# ENDPOINT ESTIMATES VIA EXTRAPOLATION THEORY

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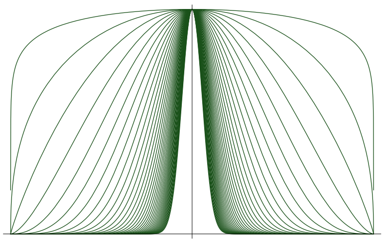

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Certifico que la present memòria ha estat
realitzada per en Carlos Domingo Salazar,
en el Departament de Matemàtica
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Barcelona, febrer del 2016
M. J. Carro Rossell

*Als avis i la yaya*

# Agraïments

*"All we have to decide is what to do*
*with the time that is given to us"*

Sembla ser que la tradició marca que els agraïments es comencin donant les gràcies al director de la tesi. En el meu cas, us puc assegurar que la tradició no hi juga cap paper, perquè independentment del que digui, no podria ser de cap altra manera. María Jesús, el apoyo que he recibido durante estos años por tu parte se merecería páginas y páginas. Eso no podrá ser, pero sí que te puedo dedicar un pequeño trocito. Podríamos decir que nuestro primer contacto fue hace casi diez años, cuando empecé la carrera. Tú eras profesora de laboratorio de Análisis I, y en una de estas pruebas de evaluación continuada, me pusiste la peor nota de todos mis años en la universidad. Probablemente (y afortunadamente) no te acuerdes de esto, pero esa primera vez que entré en tu despacho y me explicaste el desastre que había hecho con la función definida a trozos, lo tengo bien grabado! El resto de encuentros de licenciatura ya fueron menos embarazosos, pero no fue hasta que me adoptaste para hacer el trabajo final de máster sobre una transformada de Hilbert "algo más complicada" que no nos empezamos a conocer. Un año después, más unos cuantos meses de guerra con becas y ministerios, me convertí en tu alumno de tesis. Matemáticamente, has sido toda una guía, escogiendo temas, proponiendo caminos, siendo estricta pero sabiendo dar empujoncitos en los momentos oportunos para que todo tirase adelante. Todavía conservo aquel *pdf* que se titulaba *Helping Carlos*. Y a nivel más personal, que voy a decir... ha sido otro placer! Muchas gracias por todos los consejos, la paciencia que has tenido y por ser tan comprensiva, especialmente cuando accediste a que trabajase un tiempo a distancia sin pensártelo dos veces. Es cierto que luego se convirtió en una estancia oficial de lo más fructífera, pero en el momento en que subí a hablar contigo y te lo propuse, eso no lo sabíamos, y aun así, no dudaste en aceptar. Significó mucho para mi.

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Barcelona, febrer del 2016
Carlos Domingo

# Introducció en Català

A l'Anàlisi Harmònica, la pregunta de si un operador està acotat a $L^p$ sorgeix de manera natural en molts problemes. Definim els espais $L^p$ respecte d'una mesura positiva i absolutament contínua $w(x)dx$ (que anomenem pes), com el conjunt de funcions mesurables $f$ tals que

$$\|f\|_{L^p(w)} = \left( \int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right)^{1/p} < \infty.$$

Treballarem en el rang $p > 1$, i el cas $p = 1$ és el que anomenem l'*extrem*. L'acotació a $L^1$ no s'espera que sigui anàloga als casos $p > 1$, i per a provar-la, s'acostuma a fer servir tècniques específiques. Prenem, per exemple, l'operador maximal de Hardy-Littlewood

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| dy, \tag{1}$$

on el suprem es pren sobre cubs $Q \subseteq \mathbb{R}^n$ que contenen el punt $x$. Fins i tot en el cas més senzill, quan $w = 1$, sabem que, per a tot $p > 1$,

$$M : L^p \longrightarrow L^p,$$

però que, en canvi, això ja no és cert si $p = 1$. De fet, l'única funció $f \in L^1$ per a la qual $Mf$ pertany a $L^1$ és $f = 0$. Si volem que $M$ estigui acotat d'$L^1$ en algun altre espai, hem d'introduir l'anomenat espai $L^1$-dèbil, que es denota per $L^{1,\infty}$. Per a un $p \geqslant 1$ qualsevol, definim $L^{p,\infty}(w)$ com el conjunt de funcions $f$ tals que

$$\|f\|_{L^{p,\infty}(w)} = \sup_{t>0} t\, w(\{x \in \mathbb{R}^n : |f(x)| > t\})^{1/p} < \infty.$$

La desigualtat de Chebyshev ens dóna automàticament que $L^{p,\infty}$ és més gran que $L^p$, i ara sí, es pot provar que

$$M : L^1 \longrightarrow L^{1,\infty}.$$

Per a l'operador de Hardy-Littlewood, les acotacions amb pesos $M : L^p(w) \to L^p(w)$ per a $p > 1$ i $L^1(w) \to L^{1,\infty}(w)$ han estat totalment caracteritzades des del 1972, quan B. Muckenhoupt [94] va introduir les classes de pesos $A_p$ amb aquestes propietats per a $p > 1$ i $p = 1$ respectivament. No obstant això, hi ha altres operadors per als quals l'estimació a l'extrem ha resultat ser molt més difícil que la resta dels casos. Prenem, per exemple, la funció $g_2^*$ definida per

$$g_2^* f(x) = \left( \int_{\mathbb{R}_+^{n+1}} \frac{t^{n+1}}{(t + |x-y|)^{2n}} |\nabla u(y,t)|^2 dydt \right)^{1/2},$$

on $u$ és l'extensió harmònica d'$f$ al semiespai superior $\mathbb{R}_+^{n+1}$ i $\nabla u$ és el seu vector gradient (vegeu les Definicions 4.1 i 4.20). Aquest operador apareixerà al Capítol 4 i juga un paper important en problemes relacionats amb multiplicadors i espais de Sobolev (vegeu el llibre de referència d'E. Stein [112]). Al 1974, B. Muckenhoupt i R. Wheeden [97] van provar que, per a tot $p > 1$ i tot $w \in A_p$,

$$g_2^* : L^p(w) \longrightarrow L^p(w).$$

En canvi, pel que sabem, l'estimació a l'extrem $g_2^* : L^1 \to L^{1,\infty}$ continua oberta, fins i tot en un context sense pesos. Una de les majors diferències entre l'extrem i la resta dels casos rau precisament en l'espai $L^{1,\infty}$ en si. Al contrari d'$L^1$, $L^p$ o fins i tot $L^{p,\infty}$ amb $p > 1$, l'espai $L^{1,\infty}$ no es pot normar per a esdevenir espai de Banach. Totes aquestes singularitats de l'extrem són el motiu pel qual una teoria d'extrapolació és de gran interès per a moltes aplicacions. En termes generals, el nostre objectiu és obtenir informació a $p = 1$ (o en algun espai proper a $L^1$) només partint d'hipòtesis a $p > 1$. Per això, estudiarem dues teories d'extrapolació, una de Rubio de Francia i l'altra de Yano.

## Sobre l'extrapolació de Rubio de Francia

La primera d'aquestes teories es remunta a l'any 1984, i és deguda a J. L. Rubio de Francia [102]. Suposa acotació per a un únic $p_0$ però respecte tota una classe de pesos (l'anteriorment citada classe $A_{p_0}$) que ens permet treure conclusions per a tot $1 < p < \infty$. La definició d'aquestes classes $A_p$ no és important en aquest moment, però es pot trobar a la Secció 1.1. El resultat original de Rubio de Francia diu així[1]:

**Teorema 1.1** (Rubio de Francia, [102])**.** *Donat un operador sublineal $T$, si per a un cert $1 \leqslant p_0 < \infty$ i per a tot $w \in A_{p_0}$,*

$$T : L^{p_0}(w) \longrightarrow L^{p_0}(w)$$

*està acotat, aleshores, per a tot $1 < p < \infty$ i tot $w \in A_p$,*

$$T : L^p(w) \longrightarrow L^p(w)$$

*també està acotat.*

[1]La numeració dels teoremes dins d'aquesta introducció coincidirà amb la del text. Tot i així, per motius de claredat en la presentació, els enunciats poden variar una mica.

La primera cosa que hem de remarcar és que el cas $p = 1$ no es pot assolir en general, ni tan sols si només volem que $T$ porti $L^1$ en $L^{1,\infty}$ sense pesos (prenem, per exemple, $M^2 = M \circ M$ com a contraexemple). Val a dir, però, que el propòsit original d'aquest resultat era deduir estimacions per a $1 < p < \infty$ només a partir de desigualtats a $L^2$. Tot i això, avenços recents duts a terme per M. J. Carro, L. Grafakos i J. Soria [28] han provat que si canviem la classe de pesos a les hipòtesis, hi ha una manera d'assolir l'extrem. Aquest nou resultat es troba enunciat al Teorema 1.7 d'una forma més general, però la part més interessant de cara a aquesta introducció és la següent:

**Teorema 1.7** (Carro - Grafakos - Soria, [28]). *Donat un operador $T$, si per a un cert $1 < p_0 < \infty$ i tot $w \in \widehat{A}_{p_0}$, tenim*

$$\|T\chi_E\|_{L^{p_0,\infty}(w)} \leqslant C\|\chi_E\|_{L^{p_0}(w)}, \quad E \subseteq \mathbb{R}^n,$$

*aleshores, per a tot $u \in A_1$,*

$$\|T\chi_E\|_{L^{1,\infty}(u)} \leqslant C\|\chi_E\|_{L^1(u)}, \quad E \subseteq \mathbb{R}^n. \tag{2}$$

La notació $\chi_E$ representa la funció característica del conjunt $E$, i la classe de pesos d'aquest resultat es defineix com

$$\widehat{A}_p = \{(Mh)^{1-p}u : h \in L^1_{\mathrm{loc}},\, u \in A_1\},$$

on $M$ és l'operador maximal de Hardy-Littlewood de (1). La classe $\widehat{A}_p$ està íntimament relacionada amb la classe $A_p$ del Teorema 1.1 per les inclusions

$$A_p \subseteq \widehat{A}_p \subseteq A_{p+\varepsilon},$$

per a tot $1 \leqslant p < \infty$ i tot $\varepsilon > 0$. Malgrat que l'estimació a l'extrem (2) que s'aconsegueix només es pot tenir (en general) sobre funcions característiques, a la Secció 1.4 recordem que, per a una àmplia classe d'operadors, això és equivalent a l'acotació

$$T : L^1(u) \longrightarrow L^{1,\infty}(u).$$

El nostre primer objectiu serà debilitar les hipòtesis del Teorema 1.7 tant com sigui possible sense perdre informació a l'extrem $p = 1$. L'avantatge d'una extrapolació d'aquest tipus, que serà el pilar central d'aquesta tesi, és doble. D'una banda, quan s'aplica a un operador $T$, ens dóna una demostració de la seva acotació d'$L^1$ a $L^{1,\infty}$, i de l'altra, constitueix una estimació a un cert nivell $p_0 > 1$ on els espais involucrats són de Banach.

Passem a explicar els resultats principals que hem obtingut en relació a aquesta teoria i com estan organitzats a la tesi. Tractarem de donar les idees principals tot evitant detalls tècnics, pel que si el lector troba que necessita més detalls sobre algun concepte, l'índex al final hauria de resultar útil per a localitzar la seva definició dins del text.

▼ Al **Capítol 1** proporcionem totes les eines d'extrapolació que es necessitaran. Després de presentar en més detall la teoria clàssica de Rubio de Francia i la seva variant més nova de [28], a la Secció 1.3 millorem la segona d'aquestes *ad hoc* per a obtenir estimacions a l'extrem. El resultat principal d'aquest capítol es pot enunciar de la següent manera:

**Teorema 1.11.** *Sigui $T$ un operador, $E \subseteq \mathbb{R}^n$ un conjunt mesurable i $u \in A_1$. Si hi ha un cert $1 < p_0 < \infty$ tal que*

$$\|T\chi_E\|_{L^{p_0,\infty}((M\chi_E)^{1-p_0}u)} \leqslant C\|\chi_E\|_{L^{p_0}((M\chi_E)^{1-p_0}u)},$$

*aleshores*

$$\|T\chi_E\|_{L^{1,\infty}(u)} \leqslant C\|\chi_E\|_{L^1(u)}.$$

Si el comparem amb el Teorema 1.7, observem el següent:

- Donat que els pesos $A_{p_0}$ es defineixen com aquells de la forma $(Mh)^{1-p_0}u$, amb $h \in L^1_{\text{loc}}$ i $u \in A_1$, la primera simplificació que veiem al Teorema 1.11 respecte del Teorema 1.7 és que no ens cal provar l'acotació per a tot pes d'$\widehat{A}_{p_0}$. N'hi ha prou amb provar l'estimació quan $h$ és exactament la funció característica $\chi_E$ a la que estem aplicant $T$.

- La segona simplificació és que no necessitem un $1 < p_0 < \infty$ universal. Per a cada pes $u \in A_1$ podem trobar un valor diferent de $p_0 > 1$. Això serà essencial per als nostres objectius.

▼ Al **Capítol 2** presentem la primera aplicació del Teorema 1.11. L'operador que estudiarem és el de Bochner-Riesz a l'índex crític. Aquest operador es pot definir com a multiplicador de Fourier a $\mathbb{R}^n$ de la següent manera:

$$\widehat{Bf}(\xi) = (1 - |\xi|^2)_+^{\frac{n-1}{2}} \widehat{f}(\xi), \tag{3}$$

on $a_+ = \max\{a, 0\}$ és la part positiva d'$a \in \mathbb{R}$, i $\widehat{f}$ denota la transformada de Fourier d'$f$. El resultat que presentarem per a $B$ és el Teorema 2.9, i bàsicament afirma que l'operador de Bochner-Riesz es troba exactament sota les hipòtesis de l'extrapolació del Teorema 1.11:

**Teorema 2.9.** *Per a tot $u \in A_1$, existeix $1 < p_0 < \infty$ tal que, per a cada conjunt mesurable $E \subseteq \mathbb{R}^n$,*

$$\|B\chi_E\|_{L^{p_0,\infty}(w)} \leqslant C\|\chi_E\|_{L^{p_0}(w)}, \tag{4}$$

*on $w = (M\chi_E)^{1-p_0}u$.*

Sobre aquest resultat, hem de destacar que:

- L'operador $B$ es troba a la classe d'operadors que es descriuen a la Secció 1.4, i per tant, l'extrapolació del Teorema 1.11 de fet implica que $B$ està acotat d'$L^1(u)$ en $L^{1,\infty}(u)$ per a tot $u \in A_1$.

- La desigualtat $L^1 \to L^{1,\infty}$ ja havia estat establerta per M. Christ [35] en el cas sense pesos i per A. Vargas [124] per a pesos d'$A_1$. Tot i així, l'estimació d'extrapolació (4) que provem per a $B$, no només és més forta que la de $L^1 \to L^{1,\infty}$, sinó que també té l'avantatge que té lloc entre espais de Banach. Aquest fet el farem servir en el proper capítol.

▼ Tal i com acabem d'anticipar, al **Capítol 3** presentem algunes aplicacions del Teorema 2.9. El resultat principal tracta de multiplicadors radials i, ometent alguns detalls, es pot resumir així:

**Teorema 3.10.** *Fixem $n \geqslant 2$ i $\alpha = \frac{n+1}{2}$. Sigui $m$ una funció acotada a $(0,\infty)$ tal que, per a una definició de derivada fraccionària $D^\alpha$ adient,*

$$t^{\alpha-1} D^\alpha m(t) \in L^1(0,\infty).$$

*Aleshores, el multiplicador de Fourier radial $T_m$ definit com*

$$\widehat{T_m f}(\xi) = m(|\xi|^2)\widehat{f}(\xi)$$

*està acotat d'$L^1(u)$ a $L^{1,\infty}(u)$, per a tot pes $u \in A_1$.*

A continuació, expliquem la tècnica que fem servir per a provar aquest resultat, ja que il·lustra un dels principals avantatges de l'estimació d'extrapolació del Teorema 1.11. Aquests són els passos:

- Escrivim $T_m$ com a mitjana d'operadors que es comporten com el multiplicador de Bochner-Riesz. Més concretament,

$$T_m \chi_E(x) = \int_0^\infty B^s \chi_E(x) \Phi(s) ds, \quad \text{amb } \Phi \in L^1(0,\infty),$$

  on els operadors $\{B^s\}_{s>0}$ satisfan la mateixa estimació que $B$ al Teorema 2.9, uniformement en $s > 0$.

- Fem servir la desigualtat integral de Minkowski per a l'espai de Banach $L^{p_0,\infty}(w)$ per transferir l'estimació d'extrapolació (4) de $B^s$ a $T_m$, tot deduint que

$$\|T_m \chi_E\|_{L^{p_0,\infty}(w)} \leqslant C \|\Phi\|_{L^1(0,\infty)} \|\chi_E\|_{L^{p_0}(w)}.$$

- Finalment, extrapolem $T_m$ fins a $p = 1$ pel Teorema 1.11 i completem la demostració.

Cal fer notar que la conclusió per a $T_m$ no es pot deduir només d'una estimació $L^1 \to L^{1,\infty}$ per a la família $\{B^s\}_{s>0}$, donat que el rang és un espai quasi-Banach. Per a concloure el capítol, a la Secció 3.4, estudiem multiplicadors generals de tipus Hörmander a $\mathbb{R}^n$. En aquest cas, no fem servir la tècnica de les mitjanes que acabem d'explicar, sinó que ataquem el problema directament. El resultat que obtenim per a aquests operadors es pot enunciar de la següent manera:

**Teorema 3.26.** *Fixem $1 < s \leqslant 2$ i prenem $m : \mathbb{R}^n \to \mathbb{R}$ una funció acotada de classe $\mathcal{C}^n(\mathbb{R}^n \backslash \{0\})$ tal que*

$$\sup_{r>0} \left( r^{2|\alpha|-n} \int_{r \leqslant |x| \leqslant 2r} \left| \left( \frac{\partial}{\partial x} \right)^\alpha m(x) \right|^s dx \right)^{1/s} < \infty, \quad |\alpha| \leqslant n.$$

*Aleshores, el multiplicador associat $\widehat{T_m f}(\xi) = m(\xi)\widehat{f}(\xi)$ satisfà que, per a tot $u \in A_1$, existeix $1 < p_0 < \infty$ tal que*

$$\|T_m \chi_E\|_{L^{p_0,\infty}(w)} \leqslant C \|\chi_E\|_{L^{p_0}(w)}, \quad E \subseteq \mathbb{R}^n,$$

*on $w = (M\chi_E)^{1-p_0} u$.*

Les principals contribucions presentades en aquests tres primers capítols es troben recollides a [24], i han estat enviades a publicació.

▼ Al **Capítol 4** estudiem els diferents ingredients d'una teoria de Littlewood-Paley adaptada als pesos $\widehat{A}_p$. Aquesta teoria va ser iniciada als anys trenta per Littlewood i Paley en un seguit d'articles [89, 90, 91] sobre sèries de Fourier i potències, però des d'aleshores, les seves idees han resultat ser molt útils quan es treballa amb multiplicadors de Fourier $T_m$. Més concretament, en el nostre cas estarem interessats en dos tipus de desigualtats, que anomenarem estimacions inferiors i superiors, respectivament:

(*a*) $\|f\|_{L^{p,\infty}(w)} \leqslant C \|G_1 f\|_{L^{p,\infty}(w)}$,

(*b*) $\|G_2 \chi_E\|_{L^{p,\infty}(w)} \leqslant \|\chi_E\|_{L^p(w)}$.

Considerarem pesos $w \in \widehat{A}_p$, i estudiarem diferents operadors $G_1$ i $G_2$, anomenats funcions quadrat, que ja apareixien a la teoria clàssica. Provar estimacions inferiors i superiors per a funcions quadrat és interessant de per si, però a més a més, si es combinessin amb una relació del tipus

(*c*) $\|G_1(T\chi_E)\|_{L^{p,\infty}(w)} \leqslant C \|G_2 \chi_E\|_{L^{p,\infty}(w)}$,

per a un cert operador $T$, donarien una estimació en la línia del Teorema 1.11. Nosaltres estudiarem diverses funcions quadrat. Per exemple, a la Secció 4.2 obtenim la desigualtat ($a$) per a la funció d'àrea clàssica de Lusin

$$Sf(x) = \left( \int_{|x-y|<t} |\nabla u(y,t)|^2 \frac{dydt}{t^{n-1}} \right)^{1/2},$$

on, com abans, $u$ és l'extensió harmònica d'$f$ al semiespai superior $\mathbb{R}^{n+1}_+$. Un altre exemple es pot trobar a la Secció 4.3, on fem servir la recent tècnica presentada a [88] per A. Lerner i F. Nazarov sobre majoració per operadors *sparse* per obtenir la desigualtat ($b$) per a la funció quadrat

$$G_\alpha f(x) = \left( \int_0^\infty \left| \frac{\partial}{\partial t} B^t_\alpha f(x) \right|^2 t dt \right)^{1/2}.$$

Aquí

$$\widehat{B^t_\alpha f}(\xi) = (1 - |t\xi|^2)^\alpha_+ \widehat{f}(\xi)$$

no és res més que una generalització de l'operador de Bochner-Riesz $B$ definit a (3).

# Sobre l'extrapolació de Yano

La segona teoria d'extrapolació que estudiarem és deguda a S. Yano [127], i està relacionada més aviat amb l'Anàlisi Real. En aquest cas, suposem una certa acotació $L^p$ per a $p > 1$, respecte d'una mesura fixada i amb un cert control sobre les normes $C_p$ de l'operador quan $p$ s'apropa a $1^+$. A partir d'aquí, deduïm que l'operador està acotat en un cert espai que és més a prop d'$L^1$ que qualsevol altre espai $L^p$. Aquest és el resultat original de S. Yano del 1951:

**Teorema 5.6.** *Fixem espais de mesura finita $(X, \mu)$, $(Y, \nu)$, $p_0 > 1$ i $m > 0$. Si $T$ és un operador sublineal tal que, per a tot $1 < p \leqslant p_0$,*

$$T : L^p(\mu) \longrightarrow L^p(\nu)$$

*està acotat amb norma més petita o igual a $\frac{C}{(p-1)^m}$, aleshores,*

$$T : L(\log L)^m(\mu) \longrightarrow L^1(\nu)$$

*també està acotat.*

L'espai $L(\log L)^m(\mu) \subseteq L^1(\mu)$ és el conjunt de funcions $\mu$-mesurables tals que

$$\int_X |f(x)|(1 + \log_+ |f(x)|)^m d\mu(x) < \infty.$$

Aquest resultat s'ha millorat i estès posteriorment a altres tipus d'acotació. Un dels resultats més recents és degut a M. J. Carro i P. Tradacete [33] i tracta amb operadors

$$T : L^{p,\infty}(\mu) \longrightarrow L^{p,\infty}(\nu),$$

amb norma que es comporta com $\frac{1}{(p-1)^m}$ quan $p$ s'apropa a 1.

▼ Al **Capítol 5** presentem alguns resultats sobre la teoria de Yano motivats per la seva relació amb l'extrapolació de Rubio de Francia presentada al Capítol 1. Recordem que un operador $T$ sota les hipòtesis del Teorema 1.1 no necessàriament està acotat d'$L^1$ a $L^{1,\infty}$. Malgrat això, les normes $L^p \to L^p$ òptimes trobades a [48] (vegeu també [50]) ens permeten fer servir l'extrapolació de Yano per a obtenir estimacions a prop d'$L^1$. Més concretament, sabem que si, per a un cert $1 < p_0 < \infty$, un cert $\beta > 0$, i tot $w \in A_{p_0}$, $T$ és un operador sublineal tal que

$$T : L^{p_0}(w) \longrightarrow L^{p_0}(w)$$

està acotat amb norma $C_{p_0}\|w\|_{A_{p_0}}^{\beta}$, aleshores

$$T : L^p(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n)$$

està acotat per a tot $1 < p < p_0$ amb norma essencialment controlada per

$$\frac{1}{(p-1)^{\beta(p_0-1)}}, \qquad \text{quan } p \to 1^+.$$

Amb això, l'extrapolació de Yano assegura que $T$ està acotat a $L(\log L)^{\beta(p_0-1)}(\mathbb{R}^n)$, tal i com enunciem al Teorema 5.22. La conclusió només és vàlida per a la mesura de Lebesgue, ja que, en cas contrari, veurem que la norma $L^p \to L^p$ explota massa ràpidament. Tot i així, al Teorema 5.23, aconseguim treure conclusions a prop d'$L^1(u)$ per a tot $u \in A_1$ mitjançant un argument d'extrapolació diferent. Aquesta idea de buscar una bona forma de relacionar les teories d'extrapolació de Rubio de Francia i Yano per tal d'obtenir estimacions a l'extrem amb pesos ha estat recollida i desenvolupada més enllà del contingut d'aquesta tesi a [25].

L'altre escenari on podem aplicar l'extrapolació de Yano prové de la teoria de pesos $\widehat{A}_p$. Recordem que a [28] els autors proven el Teorema 1.7, i l'estimació a l'extrem que en dedueixen és

$$\|T\chi_E\|_{L^{1,\infty}(u)} \leqslant C\|\chi_E\|_{L^1(u)}, \quad u \in A_1. \tag{5}$$

Tot i així, quan $T$ és sublineal, es poden dir més coses. També demostren que en aquest cas, malgrat que no podem esperar tenir $T : L^1(u) \to L^{1,\infty}(u)$ en general, el que sí que tenim és la següent acotació, que tampoc està restringida a funcions característiques:

**Teorema 1.7** (Carro - Grafakos - Soria, [28]). *Sigui $T$ un operador sublineal tal que, per a un cert $1 < p_0 < \infty$ i tot $w \in \widehat{A}_{p_0}$, tenim*

$$\|T\chi_E\|_{L^{p_0,\infty}(w)} \leqslant C\|\chi_E\|_{L^{p_0}(w)}, \quad E \subseteq \mathbb{R}^n. \tag{6}$$

*Aleshores, per a tot $u \in A_1$, a més de (5), es compleix que*

$$T : L(\log L)^{\varepsilon}(u) \longrightarrow L^{1,\infty}_{\mathrm{loc}}(u), \quad \varepsilon > 0. \tag{7}$$

Òbviament, aquesta acotació és interessant quan l'operador $T$ no es troba a la classe d'operadors pels que (5) implica acotació d'$L^1(u)$ a $L^{1,\infty}(u)$, ja que

$$L(\log L)^{\varepsilon}(u) \subsetneq L^1(u).$$

Un altre objectiu del Capítol 5 és millorar aquesta estimació a l'extrem (7) de tipus logarítmic mitjançant la teoria d'extrapolació introduïda a [33]. Primer, ens cal calcular la norma $L^{p,\infty} \to L^{p,\infty}$ d'aquests operadors. Això es troba al següent resultat:

**Teorema 5.5.** *Sigui $T$ un operador sublineal tal que, per a un cert $1 < p_0 < \infty$ i tot $w \in \widehat{A}_{p_0}$, tenim*

$$\|T\chi_E\|_{L^{p_0,\infty}(w)} \leqslant C\|\chi_E\|_{L^{p_0}(w)}, \quad E \subseteq \mathbb{R}^n.$$

*Aleshores, per a cada pes $u \in A_1$ fixat i cada $1 < p < p_0$, es compleix que*

$$T : L^{p,\infty}(u) \longrightarrow L^{p,\infty}(u)$$

*està acotat amb norma essencialment controlada per*

$$\log\left(\frac{1}{p-1}\right)\frac{1}{p-1}, \qquad \text{quan } p \to 1^+. \tag{8}$$

Un cop tenim aquest càlcul, estenem el resultat de [33] de tal manera que admeti constants amb termes logarítmics com a (8). Amb això, al Corol·lari 5.25 som capaços de provar que un operador que satisfà (6) està acotat en un cert espai $X(u)$ tal que, per a tot $\varepsilon > 0$,

$$L(\log L)^{\varepsilon}(u) \subsetneq X(u).$$

Això ja millora l'estimació a l'extrem (7) de [33], però també ens adonem que si fem servir més informació sobre $T$ (bàsicament, que satisfà (5) sobre funcions característiques), podem obtenir una auto-millora d'aquest resultat i deduir-ne el següent:

**Corol·lari 5.29.** *Sigui $T$ un operador sublineal tal que, per a un cert $1 < p_0 < \infty$ i tot $w \in \widehat{A}_{p_0}$, tenim*

$$\|T\chi_E\|_{L^{p_0,\infty}(w)} \leqslant C\|\chi_E\|_{L^{p_0}(w)}, \quad E \subseteq \mathbb{R}^n.$$

*Aleshores, per a tot $u \in A_1$, es compleix que*

$$T : L\log\log L(u) \longrightarrow L^{1,\infty}_{\mathrm{loc}}(u).$$

Actualment, aquest és el millor resultat a l'extrem (no restringit a funcions característiques) per a operadors sublineals que satisfan les hipòtesis del Teorema 1.7, ja que

$$L(\log L)^{\varepsilon}(u) \subsetneq X(u) \subsetneq L\log\log L(u).$$

A part d'aquests resultats relacionats amb el Capítol 1, al Capítol 5 també presentem una extensió de la teoria de Yano als espais de Lorentz $L^{p,q}$. Per a $p < q < \infty$, els espais $L^{p,q}$ són espais intermedis entre $L^p$ i $L^p$-dèbil:

$$L^p \subseteq L^{p,q} \subseteq L^{p,\infty}.$$

Els resultats d'extrapolació que obtenim tracten d'operadors que porten

$$T : L^p(\mu) \longrightarrow L^{p,q}(\nu), \quad \text{o} \quad T : L^{p,q}(\mu) \longrightarrow L^{p,q}(\nu),$$

quan $p$ és proper a 1 i $1 < q < \infty$ és fix. Això es presenta als Teoremes 5.16 i 5.19, i completa la teoria de Yano en el context d'espais de Lorentz.

▼ Finalment, al **Capítol 6**, donem un seguit de resultats que ja no estan relacionats amb la teoria de pesos $A_p$ que ha estat present durant tots els capítols. Aquí fem servir les idees de l'extrapolació de Yano adaptada a funcions decreixents per tal d'obtenir cotes puntuals per a operadors integrals de la forma

$$T_K f(x) = \int_0^\infty K(x,t) f(t) dt,$$

amb $K$ un nucli positiu. El principal resultat és el Teorema 6.5, i es pot aplicar a diversos operadors com la transformada d'Abel, l'operador de Riemann-Liouville, operadors iteratius, etc. Aquestes aplicacions es troben a la Secció 6.3. El contingut d'aquest últim capítol ha estat acceptat per a publicació a [23].

# Contents

# Introduction

In Harmonic Analysis, the question of whether an operator is bounded on $L^p$ arises naturally in many problems. We define $L^p$ spaces with respect to a positive, absolutely continuous measure $w(x)dx$ (that we call weight), as the set of measurable functions $f$ such that

$$\|f\|_{L^p(w)} = \left( \int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right)^{1/p} < \infty.$$

We will work on the range $p \geqslant 1$, and the case $p = 1$ is what we call the *endpoint*. Boundedness on $L^1$ is not normally expected to be analogous to the cases $p > 1$, and to establish it, one usually requires specific techniques. Take, for instance, the Hardy-Littlewood maximal operator

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| dy, \tag{1}$$

where the supremum is taken over cubes $Q \subseteq \mathbb{R}^n$ containing $x$. Even in the easiest case, when $w = 1$, we know that, for every $p > 1$,

$$M : L^p \longrightarrow L^p,$$

but this is no longer true when $p = 1$. In fact, the only function $f \in L^1$ for which $Mf$ belongs to $L^1$ is $f = 0$. If we want $M$ to be bounded from $L^1$ into some other space, we need to introduce the so-called weak-$L^1$ space, denoted by $L^{1,\infty}$. For general $p \geqslant 1$, we define $L^{p,\infty}(w)$ as the set of measurable functions $f$ such that

$$\|f\|_{L^{p,\infty}(w)} = \sup_{t>0} t\, w(\{x \in \mathbb{R}^n : |f(x)| > t\})^{1/p} < \infty.$$

Chebyshev's inequality readily shows that $L^{p,\infty}$ is bigger than $L^p$, and it can be checked that, now,

$$M : L^1 \longrightarrow L^{1,\infty}.$$

For the Hardy-Littlewood maximal operator, the weighted estimates $M : L^p(w) \to L^p(w)$ for $p > 1$ and $L^1(w) \to L^{1,\infty}(w)$ have been completely characterized since 1972, when B. Muckenhoupt [94] introduced the classes of weights $A_p$ having these properties for

$p > 1$ and $p = 1$, respectively. However, there are other examples for which the endpoint estimate has proved to be much more difficult than the rest of the cases. Take, for instance, the $g_2^*$ function defined by

$$g_2^* f(x) = \left( \int_{\mathbb{R}_+^{n+1}} \frac{t^{n+1}}{(t + |x-y|)^{2n}} |\nabla u(y,t)|^2 dydt \right)^{1/2},$$

where $u$ is the harmonic extension of $f$ to the upper half-space $\mathbb{R}_+^{n+1}$ and $\nabla u$ is its gradient vector (see Definitions 4.1 and 4.20). This operator will appear in Chapter 4 and it plays an important role in problems related to multipliers and Sobolev spaces (see Stein's reference book [112]). In 1974, B. Muckenhoupt and R. Wheeden [97] showed that, for every $p > 1$ and every $w \in A_p$,

$$g_2^* : L^p(w) \longrightarrow L^p(w).$$

However, as far as we know, the endpoint estimate $g_2^* : L^1 \to L^{1,\infty}$ remains open, even in the unweighted setting. One of the main differences between the endpoint and the other cases stems precisely from the space $L^{1,\infty}$ itself. Unlike $L^1$, $L^p$ or even $L^{p,\infty}$ when $p > 1$, the space $L^{1,\infty}$ cannot be normed to become a Banach space. All these singularities about the endpoint are the reason why a theory of extrapolation is of great interest in many applications. Roughly speaking, our goal is to obtain information at $p = 1$ (or on some space close to $L^1$) only from assumptions at $p > 1$. To this end, we will study two different extrapolation theories, one of Rubio de Francia and the other of Yano.

## On Rubio de Francia's extrapolation

The first of these theories goes back to 1984 and is due to J. L. Rubio de Francia [102]. It assumes boundedness for a single $p_0$ but with respect to a whole class of weights (the aforementioned $A_{p_0}$ class) that allows us to draw conclusions for every $1 < p < \infty$. The definition of these $A_p$ classes is not important at the moment, but it can be found in Section 1.1. The original result by Rubio de Francia reads as follows[2]:

**Theorem 1.1** (Rubio de Francia, [102]). *Given a sublinear operator $T$, if for some fixed $1 \leqslant p_0 < \infty$ and every $w \in A_{p_0}$,*

$$T : L^{p_0}(w) \longrightarrow L^{p_0}(w)$$

*is bounded, then, for every $1 < p < \infty$ and every $w \in A_p$,*

$$T : L^p(w) \longrightarrow L^p(w)$$

*is also bounded.*

[2]Theorem numbering within this introduction will coincide with the one in the text. However, for the sake of clarity, the presentation of the results may differ.

The first thing we need to remark is that the case $p = 1$ cannot be reached in general, even if we only seek an $L^1 \to L^{1,\infty}$ boundedness without weights (take, for instance, $M^2 = M \circ M$ as a counterexample). It is fair to say, though, that the original purpose of this result was to deduce $L^p$ estimates for every $1 < p < \infty$ just from $L^2$ inequalities. However, recent developements made by M. J. Carro, L. Grafakos and J. Soria [28] have shown that if we change the class of weights in the assumption, there is a way to reach the endpoint. This new result is stated in Theorem 1.7 in a more general fashion, but the most interesting part for this introduction is the following:

**Theorem 1.7** (Carro - Grafakos - Soria, [28]). *Given an operator $T$, if for some fixed $1 < p_0 < \infty$ and every $w \in \widehat{A}_{p_0}$, it holds that*

$$\|T\chi_E\|_{L^{p_0,\infty}(w)} \leqslant C\|\chi_E\|_{L^{p_0}(w)}, \quad E \subseteq \mathbb{R}^n,$$

*then, for every $u \in A_1$,*

$$\|T\chi_E\|_{L^{1,\infty}(u)} \leqslant C\|\chi_E\|_{L^1(u)}, \quad E \subseteq \mathbb{R}^n. \tag{2}$$

The notation $\chi_E$ stands for the characteristic function of the set $E$, and the class of weights in this result is defined by

$$\widehat{A}_p = \{(Mh)^{1-p}u : h \in L^1_{\text{loc}},\ u \in A_1\},$$

where $M$ is the Hardy-Littlewood maximal operator from (1). The $\widehat{A}_p$ class is closely related to the classical $A_p$ class in Theorem 1.1 by the inclusions

$$A_p \subseteq \widehat{A}_p \subseteq A_{p+\varepsilon},$$

for every $1 \leqslant p < \infty$ and every $\varepsilon > 0$. Even though the endpoint estimate (2) that we obtain can only be expected to hold (in general) on characteristic functions, in Section 1.4 we recall that, for a large class of operators, it is equivalent to the boundedness

$$T : L^1(u) \longrightarrow L^{1,\infty}(u).$$

Our first goal will be to weaken the hypotheses in Theorem 1.7 as much as possible without losing information at the endpoint $p = 1$. The advantage of such an extrapolation, which will be the cornerstone of this thesis, is twofold. On the one hand, when applied to an operator $T$, it provides a proof of its boundedness from $L^1$ to $L^{1,\infty}$, and on the other, it constitutes an estimate at a certain level $p_0 > 1$ where the spaces involved are Banach spaces.

Let us explain the main results that we obtain related to this theory and how they are organized in this thesis. We will try to convey the main ideas avoiding technicalities, so if the reader needs further details on some notion, the index at the end should be useful to locate its definition within the text.

▼ In **Chapter 1** we provide all the extrapolation tools that will be needed. After presenting in more detail the classical theory of Rubio de Francia and its newer variant in [28], in Section 1.3 we improve the latter *ad hoc* to obtain endpoint estimates. The main result of this chapter can be stated as follows:

**Theorem 1.11.** *Let $T$ be an operator, $E \subseteq \mathbb{R}^n$ a measurable set and $u \in A_1$. If there is some $1 < p_0 < \infty$ such that*

$$\|T\chi_E\|_{L^{p_0,\infty}((M\chi_E)^{1-p_0}u)} \leqslant C\|\chi_E\|_{L^{p_0}((M\chi_E)^{1-p_0}u)},$$

*then*

$$\|T\chi_E\|_{L^{1,\infty}(u)} \leqslant C\|\chi_E\|_{L^1(u)}.$$

Comparing it to Theorem 1.7, we observe the following:

- Since $\widehat{A}_{p_0}$ weights were defined to be $(Mh)^{1-p_0}u$, with $h \in L^1_{\mathrm{loc}}$ and $u \in A_1$, the first simplification that we see in Theorem 1.11 with respect to Theorem 1.7 is that we do not have to show boundedness for every weight in $\widehat{A}_{p_0}$. It is enough to prove the estimate when $h$ is exactly the characteristic function $\chi_E$ to which we are applying $T$.
- The second simplification is that we do not need a universal $1 < p_0 < \infty$. For every weight $u \in A_1$, we can find a different value of $p_0 > 1$. This will be essential for our purposes.

▼ In **Chapter 2** we present the first application of Theorem 1.11. The operator that we will study is the Bochner-Riesz operator at the critical index. It can be defined as a Fourier multiplier on $\mathbb{R}^n$ as follows:

$$\widehat{Bf}(\xi) = (1 - |\xi|^2)_+^{\frac{n-1}{2}} \widehat{f}(\xi), \tag{3}$$

where $a_+ = \max\{a, 0\}$ is the positive part of $a \in \mathbb{R}$, and $\widehat{f}$ denotes the Fourier transform of $f$. The result that we will obtain for $B$ is Theorem 2.9, and it basically states that the Bochner-Riesz operator is exactly under the assumptions of the extrapolation in Theorem 1.11:

**Theorem 2.9.** *For every $u \in A_1$, there exists $1 < p_0 < \infty$ such that, for each measurable set $E \subseteq \mathbb{R}^n$,*

$$\|B\chi_E\|_{L^{p_0,\infty}(w)} \leqslant C\|\chi_E\|_{L^{p_0}(w)}, \tag{4}$$

*where $w = (M\chi_E)^{1-p_0}u$.*

About this result, we should emphasize that:

- The operator $B$ falls within the class of operators described in Section 1.4, and hence, the extrapolation in Theorem 1.11 actually yields that $B$ is bounded from $L^1(u)$ to $L^{1,\infty}(u)$ for every $u \in A_1$.

- The $L^1 \to L^{1,\infty}$ inequality had already been established by M. Christ [35] in the unweighted case and A. Vargas [124] for $A_1$ weights. However, the extrapolation estimate (4) that we prove for $B$, not only is stronger than the $L^1 \to L^{1,\infty}$ one, but it also has the advantage that it takes place between Banach spaces. We will use this fact in the next chapter.

▼ As we just anticipated, in **Chapter 3** we present some applications of Theorem 2.9. The main result deals with radial Fourier multipliers and, omitting some details, it can be summarized as follows:

**Theorem 3.10.** *Fix $n \geqslant 2$ and $\alpha = \frac{n+1}{2}$. Let $m$ be a bounded function on $(0,\infty)$ such that, for a suitable definition of fractional derivative $D^\alpha$,*

$$t^{\alpha-1} D^\alpha m(t) \in L^1(0,\infty).$$

*Then, the radial Fourier multiplier $T_m$ defined by*

$$\widehat{T_m f}(\xi) = m(|\xi|^2)\widehat{f}(\xi)$$

*is bounded from $L^1(u)$ into $L^{1,\infty}(u)$, for every weight $u \in A_1$.*

Let us explain the technique we use to prove this result, since it illustrates one of the main advantages of the extrapolation estimate in Theorem 1.11. These are the steps:

- We write $T_m$ as an average of operators behaving like the Bochner-Riesz multiplier. More precisely,

$$T_m \chi_E(x) = \int_0^\infty B^s \chi_E(x) \Phi(s) ds, \quad \text{with } \Phi \in L^1(0,\infty),$$

  where the operators $\{B^s\}_{s>0}$ satisfy the same estimate as $B$ in Theorem 2.9, uniformly in $s > 0$.

- We use Minkowski's integral inequality for the Banach space $L^{p_0,\infty}(w)$ to transfer the extrapolation estimate (4) from $B^s$ to $T_m$, deducing that

$$\|T_m \chi_E\|_{L^{p_0,\infty}(w)} \leqslant C \|\Phi\|_{L^1(0,\infty)} \|\chi_E\|_{L^{p_0}(w)}.$$

- Finally, we extrapolate $T_m$ down to $p = 1$ by Theorem 1.11 and complete the proof.

Notice that the conclusion for $T_m$ cannot be drawn just from an $L^1 \to L^{1,\infty}$ estimate for the family $\{B^s\}_{s>0}$, given the quasi-Banach nature of the range. To conclude the chapter, in Section 3.4, we study general multipliers of Hörmander-type on $\mathbb{R}^n$. In this case, we do not use the aforementioned averaging technique, but rather a direct approach. The main contribution for these operators can be stated as follows:

**Theorem 3.26.** *Fix $1 < s \leqslant 2$ and let $m : \mathbb{R}^n \to \mathbb{R}$ be a bounded function in $\mathcal{C}^n(\mathbb{R}^n \backslash \{0\})$ such that*

$$\sup_{r>0} \left( r^{2|\alpha|-n} \int_{r\leqslant|x|\leqslant 2r} \left| \left( \frac{\partial}{\partial x} \right)^{\alpha} m(x) \right|^s dx \right)^{1/s} < \infty, \quad |\alpha| \leqslant n.$$

*Then, the associated multiplier $\widehat{T_m f}(\xi) = m(\xi)\widehat{f}(\xi)$ satisfies that, for every $u \in A_1$, there exists $1 < p_0 < \infty$ such that*

$$\|T_m \chi_E\|_{L^{p_0,\infty}(w)} \leqslant C \|\chi_E\|_{L^{p_0}(w)}, \quad E \subseteq \mathbb{R}^n,$$

*where $w = (M\chi_E)^{1-p_0} u$.*

The main results presented in these first three chapters are gathered in the preprint [24], already submitted for publication.

