STEIN'S SQUARE FUNCTION G_α AND SPARSE OPERATORS

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Abstract. The purpose of this paper is to check that the square function G_α, introduced by E. M. Stein in 1958, can be controlled by a finite sum of sparse operators when $\alpha > \frac{n+1}{2}$. This provides a useful tool to obtain weighted estimates for G_α and related Fourier multipliers.

1. Introduction and main result

The object of study of this paper will be the square function G_α, defined for $\alpha > 0$ as

$$G_\alpha f(x) = \left(\int_0^\infty \left| \frac{\partial}{\partial t} B_\alpha^t f(x) \right|^2 t dt \right)^{1/2},$$

where B_α^t is the Bochner-Riesz multiplier

$$\widehat{B_\alpha^t f}(\xi) = \left(1 - \frac{|\xi|^2}{t^2} \right)_+^\alpha \widehat{f}(\xi).$$

This function was first introduced by E. M. Stein in [24] to study L^2 properties of the maximal Bochner-Riesz operator and deduce almost everywhere convergence for Bochner-Riesz means of Fourier integrals and series. See also [26, Chapter VII] and the detailed overview of the topic contained in [17]. It can be easily checked that

$$\frac{\partial}{\partial t} B_\alpha^t f(x) = \frac{2\alpha}{t} \int_{\mathbb{R}^n} \frac{|\xi|^2}{t^2} \left(1 - \frac{|\xi|^2}{t^2} \right)_+^{\alpha-1} \widehat{f}(\xi) e^{2\pi i x \xi} d\xi,$$

and hence, up to a constant, we have that

$$G_\alpha f(x) = \left(\int_0^\infty |K_t^\alpha * f(x)|^2 \frac{dt}{t} \right)^{1/2},$$

with $\widehat{K_t^\alpha}(\xi) = \frac{|\xi|^2}{t^2} \left(1 - \frac{|\xi|^2}{t^2} \right)_+^{\alpha-1}$. This second expression has been taken as a definition in several references, such as [2, 11, 27]. In the last one, G. Sunouchi shows that, when $\alpha > \frac{n+1}{2}$, we have that

$$G_\alpha : L^p(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n)$$

is bounded for every $1 < p \leq 2$ and

$$G_\alpha : L^1(\mathbb{R}^n) \longrightarrow L^{1,\infty}(\mathbb{R}^n).$$

2010 *Mathematics Subject Classification.* 42B25, 42B37.

Key words and phrases. Square functions, sparse operators, weighted inequalities.
The authors were supported by grants MTM2013-40985-P and 2014SGR289.

To prove it, the author relates G_α to an $L^2(0,\infty)$ vector-valued Calderón-Zygmund operator and is able to use the classical theory together with interpolation to deduce his result. However, if we want to establish weighted inequalities, it seems that the vector-valued technique in this case does not work as cleanly. Our main result basically states that the modern approach of pointwise domination by sparse operators, which has been so fruitful in Calderón-Zygmund theory, can also be applied to G_α.

Theorem 1.1. *Let $\alpha > \frac{n+1}{2}$. Then, for every Schwartz function f on $\mathbb{R}^n$, $|G_\alpha f|$ can be pointwise controlled by a finite sum of sparse operators $\{S_k\}_k$ applied to f. More precisely,*

$$|G_\alpha f(x)| \le C \sum_{k=1}^{3^n} S_k f(x),$$

for some constant C that only depends on α and the dimension n.

Despite the fact that we have not yet defined what a sparse operator is, what becomes clear at this point is that this domination allows us to deduce for G_α all the estimates that we know for arbitrary sparse operators (which are much easier to handle). The use of sparse theory to obtain weighted estimates for square functions can be found in many recent papers, such as [7, 18, 21]. However, the standard approach is to control these functions by sparse square functions (which contain the square nature of the operator in their definition). In our case, we rely on the Calderón-Zygmund properties of the $L^2(0,\infty)$-valued description of G_α, and the sparse operators appearing in the domination are completely "square-free". Some interesting consequences of Theorem 1.1 will be presented in Section 3, but first, let us focus on the proof of the theorem itself.

