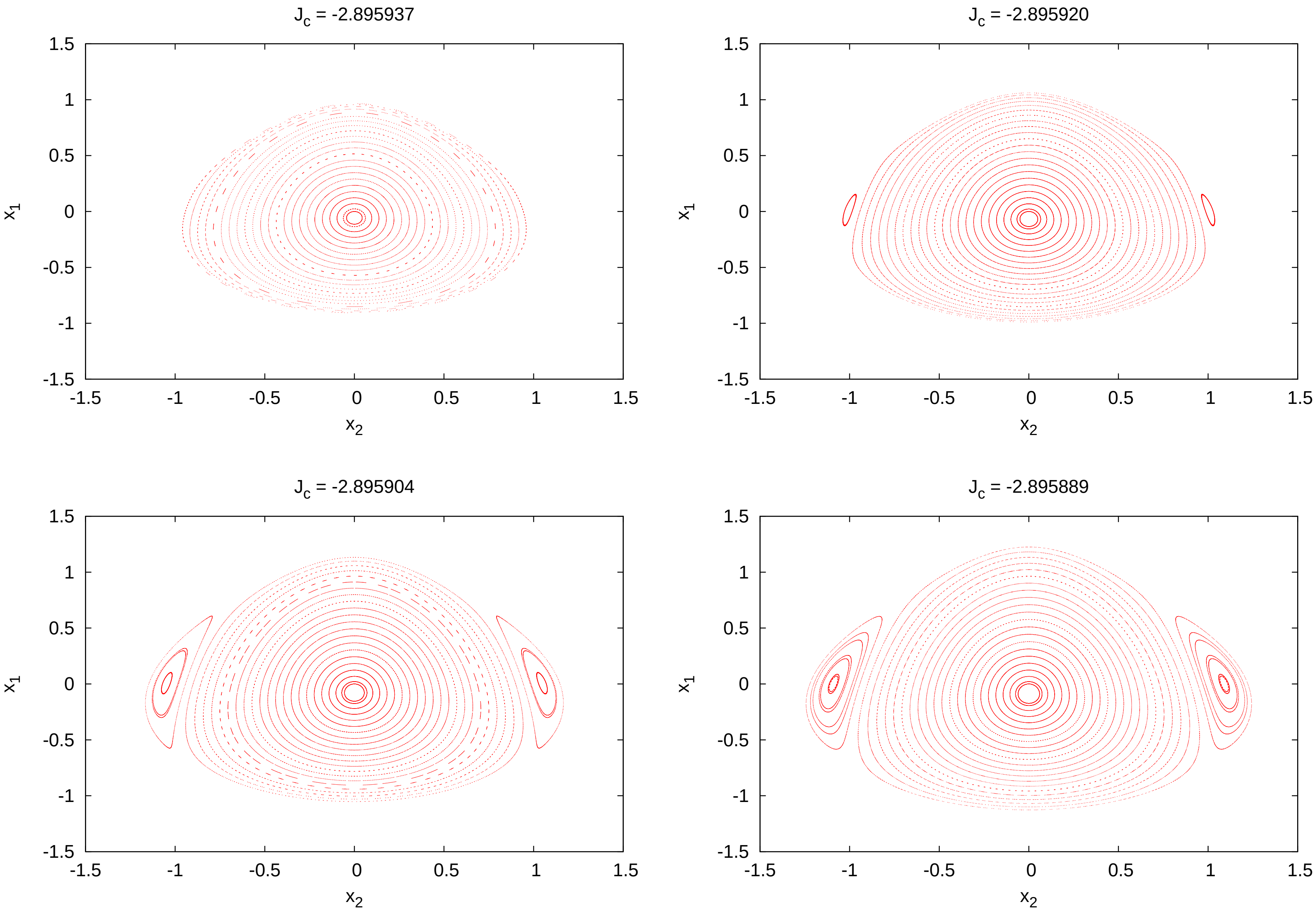
Dynamics for $\delta = 0$

Here we have a first integral:

$$J_c = \frac{1}{2}(\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2) - 2\Omega(X, Y, Z).$$

- We fix a Poincaré section $x_3 = 0$ to reduce the system to a three dimensional phase space. *(Taking $x_3 = 0$ is like taking $Z = 0$).*
- We fix the energy level to determine x_4 and reduce the system to a two dimensional phase space that is easy to visualise. *(Taking $x_4(J_c, x)$ is like taking $\dot{Z}(J_c, x)$).*
- We have taken several initial conditions and computed their successive images on the Poincaré section.