▼ In **Chapter 4** we study the different ingredients in a Littlewood-Paley theory adapted to $\widehat{A}_p$ weights. This theory was initiated in the thirties by Littlewood and Paley in a series of papers [89, 90, 91] about Fourier and power series, but since then, their ideas have proved to be really useful when dealing with Fourier multipliers $T_m$. More precisely, in our case we are interested in two types of inequalities, that we will call lower and upper estimates respectively:

(*a*) $\|f\|_{L^{p,\infty}(w)} \leqslant C\|G_1 f\|_{L^{p,\infty}(w)}$,

(*b*) $\|G_2 \chi_E\|_{L^{p,\infty}(w)} \leqslant \|\chi_E\|_{L^p(w)}$.

We will consider weights $w \in \widehat{A}_p$, and study different operators $G_1$ and $G_2$, known as square functions, that already appear in the classical theory. Establishing upper or lower estimates for square functions is interesting in its own right, but moreover, if combined with a relation of the form

(*c*) $\|G_1(T\chi_E)\|_{L^{p,\infty}(w)} \leqslant C\|G_2 \chi_E\|_{L^{p,\infty}(w)}$,

for some operator $T$, they would yield an estimate in the spirit of Theorem 1.11. We will study various square functions. For instance, in Section 4.2 we obtain inequality (*a*) for the classical Lusin area function

$$Sf(x) = \left( \int_{|x-y|<t} |\nabla u(y,t)|^2 \frac{dydt}{t^{n-1}} \right)^{1/2},$$

where, as before, $u$ is the harmonic extension of $f$ to the upper half-space $\mathbb{R}^{n+1}_+$. Another example can be found in Section 4.3, where we use the recent technique presented in [88] by A. Lerner and F. Nazarov of majorization by sparse operators to obtain inequality $(b)$ for the square function

$$G_\alpha f(x) = \left( \int_0^\infty \left| \frac{\partial}{\partial t} B^t_\alpha f(x) \right|^2 t dt \right)^{1/2}.$$

Here

$$\widehat{B^t_\alpha f}(\xi) = (1 - |t\xi|^2)^\alpha_+ \widehat{f}(\xi)$$

is just a generalization of the Bochner-Riesz operator $B$ that we defined in (3).

# On Yano's extrapolation

The second extrapolation theory that we will study is due to S. Yano [127], and it is related to the field of Real Analysis. In this case, we assume some kind of $L^p$ boundedness for $p > 1$, with respect to a fixed measure and with some control on the operator norms $C_p$ as $p$ tends to $1^+$. From here, we deduce that the operator is bounded on a certain space which is closer to $L^1$ than any other $L^p$ space. This is the original result by S. Yano from 1951:

**Theorem 5.6** (Yano, [127]). *Fix $(X, \mu)$, $(Y, \nu)$ a couple of finite measure spaces, $p_0 > 1$ and $m > 0$. If $T$ is a sublinear operator such that, for every $1 < p \leqslant p_0$,*

$$T : L^p(\mu) \longrightarrow L^p(\nu)$$

*is bounded with norm less than or equal to $\frac{C}{(p-1)^m}$, then,*

$$T : L(\log L)^m(\mu) \longrightarrow L^1(\nu)$$

*is also bounded.*

The space $L(\log L)^m(\mu) \subseteq L^1(\mu)$ is the set of $\mu$-measurable functions such that

$$\int_X |f(x)|(1 + \log_+ |f(x)|)^m d\mu(x) < \infty.$$

This result has subsequently been improved and extended to other types of boundedness. One of the latest results is due to M. J. Carro and P. Tradacete [33] and deals with operators mapping

$$T : L^{p,\infty}(\mu) \longrightarrow L^{p,\infty}(\nu),$$

with norm behaving like $\frac{1}{(p-1)^m}$ when $p$ is close to 1.

▼ In **Chapter 5** we present some results about Yano's theory motivated by its relation to the extrapolation of Rubio de Francia presented in Chapter 1. Recall that an operator $T$ under the hypotheses of Theorem 1.1 need not be bounded from $L^1$ to $L^{1,\infty}$. However, the sharp $L^p \to L^p$ norms that were derived in [48] (see also [50]) allow us to use Yano's extrapolation to obtain endpoint estimates close to $L^1$. More precisely, we know that if, for some $1 < p_0 < \infty$, some $\beta > 0$, and every $w \in A_{p_0}$, $T$ is a sublinear operator such that

$$T : L^{p_0}(w) \longrightarrow L^{p_0}(w)$$

is bounded with norm $C_{p_0} \|w\|_{A_{p_0}}^{\beta}$, then

$$T : L^p(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n)$$

is bounded for every $1 < p < p_0$ with norm essentially controlled by

$$\frac{1}{(p-1)^{\beta(p_0-1)}}, \quad \text{as } p \to 1^+.$$

With this behavior, Yano's extrapolation yields that $T$ is bounded on $L(\log L)^{\beta(p_0-1)}(\mathbb{R}^n)$, as stated in Theorem 5.22. The conclusion is only valid for the Lebesgue measure, because otherwise, we will see that the blow-up of the $L^p \to L^p$ norm is too fast. However, in Theorem 5.23, we succeed in drawing conclusions close to $L^1(u)$ for every $u \in A_1$ by means of a different extrapolation approach. This idea of finding a suitable way to relate the theories of Rubio de Francia and Yano in order to obtain weighted endpoint estimates has been gathered and developed beyond the scope of this thesis in [25].

The other setting where Yano's extrapolation can be applied comes from the theory of $\widehat{A}_p$ weights. Recall that in [28] the authors prove Theorem 1.7, and that the endpoint estimate that they obtain is

$$\|T\chi_E\|_{L^{1,\infty}(u)} \leqslant C\|\chi_E\|_{L^1(u)}, \quad u \in A_1. \tag{5}$$

However, when $T$ is sublinear, there is more to it than that. They also show that, despite the fact that we cannot expect to have $T : L^1(u) \to L^{1,\infty}(u)$ in general, what we do have is the following endpoint estimate, which is not restricted to characteristic functions either:

**Theorem 1.7** (Carro - Grafakos - Soria, [28])**.** *Let $T$ be a sublinear operator such that, for some $1 < p_0 < \infty$ and every $w \in \widehat{A}_{p_0}$, it holds that*

$$\|T\chi_E\|_{L^{p_0,\infty}(w)} \leqslant C\|\chi_E\|_{L^{p_0}(w)}, \quad E \subseteq \mathbb{R}^n. \tag{6}$$

*Then, for every $u \in A_1$, in addition to (5), we have that*

$$T : L(\log L)^{\varepsilon}(u) \longrightarrow L^{1,\infty}_{\mathrm{loc}}(u), \quad \varepsilon > 0. \tag{7}$$

Obviously, this boundedness is interesting when the operator $T$ is not in the class of operators for which (5) implies boundedness from $L^1(u)$ into $L^{1,\infty}(u)$, since

$$L(\log L)^{\varepsilon}(u) \subsetneq L^1(u).$$

Another goal of Chapter 5 is to improve this endpoint estimate (7) of logarithmic type for operators under the hypotheses of Theorem 1.7 by means of the extrapolation theory introduced in [33]. First, we need to compute the $L^{p,\infty} \to L^{p,\infty}$ norm of such operators. This is done in the following result:

**Theorem 5.5.** *Let $T$ be a sublinear operator such that, for some $1 < p_0 < \infty$ and every $w \in \widehat{A}_{p_0}$, it holds that*

$$\|T\chi_E\|_{L^{p_0,\infty}(w)} \leqslant C\|\chi_E\|_{L^{p_0}(w)}, \quad E \subseteq \mathbb{R}^n.$$

*Then, for every fixed $u \in A_1$ and every $1 < p < p_0$, we have that*

$$T : L^{p,\infty}(u) \longrightarrow L^{p,\infty}(u)$$

*is bounded with norm essentially controlled by*

$$\log\left(\frac{1}{p-1}\right)\frac{1}{p-1}, \qquad \text{as } p \to 1^+. \tag{8}$$

Once we have this computation, we extend the result in [33] in such a way that it admits constants with logarithmic terms as in (8). With this, we are able to show in Corollary 5.25 that an operator satisfying (6) is bounded on a certain space $X(u)$ such that, for every $\varepsilon > 0$,

$$L(\log L)^{\varepsilon}(u) \subsetneq X(u).$$

This already improves the endpoint estimate (7) from [28], but we also check that, by using further information about $T$ (basically, that it satisfies (5) on characteristic functions), we can self-improve this result and deduce the following:

**Corollary 5.29.** *Let $T$ be a sublinear operator such that, for some $1 < p_0 < \infty$ and every $w \in \widehat{A}_{p_0}$, it holds that*

$$\|T\chi_E\|_{L^{p_0,\infty}(w)} \leqslant C\|\chi_E\|_{L^{p_0}(w)}, \quad E \subseteq \mathbb{R}^n.$$

*Then, for every $u \in A_1$, we have that*

$$T : L \log\log L(u) \longrightarrow L^{1,\infty}_{\mathrm{loc}}(u).$$

So far, this is the best endpoint estimate (not restricted to characteristic functions) for general sublinear operators satisfying the hypotheses of Theorem 1.7, given that

$$L(\log L)^{\varepsilon}(u) \subsetneq X(u) \subsetneq L \log\log L(u).$$

In addition to these results related to Chapter 1, in Chapter 5 we also present an extension of Yano's theory to Lorentz spaces $L^{p,q}$. For $p < q < \infty$, these are intermediate spaces between $L^p$ and weak-$L^p$:

$$L^p \subseteq L^{p,q} \subseteq L^{p,\infty}.$$

The extrapolation results that we obtain deal with operators mapping

$$T : L^p(\mu) \longrightarrow L^{p,q}(\nu), \quad \text{or} \quad T : L^{p,q}(\mu) \longrightarrow L^{p,q}(\nu),$$

when $p$ is close to 1 and $1 < q < \infty$ is fixed. This is presented in Theorems 5.16 and 5.19 and completes the theory of Yano in the setting of Lorentz spaces.

▼ Finally, in **Chapter 6**, we show a series of results that are no longer related to the weighted $A_p$ theory that has been present throughout the chapters. Here we make use of Yano's extrapolation ideas adapted to decreasing functions in order to obtain pointwise bounds for integral operators of the form

$$T_K f(x) = \int_0^{\infty} K(x,t) f(t) dt,$$

with $K$ a positive kernel. The main result is contained in Theorem 6.5, and it can be applied to several integral operators such as the Abel transform, the Riemann-Liouville operator, iterative operators, etc. These applications are all gathered in Section 6.3. The content of this chapter has been accepted for publication in [23].

# Chapter 1

# Weighted Extrapolation Theory

## 1.1 The theory of Rubio de Francia

Let us start by recalling the definition of general $L^p$ spaces, which will constantly appear throughout this thesis. Given a measure space $(X,\mu)$, for every $1 \leqslant p < \infty$, $L^p(\mu)$ will denote the space of $\mu$-measurable functions satisfying

$$\|f\|_{L^p(\mu)} := \left(\int_X |f(x)|^p d\mu(x)\right)^{1/p} < \infty,$$

and $L^\infty(\mu)$ will be the space of $\mu$-measurable, bounded $\mu$-a.e functions on $X$. On many occasions, and especially in the first four chapters, we will take $X = \mathbb{R}^n$ equipped with an absolutely continuous measure $\mu$. That is, $\mu$ will satisfy $d\mu(x) = w(x)dx$, where $w$ is a non-negative, locally integrable function called weight. For these weighted $L^p$-spaces, we will write $L^p(w)$, and if $w = 1$ (i.e. $\mu$ is just the Lebesgue measure), we will use $L^p(\mathbb{R}^n)$ or simply $L^p$. Also, recall that the weak $L^p$-spaces $L^{p,\infty}(\mu)$ consist of $\mu$-measurable functions satisfying

$$\|f\|_{L^{p,\infty}(\mu)} := \sup_{t>0} t\lambda_f^\mu(t)^{1/p} < \infty,$$

where

$$\lambda_f^\mu(t) := \mu(\{x \in X : |f(x)| > t\})$$

is the distribution function of $f$ with respect to $\mu$. As usual, $\mu(E)$ denotes the $\mu$-measure of the set $E$, and if $\mu$ is the Lebesgue measure, then we write $\mu(E) = |E|$. A generalization of these spaces are the so-called Lorentz spaces. Given $1 \leqslant p < \infty$ and $1 \leqslant q < \infty$, we define $L^{p,q}(\mu)$ as the set of $\mu$-measurable functions such that

$$\|f\|_{L^{p,q}(\mu)} := \left(p\int_0^\infty (t\lambda_f^\mu(t)^{1/p})^q \frac{dt}{t}\right)^{1/q} < \infty.$$

It is immediate to check that $L^p(\mu) = L^{p,p}(\mu)$ and if $1 \leqslant q \leqslant \infty$, we have the following chain of inclusions:

$$L^{p,1} \subseteq L^{p,q} \subseteq L^{p,\infty}.$$

From now on, we will write $x \lesssim y$ when there is a positive constant $C > 0$ such that $x \leqslant Cy$. If both $x \lesssim y$ and $y \lesssim x$, then we write $x \approx y$. The constants involved are universal in their context. If there is an important dependence on some variable, we will note it with a subindex ($\lesssim_*, \approx_*$).

The extrapolation theory that we will present in this chapter will follow the ideas of Rubio de Francia [102]. First of all, let us recall the classical results. Let $M$ be the Hardy-Littlewood maximal operator, introduced by Hardy and Littlewood [66] in 1930:

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| dy, \tag{1.1}$$

where $Q \subseteq \mathbb{R}^n$ is a cube and $f$ is a locally integrable function. In 1972, B. Muckenhoupt [94] proved the following characterization for $1 < p < \infty$:

$$M : L^p(w) \longrightarrow L^p(w)$$

is bounded if, and only if $w \in A_p$, where $A_p$ is the class of weights such that

$$\|w\|_{A_p} = \sup_Q \frac{w(Q)}{|Q|} \left( \frac{w^{1-p'}(Q)}{|Q|} \right)^{p-1} < \infty.$$

Whenever an operator maps $L^p(w)$ into itself, we will say that it is of strong-type $(p, p)$ with respect to $w$. Therefore, in other words, Muckenhoupt's result states that $A_p$ weights characterize the strong-type $(p, p)$ of the maximal operator $M$. The case $p = 1$ has to be treated separately. It is clear that we cannot expect to have a strong-type $(1, 1)$ estimate of any kind for $M$, since $Mf$ is only integrable when $f = 0$. However, we do have a weaker estimate [94]:

$$M : L^1(u) \longrightarrow L^{1,\infty}(u) \tag{1.2}$$

is bounded if, and only if $u \in A_1$. This class[1] is defined by those weights $u$ such that

$$Mu(x) \leqslant Cu(x), \quad \text{a.e. } x \in \mathbb{R}^n,$$

and $\|u\|_{A_1}$ is the least constant $C > 0$ that can be taken in such an inequality. In general, an operator mapping $L^p(w)$ into $L^{p,\infty}(w)$ will be called of weak-type $(p, p)$ with respect to $w$, and hence, one could say that $A_1$ weights characterize the weak-type $(1, 1)$ of the maximal operator $M$. One can easily see that $A_p \subseteq A_q$ whenever $1 \leqslant p < q$. Indeed, given $w \in A_p$, when $p = 1$,

$$\left( \frac{w^{1-q'}(Q)}{|Q|} \right)^{q-1} \leqslant \sup_{x \in Q} w(x)^{-1} = \left( \inf_{x \in Q} w(x) \right)^{-1} \lesssim \left( \frac{w(Q)}{|Q|} \right)^{-1},$$

[1]For convenience, we will try to keep the notation $u$ only for weights in $A_1$.

and the case $p > 1$ is just Hölder's inequality. In view of these inclusions, it is natural to denote by $A_\infty$ the union

$$A_\infty = \bigcup_{1 \leqslant p < \infty} A_p.$$

This class first appeared in [95] and [38], and can be characterized (see also [49, Corollary 7.6]) by those weights $w$ for which there exists $\delta > 0$ such that

$$\sup_{E \subseteq Q} \left( \frac{|Q|}{|E|} \right)^{\delta} \frac{w(E)}{w(Q)} < \infty,$$

where the supremum is taken over all cubes $Q$ and all measurable sets $E \subseteq Q$. Even though we will not use them, we should mention that several characterizations of $A_\infty$ can be found in the literature, such as the one by N. Fujii [59] or the one by S. Hruščev [71] (and independently, by J. García-Cuerva and J. L. Rubio de Francia [60]). We also refer to the survey on this topic in [53].

The classes of $A_p$ weights have been broadly studied ever since they were introduced by B. Muckenhoupt. A basic property is that they satisfy a Reverse Hölder inequality (see, for instance, [63, Theorem 9.2.2]). More precisely, there exists an $\varepsilon > 0$, depending on $p$, $\|w\|_{A_p}$, and the dimension $n$, such that

$$\left( \frac{w^{1+\varepsilon}(Q)}{|Q|} \right)^{\frac{1}{1+\varepsilon}} \lesssim \frac{w(Q)}{|Q|}.$$

In particular, from here one can easily show that, given $w \in A_p$:

- If $1 \leqslant p < \infty$, there exists $\varepsilon > 0$ such that $w^{1+\varepsilon} \in A_p$.
- If $1 < p < \infty$, there exists $\varepsilon > 0$ such that $w \in A_{p-\varepsilon}$.

This, in some sense, represents the "openness" of these classes, an essential property in $A_p$-theory. Another consequence of the Reverse Hölder inequality is the following characterization of $A_1$ weights, introduced by R. Coifman and R. Rochberg in [41]: A weight $u$ belongs to $A_1$ if, and only if, there exist a locally integrable function $f$ and $0 \leqslant \delta < 1$ such that

$$u \approx (Mf)^\delta. \tag{1.3}$$

The last property that we want to recall about $A_p$ weights is P. Jones's factorization [73], which states that $w \in A_p$ if and only if there is a couple of $A_1$ weights $u_0$, $u_1$ such that

$$w = u_0 u_1^{1-p}. \tag{1.4}$$

However, the most important feature of $A_p$ weights for us is that they are behind Rubio de Francia's extrapolation theorem [102]. In its original version, it reads as follows:

**Theorem 1.1.** *Given a sublinear operator $T$, if for some $1 \leqslant p_0 < \infty$ and every $w \in A_{p_0}$,*

$$T : L^{p_0}(w) \longrightarrow L^{p_0}(w)$$

*is bounded, then, for every $1 < p < \infty$ and every $w \in A_p$,*

$$T : L^p(w) \longrightarrow L^p(w)$$

*is also bounded.*

Later on, simpler proofs and improvements of this result appeared. For instance, it was shown that it is still true if we have the boundedness estimates for general couples of functions $(f, g)$ instead of $(f, Tf)$, with $T$ being a sublinear operator. Also, there is a weak-type version of this result. More precisely, if we have a weak-type $(p_0, p_0)$ estimate for some $1 \leqslant p_0 < \infty$ and every weight in $A_{p_0}$, then we deduce the weak-type $(p, p)$ for every $1 < p < \infty$ and every weight in $A_p$. Moreover, in the case of sublinear operators, we can use classical interpolation to show that in fact, we have strong-type $(p, p)$. However, in all this setting, it is not possible to extrapolate down to $p = 1$, in the sense that there are operators under Rubio de Francia's hypotheses which are not of weak-type $(1, 1)$. Take, for instance, the composition $M^2 = M \circ M$. This operator trivially maps $L^p(w)$ into itself for every $w \in A_p$ and $1 < p < \infty$, but it is not of weak-type $(1, 1)$, even in the unweighted case. For further details on Rubio de Francia's extrapolation theorem and its modern variants, see [42], [43] or [50].

## 1.2 A new extrapolation to reach the endpoint $p = 1$

As we have seen, one of the drawbacks of the classical theory of extrapolation is that we cannot reach the endpoint $p = 1$ just from information at $p > 1$. In [28], however, the authors realized that if we change the class of weights in the extrapolation assumptions, there is a way to get estimates at level $p = 1$. Before we can introduce these weights and the extrapolation itself, we will need some definitions.

**Definition 1.2.** *Assume that we have an arbitrary weight $w$ on $\mathbb{R}^n$. Given $1 \leqslant p < \infty$, we say that an operator $T$ is of restricted weak-type $(p, p)$ with respect to $w$ if, for every measurable set $E$,*

$$\|T\chi_E\|_{L^{p,\infty}(w)} \leqslant C_p \|\chi_E\|_{L^p(w)} = C_p w(E)^{1/p}. \tag{1.5}$$

*In other words, if $T$ is of weak-type $(p, p)$ when restricted to characteristic functions.*

When $1 < p < \infty$ and $T$ is sublinear, it can be shown that (1.5) is equivalent to saying that

$$T : L^{p,1}(w) \longrightarrow L^{p,\infty}(w) \tag{1.6}$$

is bounded, and sometimes this is taken as a definition. However, when $p = 1$, it is not true that (1.5) is equivalent to $T : L^1(w) \to L^{1,\infty}(w)$, as we shall discuss in Section 1.4. In fact, it holds that when $p > 1$ is close to 1, if we have (1.6), then (1.5) trivially holds with the same constant, but if we have (1.5), then

$$\|T\|_{L^{p,1}(w)\to L^{p,\infty}(w)} \lesssim \frac{C_p}{p-1}.$$

For the time being, this dependence on $p$ of the constants will not be important to us and we will study restricted weak-type estimates using (1.5) or (1.6) indistinctively when $p > 1$. However, we make it explicit since we will need to take it into account when studying Yano's extrapolation theory in subsequent chapters. In this context of restricted weak-type estimates, in 1982 R. Kerman and A. Torchinsky [76] characterized the weights for which $M$ satisfied (1.5), including the case $p = 1$. More precisely, they proved that, for $1 \leqslant p < \infty$,

$$\|M\chi_E\|_{L^{p,\infty}(w)} \lesssim \|w\|_{A_p^{\mathcal{R}}} w(E)^{1/p} \tag{1.7}$$

if, and only if, $w \in A_p^{\mathcal{R}}$, where the so-called restricted $A_p$ class is the set of weights $w$ such that

$$\|w\|_{A_p^{\mathcal{R}}} = \sup_{E\subseteq Q} \frac{|E|}{|Q|}\left(\frac{w(Q)}{w(E)}\right)^{1/p} < \infty,$$

and the supremum is taken over all cubes $Q$ and all measurable sets $E \subseteq Q$. When $p = 1$, this class coincides with $A_1 = A_1^{\mathcal{R}}$, entailing that in this particular case of the maximal operator $M$, the weighted weak-type and restricted weak-type (1,1) are equivalent. For a general $1 \leqslant p < \infty$, it holds that (see [28])

$$A_p \subseteq A_p^{\mathcal{R}} \subseteq A_{p+\varepsilon}$$

for every $\varepsilon > 0$ with the following estimate:

$$\|w\|_{A_p^{\mathcal{R}}} \leqslant \|w\|_{A_p}^{1/p}.$$

Unlike for $A_p$ weights, where we know that every weight $w \in A_p$ can be written as $w = u_0 u_1^{1-p}$, with $u_0, u_1 \in A_1$, for the class $A_p^{\mathcal{R}}$ there is no factorization result so far. However, in [28, Corollary 2.8] the authors prove that, for every $u \in A_1$, every function $f \in L^1_{\text{loc}}$ and every $1 \leqslant p < \infty$, the weight $(Mf)^{1-p}u \in A_p^{\mathcal{R}}$ with

$$\|(Mf)^{1-p}u\|_{A_p^{\mathcal{R}}}^p \lesssim \|u\|_{A_1}. \tag{1.8}$$

Notice that combining (1.3) and (1.4), one has that every weight in $A_p$ is essentially of the form $(Mf)^{\delta(1-p)}u$, with $0 \leqslant \delta < 1$ and $u \in A_1$, so (1.8) states that, if we take $\delta = 1$, the resulting weight lies in $A_p^{\mathcal{R}}$. This result raises the question of whether every weight in $A_p^{\mathcal{R}}$ can be written in this way. For the time being, we will work with the (a priori) subclass for which this factorization holds.

**Definition 1.3.** *We define*

$$\widehat{A}_p = \{w : w = (Mf)^{1-p}u, \text{ for some } f \in L^1_{\mathrm{loc}} \text{ and } u \in A_1\} \subseteq A_p^{\mathcal{R}},$$

*with*

$$\|w\|_{\widehat{A}_p} = \inf \|u\|_{A_1}^{1/p},$$

*where the infimum is taken over all possible representations of $w$.*

The following lemma shows that $\widehat{A}_p$ is an intermediate class between $A_p$ and $A_p^{\mathcal{R}}$:

**Lemma 1.4.** *For every $1 < p < \infty$, we have that $A_p \subseteq \widehat{A}_p$ and $\|w\|_{\widehat{A}_p} \lesssim \|w\|_{A_p}^{2/p}$ for every $w \in A_p$.*

*Proof.* Let $w \in A_p$, factored as $w = u_0 u_1^{1-p}$, with $u_0, u_1 \in A_1$. Since $u_1 \in A_1$, we have that $u_1 \leqslant Mu_1 \leqslant \|u_1\|_{A_1} u_1$. With this and $1-p<0$, we can write

$$\|u_1\|_{A_1}^{1-p} \leqslant \frac{(Mu_1)^{1-p}}{u_1^{1-p}} =: k \leqslant 1.$$

Now, $w = u_0 u_1^{1-p} = u_0 k^{-1} (Mu_1)^{1-p}$, and it holds that

$$M(u_0 k^{-1}) \leqslant \|u_1\|_{A_1}^{p-1} M u_0 \leqslant \|u_1\|_{A_1}^{p-1} \|u_0\|_{A_1} u_0 \leqslant \|u_1\|_{A_1}^{p-1} \|u_0\|_{A_1} u_0 k^{-1}.$$

Therefore, $u_0 k^{-1} \in A_1$ and we deduce that $w \in \widehat{A}_p$. Furthermore,

$$\|w\|_{\widehat{A}_p} \leqslant \|u_0 k^{-1}\|_{A_1}^{1/p} \leqslant \left(\|u_0\|_{A_1} \|u_1\|_{A_1}^{p-1}\right)^{1/p} \lesssim \|w\|_{A_p}^{2/p},$$

using the quantitative version of the $A_p$-factorization theorem (see [39]), which states that $u_0$ and $u_1$ can be taken so that

$$\|w\|_{A_p} \leqslant \|u_0\|_{A_1} \|u_1\|_{A_1}^{p-1} \lesssim \|w\|_{A_p}^2.$$

□

**Remark 1.5.** *Even though, for a fixed $1 \leqslant p < \infty$, the classes $A_p \subseteq \widehat{A}_p \subseteq A_p^{\mathcal{R}}$ need not be the same in general, at this point it is clear that*

$$A_\infty = \bigcup_{1 \leqslant p < \infty} A_p = \bigcup_{1 \leqslant p < \infty} \widehat{A}_p = \bigcup_{1 \leqslant p < \infty} A_p^{\mathcal{R}}.$$

For later purposes, let us state the following property for weights of the form $(Mh)^\alpha$ when $\alpha < 0$:

**Lemma 1.6.** *Given a locally integrable function $h$ and $\alpha < 0$, we have that, for every cube $Q \subseteq \mathbb{R}^n$,*

$$\sup_{x \in Q} (Mh)^\alpha(x) \lesssim \frac{1}{|Q|} \int_Q (Mh)^\alpha(y) dy.$$

*In particular, if $Q \subseteq Q'$,*

$$\frac{1}{|Q|} \int_Q (Mh)^\alpha(y) dy \lesssim \frac{1}{|Q'|} \int_{Q'} (Mh)^\alpha(y) dy.$$

This property states that the weight $(Mh)^\alpha$ belongs to the Reverse Hölder class $RH_\infty$. This class was introduced by B. Franchi in [58], and in [44, Theorem 4.4], the authors prove that given an $A_1$ weight $u$, for every $p > 1$, it holds that $u^{1-p} \in RH_\infty \cap A_p$. In view of (1.3), their result shows that $(Mh)^\alpha \in RH_\infty \cap A_p$ for every $p > 1 - \alpha$ and in particular, Lemma 1.6. This estimate is also used (and proved in a different way) in [32, Corollary 2.3]. The second part of the lemma is obvious from the first.

Finally, we present the main extrapolation result obtained in [28, 32] in the context of $\widehat{A}_p$ weights:

**Theorem 1.7.** *Let $T$ be a sublinear operator such that, for some $1 < p_0 < \infty$ and every $w \in \widehat{A}_{p_0}$,*

$$T : L^{p_0,1}(w) \longrightarrow L^{p_0,\infty}(w).$$

*Then, for every $1 \leqslant p < \infty$ and every $w \in \widehat{A}_p$, $T$ is of restricted weak-type $(p, p)$ with respect to $w$. Moreover, it also satisfies that, for every $\varepsilon > 0$ and $u \in A_1$,*

$$T : L(\log L)^\varepsilon(u) \longrightarrow L^{1,\infty}_{\text{loc}}(u).$$

The details on the boundedness constants involved are gathered in [28, 32]. Regarding the sublinearity condition, we should say that it can be dropped if we want to show the restricted weak-type estimate for either $p = 1$ or $p > p_0$. It is in the range $1 < p < p_0$ (and for the $L(\log L)^\varepsilon$ estimate) where this assumption is needed. At this point we must emphasize that the main difference between this result and the classical extrapolation of Rubio de Francia is that, in this case, we can obtain estimates down to $p = 1$. In the next section we will focus on this aspect of the theory and we will see how much we can relax the hypotheses without losing the conclusion at the endpoint. Before we do that, and for later purposes, let us check what we get if we use the ideas behind Theorem 1.7 to extrapolate a restricted weak-type $(p_0, p_0)$ estimate that only holds for the classical $A_{p_0}$ class (that is, if we work with operators under the assumptions of Rubio de Francia's Theorem 1.1). Obviously, we will not be able to reach $p = 1$ in general, but in Chapter 5 we will be interested in the boundedness constant that we get for $p > 1$ when $p$ is close to 1. The result is the following:

**Theorem 1.8.** *Let $T$ be an operator such that, for some $1 < p_0 < \infty$ and every $w \in A_{p_0}$,*

$$T : L^{p_0,1}(w) \longrightarrow L^{p_0,\infty}(w)$$

*is bounded with constant $\varphi(\|w\|_{A_{p_0}})$, where $\varphi$ is an increasing function on $(0,\infty)$. Then, for every $1 < p < p_0$ and every $u \in A_1$,*

$$T : L^{p,\frac{p}{p_0}}(u) \longrightarrow L^{p,\infty}(u)$$

*is bounded with constant essentially controlled by*

$$\|u\|_{A_1}^{\frac{1}{p}-\frac{1}{p_0}} \varphi\left(\left(\frac{p_0-1}{p-1}\right)^{p_0-1} \|u\|_{A_1}\right).$$

*Proof.* We will follow the ideas in [28]. Let $\gamma > 0$ and $y > 0$. Given $f \in L^{p,\frac{p}{p_0}}(u)$, we use [28, Proposition 2.10] with $g = |Tf|$ to write

$$\lambda^u_{Tf}(y) \leqslant \lambda^u_{Mf}(\gamma y) + \gamma^{p_0-p} \frac{y^{p_0}}{y^p} \int_{\{|Tf|>y\}} (Mf)^{p-p_0}(x) u(x) dx.$$

Now, notice that $w := (Mf)^{p-p_0} u \in A_{p_0}$, since it can be factored as in (1.4). Moreover,

$$\|w\|_{A_{p_0}} = \left\| \left[ (Mf)^{\frac{p_0-p}{p_0-1}} \right]^{1-p_0} u \right\|_{A_{p_0}} \leqslant \left\| (Mf)^{\frac{p_0-p}{p_0-1}} \right\|_{A_1}^{p_0-1} \|u\|_{A_1} \lesssim \left(\frac{p_0-1}{p-1}\right)^{p_0-1} \|u\|_{A_1}. \quad (1.9)$$

Hence, we can use our assumption and deduce that

$$\lambda^u_{Tf}(y) \lesssim \lambda^u_{Mf}(\gamma y) + \gamma^{p_0-p} \frac{\varphi(\|w\|_{A_{p_0}})^{p_0}}{y^p} \left( \int_0^\infty \left( \int_{\{|f|>z\}} w(x) dx \right)^{1/p_0} dz \right)^{p_0}.$$

But, since $p - p_0 < 0$, we can bound $w = (Mf)^{p-p_0} u \leqslant z^{p-p_0} u$ on the set $\{|f| > z\}$, so we conclude that

$$\lambda^u_{Tf}(y) \lesssim \lambda^u_{Mf}(\gamma y) + \gamma^{p_0-p} \frac{\varphi(\|w\|_{A_{p_0}})^{p_0}}{y^p} \left( \int_0^\infty z^{\frac{p}{p_0}-1} \left( \int_{\{|f|>z\}} u(x) dx \right)^{1/p_0} dz \right)^{p_0}.$$

The expression in parentheses to the power $p_0$ is essentially $\|f\|^p_{L^{p,\frac{p}{p_0}}(u)}$, and using the sharp weak-type $(p,p)$ estimate for $M$ due to S. M. Buckley [9], we also know that

$$\lambda^u_{Mf}(\gamma y) \lesssim \frac{\|u\|_{A_p}}{y^p \gamma^p} \|f\|^p_{L^p(u)} \leqslant \frac{\|u\|_{A_1}}{y^p \gamma^p} \|f\|^p_{L^{p,\frac{p}{p_0}}(u)}.$$

Combining these two facts and multiplying by $y^p$ we obtain that

$$y^p \lambda_{Tf}^u(y) \lesssim \left( \frac{\|u\|_{A_1}}{\gamma^p} + \gamma^{p_0 - p} \varphi(\|w\|_{A_{p_0}})^{p_0} \right) \|f\|^p_{L^{p,\frac{p}{p_0}}(u)}.$$

Finally, we can minimize the right-hand side with respect to $\gamma > 0$ by choosing $\gamma = \|u\|_{A_1}^{1/p_0} \varphi(\|w\|_{A_{p_0}})^{-1}$, and taking supremum over $y > 0$, we get

$$\|Tf\|^p_{L^{p,\infty}(u)} \lesssim \|u\|_{A_1}^{1-p/p_0} \varphi(\|w\|_{A_{p_0}})^p \|f\|^p_{L^{p,\frac{p}{p_0}}(u)}.$$

This estimate, together with (1.9), completes the proof. □

**Remark 1.9.** *When $T$ is sublinear, in Theorem 1.8 we can also conclude that*

$$T : L^{p,1}(u) \longrightarrow L^{p,\infty}(u), \tag{1.10}$$

*since we can check that* (1.5) *holds on characteristic functions:*

$$\|T\chi_E\|_{L^{p,\infty}(u)} \leqslant C_{p,u} \|\chi_E\|_{L^{p,\frac{p}{p_0}}(u)} = C_{p,u} u(E)^{1/p}.$$

*However, as we mentioned when we introduced* (1.6) *as an alternative definition for the restricted weak-type* $(p,p)$ *of sublinear operators, the boundedness constant for* (1.10) *would have an extra factor behaving like* $\frac{1}{p-1}$ *when* $p$ *is close to 1.*

## 1.3 Extrapolating on a smaller class of weights

As we anticipated after presenting Theorem 1.7, in this section we will see how much we can relax the hypotheses of this theorem without losing information in the conclusion at $p = 1$. The following result states that if $T$ satisfies a restricted weak-type estimate as in Theorem 1.7 but only for a very particular subclass of $\widehat{A}_{p_0}$, then we obtain the analogous estimate for the whole range of $1 \leqslant p < \infty$, and at $p = 1$, we still recover the whole $A_1$ class. We will also drop the sublinearity condition on $T$, since for the weight we are considering, we can avoid the interpolation step requiring it in the original result of [28]. In fact, the results in this section could be written for couples $(\chi_E, g)$, where $g$ is a measurable function, not necessarily $T\chi_E$.

**Theorem 1.10.** *Let $1 < p_0 < \infty$. If an operator $T$ satisfies that, for every measurable set $E \subseteq \mathbb{R}^n$ and every weight $u \in A_1$,*

$$\|T\chi_E\|_{L^{p_0,\infty}((M\chi_E)^{1-p_0}u)} \leqslant \varphi(\|u\|_{A_1}) u(E)^{1/p_0}$$

*with $\varphi$ an increasing function on $(0,\infty)$, then, for every $1 \leqslant p < \infty$,*

$$\|T\chi_E\|_{L^{p,\infty}((M\chi_E)^{1-p}u)} \leqslant \varphi_p(\|u\|_{A_1}) u(E)^{1/p}$$

*with*

$$\varphi_p(t) = \begin{cases} t^{\frac{1}{p}-\frac{1}{p_0}}\varphi(t), & \text{if } 1 \leqslant p \leqslant p_0 \\ p^{\frac{2}{p_0}} t^{\frac{p+1}{pp_0}\frac{p-p_0}{p-1}} \varphi\left(\frac{p-1}{p_0-1}t\right), & \text{if } p_0 < p < \infty. \end{cases}$$

*Proof.* Let us start with $1 \leqslant p < p_0$. The argument for this case will be similar to that in Theorem 1.8, which in turn follows the ideas of [28]. We start by using [28, Proposition 2.10] with the weight $w = (M\chi_E)^{1-p}u$, $g = |T\chi_E|$, $f = \chi_E$ and $\gamma > 0$ to show that

$$\begin{aligned} \lambda^w_{T\chi_E}(y) &\leqslant \lambda^w_{M\chi_E}(\gamma y) + \gamma^{p_0-p}\frac{y^{p_0}}{y^p}\int_{\{|T\chi_E|>y\}}(M\chi_E)^{p-p_0}(x)w(x)dx \\ &= \lambda^w_{M\chi_E}(\gamma y) + \gamma^{p_0-p}\frac{y^{p_0}}{y^p}\int_{\{|T\chi_E|>y\}}(M\chi_E)^{1-p_0}(x)u(x)dx. \end{aligned}$$

Now, we apply our hypothesis, multiply by $y^p$ and use that $M$ is of restricted weak-type $(p,p)$ with respect to $w$ with constant $\|w\|_{A_p^{\mathcal{R}}} \lesssim \|u\|_{A_1}^{1/p}$ (see [28, Corollary 2.8]):

$$y^p\lambda^w_{T\chi_E}(y) \lesssim \frac{\|u\|_{A_1}u(E)}{\gamma^p} + \gamma^{p_0-p}\varphi(\|u\|_{A_1})^{p_0}u(E).$$

Finally, we take the supremum on $y$ and the infimum over $\gamma > 0$, which is attained essentially at $\gamma = \|u\|_{A_1}^{\frac{1}{p_0}}\varphi(\|u\|_{A_1})^{-1}$, to conclude that

$$\|T\chi_E\|_{L^{p,\infty}((M\chi_E)^{1-p}u)} \lesssim \|u\|_{A_1}^{\frac{1}{p}-\frac{1}{p_0}}\varphi(\|u\|_{A_1})u(E)^{1/p}. \tag{1.11}$$

The case $p_0 < p < \infty$ is a little more involved. We shall follow [32, Theorem 3.1]. Choose $\beta$ satisfying

$$1 < \beta < \frac{p_0'}{p'}, \quad \text{and} \quad \beta \leqslant 1 + \frac{1}{2^{n+1}\|u\|_{A_1}}, \tag{1.12}$$

which by [98] ensures that $u^\beta \in A_1$ and $\|u^\beta\|_{A_1} \lesssim \|u\|_{A_1}$. Let $0 < \theta < 1$ such that

$$\beta\,\frac{p_0-1}{p-1} + \theta\,\frac{p-p_0}{p-1} = 1.$$

From here we deduce that, for every $y > 0$,

$$\int_{\{|T\chi_E|>y\}}(M\chi_E)^{1-p}(x)u(x)dx \leqslant \int_{\{|T\chi_E|>y\}}(M\chi_E)^{1-p_0}(x)v(x)dx,$$

with

$$v(x) = u(x)^{\beta\frac{p_0-1}{p-1}}\left(M(u^\theta(M\chi_E)^{1-p}\chi_{\{|T\chi_E|>y\}})(x)\right)^{\frac{p-p_0}{p-1}} \in A_1,$$

and $\|v\|_{A_1} \lesssim \frac{p-1}{p_0-1}\|u\|_{A_1}$ (using [32, Lemma 2.12] for this last fact). With this, our hypothesis yields

$$\int_{\{|T\chi_E|>y\}} (M\chi_E)^{1-p}(x)u(x)dx \leqslant \frac{1}{y^{p_0}}\varphi\left(\frac{p-1}{p_0-1}\|u\|_{A_1}\right)^{p_0} v(E).$$

Finally, we need to estimate $v(E)$. Recalling that $M\chi_E \equiv 1$ on $E$ and the relation in (1.12), we can write

$$v(E) = \int_E \left(\frac{M(u^\theta (M\chi_E)^{1-p}\chi_{\{|T\chi_E|>y\}})(x)}{(M\chi_E)^{1-p}u^\theta}\right)^{\frac{p-p_0}{p-1}} u(x)dx,$$

and using Hölder,

$$\begin{aligned} v(E) &\leqslant \|\chi_E\|_{L^{\frac{p}{p_0},1}(u)} \left\| \left(\frac{M(u^\theta (M\chi_E)^{1-p}\chi_{\{|T\chi_E|>y\}})(x)}{(M\chi_E)^{1-p}u^\theta}\right)^{\frac{p-p_0}{p-1}} \right\|_{L^{\frac{p}{p-p_0},\infty}(u)} \\ &= \frac{p}{p_0}u(E)^{p_0/p} \left\| \frac{M(u^\theta (M\chi_E)^{1-p}\chi_{\{|T\chi_E|>y\}})(x)}{(M\chi_E)^{1-p}u^\theta} \right\|_{L^{p',\infty}(u)}^{\frac{p-p_0}{p-1}}. \end{aligned}$$

Here we apply [32, Lemma 2.6] and conclude that

$$v(E) \lesssim pu(E)^{p_0/p}C_{p,\theta}((M\chi_E)^{1-p}u)^{\frac{p-p_0}{p-1}}((M\chi_E)^{1-p}u)(\{|T\chi_E|>y\})^{\frac{p-p_0}{p}},$$

where the constant $C_{p,\theta}(\cdot)$ is the one appearing in [32, Lemma 2.6]. With this estimate, we obtain that

$$\|T\chi_E\|^p_{L^{p,\infty}((M\chi_E)^{1-p}u)} \lesssim p^{\frac{p}{p_0}}C_{p,\theta}((M\chi_E)^{1-p}u)^{\frac{p(p-p_0)}{p_0(p-1)}}\varphi\left(\frac{p-1}{p_0-1}\|u\|_{A_1}\right)^p u(E).$$

Using that in our case $\frac{1}{p'} < \theta < 1$, we can choose the best possible value for $\theta$ so that

$$C_{p,\theta}((M\chi_E)^{1-p}u) \lesssim p\|u\|_{A_1}^{\frac{p+1}{p}}.$$

If we plug this in the previous estimate and observe that

$$\frac{p}{p_0} + \frac{p}{p_0}\left(\frac{p-p_0}{p-1}\right) \leqslant \frac{2p}{p_0},$$

we conclude

$$\|T\chi_E\|_{L^{p,\infty}((M\chi_E)^{1-p}u)} \lesssim p^{\frac{2}{p_0}}\|u\|_{A_1}^{\frac{p+1}{pp_0}\frac{p-p_0}{p-1}}\varphi\left(\frac{p-1}{p_0-1}\|u\|_{A_1}\right)u(E)^{1/p}.$$

$\square$

Notice that the most interesting feature of this result is that the conclusion at $p = 1$ holds for the whole $A_1$ class. In fact, if our goal is just to reach the endpoint, we can make yet another simplification. Namely, we can obtain the restricted weak-type (1,1) estimate for $A_1$ weights starting from a restricted weak-type $(p_0, p_0)$ assumption in which $p_0$ may depend on the weight $u$. The key fact is that we always have $1 = p < p_0$. Therefore, regardless of the value of $p_0$, we must argue as in the first case of the proof of Theorem 1.10. Notice that in this case, to prove the estimate at level $p = 1$ for a fixed weight $u \in A_1$, we use the assumption at level $p_0$ with exactly the same weight $u$, so the dependence $p_0(u)$ does not affect the argument. The conclusion is (1.11) with $p = 1$, as we state in the following theorem. Here we make the dependence of $\varphi$ on $p_0$ explicit, since it represents dependence on $u$ and might need to be taken into account:

**Theorem 1.11.** *Let $T$ be an operator and $u \in A_1$. If there is some $1 < p_0 < \infty$ such that*

$$\|T\chi_E\|_{L^{p_0,\infty}((M\chi_E)^{1-p_0}u)} \leqslant \varphi_{p_0}(\|u\|_{A_1})u(E)^{1/p_0},$$

*then*

$$\|T\chi_E\|_{L^{1,\infty}(u)} \leqslant \|u\|_{A_1}^{1-\frac{1}{p_0}}\varphi_{p_0}(\|u\|_{A_1})u(E).$$

In the next section, we will see how an extra (mild) assumption on $T$ allows us to turn the conclusion into a weak-type (1,1) estimate rather than a restricted one. To conclude the discussion on this smaller class of weights, we present a duality result that also holds in this setting:

**Proposition 1.12.** *Let $1 < p_0 < \infty$. Assume that we have a sublinear operator $T$ with adjoint $T^*$ such that, for every measurable set $E \subseteq \mathbb{R}^n$ and $u \in A_1$,*

$$\|T^*\chi_E\|_{L^{p_0,\infty}((M\chi_E)^{1-p_0}u)} \leqslant \varphi(\|u\|_{A_1})u(E)^{1/p_0}, \tag{1.13}$$

*with $\varphi$ an increasing function on $(0,\infty)$. Then, for every $1 < p < \infty$, $u \in A_1$ and $f \in L^{p,1}(u)$,*

$$\left\|\frac{T(uf)}{u}\right\|_{L^{p,\infty}(u)} \lesssim \varphi_{p'}(\|u\|_{A_1})\|f\|_{L^{p,1}(u)},$$

*with $\varphi_{p'}$ defined as in Theorem 1.10.*

*Proof.* First, we use Theorem 1.10 to extrapolate (1.13) and deduce, for every $1 < p' < \infty$,

$$\|T^*\chi_E\|_{L^{p',\infty}((M\chi_E)^{1-p'}u)} \leqslant \varphi_{p'}(\|u\|_{A_1})u(E)^{1/p'}.$$

Now fix $1 < p < \infty$. Since we want to show a restricted weak-type estimate, it is enough to assume that $f = \chi_E$. Also, in order to compute the $L^{p,\infty}$ norm via duality, we also need to establish a restricted weak-type estimate for the duality operator $\langle \cdot, h \rangle_u$, so we take $h = \chi_F$ and

$$\begin{aligned}\left\| \frac{T(u\chi_E)}{u} \right\|_{L^{p,\infty}(u)} &\approx \sup_{u(F)^{1/p'}=1} \left| \left\langle \frac{T(u\chi_E)}{u}, \chi_F \right\rangle_u \right| = \sup_{u(F)=1} \int_{\mathbb{R}^n} T(u\chi_E)(x)\chi_F(x)dx \\ &= \sup_{u(F)=1} \int_{\mathbb{R}^n} \chi_E(x) T^* \chi_F(x) u(x) dx \\ &= \sup_{u(F)=1} \int_{\mathbb{R}^n} \chi_E(x) (M\chi_F(x))^{p'-1} T^* \chi_F(x) (M\chi_F(x))^{1-p'} u(x) dx \\ &\leqslant \sup_{u(F)=1} \|\chi_E (M\chi_F)^{p'-1}\|_{L^{p,1}((M\chi_F)^{1-p'}u)} \|T^*\chi_F\|_{L^{p',\infty}((M\chi_F)^{1-p'}u)}.\end{aligned}$$

Now, the first norm can be bounded by

$$\int_0^\infty \left( \int_{\{x \in E : M\chi_F(x)^{p'-1} > y\}} M\chi_F(x)^{1-p'} u(x) dx \right)^{1/p} dy \leqslant \int_0^1 y^{-1/p} u(E)^{1/p} dy \lesssim u(E)^{1/p}.$$

To the second norm we apply our assumption to control it by $\varphi_{p'}(\|u\|_{A_1}) u(F)^{1/p'} = \varphi_{p'}(\|u\|_{A_1})$, and this completes the proof. $\square$

## 1.4 From restricted to unrestricted weak-type (1,1)

Even though the results presented above only yield restricted weak-type $(1,1)$ estimates, it is known that for a large class of operators (as it happened for the Hardy-Littlewood maximal function $M$), this is equivalent to being of weak-type $(1,1)$. We will need to define a notion introduced in [21] that gives a sufficient condition for operators to be of weak-type $(1,1)$ just from a restricted weak-type estimate.