2. Proof of Theorem 1.1

For simplicity, from now on we will use the notation $x \lesssim y$ to denote that there exists a constant $C > 0$, independent of all non-fixed parameters, such that $x \le Cy$. If $x \lesssim y$ and $y \lesssim x$, we will write $x \approx y$.

We will start by making the definition of sparse operators precise. For convenience, we will follow the exposition in [22], but we also refer to [16] for a simpler and more effective approach to sparse domination. Given a dyadic lattice of cubes in $\mathbb{R}^n$, we will say that a family of cubes $\mathcal{S}$ is λ-sparse, with $0 < \lambda < 1$ if, for every $Q \in \mathcal{S}$, there exists a measurable subset $F_Q \subseteq Q$ such that $|F_Q| \ge (1-\lambda)|Q|$ and $\{F_Q\}_{Q\in\mathcal{S}}$ are pairwise disjoint.

Definition 2.1. *Given a λ-sparse family $\mathcal{S}$, we define the λ-sparse operator S associated with it by*

$$Sf(x) = \sum_{Q\in\mathcal{S}} \left(\frac{1}{|Q|} \int_Q |f| \right) \chi_Q(x).$$

We will also need the following definitions of the so-called local mean oscillation:

Definition 2.2. *Given a function g and a measurable set E, we define*

$$\omega(g, E) = \sup_{x\in E} g(x) - \inf_{x\in E} g(x).$$

Given $0 < \lambda < 1$ and a cube Q, we also define

$$\omega_\lambda(g, Q) = \min\{\omega(g, E) : E \subseteq Q, |E| \ge (1-\lambda)|Q|\}.$$

The next theorem is the key tool that we will need for our purposes. Its proof is contained in [22], as a consequence of a series of results therein.

Theorem 2.3. *Let f be a measurable function and let $g : \mathbb{R}^n \to \mathbb{R}$ be such that, for every $\varepsilon > 0$,*

$$|\{x \in [-R,R]^n : |g(x)| > \varepsilon\}| = o(R^n), \qquad \text{as } R \to \infty.$$

Assume that, given a dyadic cube Q and $0 < \lambda \le 2^{-n-2}$, it holds that, for some $\delta > 0$,

$$\omega_\lambda(g,Q) \le C_\lambda \sum_{k=0}^{\infty} 2^{-\delta k} \left(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |f| \right). \tag{2.1}$$

Then $|g|$ is pointwise controlled by a sum of 3^n ν-sparse operators applied to f, with ν being a universal constant only depending on the dimension n.

Fix $\alpha = \frac{n+1}{2} + \delta$, with $\delta > 0$. If we define

$$T_t f(x) = \frac{K_t^\alpha * f(x)}{\sqrt{t}},$$

it holds that,

$$G_\alpha f(x) = \|T_t f(x)\|_{L^2(0,\infty)}.$$

By [27], we know that G_α is of weak-type $(1,1)$, that is

$$y|\{x \in \mathbb{R}^n : \|T_t f(x)\|_{L^2(0,\infty)} > y\}| \le \|G_\alpha\|_{L^1 \to L^{1,\infty}} \|f\|_{L^1(\mathbb{R}^n)}. \tag{2.2}$$

Also, the author shows (see [27, Equations (3) and (4)]) that, given $r > 0$ and $s \in \mathbb{R}$ satisfying $r > 2|s|$, it holds that

$$|\mathcal{K}_t(r+s) - \mathcal{K}_t(r)| \lesssim \min\{t^{-\frac{1}{2}-\delta} r^{-n-\delta},\, |s|\, t^{\frac{1}{2}-\delta} r^{-n-\delta}\}, \tag{2.3}$$