Dynamics for $\delta = 0$ ($x_3 = 0$ section)



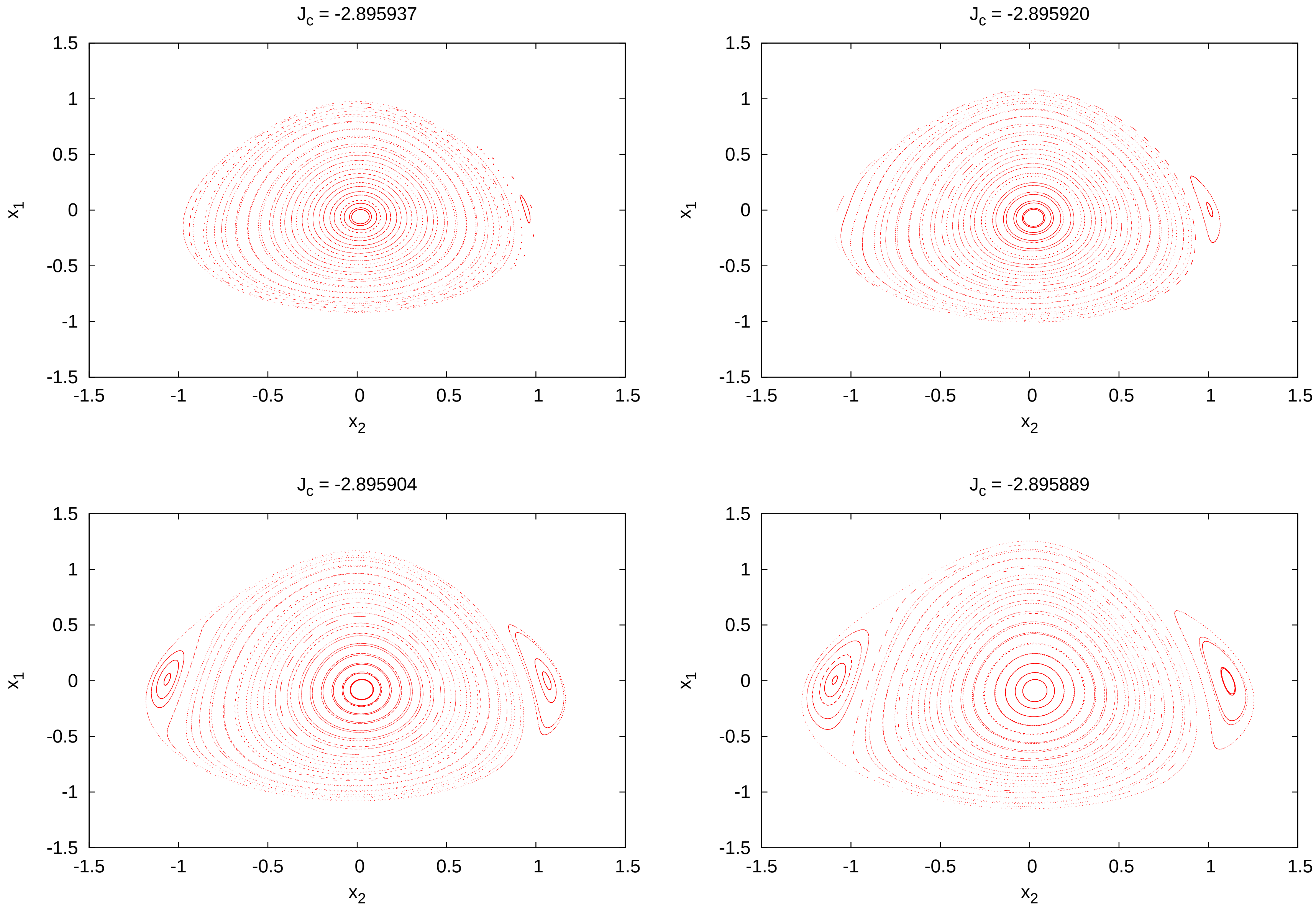
Dynamics for $\delta \neq 0$

Here we take an *“approximated first integral”* :