**Definition 1.13.** *Given $\delta > 0$, a function $a \in L^1(\mathbb{R}^n)$ is called a $\delta$-atom if it satisfies the following properties:*

*(i)* $\int_{\mathbb{R}^n} a = 0$*, and*

*(ii) there exists a cube $Q$ such that $|Q| \leqslant \delta$ and $\operatorname{supp} a \subseteq Q$.*

*With this, a sublinear operator $T$ is $(\varepsilon, \delta)$-atomic if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\|Ta\|_{L^1 + L^\infty} = \int_0^1 (Ta)^*(t) dt \leqslant \varepsilon \|a\|_1,$$

*for every $\delta$-atom $a$, and $T$ is said to be $(\varepsilon, \delta)$-atomic approximable if there exists a sequence $\{T_n\}_n$ of $(\varepsilon, \delta)$-atomic operators such that, for every measurable set $E$, $|T_n \chi_E| \leqslant |T\chi_E|$ and, for every function $f \in L^1(\mathbb{R}^n)$ with $\|f\|_\infty \leqslant 1$,*

$$|Tf(x)| \leqslant \liminf_n |T_n f(x)|, \quad a.e.\ x \in \mathbb{R}^n.$$

In [21], the author shows that this is not a strong property to assume on an operator. For instance, it is checked that if

$$Tf(x) = K * f(x), \tag{1.14}$$

with $K \in L^p(\mathbb{R}^n)$ for some $1 \leqslant p < \infty$, or $K$ measurable and uniformly continuous on $\mathbb{R}^n$, then $T$ is $(\varepsilon, \delta)$-atomic, and if $\{T_n\}_n$ is a sequence of $(\varepsilon, \delta)$-atomic operators, then both

$$T^* f(x) = \sup_n |T_n f(x)|, \quad \text{and} \quad Tf(x) = \left( \sum_n |T_n f(x)|^q \right)^{1/q},$$

are $(\varepsilon, \delta)$-atomic approximable, for every $q \geqslant 1$. We will see that this notion of approximability by $(\varepsilon, \delta)$-atomic operators is not the only possible one keeping the good properties of these operators, but for the time being we will not get into this matter. The result concerning the boundedness of this kind of operators is the following:

**Theorem 1.14.** *Let $T$ be a sublinear operator $(\varepsilon, \delta)$-atomic approximable and let $u \in A_1$. Then, if there exists a constant $C_u > 0$ such that, for every measurable set $E$,*

$$\|T\chi_E\|_{L^{1,\infty}(u)} \leqslant C_u u(E),$$

*we have that*

$$T : L^1(u) \longrightarrow L^{1,\infty}(u)$$

*with constant $2^n C_u \|u\|_{A_1}$.*

This result was proved in [21] in the unweighted case, and extended to weights in $A_1$ in [28].

## 1.5 Limited range extrapolation

Finally, let us present an extrapolation tool that will be needed in Section 3.3.1. Assume that we have a weighted $L^{p_0}$ estimate that only holds for certain powers of $A_{p_0}$ weights. Despite the fact that Rubio the Francia's extrapolation cannot be applied directly, this partial information can be used to draw conclusions for a limited range of $p$ around $p_0$, depending on the powers of the initial weights and the value of $p_0$. This idea was introduced in [51] and further developed in [26]. Let us make it precise. Its original statement is a little bit more general, but for the sake of simplicity, we will state it in a simpler way. See [26, Section 2] for more details.

**Theorem 1.15.** *Assume that, for some $1 < p_0 < \infty$ and some $\alpha \in [0, 1]$, a sublinear operator $T$ maps*

$$T : L^{p_0}(w^\alpha) \longrightarrow L^{p_0}(w^\alpha), \quad \forall w \in A_{p_0}.$$

*We define*

$$p'_- := \frac{p'_0}{1-\alpha}, \qquad p_+ := \frac{p_0}{1-\alpha},$$

*and, for every* $p \in (p_-, p_+)$*, we set* $\alpha_0(p), \alpha_1(p) \in [0,1]$ *to be such that*

$$p'_- = \frac{p'}{1-\alpha_1(p)}, \qquad p_+ = \frac{p}{1-\alpha_0(p)}.$$

*Then, it holds that, for every* $p \in (p_-, p_+)$*, and every* $u_0, u_1 \in A_1$*,*

$$T : L^p(u_0^{\alpha_0(p)} u_1^{\alpha_1(p)(1-p)}) \longrightarrow L^p(u_0^{\alpha_0(p)} u_1^{\alpha_1(p)(1-p)}).$$

Notice that the interval $(p_-, p_+)$ is built around $p_0$, and that if $\alpha = 0$, it shrinks to the singleton $\{p_0\}$ (which makes sense, because no extrapolation is possible if the initial estimate does not have weights). If $\alpha = 1$, then this result recovers Rubio de Francia's theorem, since for this particular case $(p_-, p_+) = (1, \infty)$ and $\alpha_0(p) = \alpha_1(p) = 1$, which, due to the factorization of $A_p$ weights (1.4), makes the conclusion valid for the whole $A_p$ class. If we are not interested in the weights in the conclusion, we can forget about the exponents $\alpha_i(p)$ and write the following particular case:

**Corollary 1.16.** *Assume that for some* $1 < p_0 < \infty$ *and some* $\alpha \in [0,1]$*, a sublinear operator* $T$ *maps*

$$T : L^{p_0}(w^\alpha) \longrightarrow L^{p_0}(w^\alpha), \quad \forall w \in A_{p_0}.$$

*Then, for every* $p \in (p_-, p_+)$*,*

$$T : L^p(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n),$$

*where* $p_-, p_+$ *are as in Theorem 1.15.*

# Chapter 2

# The Bochner-Riesz Operator

## 2.1 Introduction to the problem

First of all, let us recall the standard definition for the Fourier transform of an integrable function $f \in L^1(\mathbb{R}^n)$:

$$\widehat{f}(\xi) := \int_{\mathbb{R}^n} f(x) e^{-2\pi i x\cdot\xi} dx, \quad \xi \in \mathbb{R}^n.$$

This operation $f \mapsto \widehat{f}$ can be extended by duality to the class of tempered distributions, and in particular, defines an isometry on $L^2(\mathbb{R}^n)$, known as Plancherel's theorem. Its inverse transform is denoted by $f^\vee(x) := \widehat{f}(-x)$. Another essential property is that the Fourier transform of a convolution becomes a pointwise product in the following way:

$$\widehat{f * g}(\xi) = \widehat{f}(\xi)\widehat{g}(\xi).$$

A really detailed presentation of the Fourier transform and all its properties can be found in [63]. Now, we will give the general definition of the Bochner-Riesz operator. Recall that $a_+ = \max\{a, 0\}$ denotes the positive part of $a \in \mathbb{R}$.

**Definition 2.1.** *Given $\lambda > 0$ and $r > 0$, we define the Bochner-Riesz operator $B_\lambda^r$ on $\mathbb{R}^n$ by*

$$\widehat{B_\lambda^r f}(\xi) = (1 - |r\xi|^2)_+^\lambda \widehat{f}(\xi).$$

Notice that the term $(1 - |r\xi|^2)_+^\lambda$ restricts the support of $\widehat{f}$ to the ball $B(0, 1/r)$. However, the larger the value of $\lambda$, the smoother this truncation is, and thus, the better the operator $B_\lambda^r$ will behave. More precisely, it is easy to see that if $\lambda > \frac{n-1}{2}$, then $B_\lambda^r f$ is essentially controlled by the Hardy-Littlewood maximal operator $M$ (see, for instance, [63, Sec. 10.2]). However, for the so-called critical index $\lambda = \frac{n-1}{2}$, we do not have such a control. We will focus on this critical case with $r = 1$, so for the sake of simplicity, we will drop the indices $\lambda$ or $r$ whenever they are $\frac{n-1}{2}$ or 1 respectively.

Despite the fact that $B$ is no longer controlled by the Hardy-Littlewood maximal operator, when it comes to its boundedness on weighted $L^p$-spaces, it satisfies the same estimates as $M$. Namely, in 1988, M. Christ [35] showed that $B$ is of weak-type (1,1) with respect to the Lebesgue measure. Later on, in 1992, X. Shi and Q. Sun [107] proved that it was of strong-type $(p,p)$ for every weight in $A_p$ and every $1 < p < \infty$, and finally, in 1996, A. Vargas [124] extended the weak-type (1,1) estimate to $A_1$ weights. In this chapter, we will give a short proof of the strong-type $(p,p)$, then simplify A. Vargas' proof for $A_1$ weights and, finally, in Theorem 2.9, we will show that $B$ satisfies a certain restricted weak-type $(p,p)$ estimate, in the spirit of Section 1.3. The main advantage of this new estimate is that it will allow us to use extrapolation arguments on operators that can be written as an average of Bochner-Riesz operators $\{B^r\}_{r>0}$.

## 2.2 Some preliminary results

Let us consider the classical decomposition of $B$. Arguing as in [35], it is enough to study the operator (which we will call again $B$)

$$f \longmapsto \left( \sum_{j=1}^{\infty} K_j \right) * f,$$

where

$$K_j(x) = \eta \left( \frac{x}{|x|} \right) \psi(x) \varphi(2^{-j}x)|x|^{-n},$$

and:

- $\eta$ is a fixed element from a finite $\mathcal{C}^\infty$ partition of the unity on the sphere $\mathbb{S}^{n-1}$, which we can assume to have very small support.
- $\psi(x) = \cos(2\pi|x| - \pi(n-1)/4)$.
- $\varphi \in \mathcal{C}^\infty(\mathbb{R}^n)$, real-valued, radial, supported on $\{x \in \mathbb{R}^n : |x| \in [1/4, 1]\}$, and such that
$$\sum_{j \in \mathbb{Z}} \varphi(2^j x) \equiv 1, \quad \text{on } \mathbb{R}^n \backslash \{0\}.$$

Even though we will resort to some estimates from [35] for which the author needs a deep understanding of the kernels $K_j$, the only property that we will explicitly use has to do with their size and support. Namely that, for every $j \geqslant 1$,

$$|K_j(x)| \lesssim 2^{-jn} \chi_{B(0,2^j)}(x). \tag{2.1}$$

This is a direct consequence of their definition. In fact we could say that they are supported on the annulus $B(0, 2^j) \backslash B(0, 2^{j-2})$, but since we will not really use it, let us just keep estimate (2.1). Also, we will use that they are uniformly controlled by the Hardy-Littlewood maximal operator:

**Lemma 2.2.** *For every $j \geqslant 1$, and every locally integrable function $f$,*

$$|K_j * f(x)| \lesssim Mf(x).$$

*Proof.* This is a direct consequence of (2.1):

$$|K_j * f(x)| \lesssim \left(\frac{\chi_{B(0,2^j)}}{|B(0,2^j)|}\right) * f(x) \leqslant Mf(x).$$

$\square$

Once we have settled the decomposition of the kernel, we will need three more lemmas before we can reach our goal. The first one will allow us to construct a simplified Calderón-Zygmund decomposition for characteristic functions:

**Lemma 2.3.** *Let $0 < \alpha < 1$. Let $E \subseteq \mathbb{R}^n$ be a measurable set. Then there exists a family of pairwise disjoint dyadic cubes $\{Q_i\}_{i=0}^{\infty}$ such that*

$$\frac{|E \cap Q_i|}{|Q_i|} \approx \alpha,$$

*and $E \smallsetminus \mathcal{N} \subseteq \bigcup_{i=0}^{\infty} Q_i$, with $|\mathcal{N}| = 0$.*

*Proof.* We just take the Calderón-Zygmund family of dyadic cubes associated with the function $\chi_E$. By the stopping-time condition used in the decomposition, we know that these cubes satisfy, for every $i \geqslant 0$,

$$\alpha < \frac{1}{|Q_i|}\int_{Q_i} \chi_E(x)dx = \frac{|E \cap Q_i|}{|Q_i|} \leqslant 2^n\alpha.$$

Also, if we take a point $x \in \mathbb{R}^n \setminus \bigcup_{i=0}^{\infty} Q_i$, since it is not in any Calderón-Zygmund cube, we have that for each $m \geqslant 0$, there exists a unique non-selected dyadic cube $Q_x^{-m}$ with $|Q_x^{-m}| = 2^{-nm}$ that contains $x$ and

$$\frac{1}{|Q_x^{-m}|}\int_{Q_x^{-m}} \chi_E(y)dy = \frac{|E \cap Q_x^{-m}|}{|Q_x^{-m}|} \leqslant \alpha.$$

But the intersection of the closures of the cubes $\{Q_x^{-m}\}_{m \geqslant 0}$ is the singleton $\{x\}$, so using Lebesgue's differentiation theorem, we deduce that for almost every $x \in \mathbb{R}^n \setminus \bigcup_{i=0}^{\infty} Q_i$,

$$\chi_E(x) = \lim_{m\to\infty} \frac{|E \cap Q_x^{-m}|}{|Q_x^{-m}|} \leqslant \alpha < 1,$$

and hence $x \notin E$. $\square$

**Remark 2.4.** *Based on this lemma, given $0 < \alpha < 1$ and $E \subseteq \mathbb{R}^n$, we can define for every $k \geqslant 0$,*

$$E_k := E \cap \left( \bigcup_{i=0}^{\infty} Q_i^k \right),$$

*where $\{Q_i^k\}_{i=0}^{\infty}$ is the subfamily of cubes with size $|Q_i^k| = 2^{nk}$ if $k > 0$, and $|Q_i^0| \leqslant 1$. Since the set $E$ is essentially contained in the union of all the cubes $\{Q_i^k\}_{i,k=0}^{\infty}$, we have that*

$$E = \bigcup_{k=0}^{\infty} E_k,$$

*and for every $k, i \geqslant 0$:*

$$\frac{|E_k \cap Q_i^k|}{|Q_i^k|} = \frac{|E \cap Q_i^k|}{|Q_i^k|} \approx \alpha.$$

Let us illustrate this decomposition in the following picture (forgetting about the $\alpha$-ratio property). Consider a polygon $E$, in gray. We separate the cubes into three groups, one color each, depending on their size, and we look at their intersection with $E$ to find the pieces $E_1, E_2$ and $E_3$.

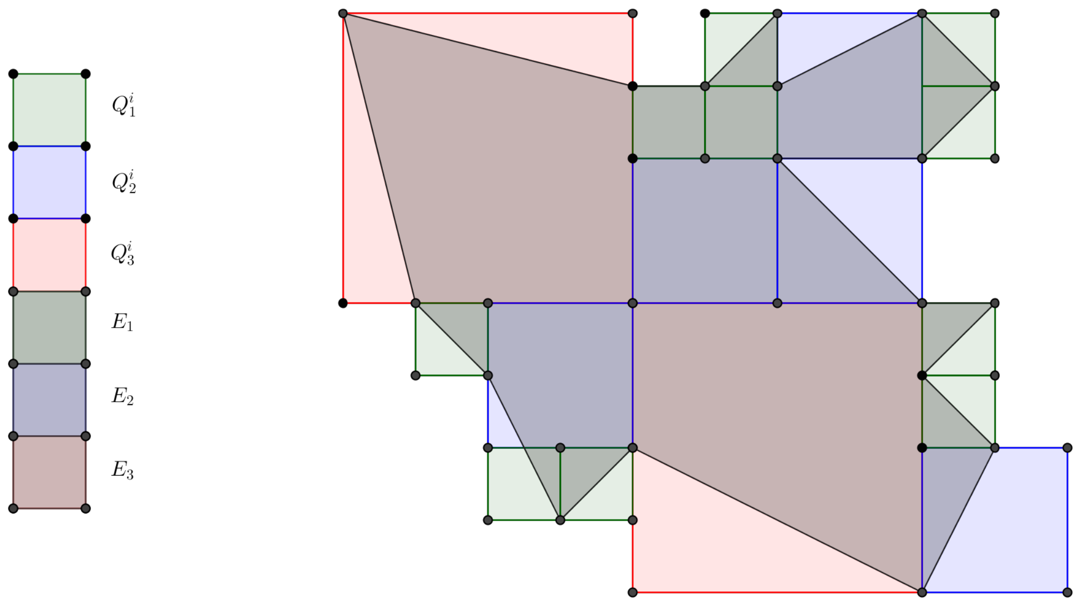


Figure 2.1: Example of decomposition of $E = E_1 \cup E_2 \cup E_3$, with $E_k = E \cap \bigcup_i Q_i^k$.

The next lemma will be the cornerstone of our argument. For technical reasons regarding interpolation, not only will we need estimates for $E$, but also for subsets $G \subseteq E$. Notice that if $G_k = G \cap E_k$, we still have the inequality $\frac{|G_k \cap Q_i^k|}{|Q_i^k|} \lesssim \alpha$, and that will suffice to get the right estimates.

**Lemma 2.5.** *Let $0 < \alpha < 1$ and let $E = \bigcup_{k=0}^{\infty} E_k$ be a measurable set decomposed as in Remark 2.4. Let $G \subseteq E$ be a measurable subset and define for every $k \geqslant 0$, $G_k = G \cap E_k$. Then for every $1 \leqslant s < \infty$:*

*(a)*

$$\left\| \sum_{j=s}^{\infty} K_j * \chi_{G_{j-s}} \right\|_2^2 \lesssim 2^{-s\frac{n-1}{2}} \alpha |G|.$$

*(b) For every weight $u \in A_1$,*

$$\left\| \sum_{j=s}^{\infty} K_j * \chi_{G_{j-s}} \right\|_{L^2(u)}^2 \lesssim \|u\|_{A_1}^2 \alpha u(G).$$

*(c)*

$$\left\| \sum_{j=s}^{\infty} K_j * \chi_{G_{j-s}} \right\|_{L^2((M\chi_E)^{-1})}^2 \lesssim |G|.$$

*Proof.* The proof of $(a)$ is exactly the same as that of [35, Estimate (3.1)], where the author proves an estimate for the bad part of a Calderón-Zygmund decomposition without using its cancellation property (which allows us to adapt it to our case). In fact, this estimate is conveniently stated in [124, Section 2, Lemma 2] in the following way:

- Let $v = \sum_{Q \in \mathcal{F}} v_Q$, where $\mathcal{F}$ is a family of disjoint dyadic cubes, with $\operatorname{supp} v_Q \subseteq Q$ and $\int |v_Q| \lesssim \alpha |Q|$. Define $\mathcal{F}_k = \{Q \in \mathcal{F} : |Q| = 2^{nk}\}$ for $k \geqslant 1$, $\mathcal{F}_0 = \{Q \in \mathcal{F} : |Q| \leqslant 1\}$ and $V_k = \sum_{Q \in \mathcal{F}_k} v_Q$. Then

$$\left\| \sum_{j=s}^{\infty} K_j * V_{j-s} \right\|_2^2 \lesssim 2^{-s\frac{n-1}{2}} \alpha \|v\|_1.$$

For our purposes, take the function $v = \chi_G$, the family $\mathcal{F} = \{Q_i^k\}_{k,i=0}^{\infty}$ and $\mathcal{F}_k = \{Q_i^k\}_{i=0}^{\infty}$. Then,

$$\chi_G = \sum_{i,k=0}^{\infty} \chi_{G \cap Q_i^k},$$

and it holds that $\operatorname{supp}\chi_{G\cap Q_i^k} \subseteq Q_i^k$ and

$$\int \chi_{G\cap Q_i^k}(x)dx = |G \cap Q_i^k| = |G_k \cap Q_i^k| \leqslant |E_k \cap Q_i^k| \approx \alpha|Q_i^k|.$$

Hence, since $V_k = \sum_{i=0}^{\infty} \chi_{G\cap Q_i^k} = \chi_{G_k}$,

$$\left\| \sum_{j=s}^{\infty} K_j * \chi_{G_{j-s}} \right\|_2^2 \lesssim 2^{-s\frac{n-1}{2}} \alpha \|\chi_G\|_1 = 2^{-s\frac{n-1}{2}} \alpha |G|.$$

Let us prove $(b)$. Writing the left-hand side as an inner product in $L^2(u)$ and using its bilinearity and symmetry, we get that it can be essentially majorized by

$$\sum_{j=s}^{\infty}\sum_{i=s}^{j} \int |K_j * \chi_{G_{j-s}}(x)||K_i * \chi_{G_{i-s}}(x)|u(x)dx.$$

Since $\chi_{G_k} = \sum_{l=0}^{\infty} \chi_{G_k\cap Q_l^k}$ for every $k \geqslant 0$, we can write the previous expression as

$$\sum_{j=s}^{\infty}\sum_{l=0}^{\infty}\left(\sum_{i=s}^{j}\sum_{m=0}^{\infty} \int |K_j| * \chi_{G_{j-s}\cap Q_l^{j-s}}(x)|K_i| * \chi_{G_{i-s}\cap Q_m^{i-s}}(x)u(x)dx\right). \tag{2.2}$$

Now, let us look at the term in parentheses, where $Q_l^{j-s}$ is fixed. Using (2.1), we know that the support of the first convolution is contained in

$$Q_l^{j-s} + B(0, 2^j) \subseteq \overline{Q}_l,$$

where $|\overline{Q}_l| = 2^{(j+2)n}$, since a cube containing the sum[1] would need to have side-length $2^{j+1} + 2^{j-s} \leqslant 2^{j+2}$, and

$$|K_j| * \chi_{G_{j-s}\cap Q_l^{j-s}}(x) \leqslant 2^{-jn}|G_{j-s} \cap Q_l^{j-s}|.$$

Similarly, for every $s \leqslant i \leqslant j$ and every $m \geqslant 0$, the support of the second convolution is contained in $\overline{Q}_m$ with $|\overline{Q}_m| = 2^{(i+2)n}$ and $Q_m^{i-s} \subseteq \overline{Q}_m$. Moreover, since $x \in \overline{Q}_l$ (for the first convolution to be non-zero), we have that

$$\begin{aligned}|K_i| * \chi_{G_{i-s}\cap Q_m^{i-s}}(x) &= \int_{G_{i-s}\cap Q_m^{i-s}} |K_i(x-z)|dz = \int_{G_{i-s}\cap Q_m^{i-s}\cap 2\overline{Q}_l} |K_i(x-z)|dz \\ &\leqslant 2^{-in}|G_{i-s} \cap Q_m^{i-s} \cap 2\overline{Q}_l|,\end{aligned}$$

which can be majorized by

$$2^{-in}|G_{i-s} \cap Q_m^{i-s}|$$

together with the fact that we only need to consider the cubes $Q_m^{i-s} \subseteq 4\overline{Q}_l$. Here we used again (2.1) to see that $z \in \overline{Q}_l + B(0, 2^i) \subseteq 2\overline{Q}_l$ and $|K_i| \leqslant 2^{-in}$. Summing up, we will use the four following facts:

[1] Actually, $\overline{Q}_l$ can be taken to be the dilation $2^{s+2}Q_l^{j-s}$, but that would just complicate the notation.

- $x \in \overline{Q}_l \cap \overline{Q}_m$,
- $|K_j| * \chi_{G_{j-s} \cap Q_l^{j-s}}(x) \leqslant 2^{-jn}|G_{j-s} \cap Q_l^{j-s}|$,
- $|K_i| * \chi_{G_{i-s} \cap Q_m^{i-s}}(x) \leqslant 2^{-in}|G_{i-s} \cap Q_m^{i-s}|$,
- $\bigcup_{i=s}^{j} \bigcup_{m=0}^{\infty} Q_m^{i-s} \subseteq 4\overline{Q}_l$.

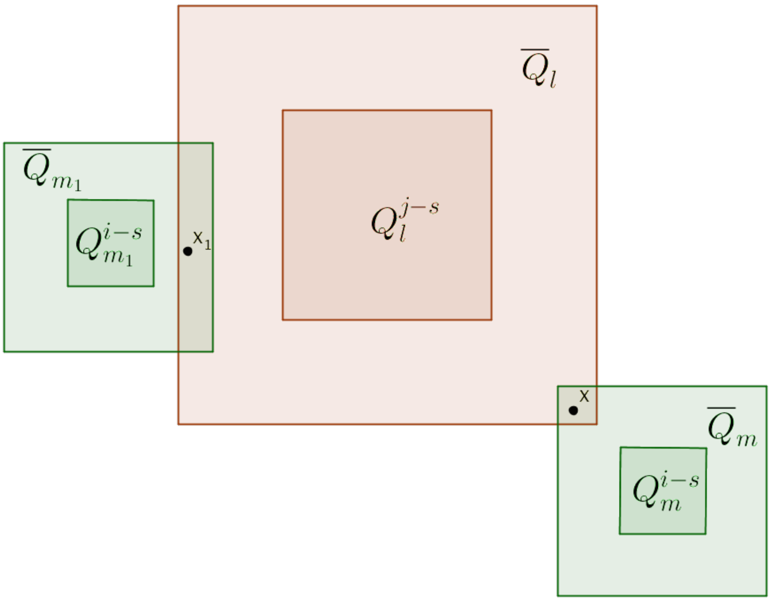


Figure 2.2: Idea of the setting when $Q_l^{j-s}$ is fixed, and we have two cubes $Q_m^{i-s}$ and $Q_{m_1}^{i-s}$.

With this, we can finish the proof of $(b)$. We bound the expression in parentheses in (2.2) by

$$\begin{aligned}
&2^{-jn}|G_{j-s} \cap Q_l^{j-s}| \sum_{i=s}^{j} \sum_{m=0}^{\infty} |G_{i-s} \cap Q_m^{i-s}| \frac{u(\overline{Q}_l \cap \overline{Q}_m)}{2^{in}} \\
&\lesssim \alpha 2^{-jn}|G_{j-s} \cap Q_l^{j-s}| \sum_{i=s}^{j} \sum_{m=0}^{\infty} |Q_m^{i-s}| \frac{u(\overline{Q}_m)}{2^{in}} \\
&\leqslant \alpha \|u\|_{A_1} 2^{-jn}|G_{j-s} \cap Q_l^{j-s}| \sum_{i=s}^{j} \sum_{m=0}^{\infty} u(Q_m^{i-s}) \\
&\leqslant \alpha \|u\|_{A_1} 2^{-jn}|G_{j-s} \cap Q_l^{j-s}| u\left(4\overline{Q}_l\right) \leqslant \alpha \|u\|_{A_1}^2 u(G_{j-s} \cap Q_l^{j-s}),
\end{aligned}$$

recalling that $|G_{i-s} \cap Q_m^{i-s}| \lesssim \alpha |Q_m^{i-s}|$ and that $|\overline{Q}_m| \approx 2^{in}$, $|4\overline{Q}_l| \approx 2^{jn}$. We can plug it in (2.2) to get the sought-after estimate:

$$\alpha \|u\|_{A_1}^2 \sum_{j=s}^{\infty} \sum_{l=0}^{\infty} u(G_{j-s} \cap Q_l^{j-s}) = \alpha \|u\|_{A_1}^2 \sum_{j=s}^{\infty} u(G_{j-s}) = \alpha \|u\|_{A_1}^2 u(G).$$

Finally we prove $(c)$. Exactly as in $(b)$, it is enough to show that

$$\sum_{j=s}^{\infty} \sum_{l=0}^{\infty} \left( \sum_{i=s}^{j} \sum_{m=0}^{\infty} \int |K_j| * \chi_{G_{j-s} \cap Q_l^{j-s}}(x) |K_i| * \chi_{G_{i-s} \cap Q_m^{i-s}}(x) (M\chi_E)^{-1}(x) dx \right) \lesssim |G|, \quad (2.3)$$

where the expression in parentheses is controlled by

$$2^{-jn} |G_{j-s} \cap Q_l^{j-s}| \sum_{i=s}^{j} \sum_{m=0}^{\infty} |G_{i-s} \cap Q_m^{i-s}| \frac{(M\chi_E)^{-1}(\overline{Q}_m)}{2^{in}}.$$

Now, since $Q_m^{i-s} \subseteq 4\overline{Q}_l$, $|\overline{Q}_l| = 2^{(j+2)n}$ and $|\overline{Q}_m| = 2^{(i+2)n}$, we deduce that $\overline{Q}_m \subseteq 5\overline{Q}_l$, and hence, by Lemma 1.6,

$$\frac{(M\chi_E)^{-1}(\overline{Q}_m)}{2^{in}} \lesssim \frac{(M\chi_E)^{-1}(5\overline{Q}_l)}{2^{jn}}.$$

Using this, we obtain

$$(2^{-jn})^2 (M\chi_E)^{-1}(5\overline{Q}_l) |G_{j-s} \cap Q_l^{j-s}| \sum_{i=s}^{j} \sum_{m=0}^{\infty} |G_{i-s} \cap Q_m^{i-s}|.$$

Now, we use the $A_2^{\mathcal{R}}$ condition of $(M\chi_E)^{-1}$ with the subset[2] $G \cap 4\overline{Q}_l \subseteq 5\overline{Q}_l$, and that $\bigcup_{i=s}^{j} \bigcup_{m=0}^{\infty} Q_m^{i-s} \subseteq 4\overline{Q}_l$ to get

$$|G \cap 4\overline{Q}_l)|^{-1} |G_{j-s} \cap Q_l^{j-s}| |G \cap 4\overline{Q}_l|,$$

which we can simplify and sum over $s \leqslant j < \infty$ and $l \geqslant 0$ to obtain that the left-hand side in (2.3) is majorized by $|G|$. □

The third and last lemma will be an interpolation argument (in the spirit of [113]) on the estimates in Lemma 2.5 that will yield the right control of the $L^2$ norm with respect to the desired weights. Let us just remark that the first estimate will be used to prove the second one, so in this case we will still need to consider subsets $G \subseteq E$.

**Lemma 2.6.** *Let $0 < \alpha < 1$ and let $E = \bigcup_{k=0}^{\infty} E_k$ be a measurable set decomposed as in Remark 2.4. Let $G \subseteq E$ be a measurable subset and define, for every $k \geqslant 0$, $G_k = G \cap E_k$. Then, for every $1 \leqslant s < \infty$ and every $u \in A_1$:*

[2] Recalling that $M\chi_E \equiv 1$ on $G \subseteq E$ and assuming $G \cap 4\overline{Q}_l$ has positive measure, otherwise the whole expression would be zero.

*(d)*

$$\left\| \sum_{j=s}^{\infty} K_j * \chi_{G_{j-s}} \right\|_{L^2(u)}^2 \lesssim \|u\|_{A_1}^2 2^{-s\varepsilon} \alpha u(G),$$

*with* $\varepsilon = \frac{n-1}{2}\left(\frac{1}{1+2^{n+1}\|u\|_{A_1}}\right)$.

*(e)*

$$\left\| \sum_{j=s}^{\infty} K_j * \chi_{E_{j-s}} \right\|_{L^2((M\chi_E)^{-\theta}u)}^2 \lesssim \|u\|_{A_1}^2 2^{-s\beta} \alpha^{1-\theta} u(E).$$

*with* $\theta = \frac{1}{1+2^{n+1}\|u\|_{A_1}}$ *and* $\beta = \frac{n-1}{2}\left(\frac{2^{n+1}\|u\|_{A_1}}{(1+2^{n+1}\|u\|_{A_1})^2}\right)$

*Proof.* For $a, b > 0$, define $w_{a,b}(x) = \min\{au(x), b\}$. Fix $t > 0$ and write

$$B^1 = \{x \in \mathbb{R}^n : \|u\|_{A_1}^2 u(x) \leqslant 2^{-s\frac{n-1}{2}} t\},$$

and $B^2 = \mathbb{R}^n \backslash B^1$. For every $k \geqslant 0$, we write $G_k = G_k^1 \cup G_k^2$, where $G_k^i = G_k \cap B^i \subseteq E_k$ and $G^i = \bigcup_{k=0}^{\infty} G_k^i = G \cap B^i$, for $i = 1, 2$. Using $(a)$ and $(b)$ in Lemma 2.5 and the definitions we just introduced, we get

$$\begin{aligned}\left\| \sum_{j=s}^{\infty} K_j * \chi_{G_{j-s}} \right\|_{L^2(w_{1,t})}^2 &\lesssim \left\| \sum_{j=s}^{\infty} K_j * \chi_{G_{j-s}^1} \right\|_{L^2(u)}^2 + t \left\| \sum_{j=s}^{\infty} K_j * \chi_{G_{j-s}^2} \right\|_2^2 \\ &\lesssim \|u\|_{A_1}^2 \alpha u(G^1) + 2^{-s\frac{n-1}{2}} t\alpha |G^2| = \alpha w_{a,bt}(G),\end{aligned}$$

with $a = \|u\|_{A_1}^2$ and $b = 2^{-s\frac{n-1}{2}}$. Now, we integrate both sides with respect to $t \in (0, \infty)$ equipped with the measure $\frac{dt}{t^{\theta+1}}$, where $0 < \theta < 1$. Using Fubini and the definition of the weight, we obtain

$$\left\| \sum_{j=s}^{\infty} K_j * \chi_{G_{j-s}} \right\|_{L^2(u^{1-\theta})}^2 \lesssim \alpha a^{1-\theta} b^{\theta} u^{1-\theta}(G).$$

But we know (see [98]) that if $u \in A_1$ and $r = 1 + \frac{1}{2^{n+1}\|u\|_{A_1}}$, then $u^r \in A_1$ and $\|u^r\|_{A_1} \lesssim \|u\|_{A_1}$, so applying what we have shown to $u^r$ and taking $\theta = (r-1)/r$, we obtain

$$\left\| \sum_{j=s}^{\infty} K_j * \chi_{G_{j-s}} \right\|_{L^2(u)}^2 \lesssim \|u\|_{A_1}^{\frac{2^{n+2}\|u\|_{A_1}}{1+2^{n+1}\|u\|_{A_1}}} 2^{-s\frac{n-1}{2}\left(\frac{1}{1+2^{n+1}\|u\|_{A_1}}\right)} \alpha u(G).$$

Notice that the exponent in $\|u\|_{A_1}$ is always less than or equal to 2, so we conclude $(d)$. The proof of $(e)$ follows the same idea but interpolating estimates $(c)$ (in Lemma 2.5) and $(d)$. Define in this case $v_{a,b}(x) = \min\{au(x), b(M\chi_E)^{-1}(x)\}$. Fix $t > 0$ and write

$$C^1 = \{x \in \mathbb{R}^n : \alpha \|u\|_{A_1}^2 2^{-s\varepsilon} u(x) \leqslant (M\chi_E)^{-1}(x) t\},$$

$C^2 = \mathbb{R}^n \backslash C^1$. Now we decompose, for every $k \geqslant 0$, $E_k = E_k^1 \cup E_k^2$, with $E_k^i = E_k \cap C^i$ and $E^i = \bigcup_{k=0}^{\infty} E_k^i = E \cap C^i$, for $i = 1, 2$. We need to use $(c)$ in Lemma 2.5 and $(d)$:

$$\left\| \sum_{j=s}^{\infty} K_j * \chi_{E_{j-s}} \right\|_{L^2(v_{1,t})}^2 \lesssim \left\| \sum_{j=s}^{\infty} K_j * \chi_{E_{j-s}^1} \right\|_{L^2(u)}^2 + t \left\| \sum_{j=s}^{\infty} K_j * \chi_{E_{j-s}^2} \right\|_{L^2((M\chi_E)^{-1})}^2$$
$$\lesssim \|u\|_{A_1}^2 2^{-s\varepsilon} \alpha u(E^1) + t|E^2| = v_{a,bt}(E),$$

with $a = \alpha \|u\|_{A_1}^2 2^{-s\varepsilon}$ and $b = 1$. Exactly as before and recalling that $M\chi_E \equiv 1$ on $E$, we deduce that for every $0 < \theta < 1$,

$$\left\| \sum_{j=s}^{\infty} K_j * \chi_{E_{j-s}} \right\|_{L^2((M\chi_E)^{-\theta} u^{1-\theta})}^2 \lesssim a^{1-\theta} b^{\theta} u^{1-\theta}(E).$$

Finally, we apply this to $u^r$ instead of $u$, take $\theta = (r-1)/r$, substitute $a, b$ and we conclude $(e)$ with the claimed values for $\theta$ and $\beta$. $\square$

## 2.3 The main results

As we mentioned at the beginning of this chapter, we will give three results concerning weighted estimates for the Bochner-Riesz operator at the critical index. The first two were already known, but we will include their proofs since they do not follow the same scheme as the ones presented in [107] and [124] respectively. We will also keep track of the boundedness constants depending on the weights.