where

$$\mathcal{K}_t(|x|) = \frac{K_t^\alpha(x)}{\sqrt{t}}.$$

Taking $g(x) = G_\alpha f(x)$, we have that the decay assumption for g in Theorem 2.3 is trivially satisfied (using, for instance, that G_α is of weak-type (1,1)). Hence, if we show (2.1), we conclude that $G_\alpha f$ is dominated by sparse operators and, consequently, finish the proof of Theorem 1.1. Fix a cube Q and $0 < \lambda \le 2^{-n-2}$. Let $x, x' \in Q$. Then,

$$\begin{aligned}
&|\|T_t f(x)\|_{L^2(0,\infty)} - \|T_t f(x')\|_{L^2(0,\infty)}| \le \|T_t f(x) - T_t f(x')\|_{L^2(0,\infty)} \\
&= \left\| T_t(f\chi_{2^{k_n}Q})(x) + \sum_{k \ge k_n} T_t(f\chi_{2^{k+1}Q \setminus 2^k Q})(x) - T_t(f\chi_{2^{k_n}Q})(x') - \sum_{k \ge k_n} T_t(f\chi_{2^{k+1}Q\setminus 2^kQ})(x') \right\|_{L^2(0,\infty)} \\
&\le I + II,
\end{aligned}$$

where $k_n \in \mathbb{N}$ is a dimensional constant to be chosen later,

$$I = \|T_t(f\chi_{2^{k_n}Q})(x)\|_{L^2(0,\infty)} + \|T_t(f\chi_{2^{k_n}Q})(x')\|_{L^2(0,\infty)},$$

and after using Minkowski's integral inequality,

$$II = \sum_{k \ge k_n} \int_{2^{k+1}Q \setminus 2^k Q} \|\mathcal{K}_t(|x-y|) - \mathcal{K}_t(|x'-y|)\|_{L^2(0,\infty)} |f(y)| dy.$$

We start by studying II. Since $x,x'\in Q$ and $y\in 2^{k+1}Q\setminus 2^kQ$, we can set $r=|x'-y|$ and observe that $|x-y|=r+s$, with $s\in(-|x-x'|,|x-x'|)$. Therefore,

$$\|\mathcal{K}_t(|x-y|)-\mathcal{K}_t(|x'-y|)\|_{L^2(0,\infty)}^2=\|\mathcal{K}_t(r+s)-\mathcal{K}_t(r)\|_{L^2(0,\infty)}^2.$$

Computing the L^2 norm and using (2.3) with the different bounds on $(0,|s|^{-1})$ and $(|s|^{-1},\infty)$ respectively, we can control the previous expression by

$$\int_0^{|s|^{-1}}|s|^2t^{1-2\delta}r^{-2n-2\delta}dt+\int_{|s|^{-1}}^{\infty}t^{-1-2\delta}r^{-2n-2\delta}dt\approx\frac{|s|^{2\delta}}{r^{2n+2\delta}}.$$

To use (2.3), we need to choose $k_n\in\mathbb{N}$ so that $r>2|s|$ for all $k\geq k_n$, keeping in mind that $r=|x'-y|\approx 2^k\ell(Q)$ and $|s|\leq|x-x'|\lesssim\ell(Q)$, where $\ell(Q)$ denotes the side-length of Q. Also, with these relations, the last quotient is essentially majorized by

$$\frac{\ell(Q)^{2\delta}}{2^{2k(n+\delta)}\ell(Q)^{2n+2\delta}}=\left(\frac{1}{2^{k(n+\delta)}|Q|}\right)^2.$$

Now we go back to II and see that

$$II\lesssim\sum_{k\geq k_n}\int_{2^{k+1}Q\setminus 2^kQ}\frac{|f(y)|}{2^{k(n+\delta)}|Q|}dy\lesssim\sum_{k\geq k_n}2^{-\delta k}\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|f(y)|dy\right).$$