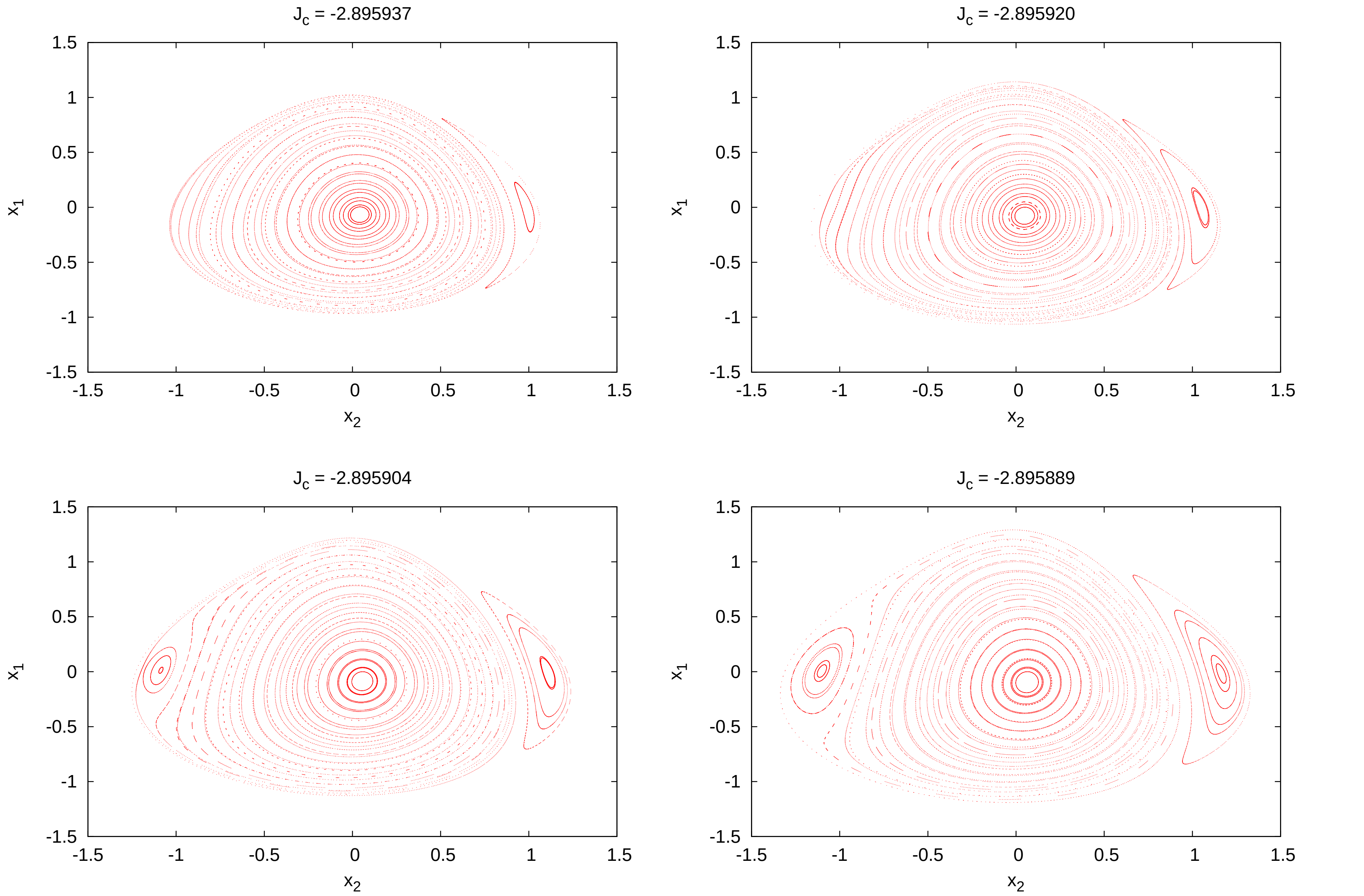
$$J_c = \frac{1}{2}(\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2) - 2\Omega(X, Y, Z) + \beta(1-\mu)\frac{Zr_2}{r_{PS}^3}\cos^2\delta\sin\delta$$

- We fix a Poincaré section $x_3 = 0$ to reduce the system to a three dimensional phase space. *(Taking $x_3 = 0$ is similar to taking $Z = Z^*$)*.
- We fix J_c to determine x_4 and reduce the system to a two dimensional phase space that is easy to visualise. *(Taking $x_4(J_c, x)$ is like taking $\dot{Z}(J_c, x)$)*.
- We have taken several initial conditions and computed their successive images on the Poincaré section.