**Theorem 2.7.** *For every $n > 1$, the Bochner-Riesz operator at the critical index $B$ is of strong-type $(p, p)$ for every weight $w \in A_p$ and every $1 < p < \infty$, with boundedness constant controlled by $\|w\|_{A_p}^{\max\{2, \frac{2}{p-1}\}}$.*

In [107], the authors follow an interpolation argument for analytic families of operators. Even though the underlying idea is simple, there are some technicalities that complicate the proof. Later, when A. Vargas went on to prove the weighted weak-type $(1, 1)$ estimate in [124], she realized that using the key inequality from the earlier paper by M. Christ [35], the strong-type $(2, 2)$ for weights in $A_2$ was just a consequence of the control $|K_j * f| \lesssim Mf$ that we have on the decomposition of the kernel. We will present this simplification with the dependence on the weight of the boundedness constant:

*Proof.* In [35, Lemma 3.1], the author shows that

$$|K_j * \widetilde{K}_j| \lesssim 2^{-jn}(1 + |x|)^{-\frac{n-1}{2}} \chi_{\{|x| \leqslant 2^{j+1}\}}(x),$$

where $\widetilde{K}_j(x) = K_j(-x)$. With this estimate, it is an easy computation to check that

$$\|K_j * \widetilde{K}_j\|_1 \lesssim 2^{-\frac{n-1}{2}j},$$

and hence, for every function $f \in L^2(\mathbb{R}^n)$, if $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $L^2(\mathbb{R}^n)$,

$$\begin{aligned}\|K_j * f\|_2 &= \left(\langle K_j * \widetilde{K}_j * f, f\rangle\right)^{1/2} \leqslant \left(\|K_j * \widetilde{K}_j * f\|_2 \|f\|_2\right)^{1/2} \leqslant \left(\|K_j * \widetilde{K}_j\|_1 \|f\|_2^2\right)^{1/2} \\ &\lesssim 2^{-\frac{n-1}{4}j}\|f\|_2.\end{aligned}$$

On the other hand, for every weight $w \in A_2$, by Lemma 2.2 and the $L^2$-boundedness of $M$:

$$\|K_j * f\|_{L^2(w)} \lesssim \|Mf\|_{L^2(w)} \lesssim \|w\|_{A_2}\|f\|_{L^2(w)},$$

so with the usual interpolation with change of measure, we deduce that, for every $0 < \theta < 1$,

$$\|K_j * f\|_{L^2(w^\theta)} \lesssim 2^{-\frac{n-1}{4}j(1-\theta)}\|w\|_{A_2}^\theta \|f\|_{L^2(w^\theta)}.$$

Since $A_2$ weights satisfy a sharp Reverse Hölder inequality (again, see [98]), for $r = 1 + \frac{1}{2^{n+5}\|w\|_{A_2}}$ we have that $w^r \in A_2$ and $\|w^r\|_{A_2} \lesssim \|w\|_{A_2}$. Hence, applying the previous estimate to this weight and choosing $\theta = 1/r < 1$, we conclude that

$$\|K_j * f\|_{L^2(w)} \lesssim 2^{-\frac{n-1}{4\left(1+2^{n+5}\|w\|_{A_2}\right)}j}\|w\|_{A_2}\|f\|_{L^2(w)}.$$

Therefore, we can sum over $j$ to deduce that

$$\|Bf\|_{L^2(w)} \leqslant \sum_{j=1}^{\infty}\|K_j * f\|_{L^2(w)} \lesssim \|w\|_{A_2}(2^c - 1)^{-1}\|f\|_{L^2(w)},$$

with $c = \frac{n-1}{4\left(1+2^{n+5}\|w\|_{A_2}\right)}$. But $(2^c-1)^{-1} \approx \|w\|_{A_2}$, so we get that $B$ is of strong-type $(2,2)$ for every weight in $A_2$ with boundedness constant controlled by $\|w\|_{A_2}^2$. By Rubio de Francia's extrapolation (see its version in [50] for the behavior of the constants), we deduce that for every $1 < p < \infty$, we have the strong-type $(p,p)$ for every weight $w \in A_p$ and with constant controlled by

$$\|w\|_{A_p}^{\max\left\{2, \frac{2}{p-1}\right\}}.$$

□

**Theorem 2.8.** *For every $n > 1$, the Bochner-Riesz operator at the critical index $B$ is of weak-type $(1,1)$ for every weight $u \in A_1$, with boundedness constant controlled by $\|u\|_{A_1}^5$.*

In this case, we present a slightly simpler proof than the one in [124]. The main difference is the fact that it is enough to show that $B$ is of restricted weak-type $(1,1)$ for weights in $A_1$. Dealing with characteristic functions allows us to avoid, by means of Remark 2.4, the Calderón-Zygmund decomposition in good and bad parts. We still use the cubes, but the only decomposition we need is[3] $\chi_E = \sum_{k=0}^{\infty} \chi_{E_k}$.

*Proof.* Using Plancherel's theorem, we know that $B$ is a convolution operator whose kernel $K$ belongs to $L^2(\mathbb{R}^n)$, so as we mentioned in (1.14), $B$ is an $(\varepsilon, \delta)$-atomic operator. Therefore, by Theorem 1.14, it is enough to show that it is of restricted weak-type $(1,1)$ for every weight in $u \in A_1$. Take $\alpha > 0$. If $\alpha \geqslant 1$, then we use Theorem 2.7:

$$\begin{aligned}\alpha u\left(\{x : |B\chi_E(x)| > \alpha\}\right) &\leqslant \alpha^2 u\left(\{x : |B\chi_E(x)| > \alpha\}\right) \leqslant \|B\chi_E\|_{L^2(u)}^2 \\ &\lesssim \|u\|_{A_2}^4 \|\chi_E\|_{L^2(u)}^2 \leqslant \|u\|_{A_1}^4 u(E).\end{aligned}$$

If $0 < \alpha < 1$, we decompose $E$ as in Remark 2.4 and

$$\alpha u\left(\{x : |B\chi_E(x)| > \alpha\}\right) \lesssim \alpha u\left(\bigcup_{i,k=0}^{\infty} 3Q_i^k\right) + \alpha u\left(\left\{x \notin \bigcup_{i,k=0}^{\infty} 3Q_i^k : |B\chi_E(x)| > \alpha\right\}\right).$$

For the first term, we use that $u$ is doubling and

$$\begin{aligned}\alpha u\left(\bigcup_{i,k=0}^{\infty} 3Q_i^k\right) &\lesssim \alpha \|u\|_{A_1} \sum_{i,k=0}^{\infty} u(Q_i^k) = \|u\|_{A_1} \sum_{i,k=0}^{\infty} \frac{u(Q_i^k)}{|Q_i^k|} \alpha |Q_i^k| \\ &\approx \|u\|_{A_1} \sum_{i,k=0}^{\infty} \frac{u(Q_i^k)}{|Q_i^k|} |E_k \cap Q_i^k| \lesssim \|u\|_{A_1}^2 u(E).\end{aligned}$$

On the other hand, looking at the intersection of the supports of $K_j$ and $\chi_{E_k}$, it is easy to see that if $x \notin \bigcup_{i,k=0}^{\infty} 3Q_i^k$, then

$$B\chi_E = \sum_{j=1}^{\infty} \sum_{k=0}^{\infty} K_j * \chi_{E_k} = \sum_{k=0}^{\infty} \sum_{j=k+1}^{\infty} K_j * \chi_{E_k} = \sum_{s=1}^{\infty} \sum_{j=s}^{\infty} K_j * \chi_{E_{j-s}},$$

[3] Recall that $E_k$ is the portion of $E$ lying in cubes of measure $2^{nk}$ if $k > 0$ or measure less than or equal to 1 if $k = 0$.

so using Chebyshev and $(d)$ in Lemma 2.6 with $G = E$,

$$\alpha u\left(\left\{x \notin \bigcup_{i,k=0}^{\infty} 3Q_i^k : |B\chi_E(x)| > \alpha\right\}\right) \leqslant \alpha u\left(\left\{x \in \mathbb{R}^n : \left|\sum_{s=1}^{\infty}\sum_{j=s}^{\infty} K_j * \chi_{E_{j-s}}\right| > \alpha\right\}\right)$$
$$\leqslant \alpha^{-1}\left\|\sum_{s=1}^{\infty}\sum_{j=s}^{\infty} K_j * \chi_{E_{j-s}}\right\|_{L^2(u)}^2 \leqslant \alpha^{-1}\left(\sum_{s=1}^{\infty}\left\|\sum_{j=s}^{\infty} K_j * \chi_{E_{j-s}}\right\|_{L^2(u)}\right)^2$$
$$\lesssim \alpha^{-1}\left(\sum_{s=1}^{\infty}\|u\|_{A_1} 2^{-s\frac{\varepsilon}{2}}\alpha^{1/2}u(E)^{1/2}\right)^2 = \|u\|_{A_1}^2(2^{\varepsilon/2}-1)^{-2}u(E) \lesssim \|u\|_{A_1}^4 u(E),$$

since $(2^{\varepsilon/2}-1)^{-2} \approx \|u\|_{A_1}^2$. So taking supremum over $\alpha > 0$, we have shown that

$$\|B\chi_E\|_{L^1(u)} \lesssim \|u\|_{A_1}^4 u(E),$$

which by Theorem 1.14, proves the weak-type $(1,1)$ for every weight $u \in A_1$ and constant controlled by $\|u\|_{A_1}^5$. □

Finally, let us present the new weighted result for the Bochner-Riesz operator at the critical index:

**Theorem 2.9.** *Given $n > 1$, the Bochner-Riesz operator at the critical index $B$ satisfies that, for every $u \in A_1$, there exists $1 < p_0 < \infty$ depending on $\|u\|_{A_1}$ such that, for each measurable set $E \subseteq \mathbb{R}^n$,*

$$\|B\chi_E\|_{L^{p_0,\infty}((M\chi_E)^{1-p_0}u)} \lesssim \|u\|_{A_1}^{4/p_0} u(E)^{1/p_0}.$$

*More precisely, the exact dependence is*

$$p_0(\|u\|_{A_1}) = 1 + \frac{1}{1 + 2^{n+1}\|u\|_{A_1}}.$$

*Proof.* We will follow the same strategy as in the proof of Theorem 2.8. Let $\theta \in (0,1)$ be as in $(e)$ from Lemma 2.6. If $\alpha \geqslant 1$ and $w_\theta := (M\chi_E)^{-\theta}u$, then by Theorem 2.7:

$$\alpha^{1+\theta} w_\theta(\{x : |B\chi_E(x)| > \alpha\}) \leqslant \alpha^2 w_\theta(\{x : |B\chi_E(x)| > \alpha\})$$
$$\leqslant \|B\chi_E\|_{L^2(w_\theta)}^2 \lesssim \|w_\theta\|_{A_2}^4 \|\chi_E\|_{L^2(w_\theta)}^2 \lesssim \|u\|_{A_1}^4 u(E).$$

In the last inequality we used that

$$\|w_\theta\|_{A_2} \leqslant \|(M\chi_E)^\theta\|_{A_1}\|u\|_{A_1} \approx \frac{\|u\|_{A_1}}{1-\theta} \approx \|u\|_{A_1},$$

since $0 < \theta = \frac{1}{1+2^{n+1}\|u\|_{A_1}} \leqslant \frac{1}{1+2^{n+1}} < 1$. If $0 < \alpha < 1$, we decompose $E$ as in Remark 2.4 and

$$\alpha^{1+\theta} w_\theta\left(\{x : |B\chi_E(x)| > \alpha\}\right) \lesssim \alpha^{1+\theta} w_\theta\left(\bigcup_{i,k=0}^{\infty} 3Q_i^k\right) + \alpha^{1+\theta} w_\theta\left(\left\{x \notin \bigcup_{i,k=0}^{\infty} 3Q_i^k : |B\chi_E(x)| > \alpha\right\}\right).$$

For the first term, we use that $w_\theta \in A_{1+\theta}^{\mathcal{R}}$ and by (1.8), $\|w_\theta\|_{A_{1+\theta}^{\mathcal{R}}}^{1+\theta} \lesssim \|u\|_{A_1}$. Also, recall that $w_\theta = u$ on $E$:

$$\begin{aligned}\alpha^{1+\theta} w_\theta\left(\bigcup_{i,k=0}^{\infty} 3Q_i^k\right) &\lesssim \alpha^{1+\theta}\|u\|_{A_1} \sum_{i,k=0}^{\infty} w_\theta(Q_i^k)\\ &\approx \|u\|_{A_1} \sum_{i,k=0}^{\infty} \frac{w_\theta(Q_i^k)}{u(E_k \cap Q_i^k)}\left(\frac{|E_k \cap Q_i^k|}{|Q_i^k|}\right)^{1+\theta} u(E_k \cap Q_i^k)\\ &\lesssim \|u\|_{A_1}^2 u(E).\end{aligned}$$

For the second term, we argue as before but now with $(e)$ in Lemma 2.6:

$$\begin{aligned}\alpha^{1+\theta} w_\theta\left(\left\{x \notin \bigcup_{i,k=0}^{\infty} 3Q_i^k : |B\chi_E(x)| > \alpha\right\}\right) &\leqslant \alpha^{1+\theta} w_\theta\left(\left\{x : \left|\sum_{s=1}^{\infty}\sum_{j=s}^{\infty} K_j * \chi_{E_{j-s}}\right| > \alpha\right\}\right)\\ \leqslant \alpha^{\theta-1}\left\|\sum_{s=1}^{\infty}\sum_{j=s}^{\infty} K_j * \chi_{E_{j-s}}\right\|_{L^2(w_\theta)}^2 &\leqslant \alpha^{\theta-1}\left(\sum_{s=1}^{\infty}\left\|\sum_{j=s}^{\infty} K_j * \chi_{E_{j-s}}\right\|_{L^2(w_\theta)}\right)^2\\ \lesssim \alpha^{\theta-1}\left(\sum_{s=1}^{\infty}\|u\|_{A_1} 2^{-s\frac{\beta}{2}}\alpha^{\frac{1-\theta}{2}} u(E)^{1/2}\right)^2 &= \|u\|_{A_1}^2 (2^{\beta/2}-1)^{-2} u(E) \lesssim \|u\|_{A_1}^4 u(E),\end{aligned}$$

since again $(2^{\beta/2}-1)^{-2} \approx \|u\|_{A_1}^2$. So taking supremum over $\alpha > 0$, we have shown that, for $p_0 = 1 + \frac{1}{1+2^{n+1}\|u\|_{A_1}} > 1$,

$$\|B\chi_E\|_{L^{p_0,\infty}((M\chi_E)^{1-p_0}u)} \lesssim \|u\|_{A_1}^{4/p_0} u(E)^{1/p_0}.$$

$\square$

For later purposes, we will also need the following fact stating that Theorem 2.9 holds for $B^r$ uniformly in $r > 0$:

**Corollary 2.10.** *For every weight $u \in A_1$, there is some $1 < p_0 < \infty$, depending on $\|u\|_{A_1}$, such that, for each measurable set $E \subseteq \mathbb{R}^n$,*

$$\|B^r \chi_E\|_{L^{p_0,\infty}((M\chi_E)^{1-p_0}u)} \lesssim \|u\|_{A_1}^{4/p_0} u(E)^{1/p_0},$$

*uniformly in $r > 0$. The dependence of $p_0$ on $\|u\|_{A_1}$ is the same as in Theorem 2.9.*

*Proof.* It is easy to check, using the formula for the Fourier transform of radial functions (see [63, Appendix B.5]), that

$$K_r(x) = r^{-n} K_1(r^{-1}x),$$

where now, for every $r > 0$, $K_r$ denotes the convolution kernel associated with $B^r$, and hence

$$B^r f(x) = K_r * f(x) = (K_1 * f(\cdot r))(r^{-1}x) = B(f(\cdot r))(r^{-1}x).$$

If we take $f = \chi_E$, we can write (here, $r^{-1}E = \{r^{-1}x \in \mathbb{R}^n : x \in E\}$):

$$B^r \chi_E(x) = B\chi_{r^{-1}E}(r^{-1}x),$$

and we can use Theorem 2.9 to choose $1 < p_0 < \infty$ depending only on $\|u\|_{A_1}$ and get that:

$$\begin{aligned}
\|B^r \chi_E\|_{L^{p_0,\infty}((M\chi_E)^{1-p_0}u)} &= \sup_{\alpha>0} \alpha \left( \int_{\{|B\chi_{r^{-1}E}(r^{-1}x)|>\alpha\}} (M\chi_E)^{1-p_0}(x) u(x)\, dx \right)^{1/p_0} \\
&= \sup_{\alpha>0} \alpha \left( \int_{\{|B\chi_{r^{-1}E}(y)|>\alpha\}} (M\chi_E)^{1-p_0}(ry)\, u(ry)\, r^n dy \right)^{1/p_0} \\
&= \sup_{\alpha>0} \alpha \left( \int_{\{|B\chi_{r^{-1}E}(y)|>\alpha\}} (M\chi_{r^{-1}E})^{1-p_0}(y) u_r(y)\, dy \right)^{1/p_0} \\
&= \|B\chi_{r^{-1}E}\|_{L^{p_0,\infty}((M\chi_{r^{-1}E})^{1-p_0}u_r)} \lesssim \|u_r\|_{A_1}^{4/p_0} u_r(r^{-1}E)^{1/p_0},
\end{aligned}$$

where $u_r(x) = r^n u(rx)$. But a simple change of variables shows that

$$\|u_r\|_{A_1} = \|u\|_{A_1} \quad \text{and} \quad u_r(r^{-1}E) = u(E),$$

as we claimed. Notice that it is essential that the dependence of $p_0$ on $u$ is in terms of $\|u\|_{A_1} = \|u_r\|_{A_1}$, so we have the same $p_0$ for both $u$ and $u_r$. $\square$

# Chapter 3

# Fourier Multipliers

In this chapter, we will try to use extrapolation results on Fourier multipliers in order to obtain weighted weak-type $(1, 1)$ estimates. The arguments in Sections 3.2 and 3.3, where we consider multipliers on $\mathbb{R}$ and radial multipliers on $\mathbb{R}^n$ respectively, will be based on a general technique that we present in Proposition 3.1. The idea is to take advantage of restricted weak-type estimates that we already know (like the one we have found for the Bochner-Riesz operator in Theorem 2.9) and transfer them to more general operators. In Section 3.4 we do not use this approach but rather we establish restricted weak-type $(p_0, p_0)$ estimates directly for multipliers of Hörmander type.

## 3.1 The averaging technique

The following proposition represents a simple idea that will turn out to be really useful to prove endpoint estimates for operators that can be written as averages.

**Proposition 3.1.** *Let $(\Omega, \mu)$ be a measure space and let $\{T_\omega\}_{\omega\in\Omega}$ be a collection of sublinear operators indexed by $\omega \in \Omega$ and such that, for every $u \in A_1$ there is some $1 < p_0 < \infty$ so that, for each $E \subseteq \mathbb{R}^n$ measurable set,*

$$\|T_\omega \chi_E\|_{L^{p_0,\infty}((M\chi_E)^{1-p_0}u)} \lesssim \varphi_{p_0}(\|u\|_{A_1}) u(E)^{1/p_0},$$

*uniformly in $\omega \in \Omega$. Then, for any given $\Phi \in L^1(\Omega, |\mu|)$, the operator*

$$Tf(x) = \int_\Omega T_\omega f(x) \Phi(\omega) d\mu(\omega)$$

*is of restricted weak-type $(1, 1)$ for every $u \in A_1$ with constant*

$$\|u\|_{A_1}^{1-\frac{1}{p_0}} \varphi_{p_0}(\|u\|_{A_1}) \|\Phi\|_{L^1(\Omega,|\mu|)}.$$

*If $T$ is in addition $(\varepsilon, \delta)$-atomic approximable, then it is of weak-type $(1,1)$ for every $u \in A_1$ with constant*

$$\|u\|_{A_1}^{2-\frac{1}{p_0}} \varphi_{p_0}(\|u\|_{A_1}) \|\Phi\|_{L^1(\Omega,|\mu|)}.$$

*Proof.* Given $u \in A_1$, take its associated $1 < p_0 = p_0(u) < \infty$ and by Minkowski's inequality

$$\begin{aligned}\|T\chi_E\|_{L^{p_0,\infty}((M\chi_E)^{1-p_0}u)} &\leqslant \int_\Omega \|T_\omega \chi_E\|_{L^{p_0,\infty}((M\chi_E)^{1-p_0}u)} |\Phi(\omega)| d|\mu|(\omega) \\ &\lesssim \varphi_{p_0}(\|u\|_{A_1}) \|\Phi\|_{L^1(\Omega,|\mu|)} u(E)^{1/p_0}.\end{aligned}$$

Then, we apply Theorem 1.11 to obtain the restricted weak-type $(1,1)$ estimate with the right constant. If $T$ is $(\varepsilon, \delta)$-atomic approximable, Theorem 1.14 completes the proof. □

**Remark 3.2.** *Notice that if we only had uniform restricted weak-type $(1,1)$ estimates for the family $\{T_\omega\}_{\omega \in \Omega}$, then the average operator $T$ would not necessarily inherit that property, since $L^{1,\infty}$ is not a Banach space. The fact that we can transfer estimates from $T_\omega$ to $T$ at level $p_0 > 1$ (where Minkowski's inequality is allowed) and then extrapolate down to $p = 1$, is the key ingredient in this result.*

## 3.2 Fourier multipliers on $\mathbb{R}$

The next application will illustrate our technique with a very simple example. The weighted estimate that will play the role of Theorem 2.9 is the following:

**Proposition 3.3.** *Given $1 < p < \infty$ and a weight $w \in A_p^{\mathcal{R}}$, the Hilbert transform $H$ satisfies the restricted weak-type estimate*

$$\|Hf\|_{L^{p,\infty}(w)} \lesssim \|w\|_{A_p^{\mathcal{R}}}^{p+1} \|f\|_{L^{p,1}(w)}.$$

This result has an easy proof based on the pointwise domination of Calderón-Zygmund operators by the so-called sparse operators, and is actually true for any operator with such a control, not just the Hilbert transform. The best result so far regarding domination by sparse operators is contained in [80], and includes all Calderón-Zygmund operators with a Dini-type condition on the modulus of continuity of the kernel. The proof of Proposition 3.3 goes as follows:

*Proof.* By the reduction to sparse operators we just pointed out, it is enough to show that

$$\|S\chi_E\|_{L^{p,\infty}(w)} \lesssim \|w\|_{A_p^{\mathcal{R}}}^{p+1} w(E)^{1/p},$$

where $S$ is a sparse operator. More precisely,

$$S\chi_E(x) := \sum_{Q \in \mathcal{S}} \frac{|E \cap Q|}{|Q|} \chi_Q(x),$$

and $\mathcal{S}$ is a sparse family of dyadic cubes, meaning that for every $Q \in \mathcal{S}$, there exists a measurable subset $F_Q \subseteq Q$ such that $|F_Q| \approx |Q|$ and $\{F_Q\}_{Q\in\mathcal{S}}$ are pairwise disjoint. We will proceed by duality. Let $h \geqslant 0$ be a function in $L^{p',1}(w)$ with $\|h\|_{L^{p',1}(w)} = 1$. Also, we know that there is a dimensional constant $c > 0$ such that, for every $Q \in \mathcal{S}$ and every $y \in Q$, we can find a cube $\widetilde{Q}_y$ centered at $y$ with

$$Q \subseteq \widetilde{Q}_y \subseteq cQ.$$

Therefore, since $|F_Q| \approx |Q| \approx |cQ|$,

$$\begin{aligned}
\int_{\mathbb{R}^n} S\chi_E(x)h(x)w(x)dx &= \sum_{Q\in\mathcal{S}} \frac{|E\cap Q|}{|Q|}\int_Q h(x)w(x)dx \approx \sum_{Q\in\mathcal{S}} \frac{|E\cap Q|}{|Q|}\frac{|F_Q|^p}{|cQ|^p}\int_Q h(x)w(x)dx \\
&\lesssim \|w\|^p_{A_p^{\mathcal{R}}} \sum_{Q\in\mathcal{S}} \frac{|E\cap Q|}{|Q|}\frac{w(F_Q)}{w(cQ)}\int_Q h(x)w(x)dx \\
&= \|w\|^p_{A_p^{\mathcal{R}}} \sum_{Q\in\mathcal{S}} \left(\frac{1}{|Q|}\int_Q \chi_E(x)dx\right)\left(\frac{1}{w(cQ)}\int_Q h(x)w(x)dx\right)\int_{F_Q} w(y)dy \\
&\leqslant \|w\|^p_{A_p^{\mathcal{R}}} \sum_{Q\in\mathcal{S}} \int_{F_Q}\left(\frac{1}{|Q|}\int_Q \chi_E(x)dx\right)\left(\frac{1}{w(\widetilde{Q}_y)}\int_{\widetilde{Q}_y} h(x)w(x)dx\right)w(y)dy \\
&\lesssim \|w\|^p_{A_p^{\mathcal{R}}} \sum_{Q\in\mathcal{S}} \int_{F_Q} M\chi_E(y)M_w^c h(y)w(y)dy,
\end{aligned}$$

where $M_w^c$ is the centered Hardy-Littlewood maximal operator associated with the measure given by $w$:

$$M_w^c h(y) = \sup_{r>0}\frac{1}{w(Q(y,r))}\int_{Q(y,r)} |h(x)|w(x)dx.$$

Here $Q(y,r)$ denotes the cube of center $y$ and side-length $r > 0$. Now, using that the sets $F_Q$ are disjoint in $Q \in \mathcal{S}$, we can sum over the cubes and, by Hölder's inequality, we conclude that

$$\int_{\mathbb{R}^n} S\chi_E(x)h(x)w(x)dx \lesssim \|w\|^p_{A_p^{\mathcal{R}}}\|M\chi_E\|_{L^{p,\infty}(w)}\|M_w^c h\|_{L^{p',1}(w)}.$$

Next, we use (1.7) for the second term and, for the third one, the fact that $M_w^c$ maps $L^{p',1}(w)$ into itself with a constant that does not depend[1] on the weight $w$. This yields that

$$\int_{\mathbb{R}^n} S\chi_E(x)h(x)w(x)dx \lesssim \|w\|^{p+1}_{A_p^{\mathcal{R}}} w(E)^{1/p}\|h\|_{L^{p',1}(w)} = \|w\|^{p+1}_{A_p^{\mathcal{R}}} w(E)^{1/p},$$

[1]The technicality of introducing the cubes $\widetilde{Q}$ is explained by the fact that we must obtain the centered maximal operator $M_w^c$. If we worked with the original cubes $Q$ instead, we would end up with the uncentered $M_w$, whose boundedness constant from $L^{p',1}(w)$ into itself does depend on $w$.

and taking supremum over $h$ we obtain $\|S\chi_E\|_{L^{p,\infty}(w)}$ on the left-hand side of the inequality, as we wanted to show. $\square$

The first result concerning Fourier multipliers that we present is the following:

**Theorem 3.4.** *Let $m$ be a function of bounded variation on $\mathbb{R}$. Then, the operator $T_m$ defined by*

$$\widehat{T_m f}(\xi) = m(\xi)\widehat{f}(\xi)$$

*is of weak-type $(1,1)$ for every weight $u \in A_1$ and with constant controlled by $\|dm\| \cdot \|u\|_{A_1}^3$, where $\|dm\|$ denotes the total variation of the measure $dm$.*

*Proof.* Since $m$ is of bounded variation on $\mathbb{R}$, the limit of $m(t)$ as $t \to -\infty$ exists, so by adding a constant to $m$ if necessary, we can assume this limit to be zero. Let $\{\varphi_j\}_j$ be a non-negative approximation to the identity as $j \to \infty$. That is:

- For every $j > 0$, it holds that $\|\varphi_j\|_1 = 1$.
- For every $r > 0$,
$$\lim_{j\to\infty} \int_{\mathbb{R}\setminus(-r,r)} \varphi_j(t)dt = 0.$$

It holds that $\|\widehat{\varphi}_j\|_\infty \leqslant \|\varphi_j\|_1 = 1$, and we can furthermore assume that the total variation $\|d\widehat{\varphi}_j\| \leqslant 2$. To this end, take for instance the approximation associated with the Poisson kernel [63, Example 1.2.17],

$$\varphi_j(t) = \frac{j}{\pi(1+j^2t^2)}, \quad j > 0,$$

which satisfies $\widehat{\varphi}_j(t) = e^{-2\pi|t|/j}$ and has this property:

$$\|d\widehat{\varphi}_j\| = \int_{\mathbb{R}} |d\widehat{\varphi}_j|(t) = 2\int_{-\infty}^{0} \frac{2\pi e^{2\pi t/j}}{j} dt = 2.$$

Now, for every $j > 0$, define

$$m_j(t) = m(t)\widehat{\varphi}_j(t).$$

This function is of bounded variation with $\|dm_j\| \leqslant 3\|dm\|$, since

$$\|dm_j\| \leqslant \|m\|_\infty \|d\widehat{\varphi}_j\| + \|\widehat{\varphi}_j\|_\infty \|dm\| \leqslant 3\|dm\|.$$

We still have that $m_j$ vanishes at $-\infty$, so we can write the Lebesgue-Stieltjes integral

$$m_j(\xi) = \int_{\mathbb{R}} \chi_{(t,\infty)}(\xi) dm_j(t).$$

The multiplier associated with $\chi_{(t,\infty)}$ is essentially a modulated Hilbert transform

$$f(x) \longmapsto e^{2\pi itx} H(e^{-2\pi it\cdot} f)(x)$$

that we will denote by $H_t$ (see [49, Estimate (3.9)]). Then,

$$T_{m_j} f(x) = \int_{\mathbb{R}} H_t f(x) dm_j(t). \tag{3.1}$$

Now we use Proposition 3.3 with the weight $w = (M\chi_E)^{1-p}u$, for some $u \in A_1$ and $1 < p < \infty$, and (1.8), to conclude

$$\begin{aligned} \|H_t \chi_E\|_{L^{p,\infty}((M\chi_E)^{1-p}u)} &= \|H(e^{2\pi it\cdot}\chi_E)\|_{L^{p,\infty}((M\chi_E)^{1-p}u)} \\ &\lesssim \|u\|_{A_1}^{1+\frac{1}{p}} \|\chi_E\|_{L^{p,1}((M\chi_E)^{1-p}u)} = \|u\|_{A_1}^{1+\frac{1}{p}} u(E)^{1/p}, \end{aligned}$$

uniformly in $t \in \mathbb{R}$. Therefore, the family $\{H_t\}_t$ is under the hypotheses of Proposition 3.1. Also, for every $j > 0$, the operator $T_{m_j}$ is $(\varepsilon, \delta)$-atomic (since $m_j$ is integrable and hence, its associated convolution kernel is uniformly continuous, as in (1.14)). With this, we conclude that $T_{m_j}$ is of weak-type $(1,1)$ for every weight $u \in A_1$ with constant[2]

$$\|u\|_{A_1}^{2-\frac{1}{p}} \|u\|_{A_1}^{1+\frac{1}{p}} \|dm_j\| \lesssim \|dm\| \|u\|_{A_1}^3.$$

Finally, since $\{\varphi_j\}_j$ is an approximation to the identity, at least for Schwartz functions $f$, there is a subsequence such that

$$T_{m_{j(i)}} f(x) = \varphi_{j(i)} * T_m f(x) \xrightarrow{i} T_m f(x) \quad \text{a.e. } x.$$

With this, we use the estimate for $T_{m_j}$ and Fatou's lemma to finish the proof:

$$\|T_m f\|_{L^{1,\infty}(u)} \leqslant \liminf_{i\to\infty} \|T_{m_{j(i)}} f\|_{L^{1,\infty}(u)} \lesssim \|dm\| \|u\|_{A_1}^3 \|f\|_{L^1(u)}.$$

□

The idea of transferring estimates on Banach spaces from $H$ to $T_m$ based on (3.1) is not new. In [49, Corollary 3.8], this method is used to show that $T_m$ is bounded on $L^p(\mathbb{R})$ for all $1 < p < \infty$. The only difference here is that the Banach estimate that we transfer from $H$ to $T_m$ is a weighted one that allows us to extrapolate down to $p = 1$ and deduce a weak-type $(1,1)$ result for $T_m$ that cannot be obtained by means of Minkowski's inequality. These multipliers are closely related to the ones appearing in the Marcinkiewicz multiplier theorem (see [49, Theorem 8.13]). In that case, the result claims that if $m$ has uniformly bounded variation on each dyadic interval in $\mathbb{R}$, then $T_m$ maps $L^p(\mathbb{R})$ into itself for every

[2]In this case, the function $\Phi$ appearing in Proposition 3.1 is the constant $\Phi \equiv 1$, and its $L^1$-norm with respect to the measure $|dm_j|$ is exactly the total variation $\|dm_j\|$.

$1 < p < \infty$. This is obtained by means of Littlewood-Paley theory, and can be extended to the weighted setting to prove the same result for $A_p$ weights [78]. However, it is known that there are operators under the hypotheses of Marcinkiewicz's theorem that fail to be of weak-type (1,1), even in the unweighted case (see [119] for sharp results near $L^1$). Therefore, we know that our assumption for $m$ to be of bounded variation on $\mathbb{R}$ cannot be relaxed to uniform bounded variation on dyadic intervals.

The next section will follow the same argument but using the estimate for the Bochner-Riesz operator in Theorem 2.9 to draw conclusions about radial Fourier multipliers on $\mathbb{R}^n$.

## 3.3 Radial Fourier multipliers on $\mathbb{R}^n$

We will start with an easy result that will be useful when $n = 3$ and will motivate the generalization to arbitrary dimensions. Notice that when $n = 3$, the critical index of the Bochner-Riesz operator is $\frac{n-1}{2} = 1$.

**Lemma 3.5.** *Let $m$ be a bounded function defined on $(0, \infty)$ such that*

*(a) The derivatives $m'$ and $m''$ are defined on $(0, \infty)$.*

*(b) The limit*

$$\lim_{t\to\infty} m(t) - tm'(t) = c \in \mathbb{R}.$$

*(c) We have that $tm''(t) \in L^1(0, \infty)$.*

*Then, the operator defined by*

$$\widehat{T_m f}(\xi) = m(|\xi|^2)\widehat{f}(\xi), \quad \xi \in \mathbb{R}^n$$

*can be written as*

$$T_m f(x) = \int_0^\infty B_1^{1/s} f(x)\Phi(s)ds + cf(x), \quad x \in \mathbb{R}^n,$$

*where $\Phi \in L^1(0, \infty)$ and, for every $r > 0$, $\widehat{B_1^r f}(\xi) = (1 - r^2|\xi|^2)_+ \widehat{f}(\xi)$.*

*Proof.* Fix $t > 0$. Since $m$ is bounded, $m(s)/s$ goes to zero as $s$ goes to infinity, so

$$m(t) = -t\int_t^\infty \left(\frac{m(s)}{s}\right)' ds = -t\int_t^\infty \frac{sm'(s) - m(s)}{s^2} ds.$$

Now, integrating by parts,

$$m(t) = -t\left(\lim_{s\to\infty}\left(-m'(s) + \frac{m(s)}{s}\right) + m'(t) - \frac{m(t)}{t} + \int_t^\infty m''(s)ds\right).$$

Using again that $m$ is bounded and property $(b)$, we get that the limit is zero, and hence

$$\begin{aligned} m(t) &= -tm'(t) + m(t) - t\int_t^\infty m''(s)ds = \int_t^\infty sm''(s)ds + c - t\int_t^\infty m''(s)ds \\ &= \int_t^\infty \left(1 - \frac{t}{s}\right) sm''(s) + c = \int_0^\infty \left(1 - \frac{t}{s}\right)_+ sm''(s)ds + c. \end{aligned}$$

Therefore, making a change of variables we get that, for every $\xi \in \mathbb{R}^n$,

$$m(|\xi|^2) = \int_0^\infty \left(1 - \frac{|\xi|^2}{s^2}\right)_+ \Phi(s)ds + c,$$

with

$$\Phi(s) = 2s^3 m''(s^2),$$

which lies in $L^1(0,\infty)$ by property $(c)$. Therefore,

$$\widehat{T_m f}(\xi) = \int_0^\infty \widehat{B_1^{1/s} f}(\xi)\Phi(s)ds + c\hat{f}(\xi),$$

and inverting the Fourier transform together with Fubini, we finish the proof. $\square$

**Proposition 3.6.** *If we have a function $m$ as in Lemma 3.5, then the operator*

$$\widehat{T_m f}(\xi) = m(|\xi|^2)\hat{f}(\xi), \quad \xi \in \mathbb{R}^3,$$

*is of weak-type $(1,1)$ for every weight $u \in A_1$ and with constant essentially controlled by $\|u\|_{A_1}^5$.*

*Proof.* When $n = 3$, the operator $B_1$ is exactly the Bochner-Riesz operator at the critical index. Now, by Corollary 2.10, we know that for every weight $u \in A_1$, there is some $1 < p_0 < \infty$ such that, for each measurable set $E \subseteq \mathbb{R}^3$,

$$\|B^{1/s}\chi_E\|_{L^{p_0,\infty}((M\chi_E)^{1-p_0}u)} \lesssim \|u\|_{A_1}^{4/p_0} u(E)^{1/p_0}, \tag{3.2}$$

uniformly in $s \in (0,\infty)$. Now, by Lemma 3.5,

$$T_m f(x) = \int_0^\infty B^{1/s} f(x)\Phi(s)ds + cf(x), \quad x \in \mathbb{R}^3,$$

with $\Phi$ an integrable function on $(0,\infty)$. The term $cf$ plays no role in the boundedness of $T_m$, so let us focus on the first one. Let $K_{1/s}$ be the convolution kernel associated with $B^{1/s}$. For every $j > 0$, define

$$K^j(x) = \int_0^j K_{1/s}(x)\Phi(s)ds = \int_0^\infty K_{1/s}(x)\Phi_j(s)ds,$$

with $\Phi_j(s) = \Phi(s)\chi_{(0,j)}(s) \in L^1(0,\infty)$ and $\|\Phi_j\|_1 \leqslant \|\Phi\|_1$. Hence

$$T^j f(x) = K^j * f(x) = \int_0^\infty B^{1/s} f(x)\Phi_j(s)ds.$$

Notice that by Minkowski and the fact that $K_1 \in L^2(\mathbb{R}^3)$,

$$\|K^j\|_2 = \left\| \int_0^\infty K_{1/s}(x)\Phi_j(s)ds \right\|_2 \leqslant \int_0^j \|s^3 K_1(sx)\|_2 |\Phi(s)| ds \leqslant j^{3/2}\|K_1\|_2\|\Phi\|_1 < \infty,$$

thus $K^j \in L^2(\mathbb{R}^3)$ and by (1.14), $T^j$ is an $(\varepsilon,\delta)$-atomic operator. Now, we use Proposition 3.1 and (3.2) to deduce that $T^j$ is of weak-type $(1,1)$ for every $u \in A_1$ and with constant

$$\|u\|_{A_1}^{2-\frac{1}{p_0}} \|u\|_{A_1}^{\frac{4}{p_0}} \|\Phi_j\|_1 \leqslant \|u\|_{A_1}^5 \|\Phi\|_1,$$

independently of $j > 0$. Using Fatou's Lemma, we conclude that for every $f \in L^1(u)$,

$$\left\| \int_0^\infty B^{1/s} f(x)\Phi(s)ds \right\|_{L^{1,\infty}(u)} \leqslant \liminf_{j\to\infty} \|T^j f\|_{L^{1,\infty}(u)} \lesssim \|\Phi\|_1 \|u\|_{A_1}^5 \|f\|_{L^1(u)}.$$

□

Finally, let us just restate Proposition 3.6 so we can see what it gives when we are dealing with radial multipliers of the form $m(|\xi|)$:

**Corollary 3.7.** *Let $T_m$ be the operator defined by*

$$\widehat{T_m f}(\xi) = m(|\xi|)\widehat{f}(\xi), \quad \xi \in \mathbb{R}^3,$$

*with $m$ a bounded function defined on $(0,\infty)$ such that*

*(a) The derivatives $m'$ and $m''$ are defined on $(0,\infty)$.*

*(b) The limit*

$$\lim_{t\to\infty} m(t) - tm'(t) = c \in \mathbb{R}.$$

*(c) We have that both $tm''(t)$, $m'(t) \in L^1(0,\infty)$.*

*Then $T_m$ is of weak-type $(1,1)$ for every weight $u \in A_1$ and with constant $\|u\|_{A_1}^5$.*

*Proof.* We just check that it is equivalent to Proposition 3.6. Notice that

$$\widehat{T_m f}(\xi) = m(|\xi|)\widehat{f}(\xi) = \widetilde{m}(|\xi|^2)\widehat{f}(\xi),$$

with $\widetilde{m}(t^2) = m(t)$, and therefore

$$\widetilde{m}'(t^2) = \frac{m'(t)}{2t}, \quad \widetilde{m}''(t^2) = \frac{m''(t)}{4t^2} - \frac{m'(t)}{4t^3}.$$

But Proposition 3.6 gives the sought-after boundedness provided that:

- The derivatives $\widetilde{m}'$ and $\widetilde{m}''$ are defined on $(0, \infty)$, which is equivalent to $(a)$.
- The limit

$$\lim_{t\to\infty} \widetilde{m}(t) - t\widetilde{m}'(t) = \lim_{t\to\infty} \widetilde{m}(t^2) - t^2\widetilde{m}'(t^2) \approx \lim_{t\to\infty} m(t) - tm'(t) = c \in \mathbb{R},$$

which is $(b)$. In the last equality we use that by $(c)$, $m' \in L^1(0, \infty)$ and hence

$$\lim_{t\to\infty} tm'(t) = 0.$$

- We have that $t\widetilde{m}''(t) \in L^1(0, \infty)$, which by a change of variables is equivalent to $t^3\widetilde{m}''(t^2)$ being integrable, and thus to

$$tm''(t),\ m'(t) \in L^1(0, \infty).$$

Hence, a direct application of Proposition 3.6 completes the proof. □

In general, it happens that if $\widetilde{m}(t^2) = m(t)$, then, for every $k \in \mathbb{N}$,

$$\widetilde{m}^{(k)}(t^2) = \sum_{j=1}^{k} c_j \frac{m^{(j)}(t)}{t^{2k-j}}, \quad \text{with } c_j \in \mathbb{R}.$$

Therefore, if we have a result for Fourier multipliers of the type $\widehat{Tf}(\xi) = \widetilde{m}(|\xi|^2)\widehat{f}(\xi)$, and the hypothesis in such a result is an integrability condition for the $k$-th derivative of $\widetilde{m}$, then, this hypothesis translates (when applied to a multiplier defined by $m(|\xi|)\widehat{f}(\xi)$) into conditions for every derivative of $m$ of order less than or equal to $k$. Therefore, in what follows, we will basically restrict our attention to radial multipliers with symbol $m(|\xi|^2)$.