To study I, we just use (2.2) to get that if

$$E^*=\left\{z\in Q:\|T_t(f\chi_{2^{k_n}Q})(z)\|_{L^2(0,\infty)}>\frac{2^{nk_n}\|G_\alpha\|_{L^1\to L^{1,\infty}}}{\lambda|2^{k_n}Q|}\int_{2^{k_n}Q}|f|\right\},$$

then

$$|E^*|\leq\|G_\alpha\|_{L^1\to L^{1,\infty}}\frac{\lambda|2^{k_n}Q|}{2^{nk_n}\|G_\alpha\|_{L^1\to L^{1,\infty}}\int_{2^{k_n}Q}|f|}\|f\chi_{2^{k_n}Q}\|_{L^1(\mathbb{R}^n)}=\lambda|Q|.$$

So defining $E=Q\setminus E^*$, we deduce that, when $x\in E$,

$$\|T_t(f\chi_{2^{k_n}Q}(x))\|_{L^2(0,\infty)}\lesssim C_\lambda\frac{1}{|2^{k_n}Q|}\int_{2^{k_n}Q}|f|,$$

and the size of E is controlled by

$$|E|\geq|Q|-|E^*|\geq(1-\lambda)|Q|.$$

Summing up, we have shown that there exists a measurable set $E\subseteq Q$ such that $|E|\geq(1-\lambda)|Q|$ and satisfying that, for every $x,x'\in E$,

$$|\|T_tf(x)\|_{L^2(0,\infty)}-\|T_tf(x')\|_{L^2(0,\infty)}|\leq I+II\lesssim C_\lambda\sum_{k=0}^{\infty}2^{-\delta k}\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|f(y)|dy\right).$$

Hence, the same bound holds for $\omega_\lambda(\|T_tf(\cdot)\|_{L^2(0,\infty)},Q)$, which proves (2.1) and, as a consequence, Theorem 1.1.

3. Consequences and applications

For this section, we will recall some notions concerning Orlicz spaces and Muckenhoupt weights. From now on, $\varphi : [0,\infty) \to [0,\infty)$ will be a Young function, that is, a convex, increasing function such that $\varphi(0) = 0$ and $\lim_{t\to\infty} \varphi(t) = \infty$. From these properties, one can deduce that its inverse φ^{-1} exists on $(0,\infty)$. Moreover, given a Young function φ, we can define its complementary function ψ by

$$\psi(s) = \sup_{t>0} \{st - \varphi(t)\}. \tag{3.1}$$

We will assume that $\lim_{t\to\infty} \varphi(t)/t = \infty$ to ensure that ψ is finite valued. Under these conditions, ψ is also a Young function, and it can be related to the dual of the Orlicz space

$$\varphi(L) = \{f : \varphi(|f|) \in L^1(\mathbb{R}^n)\}.$$

See [13] for further details.

Also, recall that we define the Hardy-Littlewood maximal operator M by

$$Mf(x) = \sup_{Q\ni x} \frac{1}{|Q|} \int_Q |f(y)|dy,$$

where the supremum is taken over all cubes $Q \subseteq \mathbb{R}^n$ containing x. Now, given a Young function φ as above, we can define the following variant of M:

$$M_{\varphi(L)}f(x) := \sup_{Q\ni x} \|f\|_{\varphi(L),Q},$$

where

$$\|f\|_{\varphi(L),Q} = \inf\left\{\lambda > 0 : \frac{1}{|Q|}\int_Q \varphi\left(\frac{|f(x)|}{\lambda}\right) dx \le 1\right\}$$

is the Luxemburg norm associated with the Orlicz space $\varphi(L)$ localized in Q. Notice that, if $\varphi(t) = t$, then $M_{\varphi(L)} = M$ is the classical Hardy-Littlewood maximal operator, and if $\varphi_1 \le \varphi_2$, then $M_{\varphi_1(L)}f \le M_{\varphi_2(L)}f$

Concerning weights, recall that for $1 < p < \infty$, A_p is the class of weights $w > 0$ such that

$$\|w\|_{A_p} = \sup_{Q\subseteq\mathbb{R}^n} \frac{w(Q)}{|Q|}\left(\frac{w^{1-p'}(Q)}{|Q|}\right)^{p-1} < \infty,$$

where $w(Q)$ denotes the measure of Q with respect to w. As p decreases to 1, the condition above strengthens to

$$\|w\|_{A_1} = \sup_{x\in\mathbb{R}^n} \frac{Mw(x)}{w(x)} < \infty,$$

and as p tends to ∞, the condition weakens to

$$\|w\|_{A_\infty} = \sup_{Q\subseteq\mathbb{R}^n} \frac{\int_Q M(w\chi_Q)}{w(Q)}.$$