Dynamics for $\delta = 0.005$ *($x_3 = 0$ section)*



Dynamics for $\delta = 0.01$ $(x_3 = 0$ section)

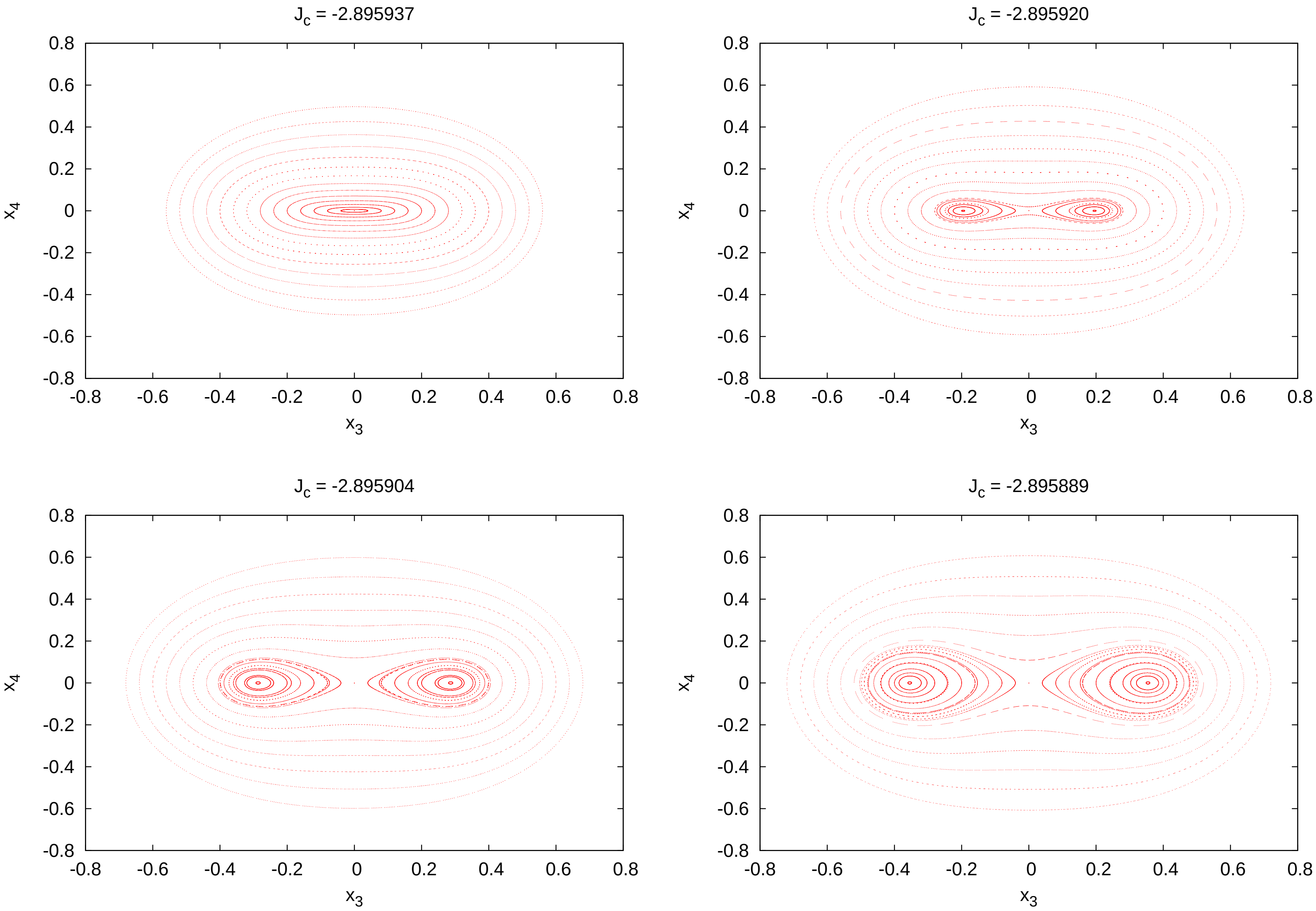


A different Poincaré section: $x_2 = 0$

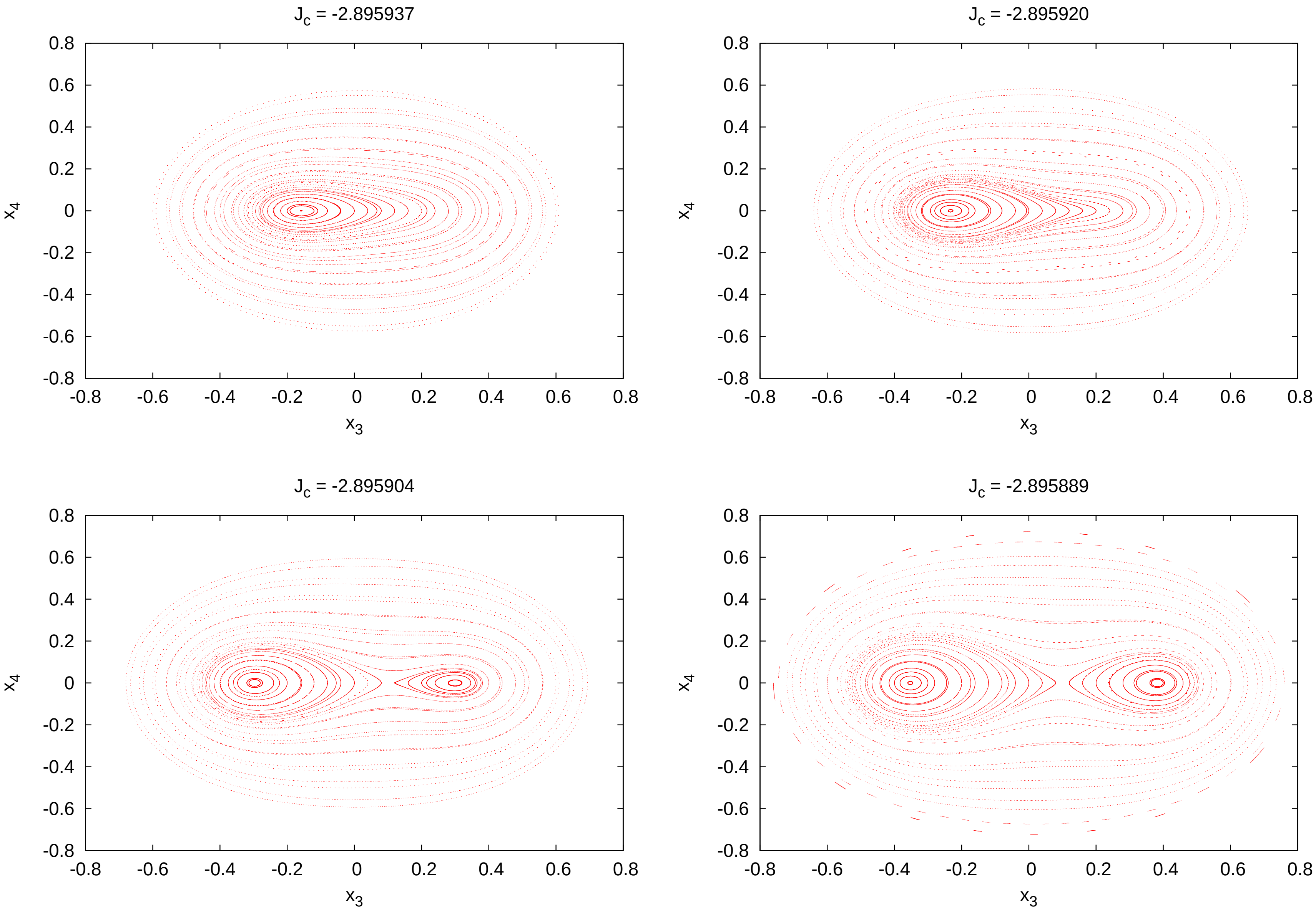
Notice that $x_3 = 0$ is not the only section one can take.

- Now we fix the Poincaré section $x_2 = 0$ to reduce the system to a three dimensional phase space. *(Taking $x_2 = 0$ is similar to taking $Y = 0$).*
- We fix J_c to determine x_1 and reduce the system to a two dimensional phase space that is easy to visualise. *(Taking $x_1(J_c, x)$ is similar to taking $\dot{Y}(J_c, x)$).*
- We have taken several initial conditions and computed their successive images on the Poincaré section.