Let us see that this technique of writing $T_m$ as an average of Bochner-Riesz operators $B^{1/s}$ at the critical index can be extended to $\mathbb{R}^n$ by means of fractional calculus. The idea of using fractional calculus to obtain results for radial Fourier multipliers was already introduced in [121] and subsequently used in [45, 61], among others. The definition that we will need is in the sense of Weyl:

**Definition 3.8.** *Given $0 \leqslant \delta < 1$ and $w > 0$, we define the truncated fractional integral of order $1 - \delta$ of a locally integrable function $f$ on $\mathbb{R}$ by*

$$I_w^{1-\delta} f(t) := \frac{1}{\Gamma(1-\delta)} \int_{-w}^{w} (s-t)_+^{-\delta} f(s)ds, \quad t < w,$$

*and 0 if $t \geqslant w$. Moreover, if $\alpha = [\alpha] + \delta > 0$, with $[\alpha]$ being its integer part and $\delta$ its fractional part, we define the fractional derivative of $f$ of order $\alpha$ by*

$$D^\alpha f(t) := -\left(\frac{d}{dt}\right)^{[\alpha]} \lim_{w\to\infty} \frac{d}{dt} I_w^{1-\delta} f(t),$$

*whenever the right-hand side exists. In particular, if $f$ has compact support, then*

$$D^\alpha f(t) := -\left(\frac{d}{dt}\right)^{[\alpha]+1} I_\infty^{1-\delta} f(t).$$

Recall that $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx$. One can define fractional derivatives in multiple ways. For instance, one can use Riemann-Liouville's fractional integral

$$J^\alpha f(t) := \frac{1}{\Gamma(\alpha)}\int_{\mathbb{R}} (t-s)_+^{\alpha-1} f(s)ds,$$

and then, if $k = \lceil\alpha\rceil$ (where this notation means that $k \in \mathbb{N}$ with $k-1 < \alpha \leqslant k$), define a fractional derivative of order $\alpha$ in the sense of Riemann-Liouville:

$$D_{RL}^\alpha f(f) = (J^{k-\alpha}f)^{(k)}(t).$$

Analogously, if we differentiate first and integrate later, we obtain the fractional derivative in the sense of Caputo:

$$D_C^\alpha f(t) = J^{k-\alpha}(f^{(k)})(t).$$

Every definition has its advantages and disadvantages, but for technical reasons, the most convenient way for us to introduce fractional calculus is in the sense of Weyl, as in Definition 3.8. For further information about fractional calculus and its different variants, we refer to [46] and [62]. We will need the following lemma:

**Lemma 3.9.** *Weyl's fractional derivative satisfies these two properties:*

*(i) Let $\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_2 \neq 0$. Then, for every $\alpha > 0$,*

$$D^\alpha(f(\lambda_1 + \lambda_2\,\cdot))(t) = \begin{cases} \lambda_2^\alpha D^\alpha f(\lambda_1 + \lambda_2 t) & \alpha \in \mathbb{N} \text{ or } \lambda_2 > 0, \\ |\lambda_2|^\alpha D^\alpha \widetilde{f}(-\lambda_1 - \lambda_2 t) & \lambda_2 < 0, \end{cases} \tag{3.3}$$

*where $\widetilde{f}(t) = f(-t)$ is the reflection of $f$ on $\mathbb{R}$.*

*(ii) If $f$ is a continuous function with compact support in $[a, b]$, then*

$$|D^\alpha f(t)| \leqslant \frac{C_{f,\alpha}}{|t|^{\alpha+1}}, \quad \text{as } t \to -\infty. \tag{3.4}$$

*Proof.* We start with $(i)$. If $\alpha \in \mathbb{N}$, the first identity is well-known, for both $\lambda_2 > 0$ and $\lambda_2 < 0$. For this reason, the second expression is also valid in the case $\lambda_2 < 0$:

$$|\lambda_2|^\alpha D^\alpha \widetilde{f}(-\lambda_1 - \lambda_2 t) = |\lambda_2|^\alpha(-1)^\alpha D^\alpha f(\lambda_1 + \lambda_2 t) = \lambda_2^\alpha D^\alpha f(\lambda_1 + \lambda_2 t).$$

If $\alpha \notin \mathbb{N}$ but $\lambda_2 > 0$, we make the change of variables $\lambda_1 + \lambda_2 s = r$ and, since $\lambda_2 > 0$, it can be easily factored out from the positive part in the denominator:

$$\begin{aligned} D^\alpha(f(\lambda_1+\lambda_2\cdot))(t) &= -\left(\frac{d}{dt}\right)^{[\alpha]} \lim_{w\to\infty} \frac{d}{dt}\left(\frac{1}{\Gamma(1-\delta)}\int_{-w}^{w} (s-t)_+^{-\delta} f(\lambda_1+\lambda_2 s)ds\right) \\ &= -\left(\frac{d}{dt}\right)^{[\alpha]} \lim_{w\to\infty} \frac{d}{dt}\left(\frac{1}{\Gamma(1-\delta)}\int_{\lambda_1-\lambda_2 w}^{\lambda_1+\lambda_2 w} \lambda_2^{\delta-1}(r-\lambda_1-\lambda_2 t)_+^{-\delta} f(r)dr\right), \end{aligned}$$

which equals

$$-\lambda_2^{\delta-1}\left(\frac{\lambda_2 d}{d(\lambda_1+\lambda_2 t)}\right)^{[\alpha]} \lim_{z\to\infty} \frac{\lambda_2 d}{d(\lambda_1+\lambda_2 t)}\left(\frac{1}{\Gamma(1-\delta)}\int_{-z}^{z} (r-\lambda_1-\lambda_2 t)_+^{-\delta} f(r)dr\right),$$

that is, $\lambda_2^\alpha D^\alpha f(\lambda_1+\lambda_2 t)$. When $\lambda_2 < 0$, we write

$$D^\alpha(f(\lambda_1+\lambda_2\,\cdot))(t) = D^\alpha(\widetilde{f}(-\lambda_1-\lambda_2\,\cdot))(t)$$

and an application of the previous case with $-\lambda_2 > 0$ yields the result:

$$D^\alpha(\widetilde{f}(-\lambda_1-\lambda_2\,\cdot))(t) = (-\lambda_2)^\alpha D^\alpha \widetilde{f}(-\lambda_1-\lambda_2 t) = |\lambda_2|^\alpha D^\alpha \widetilde{f}(-\lambda_1-\lambda_2 t).$$

To show $(ii)$, just notice that if we take $t < a$, then

$$|D^\alpha f(t)| = C_\alpha \left| \left(\frac{d}{dt}\right)^{[\alpha]+1} \int_a^b (s-t)^{[\alpha]-\alpha} f(s)ds \right|.$$

Differentiating under the integral sign and using that $f$ is bounded on $[a,b]$ yield that the previous expression can be controlled by

$$C_{f,\alpha}\int_a^b (s-t)^{-\alpha-1}ds = C_{f,\alpha}((a-t)^{-\alpha} - (b-t)^{-\alpha}).$$

But this behaves like $\frac{C_{f,\alpha}}{|t|^{\alpha+1}}$ as $t\to-\infty$, since

$$\lim_{t\to-\infty} \frac{(a-t)^{-\alpha}-(b-t)^{-\alpha}}{|t|^{-\alpha-1}} = \alpha(b-a),$$

so we finish the proof. $\square$

Now we are ready to state the main theorem of this section. $AC_{\mathrm{loc}}$ will denote the space of functions that are absolutely continuous on every compact subset of $(0,\infty)$.

**Theorem 3.10.** *Fix $n \geqslant 2$ and $\alpha = \frac{n+1}{2}$. Let $m$ be a bounded, continuous function on $(0,\infty)$ which vanishes at infinity and satisfies that*

$$D^{\alpha-j}m \in AC_{\mathrm{loc}} \quad \forall j = 1, ..., [\alpha].$$

*Then, if $D^\alpha m$ exists and*

$$\Phi(t) = t^{\alpha-1}D^\alpha m(t) \in L^1(0,\infty),$$

*the operator $T_m$ defined by*

$$\widehat{T_m f}(\xi) = m(|\xi|^2)\widehat{f}(\xi), \quad \xi \in \mathbb{R}^n,$$

*is of weak-type $(1,1)$ for every weight $u \in A_1$ with constant controlled by $C\|\Phi\|_{L^1(0,\infty)}\|u\|_{A_1}^5$.*

*Proof.* First, we will use [120, Lemma 3.14] to write

$$m(t) = \frac{(-1)^{[\alpha]}}{\Gamma(\alpha)}\int_{\mathbb{R}}(s-t)_+^{\alpha-1}D^\alpha m(s)ds = C_\alpha\int_t^\infty (s-t)^{\alpha-1}D^\alpha m(s)ds, \tag{3.5}$$

which is valid under our hypotheses for $m$. With this identity, we are able to prove that

$$T_m f(x) = \int_0^\infty B^{1/s}f(x)\overline{\Phi}(s)ds, \quad x \in \mathbb{R}^n, \tag{3.6}$$

with $\overline{\Phi} \in L^1(0,\infty)$. Indeed, it is enough to check that, for every $\xi \in \mathbb{R}^n$,

$$m(|\xi|^2) = \int_0^\infty \left(1 - \frac{|\xi|^2}{s^2}\right)_+^{\alpha-1}\overline{\Phi}(s)ds, \tag{3.7}$$

but this follows from (3.5) by the change of variables $s = r^2$, allowed for $s > 0$, and taking $t = |\xi|^2$:

$$\begin{aligned} m(|\xi|^2) =& 2C_\alpha\int_{|\xi|}^\infty (r^2 - |\xi|^2)^{\alpha-1}D^\alpha m(r^2)rdr \\ =& 2C_\alpha\int_0^\infty r^{2\alpha-1}\left(1 - \frac{|\xi|^2}{r^2}\right)_+^{\alpha-1}D^\alpha m(r^2)dr, \end{aligned}$$

which is (3.7) with $\overline{\Phi}(r) = C_\alpha r^{2\alpha-1}D^\alpha m(r^2)$ and $\|\overline{\Phi}\|_{L^1(0,\infty)} \approx \|\Phi\|_{L^1(0,\infty)}$. The second ingredient in the proof is the uniform bound given in Corollary 2.10. More precisely, that

$$\|B^{1/s}\chi_E\|_{L^{p_0,\infty}((M\chi_E)^{1-p_0}u)} \lesssim \|u\|_{A_1}^{4/p_0}u(E)^{1/p_0}, \tag{3.8}$$

uniformly in $s \in (0,\infty)$. To conclude the proof, we argue exactly as in Proposition 3.6 but with the obvious changes now that we are in $\mathbb{R}^n$ instead of $\mathbb{R}^3$. The idea was just to define

$$K^j(x) = \int_0^j K_{1/s}(x)\overline{\Phi}(s)ds = \int_0^\infty K_{1/s}(x)\overline{\Phi}_j(s)ds,$$

where $K_{1/s}$ denoted the kernel associated with $B^{1/s}$, and prove that the convolution operator $T^j$ given by $K^j$ is $(\varepsilon,\delta)$-atomic. Then we use Proposition 3.1 and the uniform bound (3.8) to deduce that $T^j$ is of weak-type $(1,1)$ for every $u \in A_1$ with constant essentially bounded by

$$\|u\|_{A_1}^5 \|\Phi\|_{L^1(0,\infty)}.$$

After this, we used Fatou's lemma and (3.6) to finish the proof. □

Let us briefly summarize how Theorem 3.10 is related to other results in the literature. The integrability condition that we require on $m$ is[3]

$$\int_0^\infty t^{\frac{n-1}{2}} |D^{\frac{n+1}{2}} m(t)| dt < \infty, \tag{3.9}$$

and we obtain a weak-type $(1,1)$ estimate with respect to every weight in $A_1$ for the Fourier multiplier with symbol $m(|\xi|^2)$. This type of condition (3.9) on $m$ is not new. For instance, in the unweighted setting, [45, 104] use Weyl's fractional calculus to obtain strong-type $(p,p)$ and weak-type $(1,1)$ results for maximal operators associated with quasiradial Fourier multipliers. The condition that they require on $m$ is also an integrability condition for $t^{\alpha-1}D^\alpha m$, but with $\alpha > \frac{n+1}{2}$ (see [104, Corollary 1]).

Another similar result to the one we presented can be found in [79]. Here the authors deal with weights, but they consider general Fourier multipliers on $\mathbb{R}^n$, not necessarily radial ones. In terms of differentiability requirements, the condition that they need on $m$ to get the weak-type $(1,1)$ for every weight in $A_1$ is up to order $n$. In our case, we only work with radial multipliers and require order $\frac{n+1}{2}$ instead. In the classical Hörmander theorem [70] without weights, it is enough to have differentiability up to order strictly larger than $\frac{n}{2}$, which is essentially optimal even for radial multipliers (see [36]). Therefore, the differentiability assumption in our result is not that far from the optimal order of the unweighted case. Another important reference is [17], where one can find sufficient conditions for radial Fourier multipliers to be bounded on $L^p(\mathbb{R}^2)$ for $4/3 \leqslant p \leqslant 4$. This limitation in the range of $p$ (which totally excludes the endpoint $p=1$) allows the authors to lower the order of differentiability of $m$ to $\alpha > 1/2$, which corresponds to $\frac{n-1}{2}$ in $\mathbb{R}^2$.

Finally, one can check that (3.9) can be controlled by an expression resembling that in Hörmander's theory. More precisely, if $\varphi$ is a $\mathcal{C}^\infty$ function, supported on $(1/2,1)$ and such that

$$\sum_{j\in\mathbb{Z}} \varphi(2^{-j}t) = \sum_{j=1}^\infty \varphi_j(t) = 1, \quad t > 0,$$

[3]Notice that when $n=3$, this is the condition $tm''(t) \in L^1(0,\infty)$ that we had in Proposition 3.6.

then it can be verified that, at least for $n$ odd,

$$\int_0^\infty t^{\frac{n-1}{2}} |D^{\frac{n+1}{2}} m(t)| dt \lesssim \sum_{j\in\mathbb{Z}} 2^{\frac{jn}{2}} \left( \int_{2^{j-1}}^{2^j} |D^{\frac{n+1}{2}} (\varphi_j m)(t)|^2 dt \right)^{1/2}. \tag{3.10}$$

The finiteness of the right-hand side is related to the classical Hörmander condition for radial multipliers, but with differentiability order $\frac{n+1}{2}$. In that case, the sum in $j$ would be replaced by a supremum. The validity of inequality (3.10), when $n$ is odd (and hence, $D^{\frac{n+1}{2}}$ is a usual derivative), is just a direct application of the Cauchy-Schwarz inequality:

$$\begin{aligned}\int_0^\infty t^{\frac{n-1}{2}} |D^{\frac{n+1}{2}} m(t)| dt &\leqslant \sum_{j\in\mathbb{Z}} \int_{2^{j-1}}^{2^j} t^{\frac{n-1}{2}} |D^{\frac{n+1}{2}} (\varphi_j m)(t)| dt \approx \sum_{j\in\mathbb{Z}} 2^{\frac{j(n-1)}{2}} \int_{2^{j-1}}^{2^j} |D^{\frac{n+1}{2}} (\varphi_j m)(t)| dt \\ &\leqslant \sum_{j\in\mathbb{Z}} 2^{\frac{jn}{2}} \left( \int_{2^{j-1}}^{2^j} |D^{\frac{n+1}{2}} (\varphi_j m)(t)|^2 dt \right)^{1/2}.\end{aligned}$$

The problem when $n$ is even is that $D^{\frac{n+1}{2}}$ is a purely fractional derivative, and in general, one can check from Definition 3.8 that if $f$ is supported on $(a, b)$, then $D^\alpha f$ is supported on $(-\infty, b)$. Hence, at the first step, where we introduce the $\varphi_j$, we would have to consider the integral on $(0, 2^j)$ instead of $(2^{j-1}, 2^j)$. If we split $(0, 2^j)$ into three intervals

$$(0, 2^{j-2}) \cup (2^{j-2}, 2^{j-1}) \cup (2^{j-1}, 2^j),$$

the terms that we get for the last two dyadic intervals can be treated as in the previous case, but the one corresponding to $(0, 2^{j-2})$ becomes a problem. Arguing as in the proof of property (3.4), one can check that for $t \in (0, 2^{j-2})$,

$$|D^{\frac{n+1}{2}} (\varphi_j m)(t)| \leqslant C_\varphi ((2^{j-1} - t)^{\frac{-n-1}{2}} - (2^j - t)^{\frac{-n-1}{2}}),$$

and with this and a change of variables, we get that

$$\int_0^{2^{j-2}} t^{\frac{n-1}{2}} |D^{\frac{n+1}{2}} (\varphi_j m)(t)| dt \lesssim \int_0^1 s^{\frac{n-1}{2}} ((2-s)^{\frac{-n-1}{2}} - (4-s)^{\frac{-n-1}{2}}) ds \approx 1,$$

which cannot be summed in $j \in \mathbb{Z}$.

Theorem 3.10 exploits the relation between Fourier multipliers $T_m$ (under a precise integrability condition on $m$) and the Bochner-Riesz operators $B^{1/s}$ at the critical index. The key idea is transferring estimates that take place in Banach spaces by means of Minkowski's inequality (in this case, to be able to extrapolate down to $p = 1$). This idea of transference of estimates motivates the next subsection, where we will consider all indices $\lambda > 0$ in an attempt to make some contribution to the so-called Bochner-Riesz conjectures. Before that, let us give a particular example of application of Theorem 3.10. It will be related to the following conjecture stated in [106]:

**Conjecture 3.11.** *Assume that $\varphi$ is a $\mathcal{C}^\infty$ function with compact support in* $(-1/2, 1/2)$ *and, for every* $0 < \delta < 1$, *set*

$$h_\delta(s) := \varphi\left(\frac{1-s}{\delta}\right).$$

*Then, for every* $1 < p < \frac{2n}{n+1}$, *it holds that the operator* $T_{h_\delta}$ *defined by*

$$\widehat{T_{h_\delta} f}(\xi) = h_\delta(|\xi|^2)\widehat{f}(\xi)$$

*satisfies*

$$\|T_{h_\delta}\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} \lesssim \delta^{-\lambda(p)}, \qquad \text{with } \lambda(p) = n\left(\frac{1}{p} - \frac{1}{2}\right) - \frac{1}{2}. \tag{3.11}$$

The result we present is the following:

**Corollary 3.12.** *Given* $n \geqslant 2$, *the operator* $T_{h_\delta}$ *is of weak-type (1,1) for every weight* $u \in A_1$ *and*

$$\|T_{h_\delta}\|_{L^1(u)\to L^{1,\infty}(u)} \lesssim \delta^{-\left(\frac{n-1}{2}\right)}\|u\|_{A_1}^5.$$

To prove this, it is enough to apply Theorem 3.10 ($h_\delta$ is under its hypotheses) together with the following computation at $\alpha = \frac{n+1}{2}$:

**Lemma 3.13.** *Given* $\alpha > 0$, *it holds that, for* $\Phi(t) = t^{\alpha-1}D^\alpha h_\delta(t)$,

$$\|\Phi\|_{L^1(0,\infty)} \leqslant C_{\varphi,\alpha}\delta^{-\alpha+1}.$$

*Proof.* First, we compute $D^\alpha h_\delta$. Using the property in (3.3), we have that

$$D^\alpha h_\delta(t) = \frac{1}{\delta^\alpha} D^\alpha \widetilde{\varphi}\left(\frac{t-1}{\delta}\right),$$

with $\widetilde{\varphi}(s) = \varphi(-s)$ being the reflection of $\varphi$ on $\mathbb{R}$. Now,

$$\int_0^\infty |\Phi(t)|dt = \delta^{-\alpha}\int_0^\infty t^{\alpha-1}\left|D^\alpha\widetilde{\varphi}\left(\frac{t-1}{\delta}\right)\right|dt = \delta^{-\alpha+1}\int_{-1/\delta}^\infty (r\delta+1)^{\alpha-1}|D^\alpha\widetilde{\varphi}(r)|dr.$$

Since $\operatorname{supp}(\widetilde{\varphi}) = \operatorname{supp}(\varphi) \subseteq (-1/2, 1/2)$, we have that $D^\alpha\widetilde{\varphi}$ is supported on $(-\infty, 1/2)$. Hence, assuming $\delta > 0$ small enough, we use (3.4) for $r < -1$ and that $|D^\alpha\widetilde{\varphi}(r)|$ is bounded on compact sets to obtain

$$\int_{-1/\delta}^\infty (r\delta+1)^{\alpha-1}|D^\alpha\widetilde{\varphi}(r)|dr \leqslant C_{\varphi,\alpha}\int_{-1/\delta}^{-1}\frac{(r\delta+1)^{\alpha-1}}{|r|^{\alpha+1}}dr + \int_{-1}^{1/2}(r\delta+1)^{\alpha-1}|D^\alpha\widetilde{\varphi}(r)|dr \lesssim C_{\varphi,\alpha},$$

which concludes the proof. □

Notice that $\lambda(1) = \frac{n-1}{2}$, and hence, Corollary 3.12 is the endpoint weighted weak-type version of estimate (3.11). In particular, taking $u = 1$ we can use our estimate in the following way:

**Proposition 3.14.** *Assume that* (3.11) *holds for some* $1 < p_0 < \frac{2n}{n+1}$. *Then, it is also true for every* $1 < p < p_0$.

*Proof.* In Corollary 3.12, we have shown that the weak-type (1,1) estimate for $T_{h_\delta}$ holds with $\delta^{-\lambda(1)}$. Then, we just use Marcinkiewicz's interpolation theorem between this endpoint estimate and our assumption to conclude that, for every $1 < p < p_0$,

$$\|T_{h_\delta}\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} \lesssim \delta^{-\lambda(1)\left(\frac{1/p-1/p_0}{1-1/p_0}\right)} \cdot \delta^{-\lambda(p_0)\left(\frac{1-1/p}{1-1/p_0}\right)} = \delta^{-\lambda(p)}.$$

□

We want to remark that the estimate in Corollary 3.12 (but with an $\varepsilon$ loss in the exponent of $\delta$) can be derived from [51, Lemma 5.2], where the authors prove that, for every $\varepsilon > 0$,

$$|T_{h_\delta} f(x)| \leqslant C_\varepsilon \delta^{-(\frac{n-1}{2}+\varepsilon)} Mf(x).$$

### 3.3.1 The Bochner-Riesz Conjectures

First, let us fix some notation. In this subsection, we will denote by $T_m$ the operator defined as a Fourier multiplier with symbol $m(|\cdot|^2)$, that is:

$$\widehat{T_m f}(\xi) = m(|\xi|^2)\widehat{f}(\xi).$$

Also, we define the maximal operator associated with $T_m$ by

$$T_m^* f(x) = \sup_{r>0} |T_m^r f(x)|,$$

where, for every $r > 0$,

$$\widehat{T_m^r f}(\xi) = m(r^2|\xi^2|)\widehat{f}(\xi).$$

We will keep $B_\lambda$ for the Bochner-Riesz operator $T_{b_\lambda}$ with $b_\lambda(t) = (1-t)_+^\lambda$ and $\lambda > 0$. We also have a maximal operator associated with it:

**Definition 3.15.** *Given* $\lambda > 0$, *we define the maximal Bochner-Riesz operator* $B_\lambda^*$ *in* $\mathbb{R}^n$ *by*

$$B_\lambda^* f(x) = \sup_{r>0} |B_\lambda^r f(x)|,$$

*where, following our notation*[4],

$$\widehat{B_\lambda^r f}(\xi) = \left(1 - r^2|\xi|^2\right)_+^\lambda \widehat{f}(\xi).$$

[4]See Definition 2.1.

Let us now state the Bochner-Riesz conjecture in two different ways. It basically deals with the Bochner-Riesz operator $B_\lambda$ below the critical index: $0 < \lambda < \frac{n-1}{2}$. Notice that $B_0$, also known as the multiplier of the ball (since its symbol $b_0$ is the characteristic function $\chi_{B(0,1)}$) is not considered. Unlike the rest of $b_\lambda$ with $\lambda > 0$, the function $b_0$ is not even continuous, and in 1971, C. Fefferman [56] showed that $B_0$ is only bounded in the trivial case $p = 2$.

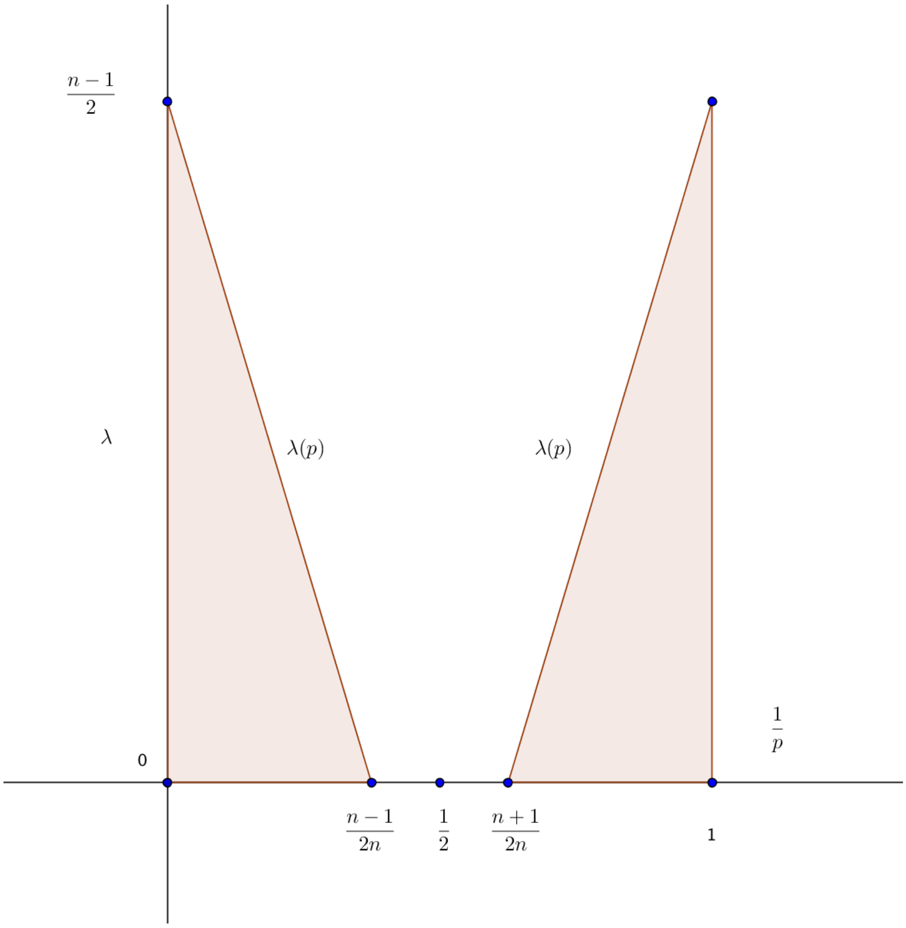


Figure 3.1: $B_\lambda$ should be bounded on $L^p(\mathbb{R}^n)$ for $(1/p, \lambda)$ outside the shaded region.

**Conjecture 3.16.** *Given* $0 < \lambda < \frac{n-1}{2}$, *then*

$$B_\lambda : L^p(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n)$$

*if and only if*

$$p_0(\lambda) := \frac{2n}{n+1+2\lambda} < p < \frac{2n}{n-1-2\lambda} =: p_1(\lambda).$$

**Conjecture 3.17.** *Given* $1 < p < \infty$, *then*

$$B_\lambda : L^p(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n)$$

*if and only if*

$$\lambda > \lambda(p) := \max\left(n\left|\frac{1}{p} - \frac{1}{2}\right| - \frac{1}{2}, 0\right).$$

It is easy to check that both statements are equivalent. See Figure 3.1 for a graphical description of the region they represent. The necessity of the condition required for $B_\lambda$ to be bounded was already proved by Herz [67] in 1954, when the author showed that for $0 < \lambda < \frac{n-1}{2}$, $B_\lambda$ is not bounded on $L^p(\mathbb{R}^n)$ if $p \notin (p_0(\lambda), p_1(\lambda))$ (the shaded region in Figure 3.1). Let us now focus on the second statement of the conjecture. By duality, it is enough to check it for either small values of $p$, $1 < p < \frac{2n}{n+1}$ or their conjugates $\frac{2n}{n-1} < p < \infty$. In dimension $n = 2$, Conjecture 3.17 was shown to be true by Carleson and Sjölin [18], but it is still open in higher dimensions. Partial results have been found over the years (see for instance [55, 118] and more recently, [82]). The best result so far for $n \geqslant 3$ is due to J. Bourgain and L. Guth [7], who showed, by an indirect argument related to the restriction problem of the Fourier transform, that the Bochner-Riesz conjecture holds whenever (written for values of $p > 2$)

$$\begin{cases} p > \frac{2(4n+3)}{4n-3}, & \text{if } n \equiv 0 \mod 3, \\ p > \frac{2n+1}{n-1}, & \text{if } n \equiv 1 \mod 3, \\ p > \frac{4(n+1)}{2n-1}, & \text{if } n \equiv 2 \mod 3. \end{cases} \tag{3.12}$$

In Figure 3.2 we illustrate this current state of the conjecture for $n \geqslant 3$ (recall that the case $n = 2$ was completely settled). The value of $p_0$ is the lower bound of $p$ appearing in (3.12), which depends on the dimension $n$ modulo 3. The segments going from $(1/2, 0)$ to $(1/p_0, \lambda(p_0))$ and $(1/p_0', \lambda(p_0'))$ respectively are obtained by analytic interpolation, so the solution in [7] actually shows that $B_\lambda$ is bounded on $L^p(\mathbb{R}^n)$ for every couple $(1/p, \lambda)$ in the green region in Figure 3.2.

There is also the corresponding conjecture for the maximal Bochner-Riesz operator $B_\lambda^*$, which initially stated that $B_\lambda^*$ should be bounded on the same region as $B_\lambda$. Here, a duality argument is no longer available, so unlike for the Bochner-Riesz operator $B_\lambda$, the cases

$$1 < p < \frac{2n}{n+1} \quad \text{and} \quad \frac{2n}{n-1} < p < \infty$$

must be considered separately. With Figure 3.3 at hand, let us explain the current state of the conjecture. For large values of $p$, the conjecture was shown to be true by A. Carbery [15] in $\mathbb{R}^2$, closing the green region of the left-hand side of the case $n = 2$. When $n \geqslant 3$, only partial results have been found. M. Christ [34] proved the conjecture for every $p \geqslant \frac{2n+2}{n-1}$, and more recently, S. Lee [82] improved it to $p > \frac{2n+4}{n}$ (as represented by

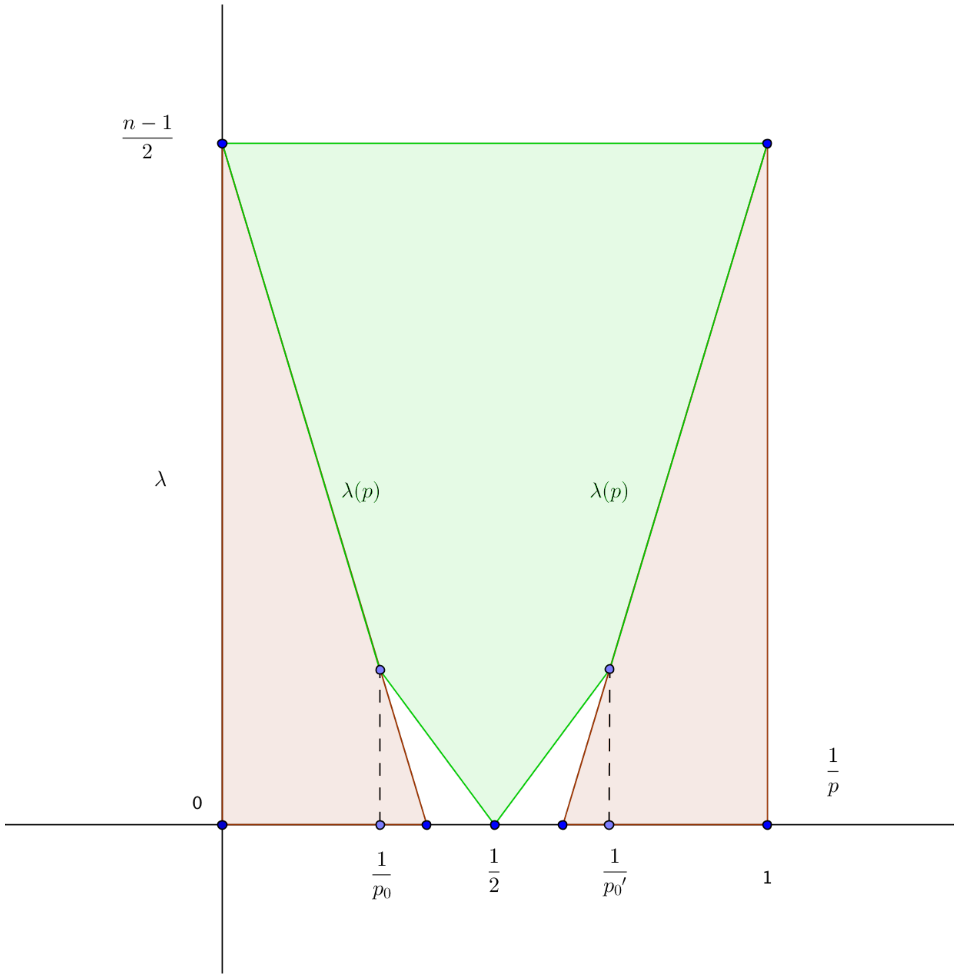


Figure 3.2: Current state of the Bochner-Riesz conjecture for $n \geqslant 3$, with $p_0$ as in (3.12).

a dotted line in the case $n \geqslant 3$). Again, in order to close the green triangle coming from Lee's result, there is an interpolation argument with the point $(1/2, 0)$. For small values of $p$ (which correspond to the right-hand side of the pictures), however, the original conjecture is known to be false, in the sense that an additional restriction has to be added if we want to have boundedness. More precisely, in [116], T. Tao showed that if $B_\lambda^*$ is bounded on $L^p(\mathbb{R}^n)$, then necessarily

$$\lambda \geqslant \frac{2n-1}{2p} - \frac{n}{2},$$

which is a stronger requirement than $\lambda > \lambda(p)$. This is represented by an additional red region with vertex at $(n/(2n-1), 0)$. Not much is known in this case except in $\mathbb{R}^2$, where the same author [117] extended the positive results to the region where

$$\lambda > \max\left(\frac{3}{4p} - \frac{3}{8}, \frac{7}{6p} - \frac{2}{3}\right), \tag{3.13}$$

adding a small, green triangle in the picture of $n = 2$.

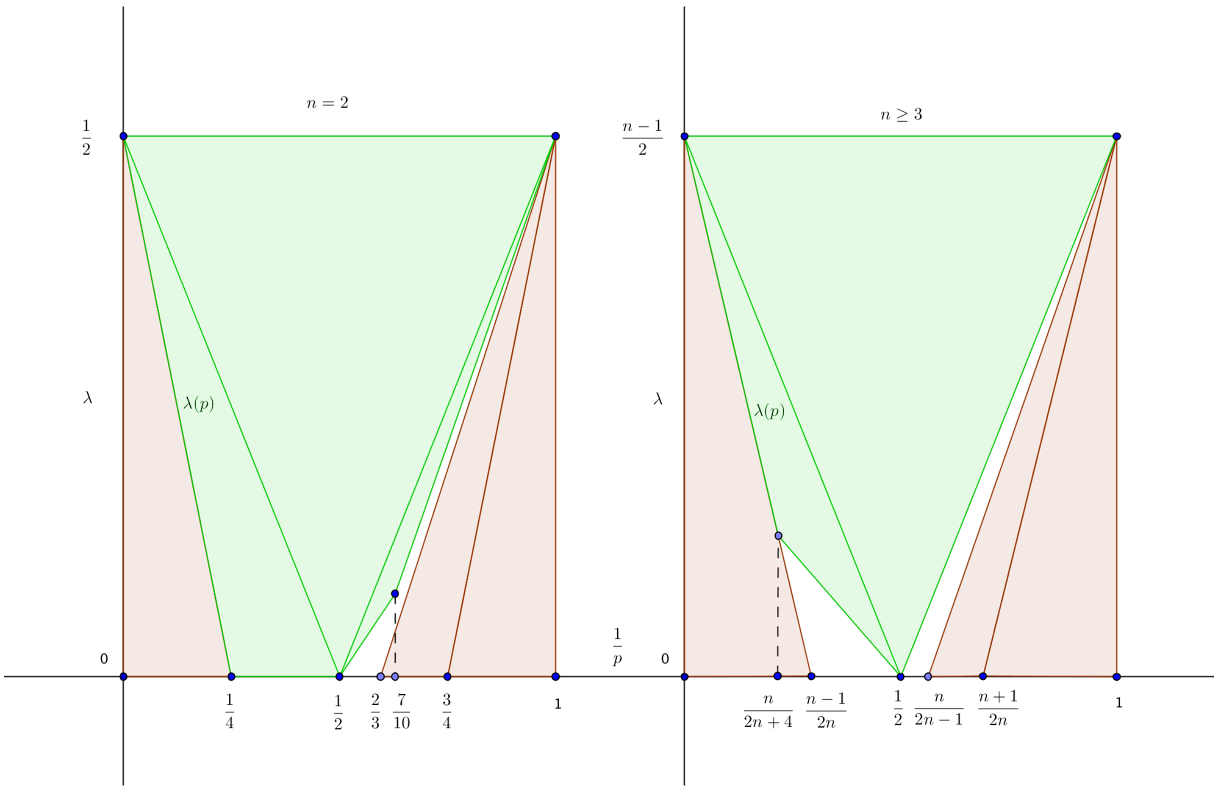


Figure 3.3: Current state of the Maximal Bochner-Riesz conjecture.

Now that both the Bochner-Riesz and maximal Bochner-Riesz conjectures have been presented, let us see how our techniques can be applied in this setting. First, we will introduce the standard decomposition of $(1-t)_+^\lambda$, which will be useful for both $B_\lambda$ and $B_\lambda^*$. Notice that in previous sections, the decomposition we made for the Bochner-Riesz operator was for the convolution kernel. Now we will need a similar one on the Fourier side (see [63, Section 10.2.2]). Take functions $\varphi \in \mathcal{C}_c^\infty(-1/2, 1/2)$ and $\psi \in \mathcal{C}_c^\infty(1/8, 5/8)$, in such a way that, for every $0 \leqslant t < 1$,

$$\varphi(t) + \sum_{k=0}^{\infty} \psi\left(\frac{1-t}{2^{-k}}\right) \equiv 1.$$

Then, for every $\xi \in \mathbb{R}^n$ and $r > 0$ such that $r|\xi| < 1$:

$$\left(1 - r^2|\xi|^2\right)^\lambda = m_{00}(r^2|\xi|^2) + \sum_{k=0}^{\infty} 2^{-k\lambda} m_k(r^2|\xi|^2),$$

where $m_{00}(r^2|\xi|^2) = \varphi(r^2|\xi|^2)(1 - r^2|\xi|^2)^\lambda$ and for $k \geqslant 0$,

$$m_k\left(r^2|\xi|^2\right) = \left(\frac{1 - r^2|\xi|^2}{2^{-k}}\right)^\lambda \psi\left(\frac{1 - r^2|\xi|^2}{2^{-k}}\right) = \Psi\left(\frac{1 - r^2|\xi|^2}{2^{-k}}\right).$$

Clearly, we also have that $\Psi(t) = t^\lambda \psi(t)$ is a function in $\mathcal{C}_c^\infty(1/8, 5/8)$. Now, this decomposition gives, for $r = 1$,

$$B_\lambda f = T_{m_{00}} f + \sum_{k=0}^{\infty} 2^{-k\lambda} T_{m_k} f, \tag{3.14}$$

and taking supremum over $r > 0$,

$$B_\lambda^* f \leqslant T_{m_{00}}^* f + \sum_{k=0}^{\infty} 2^{-k\lambda} T_{m_k}^* f. \tag{3.15}$$

The next two propositions will play an essential role in what follows, allowing us to transfer estimates from radial Fourier multipliers to Bochner-Riesz operators and vice versa:

**Proposition 3.18.** *Let $\lambda > 0$ and $X, Y$ be a couple of spaces, with $X$ quasi-Banach and $Y$ Banach. Then, if $T_{m_{00}}$(resp. $T_{m_{00}}^*$) $: X \to Y$ is bounded and, for every $k \geqslant 0$, the operators $T_{m_k}$(resp. $T_{m_k}^*$) $: X \to Y$ are bounded with constant $C_k$ in such a way that $\{C_k 2^{-k\lambda}\}_{k \geqslant 0} \in \ell^1$, then*

$$B_\lambda (\textit{resp. } B_\lambda^*) : X \longrightarrow Y$$

*is also bounded.*

*Proof.* This is just an application of Minkowski's inequality to (3.14) and (3.15) respectively. □

Notice that for every $r > 0$, $m_{00}(r^2|\xi|^2)$ is a $\mathcal{C}^\infty$ function with compact support and hence $T_{m_{00}} \leqslant T_{m_{00}}^* \lesssim M$. This means that whenever the Hardy-Littlewood maximal operator $M : X \to Y$, the boundedness assumption on $T_{m_{00}}$ and $T_{m_{00}}^*$ will automatically hold. Now, using the ideas in the proof of Theorem 3.10, we also have:

**Proposition 3.19.** *Given $\lambda > 0$, let $m$ be a bounded, continuous function on $(0, \infty)$ which vanishes at infinity and satisfies that*

$$D^{\lambda+1-j} m \in AC_{\mathrm{loc}} \quad \forall j = 1, ..., [\lambda] + 1.$$

*Then, for every $r > 0$,*

$$T_m^r f(x) = \int_0^\infty B_\lambda^{rs} f(x) \Phi_m^\lambda(s) ds, \tag{3.16}$$

*with*

$$\Phi_m^\lambda(s) = C_\lambda s^{-2\lambda-3} D^{\lambda+1} m(s^{-2}).$$

*In particular, if $Y$ is a Banach space, then*

$$\|T_m f\|_Y \leqslant \int_0^\infty \|B_\lambda^s f\|_Y |\Phi_m^\lambda(s)| ds,$$

*and*

$$\|T_m^* f\|_Y \leqslant \|B_\lambda^* f\|_Y \|\Phi_m^\lambda\|_{L^1(0,\infty)}.$$

*Proof.* Just take (3.7), write $\lambda$ instead of $\alpha - 1$, $r^2|\xi|^2$ instead of $|\xi|^2$, and conclude that

$$m(r^2|\xi|^2) = C_\lambda \int_0^\infty \left(1 - \frac{r^2|\xi|^2}{s^2}\right)_+^\lambda s^{2\lambda+1} D^{\lambda+1} m(s^2) ds = \int_0^\infty (1 - r^2 s^2 |\xi|^2)_+^\lambda \Phi_m^\lambda(s) ds,$$

which proves (3.16). Now, if we take $r = 1$ and apply Minkowski's inequality, we get the estimate for $\|T_m f\|_Y$, and taking supremum over $r > 0$ and then using Minkowski, gives the one for $\|T_m^* f\|_Y$. □

It is clear that this last result will come in handy when $m = m_k$, so let us compute the $L^1$ norm of $\Phi_{m_k}^\lambda$ just as we did for $h_\delta$ in Lemma 3.13. In fact, the computation will be analogous.