We also recall that for $1 \le p < \infty$, $A_p^{\mathcal{R}}$ is the class of weights $w > 0$ such that

$$\|w\|_{A_p^{\mathcal{R}}} = \sup_{E\subseteq Q} \frac{|E|}{|Q|}\left(\frac{w(Q)}{w(E)}\right)^{1/p} < \infty,$$

where the supremum is taken over all cubes Q and all measurable sets $E \subseteq Q$. This $A_p^{\mathcal{R}}$ class was introduced in [12] to characterize the restricted weak-type (p,p) of the Hardy-Littlewood maximal operator M, as follows:

$$\|M\chi_E\|_{L^{p,\infty}(w)} \lesssim \|w\|_{A_p^{\mathcal{R}}} w(E)^{1/p} \tag{3.2}$$

While $A_p^{\mathcal{R}}$ may not be as well-known as the Muckenhoupt A_p class, these weights possess some interesting properties related to extrapolation theory [4, 5]. We should mention that $A_p^{\mathcal{R}}$ is strictly larger than A_p when $p > 1$ since, for example, the weight $|x|^{n(p-1)} \in A_p^{\mathcal{R}} \setminus A_p$.

3.1. **Weighted estimates for G_α.** An interesting weighted estimate that we obtain as a byproduct of Theorem 1.1 is the following:

Corollary 3.1. *Let $\alpha > \frac{n+1}{2}$. Then, for every $1 < p < \infty$ and $w \in A_p$,*

$$\|G_\alpha f\|_{L^p(w)} \lesssim \|w\|_{A_p}^{\max\left\{1, \frac{1}{p-1}\right\}} \|f\|_{L^p(w)}.$$

It is known (see [1, Theorem 1] or the discussion in [17]) that if G_α is bounded on the unweighted $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$, then necessarily $\alpha \geq \frac{n+1}{2}$, so except for the critical case $\alpha = \frac{n+1}{2}$, we cannot expect to lower the value of α in Corollary 3.1. The proof of this corollary only relies on the fact that sparse operators satisfy this inequality. This is an easy computation that can be carried out for $p = 2$ by duality and then extended to every $1 < p < \infty$ by Rubio de Francia's extrapolation [23]. This strong-type (p,p) estimate for sparse operators became of great interest when it was shown [16, 19, 20] that it provided a new (and much easier) proof of the celebrated A_2 conjecture [8] for Calderón-Zygmund operators.

Also in the range $1 < p < \infty$, we have the following restricted weak-type result. The proof for sparse operators follows by duality using the same ideas as in [4, Theorem 4.1]), with the obvious modifications. We include the proof for the sake of completeness.

Corollary 3.2. *Let $\alpha > \frac{n+1}{2}$. Then, for every measurable set E, every $1 < p < \infty$ and $w \in A_p^{\mathcal{R}}$,*

$$\|G_\alpha \chi_E\|_{L^{p,\infty}(w)} \lesssim \|w\|_{A_p^{\mathcal{R}}}^{p+1} w(E)^{1/p}. \tag{3.3}$$

Proof. Let us consider the λ-sparse operator

$$Sf(x) = \sum_{Q \in \mathcal{S}} \left(\frac{1}{|Q|} \int_Q |f| \right) \chi_Q(x);$$

that is, for every $Q \in \mathcal{S}$, there exists a measurable subset $F_Q \subseteq Q$ such that $|F_Q| \geq (1-\lambda)|Q|$ and $\{F_Q\}_{Q \in \mathcal{S}}$ are pairwise disjoint. By duality, let us take $h \geq 0$ such that $\|h\|_{L^{p',1}(w)} = 1$. Then, using Theorem 1.1, we have to prove that, for every measurable set E,

$$\int_{\mathbb{R}^n} S\chi_E(x) h(x) w(x)\, dx \lesssim \|w\|_{A_p^{\mathcal{R}}}^{p+1} w(E)^{1/p}.$$

