Master: Functional Analysis and PDE

December 2015. List 3

1) Prove that $\hat{u}(\xi) = -|\xi|^{-2}$ in $\mathbb{R}^3$ is a tempered distribution and satisfies $\Delta u = \delta$. Show that u is in fact a function in $L^\infty + L^2$.

2) Let $n \geq 3$ and let $E(x) = |x|^{2-n}$. Prove that:
a) E is a tempered distribution.
b) Compute ΔE whenever exists.
c) Show that $\Delta E = \delta$ in the sense of distributions.

Recall Green's identity: If R is a bounded domain with smooth boundary S and f and g are C^1 functions on $\mathbb{R}$, then:

$$\int_R (f\Delta g - g\Delta f)dx = \int_S (f\delta_\nu g - g\delta_\nu f)d\sigma,$$

where δ_ν is the directional derivative with respect to the outward normal vector to R.

3) Let us consider the Heat equation

$$\frac{\partial u}{\partial t} = \Delta u, \qquad t > 0,\ x \in \mathbb{R}^n.$$

a) Prove that if $P(x,t) = t^{-n/2}e^{-\frac{|x|^2}{4t}}$ then P satisfies the Heat equation.
b) Let $P_\varepsilon(x,t) = \chi_{(\varepsilon,\infty)}(t)P(x,t)$. Prove that $\lim_{\varepsilon\to 0} P_\varepsilon$ is the fundamental solution of the Heat equation in $\mathbb{R}^{n+1}$.

4) Prove that if $P(x,t)$ is as in exercise 3 with $t > 0,\ x \in \mathbb{R}^n$, then:
a) $S'(\mathbb{R}^n) - \lim_{t\to 0} P(x,t) = 0$.
b) For every $\lambda > 0$,

$$F(x,\lambda) = S'(\mathbb{R}^n) - \lim_{\varepsilon\to 0} \int_\varepsilon^\infty e^{-\lambda t} P(x,t)dt$$

is the fundamental solution of $\Delta + \lambda I$.

5) a) For which $s \in \mathbb{R}$, we have that $1 \in H^s(\mathbb{R}^n)$.
b) For which $s \in \mathbb{R}$, we have that $\delta \in H^s(\mathbb{R}^n)$.
c) For which $s \in \mathbb{R}$, we have that $\chi_{[0,1]} \in H^s(\mathbb{R})$.
d) For which $s \in \mathbb{R}$, we have that $\chi_{[0,1]} \times \chi_{[0,1]} \in H^s(\mathbb{R}^2)$.

6) Prove that if $\varphi \in S(\mathbb{R}^n)$ and $f \in H^s(\mathbb{R}^n)$, then $\varphi f \in H^s(\mathbb{R}^n)$.

7) Prove, justifying all the steps, that if $f, g \in L^2(\mathbb{R})$ are such that their derivatives in the sense of distributions $f', g' \in L^2(\mathbb{R})$, then the following integration by parts formula holds

$$\int_{\mathbb{R}} fg' = -\int_{\mathbb{R}} f'g.$$

8) Given a function $f \in H^s(\mathbb{R}^n)$ prove that if f_R is such that $\hat{f}_R = \chi_{B(0,R)}\hat{f}$, then $H^2 - \lim_{R\to\infty} f_R = f$ and deduce that $C^\infty \cap L^2$ is dense in H^2.