**Lemma 3.20.** *Given $\lambda > 0$, then, for every $k \geqslant 0$,*

$$\|\Phi_{m_k}^\lambda\|_{L^1(0,\infty)} \leqslant C_{\psi,\lambda} 2^{k\lambda}.$$

*Proof.* Recall that $\Phi_{m_k}^\lambda(s) = C_\lambda s^{-2\lambda-3} D^{\lambda+1} m_k(s^{-2})$ and $m_k(s) = \Psi\left(\frac{1-s}{2^{-k}}\right)$, where $\Psi$ was a slight modification of $\psi$, still in $\mathcal{C}_c^\infty(1/8, 5/8)$. With this, we use property (3.3) to compute

$$D^{\lambda+1} m_k(s^{-2}) = 2^{k(\lambda+1)} D^{\lambda+1} \widetilde{\Psi}\left(\frac{s^{-2} - 1}{2^{-k}}\right).$$

Now, exactly as in Lemma 3.13,

$$\begin{aligned}
\int_0^\infty |\Phi_{m_k}^\lambda(s)| ds &= C_\lambda 2^{k\lambda} \int_{-2^k}^\infty (2^{-k} r + 1)^\lambda |D^{\lambda+1} \widetilde{\Psi}(r)| dr \\
&\leqslant C_{\psi,\lambda} 2^{k\lambda} \left( \int_{-2^k}^{-1} \frac{(2^{-k} r + 1)^\lambda}{|r|^{\lambda+2}} dr + \int_{-1}^{-1/8} (2^{-k} r + 1)^\lambda dr \right) \leqslant C_{\psi,\lambda} 2^{k\lambda},
\end{aligned}$$

where we need to remember that $\operatorname{supp}(\widetilde{\Psi}) \subseteq (-5/8, -1/8)$ and use the decay of $|D^{\lambda+1}\widetilde{\Psi}|$ in property (3.4) for small values of $r$. □

Now we go back to the conjectures. Take for instance the maximal Bochner-Riesz operator in $\mathbb{R}^2$ and $1 < p < 2$ (see the picture on the left in Figure 3.3). Recall that, as we mentioned in (3.13), in that case, if we define

$$\bar{\lambda}(p) := \max\left(\frac{3}{4p} - \frac{3}{8}, \frac{7}{6p} - \frac{2}{3}\right),$$

the best known result was that, when $1 < p < 2$,

$$B^*_\lambda : L^p(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n), \quad \text{if } \lambda > \bar{\lambda}(p). \tag{3.17}$$

Fix $1 < p_0 < 2$. We have that, for every $\varepsilon > 0$,

$$B^*_{\bar{\lambda}(p_0)+\varepsilon} : L^{p_0}(\mathbb{R}^n) \longrightarrow L^{p_0}(\mathbb{R}^n).$$

Now, we apply Proposition 3.19 and Lemma 3.20 to get that, for every $k \geqslant 0$,

$$\|T^*_{m_k} f\|_{p_0} \leqslant \|B^*_{\bar{\lambda}(p_0)+\varepsilon} f\|_{p_0} \|\Phi^{\bar{\lambda}(p_0)+\varepsilon}_{m_k}\|_{L^1(0,\infty)} \lesssim 2^{k(\bar{\lambda}(p_0)+\varepsilon)} \|f\|_{p_0}. \tag{3.18}$$

On the other hand, by [107] we know that, at the critical index (which for $n = 2$ corresponds to $1/2$),

$$B^*_{1/2} : L^{p_0}(w) \longrightarrow L^{p_0}(w), \quad \forall w \in A_{p_0}.$$

Again, Proposition 3.19 and Lemma 3.20 yield that, for every $k \geqslant 0$ and $w \in A_{p_0}$,

$$\|T^*_{m_k} f\|_{L^{p_0}(w)} \lesssim 2^{k/2} \|f\|_{L^{p_0}(w)}. \tag{3.19}$$

Using interpolation with change of measure between (3.18) and (3.19), one gets that, for every $k \geqslant 0$ and $\theta \in (0,1)$,

$$\|T^*_{m_k} f\|_{L^{p_0}(w^\theta)} \lesssim 2^{k(\theta/2+(1-\theta)(\bar{\lambda}(p_0)+\varepsilon))} \|f\|_{L^{p_0}(w^\theta)}.$$

Since this holds for every $\theta$ in the open interval $(0,1)$ and at the beginning, we could take any $\varepsilon > 0$, we can effectively get rid of the latter and simply write that, for every $\theta \in (0,1)$,

$$\|T^*_{m_k} f\|_{L^{p_0}(w^\theta)} \lesssim 2^{k(\theta/2+(1-\theta)\bar{\lambda}(p_0))} \|f\|_{L^{p_0}(w^\theta)}.$$

Now we use this estimate in Proposition 3.18 to conclude that, given $\lambda > 0$,

$$B^*_\lambda : L^{p_0}(w^\theta) \longrightarrow L^{p_0}(w^\theta),$$

if the sequence $\{2^{k(\theta/2+(1-\theta)\bar{\lambda}(p_0))-k\lambda}\}_{k\geqslant 0}$ belongs to $\ell^1$. The result we have proved is the following:

**Lemma 3.21.** *For every $1 < p_0 < 2$, every $w \in A_{p_0}$ and every $\theta \in (0,1)$, we have that*

$$B^*_\lambda : L^{p_0}(w^\theta) \longrightarrow L^{p_0}(w^\theta),$$

*whenever $\lambda > \frac{\theta}{2} + (1-\theta)\bar{\lambda}(p_0)$.*

Notice that the fact that we have an $A_{p_0}$ weight to a power $\theta$ does not allow the use of classical extrapolation. We will use the limited range extrapolation from [26], as presented in Corollary 1.16. A direct application of this result to the estimate in Lemma 3.21 yields that, for every $1 < p_0 < 2$ and every $\theta \in (0, 1)$,

$$B_\lambda^* : L^p(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n),$$

for every $p \in (p_-, p_+)$ and provided that $\lambda > \frac{\theta}{2} + (1-\theta)\bar{\lambda}(p_0)$. Recall that the definition of $p_-$ and $p_+$ comes from the identities

$$p'_- := \frac{p'_0}{1-\theta}, \qquad p_+ := \frac{p_0}{1-\theta}.$$

Notice that if $p > p_0$, we have that $\bar{\lambda}(p) < \bar{\lambda}(p_0) < \frac{\theta}{2} + (1-\theta)\bar{\lambda}(p_0) < \lambda$, so the boundedness of $B_\lambda^*$ on $L^p(\mathbb{R}^n)$ is already known from (3.17). Therefore, the interesting part is to study the range of $p \in (p_-, p_0)$. If we make the computation, we get that

$$p_- = \frac{p_0}{1 + \theta(p_0 - 1)}.$$

If we want to obtain the smallest $p_-$ possible, we need to pick the largest admissible value of $\theta$. Isolating $\theta$ in the inequality $\lambda > \frac{\theta}{2} + (1-\theta)\bar{\lambda}(p_0)$, we get that

$$0 < \theta < \frac{\lambda - \bar{\lambda}(p_0)}{1/2 - \bar{\lambda}(p_0)},$$

so we can just pick the upper bound[5] and, with the condition $\lambda > \bar{\lambda}(p_0)$ (so that $\theta$ is positive), we can write that, for every $1 < p_0 < \infty$ and every $\lambda > \bar{\lambda}(p_0)$, the operator $B_\lambda^*$ is bounded on $L^p$ whenever

$$p > \frac{p_0 - 2\bar{\lambda}(p_0)p_0}{1 + 2(p_0 - 1)\lambda - 2p_0\bar{\lambda}(p_0)}.$$

If we write this last inequality in terms of $\lambda$ and put it together with the condition $\lambda > \bar{\lambda}(p_0)$ we needed, everything can be summarized in the following proposition:

**Proposition 3.22.** *Given $1 < p_0 < 2$, it holds that*

$$B_\lambda^* : L^p(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n),$$

*whenever*

$$\lambda > \max\left(\bar{\lambda}(p_0), \frac{p_0 + 2\bar{\lambda}(p_0)p_0(p-1) - p}{2p(p_0 - 1)}\right).$$

[5]Technically, we should choose $\theta$ to be the upper bound minus $\varepsilon > 0$, but since all the inequalities that we will get are strict, it would not make any difference.

Unfortunately, by considering the whole possible initial values of $p_0 \in (1,2)$, one can see that the region $(1/p, \lambda)$ for which we get boundedness of $B_\lambda^*$ on $L^p(\mathbb{R}^n)$ is exactly that in (3.17), so no new estimates are obtained. The same approach was taken in the case of the maximal Bochner-Riesz conjecture for $n \geqslant 3$ and values $2 < p < \infty$, starting from Lee's best known result (see the picture on the right-hand side in Figure 3.3), but again, no new regions were found. The same idea could be used in the case of the Bochner-Riesz conjecture (with Bourgain and Guth's estimate (3.12)), but after all, it seems that the fact that we use interpolation with weighted estimates for $B_\lambda$ (or $B_\lambda^*$ respectively) at the critical index $\lambda = \frac{n-1}{2}$, prevents us from reaching any region that could not be reached by analytic interpolation in the first place. Even though we have not been able to make any new contribution to the conjectures, we wanted to include this subsection to give yet another application of transference of estimates to averages.

## 3.4 Fourier multipliers of Hörmander type on $\mathbb{R}^n$

First, let us introduce the Hörmander condition for a multiplier $m$. We will use the standard notation $|\alpha| = \alpha_1 + \cdots + \alpha_n$ for a multi-index $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n$ and if $x \in \mathbb{R}^n$,

$$\left(\frac{\partial}{\partial x}\right)^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}.$$

**Definition 3.23.** *Let $k \in \mathbb{N}$ such that $k > n/2$ and let $m : \mathbb{R}^n \to \mathbb{R}$ be a bounded function of $\mathcal{C}^k$ class on $\mathbb{R}^n \backslash \{0\}$. Given $1 < s \leqslant 2$, we say that $m \in HC(s,k)$ if*

$$\sup_{r>0} \left( r^{2|\alpha|-n} \int_{r \leqslant |x| \leqslant 2r} \left| \left(\frac{\partial}{\partial x}\right)^\alpha m(x) \right|^s dx \right)^{1/s} < \infty, \quad |\alpha| \leqslant k.$$

The classical Hörmander theorem (see for instance, the statement in [63, Theorem 5.2.7]) says that, in the unweighted case, the operator defined by

$$\widehat{T_m f}(\xi) = m(\xi) \widehat{f}(\xi), \quad \xi \in \mathbb{R}^n,$$

is of strong-type $(p,p)$ for $1 < p < \infty$, and weak-type $(1,1)$, whenever $m \in HC(2,k)$ for some $k > n/2$. The generalization of the condition to $s \neq 2$ was introduced in [11], where the authors use interpolation methods to check that the corresponding classical result for $m \in HC(s,k)$ needs $k > n/s$. In [68, 77, 122], the authors introduce power weights to the problem, but in the context of general $A_p$ weights, the best result that is known requires at least $k = n$. More precisely, it can be proved that for $m \in HC(s,n)$, $T_m$ is of strong-type $(p,p)$ for every weight in $A_p$ and $1 < p < \infty$, and weak-type $(1,1)$ for every weight in $A_1$. This can be found in [79, Theorem 1], where the authors use the function $f^\sharp$ of Fefferman and Stein introduced in [57]. Their result is the following:

**Theorem 3.24.** *Let $1 < s \leqslant 2$ and $m \in HC(s,n)$. Then, the operator defined by*

$$\widehat{T_m f}(\xi) = m(\xi)\widehat{f}(\xi), \quad \xi \in \mathbb{R}^n,$$

*is of strong-type $(p,p)$ for every $1 < p < \infty$ and every weight in $A_p$, and of weak-type $(1,1)$ for every weight in $A_1$.*

The proof of this result heavily relies on a slightly more general version of the following lemma, which translates the conditions of $m \in HC(s,k)$ into conditions on the convolution kernel $K = m^\vee$. For technical reasons, as in [70, 79], we need to work with a truncation $K_N$ of $K$. The decomposition, though, is standard: Let $\varphi$ be a non-negative $\mathcal{C}^\infty$ function, supported in $\{1/2 < |x| < 2\}$ and such that

$$\sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1, \quad \xi \neq 0.$$

For every $j \in \mathbb{Z}$, we set $m_j(\xi) = m(\xi)\varphi(2^{-j}\xi)$, which is supported in $\{2^{j-1} < |x| < 2^{j+1}\}$ and satisfies that

$$m(\xi) = \sum_{j\in\mathbb{Z}} m_j(\xi), \quad \xi \neq 0.$$

Now, for every $N \in \mathbb{N}$, if $k_j(x) = m_j^\vee(x)$, we can define

$$m^N(\xi) = \sum_{j=-N}^{N} m_j(\xi), \quad K_N(x) = (m^N)^\vee(x) = \sum_{j=-N}^{N} k_j(x).$$

We have that $\|m^N\|_\infty \leqslant C$ uniformly in $N \in \mathbb{N}$ and $m^N(\xi) \to m(\xi)$, $\xi \neq 0$, as $N \to \infty$. We define $T_m^N f := K_N * f$ and work with this approximation instead of $T_m$. The next lemma is the key estimate in [79]:

**Lemma 3.25.** *Let $1 < s \leqslant 2$, $k \in \mathbb{N}$ and $m \in HC(s,k)$. Then, for every $r > 1$ such that*

*(a)* $1 < r \leqslant s$,

*(b)* $\frac{n}{r} < k < \frac{n}{r} + 1$,

*every $1 \leqslant p \leqslant r'$ and every $R > 0$,*

$$\left(\int_{R<|x|<2R} |K_N(x-y) - K_N(x)|^p dx\right)^{1/p} \lesssim R^{-k+n/p-n/r'}|y|^{k-n/r}, \quad \text{when } |y| < \frac{R}{2},$$

*uniformly in $N$.*

In this section, we will follow the ideas in [79] to get a restricted weak-type estimate in the spirit of Theorem 2.9:

**Theorem 3.26.** *Fix $1 < s \leqslant 2$ and $m \in HC(s,n)$. The associated multiplier operator $T_m$ satisfies that, for every $u \in A_1$, there exists $1 < p_0 < \infty$ depending on $\|u\|_{A_1}$ such that, for each measurable set $E \subseteq \mathbb{R}^n$,*

$$\|T_m \chi_E\|_{L^{p_0,\infty}((M\chi_E)^{1-p_0}u)} \lesssim C_u u(E)^{1/p_0}.$$

The proof will be based on this lemma:

**Lemma 3.27.** *Let $1 < s \leqslant 2$, $m \in HC(s,n)$ and $u \in A_1$. Then, take*

$$1 < r < \min\left\{s,\ \frac{n}{n-1},\ 1 + \frac{1}{2^{n+1}\|u\|_{A_1}}\right\}$$

*and $1 < q < 2 - \frac{1}{r}$. Now, for every measurable set $E \subseteq \mathbb{R}^n$ and cube $Q \subseteq \mathbb{R}^n$, if $w := (M\chi_E)^{1-q}u$ and $c$ is the center of the cube $Q$, it holds that, for every $y \in Q$,*

$$\int_{\mathbb{R}^n \setminus 2Q} |K_N(x-y) - K_N(x-c)| w(x)dx \lesssim \frac{|Q|^q}{|E \cap Q|^q} \frac{w(E \cap Q)}{|Q|},$$

*independently of $N$.*

*Proof.* We split the integral of the left-hand side into dyadic annuli and by Hölder's inequality,

$$\begin{aligned}
&\sum_{j=1}^{\infty} \int_{2^{j+1}Q \setminus 2^j Q} |K_N(x-y) - K_N(x-c)| w(x)dx \\
&\leqslant \sum_{j=1}^{\infty} \sup_{2^{j+1}Q} (M\chi_E)^{1-q} \left( \int_{2^{j+1}Q \setminus 2^j Q} |K_N(x-y) - K_N(x-c)|^{r'} dx \right)^{1/r'} \left( \int_{2^{j+1}Q} u^r(x)dx \right)^{1/r}.
\end{aligned}$$

For the first integral, we use Lemma 3.25 with $k = n$ and $p = r'$. Conditions $(a)$ and $(b)$ are fulfilled because $r < \min\{s,\ n/(n-1)\}$. For the second one, we recall that $r < 1 + \frac{1}{2^{n+1}\|u\|_{A_1}}$ ensures that the weight $u^r$ still lies in $A_1$. With these two remarks, the previous expression can be bounded by:

$$\begin{aligned}
&\sum_{j=1}^{\infty} \sup_{x \in 2^{j+1}Q} (M\chi_E)^{1-q}(x) (2^j \ell(Q))^{-n} \ell(Q)^{n-n/r} (2^{j+1}\ell(Q))^{n/r} \left( \frac{1}{|2^{j+1}Q|} \int_{2^{j+1}Q} u^r(x)dx \right)^{1/r} \\
&\lesssim \sum_{j=1}^{\infty} \sup_{x \in 2^{j+1}Q} (M\chi_E)^{1-q}(x) (2^{jn})^{1/r-1} \inf_{x \in 2^{j+1}Q} u(x).
\end{aligned}$$

Now we use Lemma 1.6 to control the supremum of $(M\chi_E)^{1-q}$ over $2^{j+1}Q$ by its average and, inserting the weight $u$ in the integral, we get

$$\sum_{j=1}^{\infty} \frac{(2^{jn})^{1/r-1}}{|2^{j+1}Q|} \int_{2^{j+1}Q} w(x)dx = \sum_{j=1}^{\infty} \frac{w(2^{j+1}Q)}{|2^{j+1}Q|} (2^{jn})^{1/r-1}.$$

Finally, we use the $A_q^{\mathcal{R}}$ property of $w$ on the inclusion $E \cap Q \subseteq 2^{j+1}Q$,

$$\begin{aligned}\sum_{j=1}^{\infty} \frac{w(2^{j+1}Q)}{|2^{j+1}Q|}(2^{jn})^{1/r-1} &\lesssim \sum_{j=1}^{\infty} \frac{w(E\cap Q)}{|2^{j+1}Q|}\frac{|2^{j+1}Q|^q}{|E\cap Q|^q}(2^{jn})^{1/r-1} \\ &\approx \frac{|Q|^q}{|E\cap Q|^q}\frac{w(E\cap Q)}{|Q|}\sum_{j=1}^{\infty}(2^{jn})^{q+1/r-2},\end{aligned}$$

and the fact that $q < 2 - \frac{1}{r}$ to complete the proof. $\square$

*Proof of Theorem 3.26.* Let $1 < s \leqslant 2$, $m \in HC(s,n)$ and $u \in A_1$. We want to choose $1 < p_0 < \infty$ so that it satisfies the conditions of $q$ in Lemma 3.27. It is enough to take

$$1 < p_0 < \min\left\{2 - \frac{1}{s}, \frac{n+1}{n}, \frac{2+2^{n+1}\|u\|_{A_1}}{1+2^{n+1}\|u\|_{A_1}}\right\}.$$

Define $w := (M\chi_E)^{1-p_0}u$. Now we make the standard Calderón-Zygmund decomposition of $\chi_E$ at height $\alpha > 0$, obtaining a family of pairwise disjoint dyadic cubes $\{Q_k\}_k$ satisfying the stopping-time condition

$$\alpha < \frac{|E\cap Q_k|}{|Q_k|} \leqslant 2^n\alpha,$$

and a couple of functions $g, b$ such that $\chi_E = g + b$, defined by

$$g(x) = \begin{cases}\chi_E(x), & x \notin \bigcup_k Q_k,\\ \frac{|E\cap Q_k|}{|Q_k|}, & x \in Q_k,\end{cases}$$

and $b(x) = \sum_k b_k(x)$ with

$$b_k(x) = \chi_E(x) - \frac{|E\cap Q_k|}{|Q_k|}, \quad x \in Q_k.$$

Notice that when $\alpha \geqslant 1$, $\{Q_k\}_k = \varnothing$, and hence $\chi_E = g$, and when $0 < \alpha < 1$ (as we pointed out in Lemma 2.3), $E \subseteq \bigcup_k Q_k$ except for a null set, which makes $g(x) = 0$ for almost every $x \notin \bigcup_k Q_k$. Here we list the properties that we will need of this decomposition:

(i) $\int_{Q_k} b_k = \int_{\mathbb{R}^n} b = 0$ and $\|b_k\|_1 \lesssim \alpha|Q_k|$,

(ii) $\|g\|_\infty \lesssim \alpha$ and $\|g\|_{L^2(w)}^2 \lesssim \alpha^{2-p_0}w(E)$.

All the properties that do not involve the weight $w$ are well-known (see, for instance, [63, Theorem 4.3.1]). As for the weighted estimate, we only need to recall that $w \in A_{p_0}^{\mathcal{R}}$ and

$$\begin{aligned}\|g\|_{L^{p_0}(w)}^{p_0} &\lesssim w(E) + \left\|\sum_k \frac{|E\cap Q_k|}{|Q_k|}\chi_{Q_k}\right\|_{L^{p_0}(w)}^{p_0} = w(E) + \sum_k \frac{|E\cap Q_k|^{p_0}}{|Q_k|^{p_0}}w(Q_k) \\ &\lesssim w(E) + \sum_k w(E\cap Q_k) \approx w(E).\end{aligned}$$

Hence,
$$\|g\|_{L^2(w)}^2 \leqslant \|g\|_\infty^{2-p_0}\|g\|_{L^{p_0}(w)}^{p_0} \lesssim \alpha^{2-p_0}w(E).$$
With this, we can finish the proof. Clearly,

$$w(|T_m^N\chi_E|>\alpha) \lesssim w(|T_m^N g|>\alpha) + w\left(\bigcup_k 2Q_k\right) + w\left(x \notin \bigcup_k 2Q_k : |T_m^N b(x)|>\alpha\right).$$

For the first term, we use Chebyshev's inequality, the strong-type $(2,2)$ of $T_m$ for $A_2$ weights given by Theorem 3.24 (together with $w \in \widehat{A}_{p_0} \subseteq A_2$), and property $(ii)$ above:

$$w(|T_m^N g|>\alpha) \lesssim \frac{\|T_m^N g\|_{L^2(w)}^2}{\alpha^2} \lesssim \frac{\|g\|_{L^2(w)}^2}{\alpha^2} \lesssim \frac{w(E)}{\alpha^{p_0}}.$$

For the second term, we need to use that $w$ is doubling, the stopping condition of the cubes, and the $A_{p_0}^{\mathcal{R}}$ property of $w$:

$$w\left(\bigcup_k 2Q_k\right) \lesssim \sum_k w(Q_k) \approx \sum_k \frac{1}{\alpha^{p_0}}\frac{|E\cap Q_k|^{p_0}}{|Q_k|^{p_0}}w(Q_k) \lesssim \sum_k \frac{w(E\cap Q_k)}{\alpha^{p_0}} \approx \frac{w(E)}{\alpha^{p_0}}.$$

And finally, for the third term, we can use Chebyshev and reduce the problem to check if

$$\int_{\mathbb{R}^n\setminus\bigcup_k 2Q_k} |T_m^N b(x)|w(x)dx \lesssim \frac{w(E)}{\alpha^{p_0-1}} \tag{3.20}$$

holds. To see this, we use the cancellation of $b$ and Fubini,

$$\begin{aligned}\int_{\mathbb{R}^n\setminus\bigcup_k 2Q_k} |T_m^N b(x)|w(x)dx &= \int_{\mathbb{R}^n\setminus\bigcup_k 2Q_k}\left|\sum_k\int_{Q_k}(K_N(x-y)-K_N(x-y_k))b(y)dy\right|w(x)dx\\ &\leqslant \sum_k\int_{Q_k}|b(y)|\int_{\mathbb{R}^n\setminus 2Q_k}|K_N(x-y)-K_N(x-y_k)|w(x)dxdy,\end{aligned}$$

where $y_k$ is the center of $Q_k$. Now we use Lemma 3.27 with $Q = Q_k$ and $q = p_0$ and, recalling that $|E\cap Q_k|/|Q_k| \approx \alpha$, we get

$$\sum_k \|b_k\|_1 \frac{w(E\cap Q_k)}{\alpha^{p_0}|Q_k|}.$$

But we know that $\|b_k\|_1 \lesssim \alpha|Q_k|$, so we obtain (3.20). Bringing the three estimates together, we show that

$$\alpha^{p_0}w(|T_m^N\chi_E|>\alpha) \lesssim w(E) = u(E),$$

so taking supremum over $\alpha>0$ we finish the proof for $T_m^N$. Since all the estimates are independent of $N\in\mathbb{N}$, we can use Fatou's lemma to deduce the result for $T_m$. □

**Remark 3.28.** *Notice that, even though the value of $p_0$ heavily depends on $u$ (and hence we cannot prove an estimate for the whole $\widehat{A}_{p_0}$ class and some $1 < p_0 < \infty$), we have not used that the weight $w = (M\chi_E)^{1-p_0}u$ has the characteristic function of $E$. Therefore, it is worth pointing out that the estimate that we have for $T_m$ would still be true if we considered a weight of the form $(Mh)^{1-p_0}u$, for some $h \in L^1_{\mathrm{loc}}$.*

**Remark 3.29.** *Notice also that in the proof of Lemma 3.27, when we estimate the integral*

$$\int_{2^{j+1}Q\setminus 2^jQ} |K_N(x-y) - K_N(x-c)|u(x)dx,$$

*we need to use Hölder's inequality to separate the weight from the kernel and be able to use Lemma 3.25. Moreover, since we need $u^r$ to remain in $A_1$, we pay the price of having an $L^{r'}$ norm on $|K_N(x-y) - K_N(x-c)|$ with a large $r'$ . Let us see that, if we assume $u = 1$, we can improve the differentiability conditions on $m$.*

**Lemma 3.30.** *Let $1 < s \leqslant 2$, $k \in \mathbb{N}$ with $k > \frac{n}{s}$ and $m \in HC(s,k)$. There exists $q > 1$ such that, for every measurable set $E \subseteq \mathbb{R}^n$ and cube $Q \subseteq \mathbb{R}^n$, if $w := (Mh)^{1-q}$ for some $h \in L^1_{\mathrm{loc}}$ and $c$ is the center of the cube $Q$, then, for every $y \in Q$,*

$$\int_{\mathbb{R}^n\setminus 2Q} |K_N(x-y) - K_N(x-c)|w(x)dx \lesssim \frac{|Q|^q}{|E\cap Q|^q}\frac{w(E\cap Q)}{|Q|},$$

*independently of $N$.*

*Proof.* Set $\varepsilon = k - \frac{n}{s}$. Since $HC(s,k_1) \subseteq HC(s,k_2)$ when $k_2 \leqslant k_1$, we can assume that $k$ is the smallest integer such that $k > \frac{n}{s}$ and hence, $0 < \varepsilon \leqslant 1$. We start exactly as in the proof of Lemma 3.27, but now we do not use Hölder's inequality and simply write

$$\begin{aligned}&\sum_{j=1}^{\infty}\int_{2^{j+1}Q\setminus 2^jQ} |K_N(x-y) - K_N(x-c)|w(x)dx\\ &\leqslant \sum_{j=1}^{\infty}\sup_{x\in 2^{j+1}Q}(Mh)^{1-q}(x)\int_{2^{j+1}Q\setminus 2^jQ} |K_N(x-y) - K_N(x-c)|dx.\end{aligned}$$

Here we use Lemma 3.25 with $p = 1$ to control the integral. If $0 < \varepsilon < 1$, we can take $r = s$ so that $(a)$ and $(b)$ from the lemma hold, because

$$\frac{n}{s} < k = \frac{n}{s} + \varepsilon < \frac{n}{s} + 1.$$

With this, we can bound the previous expression by

$$\sum_{j=1}^{\infty}\sup_{x\in 2^{j+1}Q}(Mh)^{1-q}(x)(2^j\ell(Q))^{-k+n-n/s'}\ell(Q)^{k-n/s} = \sum_{j=1}^{\infty}\sup_{x\in 2^{j+1}Q}(Mh)^{1-q}(x)2^{-j\varepsilon}.$$

Here we use Lemma 1.6 followed by the $A_q^{\mathcal{R}}$ property of $(Mh)^{1-q}$ as before and the only thing we need to finish the proof is to make sure that

$$\sum_{j=1}^{\infty} 2^{-j(\varepsilon-qn+n)} < \infty.$$

But this is guaranteed if we choose $1 < q < 1 + \frac{\varepsilon}{n}$, so we are done for this first case. If $\varepsilon = 1$, we cannot choose $r = s$ when applying Lemma 3.25, but we can take $r < s$ close enough to $s$ so that

$$\frac{n}{r} < \frac{n}{s} + 1 < \frac{n}{r} + 1,$$

and now the series we need to converge is

$$\sum_{j=1}^{\infty} 2^{-j(k-n/r-qn+n)}.$$

Choosing $1 < q < 1 + \frac{k-n/r}{n}$ we complete the proof. □

It is clear that, in the same way that we obtained Theorem 3.26 from Lemma 3.27, from here we can deduce the following weighted estimate for multipliers $m \in HC(s,k)$ with $k > n/s$, which is the condition of the classical Hörmander theorem without weights:

**Theorem 3.31.** *Fix $1 < s \leqslant 2$, $k \in \mathbb{N}$ with $k > \frac{n}{s}$ and $m \in HC(s,k)$. Then, there exists $1 < p_0 < \infty$ so that, for every weight of the form $w = (Mh)^{1-p_0}$, $h \in L^1_{\text{loc}}$, the multiplier operator $T_m$ satisfies*

$$\|T_m \chi_E\|_{L^{p_0,\infty}(w)} \lesssim w(E)^{1/p_0},$$

*for every measurable set $E \subseteq \mathbb{R}^n$.*

Using an extrapolation argument, from Theorems 3.26 and 3.31 we can deduce the weak-type (1,1) with no weights (when $m \in H(s,k)$ and $k > n/s$) and for every weight in $A_1$ (when $m \in HC(s,n)$). To be precise, the extrapolation of Theorem 1.11 yields restricted weak-type estimates, but if we show that the family of operators $\{T_m^N\}_{N\in\mathbb{N}}$ are $(\varepsilon,\delta)$-atomic, then we can prove the unrestricted estimates for each $T_m^N$ and passing to the limit when $N \to \infty$, deduce the result for $T_m$:

**Corollary 3.32.** *Let $1 < s \leqslant 2$, $k \in \mathbb{N}$ and $m \in HC(s,k)$.*

- *If $k > \frac{n}{s}$, then $T_m$ is of weak-type (1,1).*
- *If $k = n$, then $T_m$ is of weak-type (1,1) for every weight in $A_1$.*

*Proof.* As we mentioned, it all boils down to proving that, for every $N \in \mathbb{N}$, $T_m^N$ is an $(\varepsilon, \delta)$-atomic operator. Using an estimate from [79, p. 349], we have that if $k > \frac{n}{s}$ (which happens in both cases), then, for every $R > 0$,

$$\int_{R<|x|<2R} |K_N(x)|^2 dx \lesssim R^{-n}.$$

Hence,

$$\int_{\mathbb{R}^n \setminus B(0,1)} |K_N(x)|^2 dx = \sum_{j \geqslant 0} \int_{2^j<|x|<2^{j+1}} |K_N(x)|^2 dx \lesssim \sum_{j \geqslant 0} 2^{-jn} < \infty.$$

Moreover, since $m_j \in L^1$, we have that $k_j = m_j^\vee \in L^\infty$ and $K_N = \sum_{|j| \leqslant N} k_j \in L^\infty$. Therefore,

$$\int_{B(0,1)} |K_N(x)|^2 dx < \infty,$$

and we conclude that $K_N \in L^2(\mathbb{R}^n)$ for every $N \in \mathbb{N}$. By (1.14), we have that $T_m^N$ is $(\varepsilon, \delta)$-atomic. With this, we need to combine Theorems 3.26 and 3.31 with Theorem 1.11 and Theorem 1.14 to prove the result for $T_m^N$. We finish the proof for $T_m$ by Fatou's lemma when $N \to \infty$. □

### 3.4.1 A brief remark on the singular integral $T_\Omega$

The argument that we used in this section to obtain weighted results for multipliers relied on estimates concerning their associated convolution kernels. For this reason, in [79] the authors can deduce analogous weighted inequalities for convolution operators without much effort. We will see what happens if we try to replicate the argument in our case. First of all, let us introduce the problem and explain what is known. Let $\mathbb{S} := \mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ be the unit sphere in $\mathbb{R}^n$, equipped with the surface measure $\sigma$. For every $x \neq 0$, we denote by $x'$ its normalization $x' = x/|x| \in \mathbb{S}$. Also, given a rotation of the sphere $\rho : \mathbb{S} \to \mathbb{S}$, we define its magnitude $|\rho|$ by

$$|\rho| = \sup_{x \in \mathbb{S}} |\rho(x) - x|.$$

Let $\Omega \in L^1(\mathbb{S})$ be a function on $\mathbb{S}$ such that

$$\int_{\mathbb{S}} \Omega(x) d\sigma(x) = 0.$$

For every $1 \leqslant r \leqslant \infty$, if $\Omega \in L^r(\mathbb{S})$, we say that it satisfies the $L^r$-Dini condition when

$$\int_0^1 \omega_r(t) \frac{dt}{t} < \infty,$$

where $\omega_r(t) = \sup_{|\rho|<t} \|\Omega \circ \rho - \Omega\|_{L^r(\mathbb{S})}$. Clearly, the weakest of these conditions is when $r = 1$ and the strongest, when $r = \infty$. We define the singular integral $T_\Omega$ as the convolution operator with kernel $K(x) = \Omega(x')/|x|^n$, in the principal value sense. That is, for Schwartz functions $f$,

$$T_\Omega f(x) = \lim_{\varepsilon \to 0} \int_{|y|>\varepsilon} \frac{\Omega(y')}{|y|^n} f(x-y)dy.$$

It can be checked (see [13] or [49, p. 73]) that for $T_\Omega$ to be bounded on $L^2(\mathbb{R}^n)$ (or equivalently, for $\widehat{K}$ to be in $L^\infty(\mathbb{R}^n)$), it suffices that the even part of $\Omega$ belongs to $L \log L(\mathbb{S})$. In [12, 14], the authors show that $\Omega$ satisfying the $L^1$-Dini condition is equivalent to saying that the convolution kernel $K(x) = \Omega(x')/|x|^n$ is under the hypotheses of the classical Calderón-Zygmund kernels (with an $L^1$-Hörmander condition), and in particular, it means that $T_\Omega$ is of strong-type $(p,p)$ for $1 < p < \infty$ and weak-type $(1,1)$, without weights[6]. They also show that the $L^1$-Dini condition on $\Omega$ implies that $\Omega \in L \log L(\mathbb{S})$, and several years later, in 1996, A. Seeger [105] shows that, in fact, assuming $\Omega \in L \log L(\mathbb{S})$ is enough for $T_\Omega$ to be of weak-type (1,1). This weak-type $(1,1)$ estimate had already been established in dimension $n \leqslant 5$ with different techniques in [37], which improved the almost simultaneous result by S. Hofmann [69], that worked only for $n = 2$ and assumed the stronger condition $\Omega \in L^r(\mathbb{S})$ for some $r > 1$. In the weighted setting, J. Duoandikoetxea and J. L. Rubio de Francia proved in [52] that if $\Omega \in L^\infty(\mathbb{S})$, then $T_\Omega$ is of strong-type $(p,p)$ for every $1 < p < \infty$ and every weight in $A_p$. Moreover, the hypothesis $\Omega \in L^\infty(\mathbb{S})$ cannot be relaxed to $\Omega \in L^r(\mathbb{S})$ for some $r > 1$, as was shown by B. Muckenhoupt and R. Wheeden in [96]. In particular, this means that an $A_1$ weighted analogue of Hofmann's result [69] in $n = 2$ cannot hold. However, in [124], A. Vargas proved that when $n = 2$, for every $u \in A_1$, we have

$$T_\Omega : L^1(u) \longrightarrow L^{1,\infty}(u),$$

provided that $\Omega \in \bigcap_{1 \leqslant r < \infty} L^r(\mathbb{S})$. By extrapolation, we also get the strong-type $(p,p)$ for $A_p$ weights, and as pointed out in [124], by testing with power weights it can be seen that the condition $\Omega \in \bigcap_{1 \leqslant r < \infty} L^r(\mathbb{S})$ is the best possible within the scale of $L^r$ spaces. Using A. Seeger's [105] techniques to obtain weak-type $(1,1)$ estimates without any restriction on the dimension together with A. Vargas' [124] ideas to introduce weights, in 2004, D. Fan and S. Sato [54] were able to extend this last weighted weak-type $(1,1)$ result to every $n \in \mathbb{N}$. After this short summary of the state of the art, we go back to the paper of D. Kurtz and R. Wheeden [79]. Back then, what was known [75] was that, if $\Omega$ satisfies the $L^\infty$-Dini condition, then $T_\Omega$ is of weak-type $(1,1)$ for every weight in $A_1$. In [79], the authors give a different proof of this result by means of a lemma analogous to Lemma 3.25:

[6] Recall that the $A_p$ weighted theory for Calderón-Zygmund operators assumes the stronger pointwise Hörmander-type condition of standard kernels (following the terminology of Coifman and Meyer [40]). See [49, p. 99] for a clear presentation of this notion of standard kernel.

**Lemma 3.33.** *Let $1 \leqslant r < \infty$ and assume that $\Omega \in L^r(\mathbb{S})$ satisfies the $L^r$-Dini condition. There exists a constant $\alpha_0 > 0$ such that, if $|y| < \alpha_0 R$, then*

$$\left(\int_{R<|x|<2R} |K(x-y) - K(x)|^r dx\right)^{1/r} \lesssim R^{n/r-n} \left(\frac{|y|}{R} + \int_{\frac{|y|}{2R}}^{\frac{|y|}{R}} \omega_r(t)\frac{dt}{t}\right),$$

*where $K(x) = \Omega(x')/|x|^n$.*

With this, we would like to show an estimate in the spirit of Lemma 3.27 as we did for multipliers. However, this last integral term related to the Dini condition will become a problem. Let $\Omega$ be a function on $\mathbb{S}$ with $\int_{\mathbb{S}} \Omega = 0$, and assume it satisfies the $L^\infty$-Dini condition. Take $u \in A_1$ and $h \in L^1_{\text{loc}}$. We would like to show that there exists $q > 1$ such that, for every measurable set $E \subseteq \mathbb{R}^n$ and cube $Q \subseteq \mathbb{R}^n$, if $w := (Mh)^{1-q}u$ and $c$ is the center of the cube $Q$, it holds that, for $y \in Q$,

$$\int_{\mathbb{R}^n \setminus 2Q} |K(x-y) - K(x-c)| w(x) dx \lesssim \frac{|Q|^q}{|E \cap Q|^q} \frac{w(E \cap Q)}{|Q|}, \tag{3.21}$$

where $K(x) = \Omega(x')/|x|^n$. Fix $q > 1$ to be chosen later. We mimic the argument in Lemma 3.27 and bound the left-hand side of (3.21) by

$$\sum_{j=1}^\infty \sup_{x \in 2^{j+1}Q} (Mh)^{1-q}(x) \left(\int_{2^{j+1}Q \setminus 2^jQ} |K(x-y) - K(x-c)|^{r'} dx\right)^{1/r'} \left(\int_{2^{j+1}Q} u^r(x) dx\right)^{1/r},$$

for some $1 < r < 1 + \frac{1}{2^{n+1}\|u\|_{A_1}}$ that ensures $u^r \in A_1$. Since $\Omega$ satisfies the $L^\infty$-Dini condition, it also satisfies the $L^{r'}$-Dini condition and we can use Lemma 3.33 with $R = 2^j\ell(Q)$ and large $j \geqslant 1$ so that $2^{-j} < \alpha_0$. This is because, for the lemma, we need that $|y-c| < \alpha_0 R$, which holds with this restriction on $j$ and recalling that $|y - c| < \ell(Q)$. Since we only need to worry about large values of $j \geqslant 1$, for simplicity assume that we can use it for every $j \geqslant 1$, and what we get is

$$\sum_{j=1}^\infty \sup_{x \in 2^{j+1}Q} (Mh)^{1-q}(x) (2^j\ell(Q))^{n/r'-n} \left(\frac{|y-c|}{2^j\ell(Q)} + \int_{\frac{|y-c|}{2^{j+1}\ell(Q)}}^{\frac{|y-c|}{2^j\ell(Q)}} \omega_{r'}(t)\frac{dt}{t}\right) \left(\int_{2^{j+1}Q} u^r(x) dx\right)^{1/r}$$
$$\lesssim \sum_{j=1}^\infty \sup_{x \in 2^{j+1}Q} (Mh)^{1-q}(x) \left(\frac{1}{2^j} + \int_{\frac{|y-c|}{2^{j+1}\ell(Q)}}^{\frac{|y-c|}{2^j\ell(Q)}} \omega_{r'}(t)\frac{dt}{t}\right) \inf_{x \in 2^{j+1}Q} u(x).$$

Now we had to use Lemma 1.6 to control the supremum by an average, and the $A_q^{\mathcal{R}}$ property of $w$ on the inclusion $E \cap Q \subseteq 2^{j+1}Q$. After these two steps, we are left with

$$\frac{|Q|^q}{|E \cap Q|^q} \frac{w(E \cap Q)}{|Q|} \sum_{j=1}^\infty (2^{jn})^{q-1} \left(\frac{1}{2^j} + \int_{\frac{|y-c|}{2^{j+1}\ell(Q)}}^{\frac{|y-c|}{2^j\ell(Q)}} \omega_{r'}(t)\frac{dt}{t}\right).$$

At this point, we would like to find $q > 1$ so that the series is finite. However, this cannot be achieved in general. We know that the integral itself is summable, since

$$\sum_{j=1}^{\infty} \int_{\frac{|y-c|}{2^{j+1}\ell(Q)}}^{\frac{|y-c|}{2^{j}\ell(Q)}} \omega_{r'}(t)\frac{dt}{t} \leqslant \int_0^1 \omega_{r'}(t)\frac{dt}{t} < \infty,$$

but when multiplied by $(2^{jn})^{q-1}$ it need not be. Clearly, if we assumed an extra (and somewhat artificial) hypothesis of Dini condition on dyadic intervals, we could finish this proof and, with it, show a restricted weak-type estimate for $T_\Omega$ analogous to Theorem 3.26. The conclusion is that, unlike for the case of Hörmander type multipliers, where the ideas in [79] could be carried over to the setting of $\widehat{A}_q$ weights and restricted weak-type $(q, q)$ estimates with $q > 1$, for the singular integral $T_\Omega$ it cannot be done as simply as the authors in [79] did for $q = 1$. In any case, the result that we would get if we followed the previous scheme would be this:

**Theorem 3.34.** *Let $\Omega$ be a function on $\mathbb{S}$ with $\int_{\mathbb{S}} \Omega = 0$, and assume that, for every $r > 1$ and $0 < a < 1$, there exists $\varepsilon > 0$ such that*

$$\int_a^{2a} \omega_r(t)\frac{dt}{t} \lesssim a^{\varepsilon}.$$

*Then, the singular integral $T_\Omega$ satisfies that, for every $u \in A_1$, there exists $1 < p_0 < \infty$ such that, for each measurable set $E \subseteq \mathbb{R}^n$,*

$$\|T_\Omega \chi_E\|_{L^{p_0,\infty}((M\chi_E)^{1-p_0}u)} \lesssim C_u u(E)^{1/p_0}.$$

The condition that we assume on the function $\Omega$ could be interpreted as an $L^r$-Dini condition for every $1 \leqslant r < \infty$ with an extra size condition for $\omega_r$ on dyadic intervals. This is obviously far from the hypotheses in the weighted weak-type $(1,1)$ result of D. Fan and S. Sato [54], where the authors only assume that $\Omega \in L^r(\mathbb{S})$ for every $1 \leqslant r < \infty$. The next natural step would be to check if their ideas could be adapted to our setting to show restricted weak-type $(q, q)$ estimates for some $q > 1$ and weights in $\widehat{A}_q$. This seems likely to be true, but we have decided to leave it as future work until we find an interesting application, such as the ones presented in Sections 3.2 and 3.3 for the Hilbert transform or the Bochner-Riesz operator based on the averaging technique.

# Chapter 4

# Weighted Littlewood-Paley Theory

## 4.1 The general setting

In this chapter we will study different estimates related to a weighted Littlewood-Paley theory for multipliers. This theory was initiated by J. E. Littlewood and R. E. A. C. Paley in the thirties in a series of papers [89, 90, 91] dealing with Fourier and power series. The general scheme is the following: Assume that we have a certain operator $T$ for which we know that there is an estimate of the form

$$G_1(Tf)(x) \lesssim G_2 f(x), \qquad \text{a.e. } x \in \mathbb{R}^n, \tag{4.1}$$

where $G_1$ and $G_2$ are certain operators called square functions. If we combine (4.1) with a lower estimate for $G_1$ and an upper estimate for $G_2$, say

$$\|f\|_X \lesssim \|G_1 f\|_X, \qquad \text{and} \qquad \|G_2 f\|_X \lesssim \|f\|_Y,$$

with $X$ and $Y$ being a couple of quasi-Banach spaces[1], then we can deduce that

$$\|Tf\|_X \lesssim \|f\|_Y.$$

In our case, we will consider the spaces $X = L^{p,\infty}(v)$ and $Y = L^{p,1}(v)$, with $v \in \widehat{A}_p$, that correspond to a weighted Littlewood-Paley theory seeking the inequalities that appear in the extrapolation of Section 1.2. We will investigate lower and upper estimates for different square functions independently, which are interesting in their own right. Finally, in Section 4.4, we will see how they can be related when introducing pointwise estimates as in (4.1).

[1] We also need that $X$ satisfies the lattice property, that is, $0 \leqslant f \leqslant g \Rightarrow \|f\|_X \leqslant \|g\|_X$, for every $f, g \in X$.

## 4.2 Lower estimates

Our first goal is to prove lower Littlewood-Paley inequalities of the form

$$\|f\|_{L^{p,\infty}(v)} \lesssim \|Gf\|_{L^{p,\infty}(v)}, \tag{4.2}$$

with $v \in \widehat{A}_p$ and $G$ being a certain square function. A nice presentation of some of the different square functions that we will consider can be found in [126].