Let $c > 0$ such that, for every $y \in Q$, there exists another cube Q_y centered at y such that $Q \subset Q_y \subset cQ$. Then, since $|Q|/|F_Q| \leq \frac{1}{1-\lambda}$, it holds that $|cQ|/|F_Q| \leq \frac{c^n}{1-\lambda}$ and thus

$w(cQ)/w(F_Q) \lesssim \|w\|^p_{A_p^{\mathcal{R}}}$. Hence, for every $y \in Q$,

$$\begin{aligned}\frac{|E\cap Q|}{|Q|}\int_Q h(x)w(x)\,dx &\lesssim \|w\|^p_{A_p^{\mathcal{R}}}\frac{|E\cap Q|}{|Q|}\left(\frac{1}{w(cQ)}\int_Q h(x)w(x)\,dx\right)w(F_Q)\\ &\lesssim \|w\|^p_{A_p^{\mathcal{R}}}\frac{|E\cap Q|}{|Q|}\left(\frac{1}{w(Q_y)}\int_{Q_y} h(x)w(x)\,dx\right)w(F_Q)\\ &\lesssim \|w\|^p_{A_p^{\mathcal{R}}} M\chi_E(y)M_w^c h(y)\,w(F_Q),\end{aligned}$$

where

$$M_w^c h(z) = \sup_{Q_z}\frac{1}{w(Q_z)}\int_{Q_z}|h(x)|w(x)\,dx,$$

being Q_z cubes centered at z. Consequently,

$$\begin{aligned}\frac{|E\cap Q|}{|Q|}\int_Q h(x)w(x)\,dx &\lesssim \|w\|^p_{A_p^{\mathcal{R}}}\inf_{y\in Q}\big(M\chi_E(y)M_w^c h(y)\big)\,w(F_Q)\\ &\lesssim \|w\|^p_{A_p^{\mathcal{R}}}\int_{F_Q} M\chi_E(x)M_w^c h(x)w(x)\,dx.\end{aligned}$$

Summing in $Q\in\mathcal{S}$ and using that $\{F_Q\}_Q$ are pairwise disjoint and (3.2), we obtain that

$$\begin{aligned}\int_{\mathbb{R}^n} S\chi_E(x)h(x)w(x)\,dx &\lesssim \|w\|^p_{A_p^{\mathcal{R}}}\int_{\mathbb{R}^n} M\chi_E(x)M_w^c h(x)w(x)\,dx\\ \lesssim \|w\|^p_{A_p^{\mathcal{R}}}\|M\chi_E\|_{L^{p',\infty}(w)}\|M_w^c h\|_{L^{p,1}(w)} &\lesssim \|w\|^{p+1}_{A_p^{\mathcal{R}}} w(E)^{1/p},\end{aligned}$$

as we wanted to prove. □

The important property of inequality (3.3) is that, unlike the one in Corollary 3.1, it can be extrapolated down to $p=1$ by means of the techniques presented in [4], and deduce the weak-type (1,1) of G_α for A_1 weights as a consequence. We also want to point out that this estimate takes place between Banach spaces and this might be useful for applications. See [3] for examples that illustrate how we can take advantage of this fact when dealing with operators that can be written as averages.