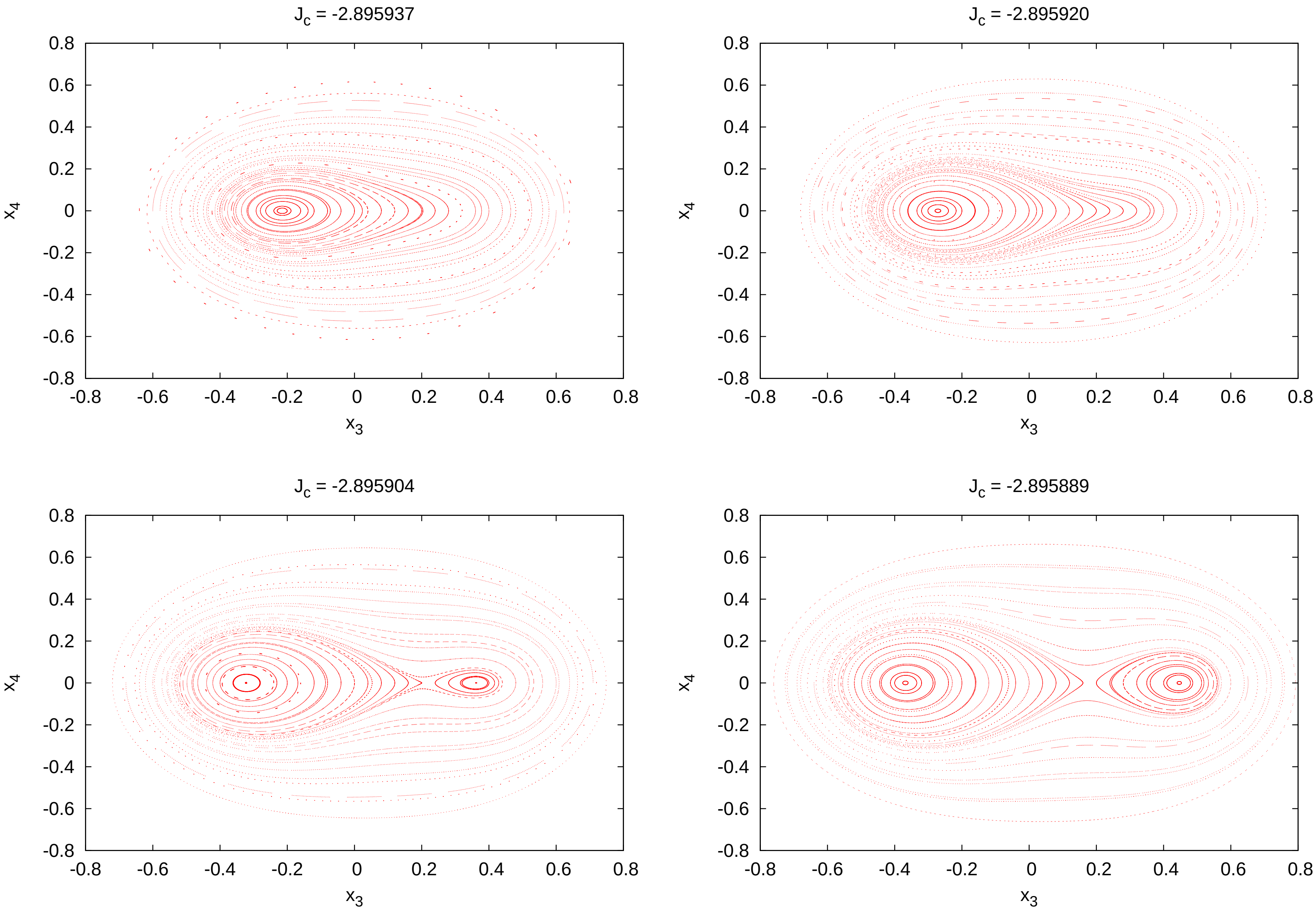
Dynamics for $\delta = 0$ ($x_2 = 0$ section)



Dynamics for $\delta = 0.005$ ($x_2 = 0$ section)



Dynamics for $\delta = 0.01$ $(x_2 = 0$ section)



Conclusions & Future Work

Conclusions:

- We have understood the dynamics around an equilibrium point for a solar sail with $\alpha = 0$ and small δ.
- We have designed strategies for the station keeping and surfing around equilibria for a Solar Sail.

Future Work:

- Understand the dynamics for the $\alpha \neq 0$ close to the SL_1 family (the system is no longer reversible).
- Extend the station keeping strategies to periodic orbits.
- Consider more complex models, for example adding the effect of the Moon, non-perfectly reflecting sail, ...

The End

Thank You !!!