### 4.2.1 The Lusin area function $S$

The first function for which we will seek a lower estimate is the classical Lusin area function $S$. First, we will need a list of definitions concerning the upper half-space $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times \mathbb{R}_+$.

**Definition 4.1.**

- *Given a fixed aperture $a > 0$, we define the cone centered at $x \in \mathbb{R}^n$ by*

$$\Gamma(x) = \Gamma_a(x) = \{(y,t) \in \mathbb{R}^{n+1}_+ : |x-y| < at\}.$$

- *Given $(x,t) \in \mathbb{R}^{n+1}_+$, we define the Poisson kernel*

$$P_t(x) = \frac{c_n t}{(t^2+|x|^2)^{\frac{n+1}{2}}},$$

*with $c_n > 0$ such that $\|P_1\|_1 = 1$. Since $P_t$ is a dilation of $P_1$, this normalization holds for every $t > 0$ and $\{P_t\}_{t>0}$ forms an approximation to the identity. With this, we define the harmonic extension (or Poisson integral) of a function $f$ to the upper half-space by*

$$u(x,t) = P_t * f(x).$$

- *Now, we can define the Lusin area function as*

$$Sf(x) = \left( \int_{\Gamma(x)} |\nabla u(y,t)|^2 \frac{dydt}{t^{n-1}} \right)^{1/2},$$

*where $\nabla u(y,t) = \left( \frac{\partial u}{\partial y_1}, \ldots, \frac{\partial u}{\partial y_n}, \frac{\partial u}{\partial t} \right)$ is the gradient vector.*

- *The non-tangential maximal function is given by*

$$Nf(x) = \sup_{(y,t)\in\Gamma(x)} |u(y,t)|.$$

- *We will also need an auxiliary function, namely*
$$Df(x) = \sup_{(y,t)\in\Gamma(x)} t|\nabla u(y,t)|.$$

- *For technical reasons, we will also work with the local versions of $S, N$ and $D$. Given a measurable set $R \subseteq \mathbb{R}^{n+1}_+$, we will denote them by $S_R$, $N_R$ and $D_R$ respectively and define them exactly as $S, N$ and $D$ but replacing $\Gamma(x)$ by $\Gamma(x) \cap R$.*

- *Finally*[2], *$N^0_R$ will denote the following variant of $N_R$:*
$$N^0_R f(x) = \sup_{(y,t)\in\Gamma(x)\cap R} |u(y,t) - u(y,t_y)|$$
*if $\Gamma(x) \cap R \neq \varnothing$ and 0 otherwise. Here,*
$$t_y = \sup\{t' > 0 : (y,t') \in R\} \in \mathbb{R}_+ \cup \{+\infty\}.$$

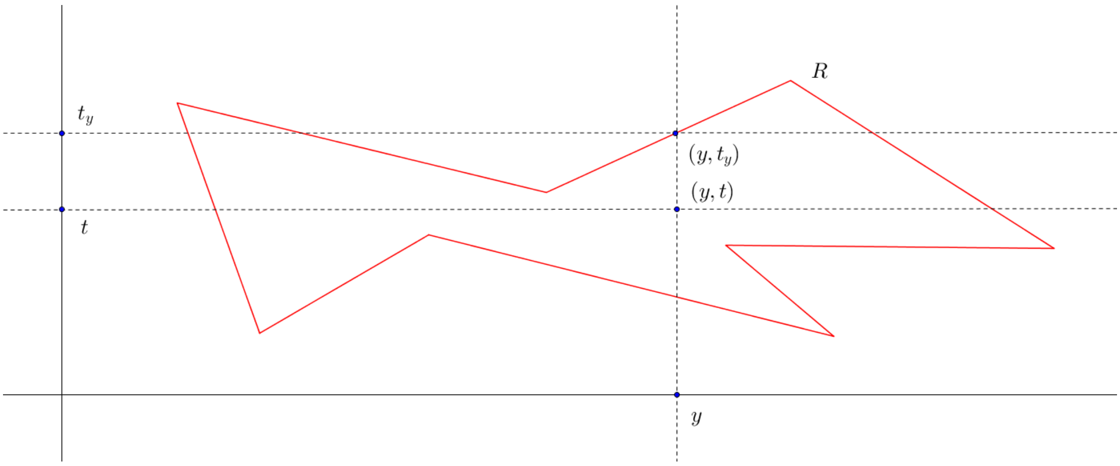


Figure 4.1: Idea of the definition of $t_y$.

The main result of this subsection is the following:

**Theorem 4.2.** *Let $w \in A_\infty$ and $f$ a function such that its Poisson integral $u(x,t)$ satisfies*
$$\lim_{t\to\infty} u(x,t) = 0,$$
*for every $x \in \mathbb{R}^n$, then for every $1 < p < \infty$,*
$$\|f\|_{L^{p,\infty}(w)} \lesssim \|Sf\|_{L^{p,\infty}(w)}.$$

[2] All these operators $S, N, D$ and their local versions, could be defined on harmonic functions $u$ on the upper half-space, not necessarily being the Poisson integral of a function $f$, and we would simply write $Su, Nu$, etc.

We will need the following result proved in [64, Theorem 4]:

**Theorem 4.3.** *Let $G$ be a bounded open subset of $\mathbb{R}^n$, and let $R$ be the interior of the complement of $\bigcup_{x \notin G} \Gamma(x)$ in $\mathbb{R}^{n+1}_+$. Given $w \in A_\infty$, $\alpha > 1$ and $\beta > 1$, there exist constants $\gamma, \delta > 0$ such that*

$$\alpha w(\{N_R^0 f > \beta\lambda, S_R f \leqslant \gamma\lambda, D_R f \leqslant \delta\lambda\}) \leqslant w(\{N_R^0 f > \lambda\}),$$

*for every $\lambda > 0$. The conclusion also holds for $R = \mathbb{R}^{n+1}_+$ by passage to the limit.*

From this, we can deduce the following corollary:

**Corollary 4.4.** *If $w \in A_\infty$ and $f$ is a function such that its Poisson integral $u(x,t)$ satisfies*

$$\lim_{t \to \infty} u(x,t) = 0, \tag{4.3}$$

*for every $x \in \mathbb{R}^n$, then, for every $1 < p < \infty$,*

$$\|Nf\|_{L^{p,\infty}(w)} \lesssim \|Sf\|_{L^{p,\infty}(w)} + \|Df\|_{L^{p,\infty}(w)}.$$

*Proof.* Let us take $G$ and $R$ as in Theorem 4.3, $\alpha = 2^{p+1}$ and $\beta = 2$. Then we have constants $\gamma, \delta > 0$ such that $w(\{N_R^0 f > 2\lambda\})$ can be bounded by:

$$\begin{aligned} &w(\{N_R^0 f > 2\lambda, S_R f \leqslant \gamma\lambda, D_R f \leqslant \delta\lambda\}) + w(\{S_R f > \gamma\lambda\}) + w(\{D_R f > \delta\lambda\}) \\ &\leqslant \frac{1}{2^{p+1}} w(\{N_R^0 f > \lambda\}) + w(\{S_R f > \gamma\lambda\}) + w(\{D_R f > \delta\lambda\}). \end{aligned}$$

If we multiply by $2^p \lambda^p$ and take supremum over $\lambda > 0$, we conclude that

$$\|N_R^0 f\|^p_{L^{p,\infty}(w)} \leqslant \frac{1}{2} \|N_R^0 f\|^p_{L^{p,\infty}(w)} + C(\|S_R f\|^p_{L^{p,\infty}(w)} + \|D_R f\|^p_{L^{p,\infty}(w)}).$$

Now, as in [64], $N_R^0$ is bounded with compact support (just like $S_R$ and $D_R$), so all the quantities in the previous inequality are finite and we can subtract to obtain the desired estimate for the local versions:

$$\|N_R^0 f\|_{L^{p,\infty}(w)} \lesssim \|S_R f\|_{L^{p,\infty}(w)} + \|D_R f\|_{L^{p,\infty}(w)}.$$

Finally we let $R$ increase to $\mathbb{R}^{n+1}_+$ (by making $G$ increase to $\mathbb{R}^n$). By the monotone convergence theorem, it is clear that the right-hand side of the last inequality tends to $\|Sf\|_{L^{p,\infty}(w)} + \|Df\|_{L^{p,\infty}(w)}$. On the other hand, in [10, p. 533], the authors show that assuming (4.3) and taking $R = R_\rho$ associated with the open ball $G_\rho = B(0, a\rho)$ (where $a$ is aperture of the cones and $\rho > 0$), it holds that

$$Nf(x) \leqslant \lim_{\rho \to \infty} N_{R_\rho}^0 f(x),$$

so by Fatou's lemma, we conclude that

$$\|Nf\|_{L^{p,\infty}(w)} \leqslant \lim_{\rho \to \infty} \|N_{R_\rho}^0 f\|_{L^{p,\infty}(w)}$$

and finish the proof. □

Once we have this, we are ready to prove our main result:

*Proof of Theorem 4.2.* Recall that the cone $\Gamma(x)$ we have been working with has a fixed aperture $a > 0$. Take now a smaller parameter $0 < a_0 < a$ and let $S_{a_0}f$, $D_{a_0}f$ and $N_{a_0}f$ be the analogous functions on the smaller cone $\Gamma_{a_0}(x) \subseteq \Gamma(x)$. The following holds:

(i) By [64, Lemma 1], we have[3] $\|Nf\|_{L^{p,\infty}(w)} \lesssim \|N_{a_0}f\|_{L^{p,\infty}(w)}$.

(ii) Trivially, $S_{a_0}f \leqslant Sf$.

(iii) By [112, p. 207, Lemma (ii)], $D_{a_0}f(x) \lesssim Sf(x)$.

Combining these three facts and Corollary 4.4 (this time, with aperture $a_0$), we get that

$$\begin{aligned}\|Nf\|_{L^{p,\infty}(w)} &\lesssim \|N_{a_0}f\|_{L^{p,\infty}(w)} \lesssim \|S_{a_0}f\|_{L^{p,\infty}(w)} + \|D_{a_0}f\|_{L^{p,\infty}(w)}\\ &\lesssim \|Sf\|_{L^{p,\infty}(w)} + \|Sf\|_{L^{p,\infty}(w)} \approx \|Sf\|_{L^{p,\infty}(w)}.\end{aligned}$$

Using now that $f(x) = \lim_{\substack{(y,t)\to(\cdot,0)\\(y,t)\in\Gamma(x)}} u(y,t)$ a.e. $x \in \mathbb{R}^n$, we complete the proof:

$$\|f\|_{L^{p,\infty}(w)} \leqslant \left\| \sup_{(y,t)\in\Gamma(x)} |u(y,t)| \right\|_{L^{p,\infty}(w)} = \|Nf\|_{L^{p,\infty}(w)} \lesssim \|Sf\|_{L^{p,\infty}(w)}.$$

□

To finish this subsection, we want to point out that if we want to apply Theorem 4.2 to show restricted weak-type estimates for a Fourier multiplier

$$\widehat{T_m f}(\xi) = m(\xi)\widehat{f}(\xi),$$

the vanishing assumption on $u$ is not a limitation.

**Corollary 4.5.** *Let $m : \mathbb{R}^n \to \mathbb{R}$ be a bounded function (that is, $T_m$ maps $L^2(\mathbb{R}^n)$ into itself), then for every $w \in A_\infty$ and $1 < p < \infty$,*

$$\|T_m\chi_E\|_{L^{p,\infty}}(w) \lesssim \|S(T_m\chi_E)\|_{L^{p,\infty}(w)}.$$

*Proof.* By Theorem 4.2, it is enough to see that, for every $x \in \mathbb{R}^n$,

$$\lim_{t\to\infty} u(x,t) = \lim_{t\to\infty} P_t * (T_m\chi_E)(x) = 0.$$

[3]Even though the authors in [64] work with $L^p(w)$, their Lemma 1 gives an estimate for the measure of level sets, so we can use it to compare weak norms as well.

It is easy to check that $P_t(y)$ as a function of $t$ has a maximum at $t = \frac{|y|}{\sqrt{n}}$. With this in mind, and looking only at $t > 1$, we have that

$$P_t(y) \lesssim \frac{1}{1+|y|^n} =: F(y) \in L^2(\mathbb{R}^n),$$

for every $y \in \mathbb{R}^n$, $t > 1$. Now, using Cauchy-Schwarz and the fact that $T_m$ is of strong-type (2,2), we get that, for every $t > 1$,

$$|T_m\chi_E(x-y)P_t(y)| \leqslant |T_m\chi_E(x-y)|F(y) \in L^1(\mathbb{R}^n),$$

so using the dominated convergence theorem,

$$\lim_{t\to\infty} u(x,t) = \int_{\mathbb{R}^n} \lim_{t\to\infty} T_m\chi_E(x-y)P_t(y)dy = 0.$$

□

### 4.2.2 The $S_{\psi,\alpha}$ function

Now we will present a different approach that yields the lower estimate corresponding to a modern version of the area function $S$, the $S_{\psi,\alpha}$ function. Here we will follow the ideas in [125], where weighted $L^p$ inequalities for $S_{\psi,\alpha}$ are studied by means of dyadic techniques.

**Definition 4.6.** *Let $\mathcal{D}$ be the standard dyadic lattice*[4] *in $\mathbb{R}^n$. Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$. We set, for every $k \in \mathbb{Z}$,*

$$f_k := \sum_{\substack{Q\in\mathcal{D}\\ l(Q)=2^{-k}}} \left(\frac{1}{|Q|}\int_Q f\right)\chi_Q,$$

*and*

$$M_{\mathcal{D}}f(x) = \sup_{k\in\mathbb{Z}} |f_k(x)|$$

*the dyadic maximal function*[5] *of $f$.*

In [125, p. 665], the author shows that every function $f \in \mathcal{C}_0^\infty$ can be written as

$$f(x) = \sum_{k=1}^{3^n} f_{(k)}(x),$$

where, for every $k = 1, ..., 3^n$,

$$f_{(k)}(x) = \sum_{Q\in\mathcal{G}_k} \lambda_Q a_Q(x), \tag{4.4}$$

and

[4] It is defined by $\mathcal{D} = \{[2^k m_1, 2^k(m_1+1)) \times \cdots \times [2^k m_n, 2^k(m_n+1)) : k, m_1, ..., m_n \in \mathbb{Z}\}$.

[5] Notice that the Lebesgue differentiation theorem gives that $f(x) \leqslant M_{\mathcal{D}}f(x)$ for almost every $x \in \mathbb{R}^n$.

(i) $\mathcal{G}_k$ is a collection of cubes (not necessarily from the standard dyadic lattice $\mathcal{D}$) satisfying that for all $Q, Q' \in \mathcal{G}_k$, either $Q \cap Q' = \varnothing$ or one is contained in the other, and that $Q \subseteq Q'$ with $Q \neq Q'$ implies $l(Q) \leqslant \frac{1}{2} l(Q')$.

(ii) For every $Q \in \mathcal{G}_k$, it holds that $\operatorname{supp} a_Q \subseteq Q$, $\int a = 0$, $\|a\|_\infty \leqslant |Q|^{-1/2}$ and $\|\nabla a\|_\infty \leqslant l(Q)^{-1}|Q|^{-1/2}$.

(iii) The families $\{\mathcal{G}_k\}_{k=1}^{3^n}$ are pairwise disjoint.

Even though the cubes $\mathcal{G}_k$ may not belong to $\mathcal{D}$, we can assume without loss of generality that they are dyadic (as the author points out in [125, p. 666]), since the only properties that are required are the ones described in $(i)$. If a function can be written as in (4.4) with respect to some family of cubes $\mathcal{G}$ satisfying $(i)$ and $(ii)$, we will say that it is of *special form* with respect to $\mathcal{G}$. Once this is settled, let us give the following definition:

**Definition 4.7.** *Given a subfamily of dyadic cubes $\mathcal{G} \subseteq \mathcal{D}$, and a function $f$ of special form with respect to $\mathcal{G}$, we define*

$$S_\Lambda f(x) = \left( \sum_{x \in Q \in \mathcal{G}} \frac{|\lambda_Q|^2}{|Q|} \right)^{1/2}.$$

With this, we have the following lemma:

**Lemma 4.8.** *Let $0 < p < \infty$, $0 < \eta \leqslant 1$ and $A > 0$. Let $\mathcal{G} \subseteq \mathcal{D}$ be a subfamily of dyadic cubes. Let $f$ be of special form with respect to $\mathcal{G}$ and such that $M_{\mathcal{D}} f \in L^{p,\infty}(v)$, where $v$ is a weight for which the following quantity*

$$Y_\eta(Q, v) = \begin{cases} v(Q)^{-1} \int_Q v(x) \log^\eta \left(1 + \frac{v(x)}{|Q|^{-1} v(Q)}\right) dx & \text{if } v(Q) > 0, \\ 1 & \text{if } v(Q) = 0, \end{cases}$$

*is controlled by $A$ for all $Q \in \mathcal{G}$. Then, there exists a constant $C(p, n, \eta) < \infty$ such that*

$$\|M_{\mathcal{D}} f\|_{L^{p,\infty}(v)} \leqslant C(p, n, \eta) A^{1/2\eta} \|S_\Lambda f\|_{L^{p,\infty}(v)}.$$

The proof of this lemma is based on a good-$\lambda$ inequality that the author shows in the proof of [125, Lemma 2.3]. More precisely:

**Lemma 4.9.** *Under the hypotheses of Lemma 4.8, it holds that for every $\lambda > 0$,*

$$v(\{M_{\mathcal{D}} f > 2\lambda, S_\Lambda f \leqslant \gamma\lambda\}) \leqslant \varepsilon(p) v(\{M_{\mathcal{D}} f > \lambda\}),$$

*with $\gamma > C(p, n, \eta) A^{-1/2\eta}$ and $2\varepsilon(p)^{1/p} \leqslant 1/2$.*

Now, our result is an easy consequence:

*Proof of Lemma 4.8.* With the previous inequality,

$$\begin{aligned}
\|M_{\mathcal{D}}f\|_{L^{p,\infty}(v)} &= \sup_{\lambda>0} 2\lambda v(\{M_{\mathcal{D}}f > 2\lambda\})^{1/p} \\
&\leqslant 2\sup_{\lambda>0} \lambda v(\{M_{\mathcal{D}}f > 2\lambda, S_\Lambda f \leqslant \gamma\lambda\})^{1/p} + 2\sup_{\lambda>0} \lambda v(\{S_\Lambda f > \gamma\lambda\})^{1/p} \\
&\leqslant 2\varepsilon(p)^{1/p} \sup_{\lambda>0} \lambda v(\{M_{\mathcal{D}}f > \lambda\})^{1/p} + \frac{2}{\gamma}\|S_\Lambda f\|_{L^{p,\infty}(v)} \\
&\leqslant \frac{1}{2}\|M_{\mathcal{D}}f\|_{L^{p,\infty}(v)} + C(p,n,\eta)A^{1/2\eta}\|S_\Lambda f\|_{L^{p,\infty}(v)}.
\end{aligned}$$

Isolating the term $\|M_{\mathcal{D}}f\|_{L^{p,\infty}(v)}$, we finish the proof. □

**Definition 4.10.** *Let $\psi \in \mathcal{C}^k(\mathbb{R}^n)$ be a real, radial, non-trivial function such that $\int \psi = 0$, and whose support lies inside the closed ball $\overline{B(0,1)}$. We can assume that $\psi$ is normalized so that*

$$\int_0^\infty |\widehat{\psi}(\xi t)|^2 \frac{dt}{t} = 1,$$

*for all $\xi \neq 0$. As usual, for $t > 0$ we define the dilation $\psi_t(x) = t^{-n}\psi(x/t)$. For $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $\alpha > 0$, we define the square function of $f$ with respect to $\psi$ of aperture $\alpha$:*

$$S_{\psi,\alpha}f(x) = \left(\iint_{|x-y|<\alpha t} |f * \psi_t(y)|^2 \frac{dydt}{t^{n+1}}\right)^{1/2}.$$

**Remark 4.11.** *At the beginning of this section, we said that the author in [125] shows that every function in $\mathcal{C}_0^\infty$ can be written as a finite sum of $3^n$ functions $f_{(k)}$ of special form. From his construction, one can check that for*[6] *$\alpha \geqslant 3\sqrt{n}$,*

$$\sum_{k=1}^{3^n} S_\Lambda(f_{(k)}) \lesssim S_{\psi,\alpha}f. \tag{4.5}$$

*This fact is explicitly stated at the end of the proof of [125, Theorem 2.5].*

Finally, we state our main result:

**Theorem 4.12.** *Let $1 \leqslant p < \infty$ and $v = (Mh)^{1-p}u \in \widehat{A}_p$. Then, for every $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ and $\alpha \geqslant 3\sqrt{n}$,*

$$\|f\|_{L^{p,\infty}(v)} \lesssim \|u\|_{A_1}\|S_{\psi,\alpha}f\|_{L^{p,\infty}(v)},$$

*and the implicit constant only depends on $p$ and $n$.*

[6] Note that, whenever we write $S_\Lambda f_{(k)}$, the dyadic sum involved in its definition is with respect to the corresponding family $\mathcal{G}_k$.

*Proof.* Take $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ and write

$$f = \sum_{k=1}^{3^n} f_{(k)},$$

where each $f_{(k)}$ is of special form with respect to a dyadic subfamily $\mathcal{G}_k \subseteq \mathcal{D}$, the families $\mathcal{G}_k$ are pairwise disjoint and we have (4.5). Moreover, we know that for every cube $Q$ with $v(Q) > 0$, the weight $v = (Mh)^{1-p}u$ satisfies that

$$\begin{aligned} Y_1(Q,v) &\approx v(Q)^{-1}\int_Q M[(Mh)^{1-p}u\chi_Q](x)dx \leqslant v(Q)^{-1}\sup_{x\in Q}(Mh)^{1-p}(x)\int_Q Mu(x)dx \\ &\lesssim \|u\|_{A_1}v(Q)^{-1}\frac{1}{|Q|}\int_Q (Mh)^{1-p}(x)dx\int_Q u(x)dx \leqslant \|u\|_{A_1}^2 v(Q)^{-1}v(Q) = \|u\|_{A_1}^2. \end{aligned}$$

The first equivalence is stated in [125, p. 668], and then we used Lemma 1.6 to control the supremum and the $A_1$ property of $u$ to finish the estimate. Therefore, for every $k = 1, ..., 3^n$, we can apply Lemma 4.8 with $\eta = 1$, $A = \|u\|_{A_1}^2$ and the pair $(f_{(k)}, \mathcal{G}_k)$ to obtain that

$$\|M_{\mathcal{D}} f_{(k)}\|_{L^{p,\infty}(v)} \lesssim \|u\|_{A_1}\|S_\Lambda f_{(k)}\|_{L^{p,\infty}(v)}.$$

But the families $\mathcal{G}_k$ are pairwise disjoint, so

$$M_{\mathcal{D}} f(x) = \sum_{k=1}^{3^n} M_{\mathcal{D}} f_{(k)}(x),$$

and hence, by (4.5) and exploiting the finiteness of the sum and that $S_\Lambda f_{(k)} \geqslant 0$, we finish the proof:

$$\begin{aligned} \|f\|_{L^{p,\infty}(v)} &\leqslant \|M_{\mathcal{D}} f\|_{L^{p,\infty}(v)} \lesssim \sum_{k=1}^{3^n}\|M_{\mathcal{D}} f_{(k)}\|_{L^{p,\infty}(v)} \lesssim \|u\|_{A_1}\sum_{k=1}^{3^n}\|S_\Lambda f_{(k)}\|_{L^{p,\infty}(v)} \\ &\lesssim 3^n\|u\|_{A_1}\left\|\sum_{k=1}^{3^n} S_\Lambda f_{(k)}\right\|_{L^{p,\infty}(v)} \lesssim \|u\|_{A_1}\|S_{\psi,\alpha} f\|_{L^{p,\infty}(v)}. \end{aligned}$$

□

**Remark 4.13.** *Notice that this last theorem is stated in view of our goal (4.2) in this section. However, from its proof one can check that the same would hold for any weight $v \in A_\infty$ which we know that can be written as $v = (Mh)^{1-q}u$, for some locally integrable $h$, $1 \leqslant q < \infty$ and $u \in A_1$. Hence, for every $1 \leqslant p < \infty$, $f \in \mathcal{C}_0^\infty$ and $\alpha \geqslant 3\sqrt{n}$, the corresponding estimate would be*

$$\|f\|_{L^{p,\infty}(v)} \lesssim \|u\|_{A_1}\|S_{\psi,\alpha} f\|_{L^{p,\infty}(v)},$$

*with the implicit constant depending on $p$, $n$ and $q$.*

### 4.2.3 The $g_\Phi$ square function

Let $\Phi$ be a fixed, non-negative smooth bump function with support in $[1,2]$. Let $\psi$ be the function on $\mathbb{R}^n$ defined by

$$\widehat{\psi}(\xi) = |\xi|\Phi(|\xi|).$$

Notice that $\psi$ satisfies

$$\int_{\mathbb{R}^n} \psi(x)dx = \widehat{\psi}(0) = 0.$$

With this, we introduce the $g_\Phi$ (vertical) square function associated with $\Phi$ as follows:

$$g_\Phi f(x) = \left(\int_0^\infty |\psi_t * f(x)|^2 \frac{dt}{t}\right)^{1/2},$$

where, as usual, $\psi_t(x) = t^{-n}\psi(t^{-1}x)$. This function will appear when dealing with radial multipliers. This is a generalization of the classical Littlewood-Paley $g$-function defined by

$$gf(x) = \left(\int_0^\infty \left|\frac{\partial}{\partial t}P_t * f(x)\right|^2 tdt\right)^{1/2},$$

where $P$ is the standard Poisson kernel (see [112, Chapter IV]) and $P_t * f(x) = u(x,t)$ is the harmonic extension of $f$ to the upper half-space. Introducing different functions $\Phi$ will allow us to define different classes of radial multipliers associated with them and, for each class, we will have pointwise inequalities involving the corresponding $g_\Phi$. Just like for the classical $g$-function, it holds that, for some constant $C_\Phi > 0$,

$$\|g_\Phi f\|_{L^2(\mathbb{R}^n)} = C_\Phi \|f\|_{L^2(\mathbb{R}^n)}. \tag{4.6}$$

To check this, we use Fubini and Plancherel's identity as follows:

$$\begin{aligned}\|g_\Phi f\|^2_{L^2(\mathbb{R}^n)} &= \int_0^\infty \int_{\mathbb{R}^n} |\psi_t * f(x)|^2 dx \frac{dt}{t} = \int_0^\infty \int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 |\widehat{\psi}_t(\xi)|^2 d\xi \frac{dt}{t} \\ &= \int_{\mathbb{R}^n} \int_0^\infty |\widehat{\psi}(t\xi)|^2 \frac{dt}{t} |\widehat{f}(\xi)|^2 d\xi.\end{aligned}$$

But using the definition of $\widehat{\psi}$ and the support of $\Phi$, for every $\xi \in \mathbb{R}^n$,

$$\int_0^\infty |\widehat{\psi}(t\xi)|^2 \frac{dt}{t} = \int_1^2 s^2 \Phi(s)^2 \frac{ds}{s} = C_\Phi^2,$$

so we get the equality in (4.6).

**Proposition 4.14.** *It holds that, for every $1 < p < \infty$ and $w \in A_p$,*

$$\|f\|_{L^p(w)} \lesssim \|g_\Phi f\|_{L^p(w)}.$$

*Proof.* It follows the same idea as in the unweighted case (see [112, p. 85]). We know by (4.6) that $\|g_\Phi f\|_{L^2(\mathbb{R}^n)} = C_\Phi \|f\|_{L^2(\mathbb{R}^n)}$. Using the polarization identity of $L^2(\mathbb{R}^n)$ and introducing the weight in a simple way, we have that

$$\left|\int_{\mathbb{R}^n} f(x)h(x)dx\right| \approx \left|\int_{\mathbb{R}^n} g_\Phi f(x) g_\Phi h(x) w^{-1}(x) w(x) dx\right|.$$

Now we use Hölder's inequality to bound the previous expression by

$$\|g_\Phi f\|_{L^p(w)} \|g_\Phi h\|_{L^{p'}(w^{1-p'})}.$$

But $w^{1-p'} \in A_{p'}$, and the operator $g_\Phi$ is bounded on $L^q(v)$ for every $1 < q < \infty$ and $v \in A_q$ (see, for instance, [84]), so using this fact, we conclude that

$$\left|\int_{\mathbb{R}^n} f(x)h(x)dx\right| \lesssim \|g_\Phi f\|_{L^p(w)} \|h\|_{L^{p'}(w^{1-p'})}.$$

Dividing by $\|h\|_{L^{p'}(w^{1-p'})} \neq 0$ and taking supremum over $h \in L^{p'}(w^{1-p'})$, duality yields

$$\|f\|_{L^p(w)} \lesssim \|g_\Phi f\|_{L^p(w)}.$$

$\square$

**Remark 4.15.** *So far, we have not been able to find a proof of the inequality*

$$\|f\|_{L^{p,\infty}(w)} \lesssim \|g_\Phi f\|_{L^{p,\infty}(w)},$$

*for $w \in \widehat{A}_p$. In the previous cases (of the functions $S$ and $S_{\psi,\alpha}$) where we were seeking this lower estimate, we had a certain good-$\lambda$ inequality that worked for $A_\infty$ weights and, therefore, we could deduce the $L^{p,\infty}(w) \to L^{p,\infty}(w)$ estimate for $\widehat{A}_p$ weights similarly to the $L^p(w) \to L^p(w)$ estimate for $A_p$. For $g_\Phi$, however, we used a duality argument that, despite being really simple, does not work beyond the $A_p$ classes.*

## 4.3 Upper estimates

In this section, we want to study upper Littlewood-Paley inequalities of the form

$$\|Gf\|_{L^{p,\infty}(v)} \lesssim \|f\|_{L^{p,1}(v)}, \quad v \in \widehat{A}_p. \tag{4.7}$$

### 4.3.1 The $G_\alpha$ function

We define the following square function

$$G_\alpha f(x) = \left(\int_0^\infty \left|\frac{\partial}{\partial t} B_\alpha^t f(x)\right|^2 t dt\right)^{1/2},$$

where $B^t_\alpha$ is the Bochner-Riesz operator as in Definition 2.1. This function was first introduced by E. M. Stein in [111] to study $L^2$ properties of the maximal Bochner-Riesz operator. It can be easily checked that

$$\frac{\partial}{\partial t} B^t_\alpha f(x) = \frac{2\alpha}{t} \int_{\mathbb{R}^n} \frac{|\xi|^2}{t^2} \left(1 - \frac{|\xi|^2}{t^2}\right)_+^{\alpha-1} \widehat{f}(\xi) e^{2\pi i x\xi} d\xi,$$

and from here deduce that

$$G_\alpha f(x) \approx \left( \int_0^\infty |K^\alpha_t * f(x)|^2 \frac{dt}{t} \right)^{1/2},$$

with $\widehat{K^\alpha_t}(\xi) = \frac{|\xi|^2}{t^2} \left(1 - \frac{|\xi|^2}{t^2}\right)_+^{\alpha-1}$. This is the way that $G_\alpha$ was defined in [16], [17] and [115], some of the references that we will use for this part. See also the expository introduction of [83]. The proof that for $\alpha > \frac{n+1}{2}$, the operator $G_\alpha$ is of (unweighted) strong-type $(p,p)$ for every $1 < p < \infty$ and of weak-type $(1,1)$ is due to S. Sunouchi [115]. Here, the author relates $G_\alpha$ to an $L^2(0,\infty)$ vector-valued Calderón-Zygmund operator and is able to use the classical theory to obtain his result. However, if we want to establish weighted inequalities, it seems that the vector-valued theory in this case does not work as cleanly. Our main result is the following:

**Theorem 4.16.** *Let $\alpha > \frac{n+1}{2}$. Then $G_\alpha$ is*

*(i) of strong-type $(p,p)$ for every weight in $A_p$ and $1 < p < \infty$,*

*(ii) of restricted weak-type $(p,p)$ for every weight in $A_p^{\mathcal{R}}$ and $1 < p < \infty$,*

*(iii) of weak-type $(1,1)$ for every weight in $A_1$.*

The proof of this theorem is based on the fact that we will be able to control $G_\alpha f$ by a finite sum of sparse operators, which are much easier to handle and known to satisfy these three properties[7]. The notion of sparse operator already appeared in the proof of Proposition 3.3, where we actually showed that they satisfy the corresponding restricted weak-type $(p,p)$ estimate in *(ii)*. For other examples of the use of sparse theory to obtain weighted estimates for square functions, see [47, 84, 87]. Now, let us recall their definition in a little more detail. For convenience, we will follow the exposition in [88]. Given a dyadic lattice of cubes in $\mathbb{R}^n$, we will say that a family of cubes $\mathcal{S}$ is $\lambda$-sparse, with $0 < \lambda < 1$ if, for every $Q \in \mathcal{S}$, there exists a measurable subset $F_Q \subseteq Q$ such that $|F_Q| \geqslant (1-\lambda)|Q|$ and $\{F_Q\}_{Q\in\mathcal{S}}$ are pairwise disjoint.

[7] The strong-type $(p,p)$ for $A_p$ weights was of great interest when it was seen [85, 86] that it gave a new (and easier) proof of the celebrated $A_2$ theorem [72] for Calderón-Zygmund operators.

**Definition 4.17.** *The sparse operator $S$ associated with the sparse family $\mathcal{S}$ is defined by*

$$Sf(x) := \sum_{Q \in \mathcal{S}} \left( \frac{1}{|Q|} \int_Q |f| \right) \chi_Q(x).$$

We will also need the following definitions of the so-called local mean oscillation:

**Definition 4.18.** *Given a function $g$ and a measurable set $E$, we define*

$$\omega(g, E) := \sup_{x \in E} g(x) - \inf_{x \in E} g(x).$$

*Given $0 < \lambda < 1$ and a dyadic cube $Q$, we also define*

$$\omega_\lambda(g, Q) := \min\{\omega(g, E) : E \subseteq Q, |E| \geqslant (1-\lambda)|Q|\}.$$

The key result that we will need is the following, and it can be found in [88]:

**Theorem 4.19.** *Let $f$ be a measurable function and let $F : \mathbb{R}^n \to \mathbb{R}$ be such that, for every $\varepsilon > 0$,*

$$|\{x \in [-R, R]^n : |F(x)| > \varepsilon\}| = o(R^n), \qquad \textit{as } R \to \infty.$$

*If, given a dyadic cube $Q$ and $0 < \lambda \leqslant 2^{-n-2}$, it holds that, for some $\delta > 0$*

$$\omega_\lambda(F, Q) \leqslant C_\lambda \sum_{k=0}^{\infty} 2^{-\delta k} \left( \frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |f| \right), \tag{4.8}$$

*then $|F|$ is pointwise controlled by a finite sum of sparse operators applied to $f$.*

*Proof of Theorem 4.16.* Fix $\alpha = \frac{n+1}{2} + \delta$, with $\delta > 0$. If we define

$$T_t f(x) = \frac{K_t^\alpha * f(x)}{\sqrt{t}},$$

it holds that,

$$G_\alpha f(x) = \|T_t f(x)\|_{L^2(0,\infty)}.$$

By [115], we know that $G_\alpha$ is of weak-type $(1,1)$, that is

$$y|\{x \in \mathbb{R}^n : \|T_t f(x)\|_{L^2(0,\infty)} > y\}| \leqslant \|G_\alpha\|_{L^1 \to L^{1,\infty}} \|f\|_{L^1(\mathbb{R}^n)}, \tag{4.9}$$

and the author also shows (see [115, Equations (3) and (4)]) that, given $r > 0$ and $s \in \mathbb{R}$ such that $r > 2|s|$,

$$|\mathcal{K}_t(r+s) - \mathcal{K}_t(r)| \lesssim \min\{t^{-\frac{1}{2}-\delta} r^{-n-\delta},\ |s|\, t^{\frac{1}{2}-\delta} r^{-n-\delta}\}, \tag{4.10}$$

where

$$\mathcal{K}_t(|x|) = \frac{K_t^\alpha(x)}{\sqrt{t}}.$$

Taking $F(x) = G_\alpha f(x)$, we have that the decay assumption for $F$ in Theorem 4.19 is trivially satisfied (using, for instance, that $G_\alpha$ is of weak-type (1,1)), so if we show (4.8), then we conclude that $G_\alpha f$ is dominated by sparse operators and, hence, finish the proof. Fix a cube $Q$ and $0 < \lambda \leqslant 2^{-n-2}$. Let $x, x' \in Q$. Then,

$$\begin{aligned}
&|\|T_t f(x)\|_{L^2(0,\infty)} - \|T_t f(x')\|_{L^2(0,\infty)}| \leqslant \|T_t f(x) - T_t f(x')\|_{L^2(0,\infty)} \\
&= \left\| T_t(f\chi_{2Q})(x) + \sum_{k\geqslant 1} T_t(f\chi_{2^{k+1}Q\setminus 2^k Q})(x) - T_t(f\chi_{2Q})(x') - \sum_{k\geqslant 1} T_t(f\chi_{2^{k+1}Q\setminus 2^k Q})(x') \right\|_{L^2(0,\infty)} \\
&\leqslant I + II,
\end{aligned}$$

where

$$I = \|T_t(f\chi_{2Q})(x)\|_{L^2(0,\infty)} + \|T_t(f\chi_{2Q})(x')\|_{L^2(0,\infty)},$$

and after using Minkowski's integral inequality,

$$II = \sum_{k\geqslant 1} \int_{2^{k+1}Q\setminus 2^k Q} \|\mathcal{K}_t(|x-y|) - \mathcal{K}_t(|x'-y|)\|_{L^2(0,\infty)} |f(y)| dy.$$

We start by studying $II$. Since $x, x' \in Q$ and $y \in 2^{k+1}Q\setminus 2^k Q$, we can set $r := |x'-y|$ and observe that $|x-y| = r+s$, with $s \in (-|x-x'|, |x-x'|)$. Therefore,

$$\|\mathcal{K}_t(|x-y|) - \mathcal{K}_t(|x'-y|)\|^2_{L^2(0,\infty)} = \|\mathcal{K}_t(r+s) - \mathcal{K}_t(r)\|^2_{L^2(0,\infty)}.$$

Computing the $L^2$ norm and using (4.10) with the different bounds on $(0, |s|^{-1})$ and $(|s|^{-1}, \infty)$ respectively, we can control the previous expression by

$$\int_0^{|s|^{-1}} |s|^2 t^{1-2\delta} r^{-2n-2\delta} dt + \int_{|s|^{-1}}^\infty t^{-1-2\delta} r^{-2n-2\delta} dt \approx \frac{|s|^{2\delta}}{r^{2n+2\delta}}.$$

But $r = |x'-y| \approx 2^k \ell(Q)$ and $|s| \leqslant |x-x'| \leqslant \ell(Q)$, so again, the last term is majorized by

$$\frac{\ell(Q)^{2\delta}}{2^{2k(n+\delta)}\ell(Q)^{2n+2\delta}} = \left(\frac{1}{2^{k(n+\delta)}|Q|}\right)^2.$$

With this estimate, we go back to $II$ and see that

$$II \lesssim \sum_{k\geqslant 1} \int_{2^{k+1}Q\setminus 2^k Q} \frac{|f(y)|}{2^{k(n+\delta)}|Q|} dy \lesssim \sum_{k\geqslant 1} 2^{-\delta k} \left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q} |f(y)| dy\right).$$

To study $I$, we just use (4.9) to get that if

$$E^* := \left\{ z \in Q : \|T_t(f\chi_{2Q})(z)\|_{L^2(0,\infty)} > \frac{2^n \|G_\alpha\|_{L^1\to L^{1,\infty}}}{\lambda|2Q|} \int_{2Q} |f| \right\},$$

then

$$|E^*| \leqslant \|G_\alpha\|_{L^1\to L^{1,\infty}} \frac{\lambda|2Q|}{2^n\|G_\alpha\|_{L^1\to L^{1,\infty}} \int_{2Q}|f|} \|f\chi_{2Q}\|_{L^1(\mathbb{R}^n)} = \lambda|Q|.$$

So defining $E := Q\backslash E^*$, we deduce that, when $x \in E$,

$$\|T_t(f\chi_{2Q}(x))\|_{L^2(0,\infty)} \lesssim C_\lambda \frac{1}{|2Q|} \int_{2Q} |f|,$$

and the size of $E$ is controlled by

$$|E| \geqslant |Q| - |E^*| \geqslant (1-\lambda)|Q|.$$

Summing up, bringing it all together, we have shown that there exists a measurable set $E \subseteq Q$ such that $|E| \geqslant (1-\lambda)|Q|$ and satisfying that, for every $x, x' \in E$,

$$|\|T_tf(x)\|_{L^2(0,\infty)} - \|T_tf(x')\|_{L^2(0,\infty)}| \leqslant I + II \lesssim C_\lambda \sum_{k=0}^{\infty} 2^{-\delta k} \left( \frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |f(y)|dy \right).$$

Hence, the same bound holds for $\omega_\lambda(\|T_tf(\cdot)\|_{L^2(0,\infty)}, Q)$, and we finish the proof. □

# 4.4 Pointwise estimates and consequences

Even though the main goal of this chapter was to study lower and upper estimates independently one from another, for the sake of completeness we will devote this last section to see if some of them can be related by means of pointwise estimates. We will consider two kinds of multipliers. First, we will study general multipliers of Hörmander type like the ones appearing in Section 3.4, and then we will turn our attention to radial ones.

## 4.4.1 General multipliers

In Section 3.4, we showed a restricted weak-type estimate that extended the results of D. Kurtz and R. Wheeden [79] about multipliers of Hörmander type. The technique behind those results avoided the use of Littlewood-Paley theory. However, in [78, Theorem 4], the author resorts to this theory to tackle the same problem as in [79]. When $1 < p < \infty$, he succeeds in showing the strong-type $(p,p)$ estimates with respect to $A_p$ weights for multipliers satisfying $m \in HC(2,n)$, but he cannot prove the weighted weak-type (1,1) part due to limitations regarding the square function $g_\lambda^*$ involved. Let us introduce this function and state in a lemma the pointwise inequality that one has in this setting for Fourier multipliers with $m \in HC(2,n)$. It relates the Lusin area function from Definition 4.1 and $g_2^*$:

**Definition 4.20.** *We define, for $\lambda > 1$,*

$$g_\lambda^* f(x) = \left( \int_{\mathbb{R}^{n+1}_+} \frac{t^{(\lambda-1)n+1}}{(t + |x-y|)^{\lambda n}} |\nabla u(y,t)|^2 dydt \right)^{1/2},$$

*where $u(x,t) = P_t * f(x)$ is the Poisson integral of $f$.*

**Lemma 4.21.** *Given $m \in HC(2,n)$, we have that*

$$S(T_m f)(x) \lesssim g_2^* f(x).$$

This inequality can be found in [112, Lemma, p. 233] (see also [78, p. 239]), stated for $m \in HC(2, n+1)$ and $g_\lambda^*$ with $\lambda = \frac{2n+2}{n}$. Even though one cannot deduce Lemma 4.21 directly from here, in the proof, the author assumes $m \in HC(2,k)$ and obtains the estimate involving $g_\lambda^*$ with $\lambda = \frac{2k}{n}$. He concludes the argument taking $k = n+1$, but if we take $k = n$ instead, we get Lemma 4.21. Even though we do have the lower estimate

$$\|f\|_{L^{p,\infty}(w)} \lesssim \|Sf\|_{L^{p,\infty}(w)}$$

for $\widehat{A}_p$ weights, we have not been able to establish the corresponding upper estimate

$$\|g_2^* f\|_{L^{p,\infty}(w)} \lesssim \|f\|_{L^{p,1}(w)}$$

for these weights, and hence, we cannot deduce the restricted weak-type $(p,p)$ for multipliers $T_m$ with $m \in HC(2,n)$. The function $g_2^*$, however, is known to be of strong-type $(p,p)$ for the smaller class $A_p$ (see [97]), and this is what allows the author in [78] to use the Littlewood-Paley approach to show that $T_m$ with $m \in HC(2,n)$ is of strong-type $(p,p)$ for $A_p$ weights and $1 < p < \infty$. The weighted weak-type $(1,1)$ endpoint result for $T_m$ and $A_1$ weights is also true (see Section 3.4) but, as far as we know, it is an open problem whether the function $g_2^*$ is of weak-type $(1,1)$ or not, even in the unweighted case.