In Corollary 3.4 below, we will see another approach to establish the weak-type (1,1) of G_α for A_1 weights with better constants than with the previous extrapolation argument. It will come as a consequence of the endpoint result in Corollary 3.3, which was proved for sparse operators in [7, Theorem 1.6]. Its original purpose was to obtain a borderline variant of the Muckenhoupt-Wheeden inequality for Calderón-Zygmund operators. In our case, it yields the following weak-type (1,1) estimate for G_α with respect to general weights on $\mathbb{R}^n$:

Corollary 3.3. *Suppose that φ is a Young function satisfying*

$$C_\varphi = \sum_{k=1}^{\infty}\frac{1}{\psi^{-1}(2^{2^k})}<\infty,$$

where ψ is its complementary function as in (3.1)*. Then, for every weight w on $\mathbb{R}^n$ and every $\alpha>\frac{n+1}{2}$,*

$$\|G_\alpha f\|_{L^{1,\infty}(w)} \lesssim C_\varphi\|f\|_{L^1(M_{\varphi(L)}w)}.$$

For instance, if we denote

$$\log_1(x) := 1 + \log_+(x) \quad \text{and} \quad \log_k(x) := \log_1 \log_{k-1}(x), \text{ for } k > 1,$$

one can take $\varphi(t) = t \log_2 t (\log_3 t)^\delta$ for some $1 < \delta < 2$ and check that

$$C_\varphi \lesssim \frac{1}{\delta - 1}.$$

Also, we have the following particular case if we assume that the weight is in A_1:

Corollary 3.4. *For every weight $u \in A_1$ and every $\alpha > \frac{n+1}{2}$,*

$$\|G_\alpha f\|_{L^{1,\infty}(u)} \lesssim (\|u\|_{A_1} \log_1 \|u\|_{A_\infty}) \|f\|_{L^1(u)}.$$

This can be proved by taking $\varphi(t) = t^r$ for some $r > 1$, checking that $C_\varphi \approx \log_1 r'$ and then choosing a suitable r by means of the sharp Reverse Hölder property for A_1 weights proved in [9]. See also [10] for a different proof of this A_1 weighted estimate for sparse operators.

3.2. **Application to maximal radial multipliers.** For $\alpha > 0$, whenever we write $\left(\frac{d}{dt}\right)^\alpha$ we will be referring to the derivative defined by

$$\widehat{\left(\frac{d}{dt}\right)^\alpha h}(\xi) = (-2\pi i \xi)^\alpha \widehat{h}(\xi),$$

in the distributional sense if needed. In [1, Theorem 4], the author proves the following pointwise estimate for maximal radial Fourier multipliers (see also [6] for related results in the quasiradial setting):

Theorem 3.5. *For $\alpha > \frac{1}{2}$, let $m : [0, \infty) \to \infty$ be a bounded function satisfying*

$$\int_0^\infty \left| s^{\alpha+1} \left(\frac{d}{ds}\right)^\alpha \frac{m(s)}{s} \right|^2 \frac{ds}{s} < \infty. \tag{3.4}$$

Then,

$$T_m^* f(x) \lesssim G_\alpha f(x), \quad \text{a.e. } x \in \mathbb{R}^n,$$

where $T_m^* f(x) = \sup_{t>0} |T_m^t f(x)|$ is the maximal operator associated with the family of multipliers $\{T_m^t\}_{t>0}$ defined by

$$\widehat{T_m^t f}(\xi) = m(t|\xi|) \widehat{f}(\xi).$$

Using Theorem 1.1 and its consequences, we deduce the following:

Corollary 3.6. *Let $\alpha > \frac{n+1}{2}$ and $m : [0, \infty) \to \infty$ be a bounded function satisfying* (3.4). *Then T_m^* can be controlled by a finite sum of sparse operators. In particular, for every $1 < p < \infty$ and $w \in A_p$,*

$$\|T_m^* f\|_{L^p(w)} \lesssim \|w\|_{A_p}^{\max\left\{1, \frac{1}{p-1}\right\}} \|f\|_{L^p(w)},$$

and for every weight $u \in A_1$,

$$\|T_m^* f\|_{L^{1,\infty}(u)} \lesssim (\|u\|_{A_1} \log_1 \|u\|_{A_\infty}) \|f\|_{L^1(u)}.$$

Weighted estimates for multipliers have also been studied in [14, 15], although the authors deal with general multipliers satisfying a Hörmander type condition. The class of multipliers that satisfy our assumption (3.4) can be related to the Bessel potential spaces introduced in [25, Chapter V]. See [1, Section III] for more details on this relation and particular examples of this class.

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