### 4.4.2 Radial multipliers

Here we fix a non-negative, smooth bump function $\Phi$ with support in $[1,2]$, just as we did when we defined $g_\Phi$ in Subsection 4.2.3. Now, the parameter $\alpha > 0$ will be a positive real number, and whenever we write $\left(\frac{d}{dt}\right)^\alpha$ for $\alpha \notin \mathbb{N}$, we will be referring to

$$\widehat{\left(\frac{d}{dt}\right)^\alpha h}(\xi) = (-2\pi i \xi)^\alpha \widehat{h}(\xi),$$

in the distributional sense if needed.

**Definition 4.22.** *Given a bounded function $m : [0,\infty) \to \mathbb{R}$, extended by zero to the whole line $\mathbb{R}$, we say that $m \in R_\Phi(2,\alpha)$ if*

$$\sup_{r>0} \left( r^{2\alpha-1} \int_{\mathbb{R}} \left| \left(\frac{d}{dt}\right)^{\alpha} \left( \Phi\left(\frac{t}{r}\right) m(t) \right) \right|^2 dt \right)^{1/2} < \infty.$$

A simple change of variables shows that this condition is equivalent to

$$\sup_{r>0} \left\| \left(\frac{d}{dt}\right)^{\alpha} \Phi(t) m(rt) \right\|_{L^2(\mathbb{R})} < \infty,$$

and by [16, Theorem 2], we have that:

**Theorem 4.23.** *Given $\alpha > \frac{1}{2}$ and $m \in R_\Phi(2,\alpha)$, the multiplier defined by*

$$\widehat{T_m f}(\xi) = m(|\xi|)\widehat{f}(\xi),$$

*satisfies*

$$g_\Phi(T_m f)(x) \lesssim G_\alpha f(x), \quad a.e.\ x \in \mathbb{R}^n.$$

With this estimate together with Proposition 4.14 and (i) in Theorem 4.16, we obtain the following multiplier result:

**Theorem 4.24.** *Given a non-negative, smooth bump function $\Phi$ supported in $[1,2]$ and a bounded function $m : [0,\infty) \to \mathbb{R}$ in $R_\Phi(2,\alpha)$ for some $\alpha > \frac{n+1}{2}$, we have that the associated radial multiplier $T_m$ on $\mathbb{R}^n$ satisfies*

$$T_m : L^p(w) \longrightarrow L^p(w),$$

*for every $1 < p < \infty$ and $w \in A_p$.*

Here we have the opposite problem to the one we had for general multipliers. In this case, we do have Theorem 4.16 (an upper estimate) that gives restricted weak-type inequalities for $\widehat{A}_p$ and the function $G_\alpha$ ($\alpha > \frac{n+1}{2}$), but we lack the corresponding lower estimate for $g_\Phi$,

$$\|f\|_{L^{p,\infty}(w)} \lesssim \|g_\Phi f\|_{L^{p,\infty}(w)}, \quad w \in \widehat{A}_p,$$

as mentioned in Remark 4.15. This is the reason why Theorem 4.24 only applies to the $A_p$ setting. In [16, Theorem 4], however, the author gives yet another related pointwise estimate, but this time for the maximal operator associated with $T_m$.

**Theorem 4.25.** *Let $m : [0,\infty) \to \infty$ be a bounded function satisfying, for $\alpha > \frac{1}{2}$,*

$$\int_0^\infty \left| s^{\alpha+1} \left(\frac{d}{ds}\right)^{\alpha} \frac{m(s)}{s} \right|^2 \frac{ds}{s} < \infty.$$

*Then,*

$$T_m^* f(x) \lesssim G_\alpha f(x), \quad a.e. \ x \in \mathbb{R}^n,$$

*where* $T_m^* f(x) = \sup_{t>0} |T_m^t f(x)|$ *is the maximal operator associated with the family* $\{T_m^t\}_{t>0}$ *defined by*

$$\widehat{T_m^t f}(\xi) = m(t|\xi|)\widehat{f}(\xi).$$

In contrast with Theorem 4.23, this pointwise inequality is for the operator $T_m^*$ itself, so we do not have to rely on a lower estimate in order to obtain boundedness results for $T_m^*$. In fact, we can use the full potential of Theorem 4.16 to deduce the following:

**Corollary 4.26.** *Let* $\alpha > \frac{n+1}{2}$ *and* $m : [0,\infty) \to \infty$ *be a bounded function such that*

$$\int_0^\infty \left| s^{\alpha+1} \left(\frac{d}{ds}\right)^\alpha \frac{m(s)}{s} \right|^2 \frac{ds}{s} < \infty.$$

*Then* $T_m^*$ *is*

*(i) of strong-type* $(p,p)$ *for every weight in* $A_p$ *and* $1 < p < \infty$*,*

*(ii) of restricted weak-type* $(p,p)$ *for every weight in* $A_p^{\mathcal{R}}$ *and* $1 < p < \infty$*,*

*(iii) of weak-type* $(1,1)$ *for every weight in* $A_1$*.*

For more details on the class of multipliers $m$ satisfying the condition in Theorem 4.25 (or its corollary), see [16, Section III] and how the author relates this class to the Bessel potential spaces introduced in [112, Chapter VI]. See also [83, (4) and (5)] for another presentation of the pointwise estimates that we have used from [16].

# Chapter 5

# Yano's Extrapolation Theory

## 5.1 A connection between two theories

The theory of extrapolation we have presented so far follows the ideas introduced by J. L. Rubio de Francia. As we have seen, in the context of $L^p$ spaces, the goal is to find an estimate at a fixed level $p_0$ that holds for a whole class of weights and deduce new estimates at other levels of $p$. Yano's extrapolation, on the other hand, works in a different way. In this case, one would fix the measure (not necessarily a weight) and find estimates for a whole range of $p \in (1, p_0)$, with a boundedness constant that blows up in a precise way when $p \to 1^+$. The extrapolation argument, then, would seek boundedness in a suitable space, closer to $L^1$ than any of the initial $L^p$ with $p \in (1, p_0)$. Even though these two theories are different, Yano's extrapolation can be used to, in some sense, complete the information that we have for operators of Rubio de Francia type at the endpoint. Let us explain this relation to motivate this chapter.

We know that an operator $T$ under the hypotheses of Rubio de Francia's extrapolation theorem need not be bounded from $L^1$ to $L^{1,\infty}$. However, the sharp $L^p$ constants obtained in [48] provide useful information to obtain endpoint estimates for these operators. In particular, we know that, if for some $1 < p_0 < \infty$ and every $w \in A_{p_0}$,

$$T : L^{p_0}(w) \longrightarrow L^{p_0}(w)$$

is bounded with constant $\varphi(\|w\|_{A_{p_0}})$, with $\varphi$ an increasing function on $(0, \infty)$, then, given $u \in A_1$,

$$T : L^p(u) \longrightarrow L^p(u)$$

is bounded for every $1 < p < p_0$ with constant essentially controlled by

$$\varphi\left(\frac{C\|u\|_{A_1}^{\frac{p_0-1}{p-1}}}{(p-1)^{p_0-1}}\right), \qquad \text{as } p \to 1^+. \tag{5.1}$$

As we just mentioned[1], the starting point in Yano's theory is, precisely, having an $L^p$ boundedness on a range $(1, p_0)$ with some control on how the boundedness constant explodes when $p$ is close to 1. In fact, the blow-up that we would like to have in order to extrapolate is of the order of $\frac{1}{(p-1)^m}$, for some $m > 0$. Therefore, examining (5.1), we see that if we assume $\varphi(t) = t^\beta$ for some $\beta > 0$ and $u = 1$, we obtain that $T : L^p \to L^p$ is bounded with constant essentially controlled by

$$\frac{1}{(p-1)^{\beta(p_0-1)}}, \qquad \text{as } p \to 1^+.$$

With this information, one can show (as we will see in Theorem 5.22) endpoint results close to $L^1(\mathbb{R}^n)$ for sublinear operators under the hypotheses of Rubio de Francia's theorem. A converse argument can be used to find optimal values of $\beta$ (see [92]), but we will not get into this particular problem.

In Yano's theory, as one would expect, the slower the blow-up of the constant is, the better the conclusions are, so one could try to start with the boundedness constant associated with the restricted weak-type $(p_0, p_0)$ of $T$ instead. Take, for instance, the Hardy-Littlewood maximal operator. It is known [9] that, for $M$,

$$\|M\|_{L^{p_0}(w) \longrightarrow L^{p_0}(w)} \lesssim \frac{\|w\|_{A_{p_0}}^{\frac{1}{p_0-1}}}{p_0 - 1}, \tag{5.2}$$

whereas

$$\|M\|_{L^{p_0,1}(w) \longrightarrow L^{p_0,\infty}(w)} \lesssim \|w\|_{A_{p_0}}^{1/p_0}. \tag{5.3}$$

Since we want to work with constants that have the least possible blow-up when $p$ is close to 1, it makes sense to start with this weaker assumption. The extrapolation of restricted weak-type $(p_0, p_0)$ estimates for $A_{p_0}$ weights was carried out in Theorem 1.8 avoiding the use of Rubio de Francia's classical theory. What we showed is that, if for every $w \in A_{p_0}$,

$$T : L^{p_0,1}(w) \longrightarrow L^{p_0,\infty}(w)$$

is bounded with constant $\varphi(\|w\|_{A_{p_0}})$, then, given $u \in A_1$,

$$T : L^{p,\frac{p}{p_0}}(u) \longrightarrow L^{p,\infty}(u) \tag{5.4}$$

is bounded for $1 < p < p_0$ with constant essentially controlled by

$$\|u\|_{A_1}^{\frac{1}{p}-\frac{1}{p_0}} \varphi\left(\left(\frac{p_0-1}{p-1}\right)^{p_0-1} \|u\|_{A_1}\right). \tag{5.5}$$

[1] In this introductory section we will not make the classical results in Yano's theory precise. We refer to Section 5.2 for a detailed presentation.

Notice that now, if we want to have a blow-up of the form $\frac{1}{(p-1)^m}$, we are allowed to consider[2] any fixed $u \in A_1$. Therefore, if $\varphi(t) = t^\alpha$ with $\alpha > 0$, for every $u \in A_1$ we get that the constant for (5.4) behaves like

$$\frac{1}{(p-1)^{\alpha(p_0-1)}}, \qquad \text{as } p \to 1^+.$$

The extrapolation that we will use is not the classical one, but a newer version that assumes boundedness from $L^{p,\infty}$ into itself on a range $(1, p_0)$. This variant was developed in [33] and will be presented in Section 5.3.

For the time being, the only goal of this first section will be to compute the $L^{p,\infty} \to L^{p,\infty}$ constants of operators under the assumptions of Rubio de Francia's theory ($A_p$ theory) and under the assumptions of the theory presented in Section 1.2 ($\widehat{A}_p$ theory). To do so, we will need the following interpolation result:

**Lemma 5.1.** *Let $0 < s_0, s_1 \leqslant 1 < r_0 < r_1 < \infty$ and let $T$ be a sublinear operator such that*

$$T : L^{r_j,s_j}(u) \longrightarrow L^{r_j,\infty}(u)$$

*is bounded with constant $M_j$, for $j = 0, 1$. Then, for every $0 < \theta < 1$, if $\frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}$, we have that*

$$T : L^{r,\infty}(u) \longrightarrow L^{r,\infty}(u)$$

*is bounded with constant controlled by $BM_0^{1-\theta}M_1^{\theta}$, where*

$$B = \left(\frac{r_0 r}{s_0(r-r_0)}\right)^{1/s_0} + \left(\frac{r_1 r}{s_1(r_1-r)}\right)^{1/s_1} + \left(\frac{r_1}{s_1}\right)^{1/s_1}$$

*Proof.* The proof of this result can be found, for instance, in [6, Theorem 5.3.2], but we need to see how the constant behaves and this is not included in classical books. We will proceed as in [28, Lemma 3.10]. By the real interpolation $K$-method (see [5, Chapter 5]), we have that

$$T : (L^{r_0,s_0}(u), L^{r_1,s_1}(u))_{\theta,\infty} \longrightarrow (L^{r_0,\infty}(u), L^{r_1,\infty}(u))_{\theta,\infty},$$

with constant less than or equal to $M_0^{1-\theta}M_1^{\theta}$, where

$$(A_1, A_2)_{\theta,\infty} = \left\{ f \in A_1 + A_2 : \sup_{t>0} t^{-\theta} K(t, f; A_1, A_2) < \infty \right\},$$

and

$$K(t, f; A_1, A_2) = \inf \left\{ \|f_0\|_{A_1} + t\|f_1\|_{A_2} : f = f_0 + f_1, f_0 \in A_1, f_1 \in A_2 \right\}.$$

Therefore, it is enough to show that:

[2]The exponents of $\|u\|_{A_1}$ in (5.5), unlike in (5.1), do not explode when $p$ is close to 1, and hence having $\|u\|_{A_1} > 1$ is no longer a problem. We will see that an exponential blow-up in $p$ is hopeless if we want to extrapolate in the sense of Yano.

(i) $\|f\|_{L^{r,\infty}(u)} \leqslant 2\|f\|_{(L^{r_0,\infty}(u),L^{r_1,\infty}(u))_{\theta,\infty}}$,

(ii) $\|f\|_{(L^{r_0,s_0}(u),L^{r_1,s_1}(u))_{\theta,\infty}} \leqslant B\|f\|_{L^{r,\infty}(u)}$.

The proof of (i) goes as follows: define $\gamma := \frac{r_0r_1}{r_1-r_0}$, fix $t > 0$ and let $f = f_0 + f_1$ be a decomposition of $f$ in $L^{r_0,\infty}(u) + L^{r_1,\infty}(u)$. Then,

$$\begin{aligned}
\sup_{y\leqslant t^\gamma} y^{1/r_0} f_u^*(y) &\leqslant \sup_{y\leqslant t^\gamma} y^{1/r_0}\left((f_0)_u^*\left(\frac{y}{2}\right) + (f_1)_u^*\left(\frac{y}{2}\right)\right)\\
&\leqslant \sup_{y\leqslant t^\gamma} 2^{1/r_0}\|f_0\|_{L^{r_0,\infty}(u)} + y^{\frac{1}{r_0}-\frac{1}{r_1}} 2^{1/r_1}\|f_1\|_{L^{r_1,\infty}(u)}\\
&\leqslant 2(\|f_0\|_{L^{r_0,\infty}(u)} + t\|f_1\|_{L^{r_1,\infty}(u)}).
\end{aligned}$$

Taking infimum over all possible decompositions of $f$, we conclude that

$$\sup_{y\leqslant t^\gamma} y^{1/r_0} f_u^*(y) \leqslant 2K(t,f;L^{r_0,\infty}(u),L^{r_1,\infty}(u)),$$

and with this estimate,

$$\begin{aligned}
2\|f\|_{(L^{r_0,\infty}(u),L^{r_1,\infty}(u))_{\theta,\infty}} &= \sup_{t>0} 2t^{-\theta}K(t,f;L^{r_0,\infty}(u),L^{r_1,\infty}(u))\\
&\geqslant \sup_{t>0}\sup_{y\leqslant t^\gamma} t^{-\theta}y^{1/r_0}f_u^*(y) = \sup_{y>0} y^{1/r_0}f_u^*(y)\sup_{t\geqslant y^{1/\gamma}} t^{-\theta}\\
&= \sup_{y>0} y^{\frac{-\theta}{\gamma}+\frac{1}{r_0}} f_u^*(y) = \|f\|_{L^{r,\infty}(u)}.
\end{aligned}$$

For (ii), let $f \in L^{r,\infty}(u)$ and $\gamma$ as before. For every $t > 0$, we write $f = f_0 + f_1$ with

$$f_0 = f\chi_{\{|f|>f_u^*(t^\gamma)\}} \quad \text{and} \quad f_1 = f\chi_{\{|f|\leqslant f_u^*(t^\gamma)\}}.$$

Now,

$$\begin{aligned}
\|f_0\|_{L^{r_0,s_0}(u)} &\leqslant \left(\int_0^{t^\gamma} (f_u^*(y)y^{1/r})^{s_0} y^{\frac{s_0}{r_0}-\frac{s_0}{r}-1}dy\right)^{1/s_0} \leqslant \|f\|_{L^{r,\infty}(u)} \frac{t^{\gamma\left(\frac{1}{r_0}-\frac{1}{r}\right)}}{\left(\frac{s_0}{r_0}-\frac{s_0}{r}\right)^{1/s_0}}\\
&= t^\theta\left(\frac{r_0r}{s_0(r-r_0)}\right)^{1/s_0}\|f\|_{L^{r,\infty}(u)},
\end{aligned}$$

by the definition of $\gamma$ and $\theta = \frac{r_0r_1-rr_1}{rr_0-rr_1}$. Also,

$$\|f_1\|_{L^{r_1,s_1}(u)} \leqslant f_u^*(t^\gamma)\left(\int_0^{t^\gamma} y^{\frac{s_1}{r_1}-1}dy\right)^{1/s_1} + \left(\int_{t^\gamma}^\infty f_u^*(y)^{s_1}y^{\frac{s_1}{r_1}-1}dy\right)^{1/s_1}.$$

For the first term, we multiply and divide by $t^{\gamma/r}$, compute the integral and the bound we get is

$$t^{\theta-1}\left(\frac{r_1}{s_1}\right)^{1/s_1}\|f\|_{L^{r,\infty}(u)}.$$

For the second term, we proceed exactly as for $\|f_0\|_{L^{r_0,s_0}(u)}$ and control it by

$$t^{\theta-1}\left(\frac{r_1 r}{s_1(r_1-r)}\right)^{1/s_1}\|f\|_{L^{r,\infty}(u)}.$$

Bringing the estimates together, we conclude that

$$\begin{aligned}\|f\|_{(L^{r_0,s_0}(u),L^{r_1,s_1}(u))_{\theta,\infty}} &= \sup_{t>0} t^{-\theta}K(t,f;L^{r_0,s_0}(u),L^{r_1,s_1}(u)) \\ &\leqslant \sup_{t>0} t^{-\theta}(\|f_0\|_{L^{r_0,s_0}}(u)+t\|f_1\|_{L^{r_1,s_1}}(u)) \leqslant B\|f\|_{L^{r,\infty}(u)}.\end{aligned}$$

□

Next, we will use this interpolation to study the behavior in $p$ of the $L^{p,\infty} \to L^{p,\infty}$ constant for operators under the hypotheses of Rubio de Francia's Theorem 1.1. The boundedness from which we will start will be of restricted weak-type $(p_0,p_0)$, instead of strong-type. The result we get is the following:

**Theorem 5.2.** *Let $1<p_0<\infty$, and let $T$ be a sublinear operator such that*

$$T : L^{p_0,1}(w) \longrightarrow L^{p_0,\infty}(w)$$

*is bounded for every $w \in A_{p_0}$ with constant $\varphi(\|w\|_{A_{p_0}})$, where $\varphi$ is an increasing function on $(0,\infty)$. Then, for every $u \in A_1$ and $1<p<p_0$,*

$$T : L^{p,\infty}(u) \longrightarrow L^{p,\infty}(u) \tag{5.6}$$

*is bounded with constant*

$$\left[\left(\frac{2pp_0}{p-1}\right)^{\frac{2p_0}{p+1}}+\frac{p_0^2}{p_0-p}\right]\varphi\left(\left(\frac{2(p_0-1)}{p-1}\right)^{p_0-1}\|u\|_{A_1}\right)^{\frac{(p+1)(p_0-p)}{p(2p_0-p-1)}}\varphi\left(\|u\|_{A_1}\right)^{\frac{p_0(p-1)}{p(2p_0-p-1)}}\|u\|_{A_1}^{\frac{1}{p}-\frac{1}{p_0}}.$$

*In particular, if $\varphi(t)=t^{\alpha}$ for some $\alpha>0$ and $u \in A_1$ is fixed, then the boundedness constant behaves like*

$$\frac{1}{(p-1)^{\alpha(p_0-1)+p_0}}$$

*when $p$ is close to one.*

*Proof.* Let $1 < p < p_0$ and $u \in A_1 \subseteq A_{p_0}$. If we extrapolate down to $\frac{p+1}{2}$ by means of Theorem 1.8, we get that

$$T : L^{\frac{p+1}{2}, \frac{p+1}{2p_0}}(u) \longrightarrow L^{\frac{p+1}{2}, \infty}(u)$$

with constant less than or equal to

$$M_0 = \|u\|_{A_1}^{\frac{2}{p+1} - \frac{1}{p_0}} \varphi\left(\left(\frac{2(p_0-1)}{p-1}\right)^{p_0-1} \|u\|_{A_1}\right).$$

Moreover, our hypothesis is that

$$T : L^{p_0,1}(u) \longrightarrow L^{p_0,\infty}(u)$$

with constant

$$M_1 = \varphi(\|u\|_{A_{p_0}}) \leqslant \varphi(\|u\|_{A_1}).$$

Therefore, we can interpolate by Lemma 5.1 with

$$\begin{array}{ll} r_0 = \frac{p+1}{2}, & s_0 = \frac{p+1}{2p_0} \\ r_1 = p_0, & s_1 = 1, \end{array}$$

and the corresponding boundedness constants $M_0$ and $M_1$. We obtain (5.6) for the fixed $p$, which lies in $(r_0, r_1) = \left(\frac{p+1}{2}, 1\right)$, with constant

$$\left[\left(\frac{2pp_0}{p-1}\right)^{\frac{2p_0}{p+1}} + \frac{p_0^2}{p_0 - p}\right] M_0^{1-\theta} M_1^{\theta},$$

where

$$\theta = \frac{p_0(p-1)}{p(2p_0 - p - 1)}, \quad 1 - \theta = \frac{(p+1)(p_0 - p)}{p(2p_0 - p - 1)}.$$

If we replace the expressions of $\theta$ and $1-\theta$, we get the sought-after constant. Finally, if we consider $\varphi(t) = t^\alpha$ and $u \in A_1$ fixed, it is easy to check that the behavior of the constant is like

$$\frac{1}{(p-1)^{\alpha(p_0-1)+p_0}}$$

when $p$ is close to 1. □

For simplicity, from now on we will adopt the following notation:

$$\log_1(x) = 1 + \log_+(x) \quad \text{and} \quad \log_k(x) = \log_1 \log_{k-1}(x), \text{ for } k > 1,$$

where $\log_+$ denotes the positive part of the logarithm. Let us state a lemma that will become a useful computation for the rest of this chapter.

**Lemma 5.3.** *Let $1 < p_0 < \infty$, $m > 0$ and $A > 0$. We have that*

$$I = \inf_{1<p\leqslant p_0} \frac{A^{1/p}}{(p-1)^m} \lesssim A \log_1^m \frac{1}{A},$$

*where the constant in the inequality depends on $p_0$.*

*Proof.* For the sake of simplicity, we will prove it for $m = 1$, though the general case is identical. Notice that

$$I = \inf_{1<p\leqslant p_0} \frac{A^{1/p}}{p(1-1/p)} \approx \inf_{1<p\leqslant p_0} \frac{A^{1/p}}{1-1/p} = \inf_{1/p_0\leqslant x<1} \frac{A^x}{1-x},$$

since $1/p_0 \leqslant 1/p < 1$ (i.e., $1/p \approx 1$). Now, let us consider two cases:

- $A \geqslant 1$:

  We have that $s(x) = \frac{A^x}{1-x}$ and $s'(x) = \frac{A^x(1-x)\log A + A^x}{(1-x)^2} > 0$ for every $0 < x < 1$. Hence,

  $$I \approx \inf_{1/p_0\leqslant x<1} \frac{A^x}{1-x} = \frac{A^{1/p_0}}{1-1/p_0} \approx A^{1/p_0} \leqslant A = A \log_1 \frac{1}{A}.$$

- $A < 1$:

  Now, $s'(\widetilde{x}) = 0$ for $\widetilde{x} = 1 + \frac{1}{\log A} < 1$. This is a minimum and therefore,

  $$I \approx \begin{cases} \frac{A^{\widetilde{x}}}{1-\widetilde{x}}, & \text{if } \widetilde{x} \geqslant 1/p_0, \\ \frac{A^{1/p_0}}{1-1/p_0}, & \text{if } \widetilde{x} < 1/p_0. \end{cases}$$

  We have $\widetilde{x} \geqslant 1/p_0$ if and only if $A \leqslant e^{\frac{p_0}{1-p_0}} =: C_0$, with $0 < C_0 < 1$.

  If $0 < A \leqslant C_0 < 1$, then

  $$I \approx \frac{A^{1+\frac{1}{\log A}}}{-\frac{1}{\log A}} \approx A \log \frac{1}{A} = A \log_+ \frac{1}{A} \leqslant A \log_1 \frac{1}{A}.$$

  If $C_0 < A < 1$, then $\frac{A}{C_0} > 1$ and

  $$\begin{aligned} I &\approx \inf_{1/p_0\leqslant x<1} \frac{(A/C_0)^x}{1-x} C_0^x \leqslant C_0^{1/p_0} \inf_{1/p_0\leqslant x<1} \frac{(A/C_0)^x}{1-x} \\ &\lesssim C_0^{1/p_0} \frac{A}{C_0} \lesssim A \log_1 \frac{1}{A}, \end{aligned}$$

  using the estimate in Case $A \geqslant 1$ with $\frac{A}{C_0}$.

□

**Remark 5.4.** *At some point, we will also need the following, similar estimate:*

$$I = \inf_{1<p\leqslant p_0} \log_1\left(\frac{1}{p-1}\right)\frac{A^{1/p}}{p-1} \lesssim A\log_1\frac{1}{A}\log_2\frac{1}{A}.$$

*Proof.* Let us go over the proof of the previous lemma and see what changes we have to make. As before, we compute

$$I \approx \inf_{1/p_0\leqslant x<1} \log_1\left(\frac{1}{1-x}\right)\frac{A^x}{1-x}.$$

The case $A \geqslant 1$ is the same, with $I \lesssim A$. Now, consider again the point $\widetilde{x} = 1 + \frac{1}{\log A} < 1$, which we know lies in $[1/p_0, 1)$ if, and only if $A \leqslant C_0$. In such a case, we clearly have that

$$I \lesssim \log_1\left(\frac{1}{1-\widetilde{x}}\right)\frac{A^{\widetilde{x}}}{1-\widetilde{x}} = \log_1\left(\log\frac{1}{A}\right)A^{1+\frac{1}{\log A}}\log\frac{1}{A} \lesssim A\log_1\frac{1}{A}\log_2\frac{1}{A}.$$

On the other hand, if $C_0 < A < 1$,

$$I \lesssim C_0^{1/p_0}\inf_{1/p_0\leqslant x<1}\log_1\left(\frac{1}{1-x}\right)\frac{(A/C_0)^x}{1-x} \lesssim C_0^{1/p_0-1}A \lesssim A\log_1\frac{1}{A}\log_2\frac{1}{A}.$$

□

Let us go back to the computation of $L^{p,\infty} \to L^{p,\infty}$ norms. Now, we will show a result in the spirit of Theorem 5.2 but, this time, considering operators under the hypotheses of Theorem 1.7 instead. We know that this is a stronger condition to assume on an operator (see [28, Theorem 3.11]), so the $L^{p,\infty} \to L^{p,\infty}$ constant that we will get should be better behaved than the one in Theorem 5.2.

**Theorem 5.5.** *Let $1 < p_0 < \infty$, and let $T$ be a sublinear operator such that*

$$T : L^{p_0,1}(w) \longrightarrow L^{p_0,\infty}(w)$$

*is bounded for every $w \in \widehat{A}_{p_0}$ with constant $\varphi(\|w\|_{\widehat{A}_{p_0}})$, where $\varphi$ is an increasing function on $(0,\infty)$. Then, for every $u \in A_1$, $1 < p < p_0$ and $0 < \varepsilon \leqslant p_0 - 1$,*

$$T : L^{p,\infty}(u) \longrightarrow L^{p,\infty}(u) \tag{5.7}$$

*is bounded with constant*

$$\left[\left(\frac{p(1+\varepsilon)}{p-1}\right)^{1+\varepsilon} + \frac{p_0^2}{p_0-p}\right]\left(\frac{1}{\varepsilon}\right)^{\frac{p_0-p}{p(p_0-1)}}\|u\|_{A_1}^{\frac{p_0-p}{p}}\varphi\left(\|u\|_{A_1}^{1/p_0}\right).$$

*In particular, if $u \in A_1$ is fixed and $p$ is close to one, then the boundedness constant behaves like*

$$\log_1\left(\frac{1}{p-1}\right)\frac{1}{p-1}.$$

*Proof.* In [28], the authors prove that from these hypotheses we can deduce that, for $1 < p_1 \leqslant p_0$ and $u \in A_1$,

$$T : L^{1,\frac{1}{p_1}}(u) \longrightarrow L^{1,\infty}(u)$$

with constant

$$M_0 = \frac{1}{p_1 - 1}\|u\|_{A_1}^{1-\frac{1}{p_0}} \varphi\left(\|u\|_{A_1}^{1/p_0}\right).$$

Moreover, our hypothesis is

$$T : L^{p_0,1}(u) \longrightarrow L^{p_0,\infty}(u)$$

with constant

$$M_1 = \varphi(\|u\|_{\widehat{A}_{p_0}}) \leqslant \varphi(\|u\|_{A_1}^{1/p_0}).$$

Therefore, we can interpolate using Lemma 5.1 with

$$\begin{aligned} r_0 &= 1, \quad s_0 = \tfrac{1}{p_1} \\ r_1 &= p_0, \quad s_1 = 1, \end{aligned}$$

and the corresponding boundedness constants $M_0$ and $M_1$. We obtain (5.7) for every $p \in (r_0, r_1) = (1, p_0)$ with constant

$$\left[\left(\frac{pp_1}{p-1}\right)^{p_1} + \frac{p_0 p}{p_0 - p} + p_0\right]\left(\frac{1}{p_1 - 1}\right)^{1-\theta} \|u\|_{A_1}^{\left(1-\frac{1}{p_0}\right)(1-\theta)} \varphi_{p_0}\left(\|u\|_{A_1}^{1/p_0}\right)^{1-\theta+\theta}.$$

Since

$$\theta = \frac{p_0(p-1)}{p(p_0-1)}, \quad 1 - \theta = \frac{p_0 - p}{p(p_0 - 1)},$$

we can rewrite the constant as

$$\left[\left(\frac{pp_1}{p-1}\right)^{p_1} + \frac{p_0^2}{p_0 - p}\right]\left(\frac{1}{p_1 - 1}\right)^{\frac{p_0 - p}{p(p_0-1)}} \|u\|_{A_1}^{\frac{p_0-p}{pp_0}} \varphi_{p_0}\left(\|u\|_{A_1}^{1/p_0}\right).$$

If we set $p_1 = 1 + \varepsilon$, we get the first part of the result, since the condition $1 < p_1 \leqslant p_0$ is equivalent to $0 < \varepsilon \leqslant p_0 - 1$. Now, if we fix $u \in A_1$ and take $p$ close to one, notice that the previous constant is equivalent to

$$\frac{C}{\varepsilon(p-1)^{1+\varepsilon}},$$

with a constant $C$ independent of $\varepsilon$ and $p$. In particular, we have that $T$ satisfies (5.7) with constant equivalent to the infimum of the previous expression over $\varepsilon \in (0, p_0 - 1)$. Without loss of generality, assume that

$$\varepsilon_0 = \frac{1}{\log \frac{1}{p-1}} < p_0 - 1.$$

If we write $A = \frac{1}{p-1}$, we want to compute the infimum of $\frac{A^{1+\varepsilon}}{\varepsilon}$. This can be computed by differentiation (exactly as we did in Lemma 5.3) and it is attained at $\varepsilon_0 = \frac{1}{\log A}$, which lies in $(0, p_0 - 1)$ by assumption. Hence, we can take as a boundedness constant

$$\frac{A^{1+\varepsilon_0}}{\varepsilon_0} = \left(\log \frac{1}{p-1}\right)\left(\frac{1}{p-1}\right)^{1+\frac{1}{\log \frac{1}{p-1}}} \lesssim \log_1\left(\frac{1}{p-1}\right)\frac{1}{p-1}.$$

□

We see that in this case we can consider any function $\varphi$ (not necessarily a power of $t$) and the blow-up that we obtain is independent of $\varphi$. Even though the constant is not of the form $\frac{1}{(p-1)^m}$, we will see that the extrapolation can be easily modified to admit a logarithmic factor. The rest of the chapter is organized as follows. First we will introduce, in a more precise way, Yano's classical theory. Then we will explain the new results that have been found in [33] for operators mapping $L^{p,\infty}$ into itself. After this, we will make some contributions in the setting of Lorentz spaces $L^{p,q}$, and finally, we will come back to this connection with Rubio de Francia's theory to see what we get from this behavior of the constants when $p$ tends to 1.

## 5.2 Classical results

Yano's extrapolation theory goes back to 1951, when S. Yano published a result [127] for sublinear operators of strong-type $(p,p)$:

**Theorem 5.6.** *Fix $(X,\mu)$, $(Y,\nu)$ a couple of finite measure spaces, $p_0 > 1$ and $m > 0$. If $T$ is a sublinear operator such that, for every $1 < p \leqslant p_0$,*

$$T : L^p(\mu) \longrightarrow L^p(\nu)$$

*is bounded with norm controlled by $\frac{C}{(p-1)^m}$, then,*

$$T : L(\log L)^m(\mu) \longrightarrow L^1(\nu).$$

Recall that $L(\log L)^m(\mu)$ is the space of $\mu$-measurable functions such that

$$\|f\|_{L(\log L)^m(\mu)} = \int_0^\infty f_\mu^*(t) \log_1^m \frac{1}{t} dt < \infty.$$

As usual, $f_\mu^*$ denotes the decreasing rearrangement of $f$ with respect to $\mu$, defined by

$$f_\mu^*(t) = \inf\left\{y > 0 : \lambda_f^\mu(y) \leqslant t\right\},$$

where $\lambda_f^\mu(y) = \mu(\{x : |f(x)| > y\})$ was the distribution function of $f$ with respect to $\mu$. In general, given natural numbers $1 \leqslant j_1 < j_2 < \cdots < j_n$ and positive real numbers $m_1, ..., m_n > 0$, we define the associated log-type space as follows:

$$L(\log_{j_1} L)^{m_1} \cdots (\log_{j_n} L)^{m_n}(\mu) = \{f \ \mu\text{-measurable} : \|f\|_{L(\log_{j_1} L)^{m_1} \cdots (\log_{j_n} L)^{m_n}(\mu)} < \infty\},$$

where

$$\|f\|_{L(\log_{j_1} L)^{m_1} \cdots (\log_{j_n} L)^{m_n}(\mu)} = \int_0^\infty f_\mu^*(t) \log_{j_1}^{m_1} \frac{1}{t} \cdots \log_{j_n}^{m_n} \frac{1}{t} dt.$$

Unlike in previous chapters, now, it will be more convenient to work with the decreasing rearrangement when dealing with $L^p$ spaces. More precisely, we will use the following equivalent definition for the $L^p$ norm:

$$\|f\|_{L^p(\mu)} = \left( \int_{\mathbb{R}^n} |f(x)|^p d\mu(x) \right)^{1/p} = \left( \int_0^\infty f_\mu^*(t)^p dt \right)^{1/p}.$$

Even though Yano's original statement was for finite measures, it can actually be extended to $\sigma$-finite measures (that is, measures defined on a $\sigma$-algebra $\Sigma$ of subsets of a set $\Omega$ with the latter being a countable union of measurable sets with finite measure) and improved in order to have weaker hypotheses and a better range space. More precisely, one can prove that for $\sigma$-finite measures $\mu$ and $\nu$, if a sublinear operator $T$ satisfies

$$T : L^{p,1}(\mu) \longrightarrow L^p(\nu)$$

with constant essentially controlled by $\frac{1}{(p-1)^m}$, then

$$T : L(\log L)^m(\mu) \longrightarrow E_m(\nu),$$

where $E_m(\nu)$ is the space of $\nu$-measurable functions such that

$$\|f\|_{E_m(\nu)} = \sup_{t>0} \frac{t f_\nu^{**}(t)}{\log_1^m t} < \infty,$$

and $f_\nu^{**}(t) = \frac{1}{t}\int_0^t f_\nu^*(s)ds$ . See [19] and [20] for more details on this extension.

Later, N. Yu Antonov [3] proved that there is almost everywhere convergence for the Fourier series of every function in $L \log L \log_3 L(\mathbb{T})$. To do so, he checked that the Carleson maximal operator satisfied a certain estimate that ensured its boundedness on $L \log L \log_3 L(\mathbb{T})$ and hence, the almost everywhere convergence for the Fourier series of every function in this space. Further study (see [4], [30], [31], [109]) showed that with Antonov's ideas, it is possible to write an extrapolation result that we will refer to as *Antonov's extrapolation theorem*:

**Theorem 5.7.** *Fix $\sigma$-finite measures $\mu$ and $\nu$, $1 < p_0 < \infty$ and $m > 0$. If $T$ is a sublinear operator mapping*

$$L^p(\mu) \longrightarrow L^{p,\infty}(\nu)$$

*with constant controlled by $\frac{C}{(p-1)^m}$ for every $1 < p \leqslant p_0$, then*

$$T : L(\log L)^m \log_3 L(\mu) \longrightarrow R_m(\nu), \tag{5.8}$$

*where $R_m(\nu)$ is the space of $\nu$-measurable functions such that*

$$\|f\|_{R_m(\nu)} = \sup_{t>0} \frac{t f_\nu^*(t)}{\log_1^m t} < \infty.$$

For the weak-$L^p$ space, $L^{p,\infty}(\nu)$, we also have an equivalent definition in terms of the decreasing rearrangement that will be used:

$$\|f\|_{L^{p,\infty}(\nu)} = \sup_{t>0} t^{1/p} f_\nu^*(t) = \sup_{y>0} y \lambda_f^\nu(y)^{1/p}.$$

## 5.3 Extrapolation on $L^{p,\infty}$ spaces

In the context of $L^p$ and weak-$L^p$ spaces, another result has been recently obtained in [33] for operators mapping

$$T : L^{p,\infty}(\mu) \longrightarrow L^{p,\infty}(\nu)$$

with constant controlled by $\frac{C}{(p-1)^m}$ near $p = 1$, yielding a better estimate than if we simply apply Antonov's Theorem 5.7. Before stating it, we need the following definition, as in [33]:

**Definition 5.8.** *Given a quasi-Banach rearrangement invariant space $X$ over a measure space $(\Omega, \mu)$, for each $p \geqslant 1$ we denote*

$$[X]_p = \left\{ g \in L^{p,\infty}(\mu) : \frac{\sup_{t \leqslant y} t^{1/p} g_\mu^*(t)}{y} \chi_{[0,1]}(y) \in \widetilde{X} \right\},$$

*endowed with the quasi-norm*

$$\|g\|_{[X]_p} = \|g\|_{L^{p,\infty}(\mu)} + \left\| \frac{\sup_{t \leqslant y} t^{1/p} g_\mu^*(t)}{y} \chi_{[0,1]}(y) \right\|_{\widetilde{X}},$$

*and where $\widetilde{X}$ denotes the canonical representation of the space $X$ on the line $(0, \infty)$ by means of $f_\mu^*$ (see [5, Chapter 2]).*

**Definition 5.9.** *Let $X$ be a quasi-Banach rearrangement invariant space over a measure space $(\Omega, \mu)$ such that its quasi-norm can be written by means of an integral over $(0, \infty)$. That is, for every $f \in X$,*

$$\|f\|_X = \int_0^\infty \Phi_X(f_\mu^*(t), t)dt.$$

*Then, we define the space*

$$\underline{X} = \left\{ f \ \mu\text{-measurable} : \int_0^1 \Phi_X(f_\mu^*(t), t)dt < \infty \right\}.$$

**Example 5.10.** *If $X = L\log L(\mu)$, then $\underline{L\log L(\mu)}$ is the set of $\mu$-measurable functions such that*

$$\int_0^1 f_\mu^*(t) \log_1 \frac{1}{t} dt < \infty.$$

Also, recall that we say that a measure space $(\Omega, \mu)$ is non-atomic (or simply $\mu$ is a non-atomic measure) if, for any $\mu$-measurable set $E \subseteq \Omega$ with $\mu(E) > 0$, there exists a $\mu$-measurable subset $F \subseteq E$ such that $\mu(E) > \mu(F) > 0$. This is the main result in [33]:

**Theorem 5.11.** *Fix a couple of $\sigma$-finite, non-atomic measures $\mu$ and $\nu$, $1 < p_0 < \infty$, $m > 0$, and let*

$$T : L^{p,\infty}(\mu) \longrightarrow L^{p,\infty}(\nu)$$

*be a bounded sublinear operator with constant controlled by $\frac{C}{(p-1)^m}$ for every $1 < p \leqslant p_0$. Then,*

$$T : \left[L(\log L)^{m-1} \log_3 L(\mu)\right]_1 \longrightarrow R_m(\nu),$$

*where $R_m(\nu)$ is the space of $\nu$-measurable functions such that*

$$\|f\|_{R_m(\nu)} = \sup_{t>0} \frac{t f_\nu^*(t)}{\log_1^m t} < \infty.$$

As we anticipated at the beginning of this section, this result is better than if we just apply Antonov's theorem to $T$ (which would give that $T$ satisfies (5.8)). This is due to the fact that

$$L(\log L)^m \log_3 L \subsetneq \left[L(\log L)^{m-1} \log_3 L(\mu)\right]_1,$$

as shown in [33].

## 5.4 An extension to $L^{p,q}$ spaces

In this section, we will see what happens if we introduce Lorentz spaces $L^{p,q}$, with values $p < q < \infty$, instead of $L^{p,\infty}$. As we did for $L^p$ and $L^{p,\infty}$, we point out that the spaces $L^{p,q}(\mu)$ can be written in terms of the decreasing rearrangement, since

$$\|f\|_{L^{p,q}(\mu)} = \left(p\int_0^\infty (t\lambda_f^\mu(t)^{1/p})^q \frac{dt}{t}\right)^{1/q} = \left(\int_0^\infty f_\mu^*(t)^q t^{q/p-1} dt\right)^{1/q}.$$

We will present two extrapolation results, for operators:

- $T : L^p(\mu) \longrightarrow L^{p,q}(\nu)$,
- $T : L^{p,q}(\mu) \longrightarrow L^{p,q}(\nu)$.

In the first case we will follow the ideas of Antonov's theorem (as presented in [27]) and in the second one, we will follow [33].

### 5.4.1 Extrapolation of $T : L^p \to L^{p,q}$ near $p = 1$

Here, we will fix a couple of $\sigma$-finite measures $\mu$ and $\nu$, $1 < p_0 \leqslant q < \infty$ and $m > 0$, and we will assume that we have a bounded sublinear operator

$$T : L^p(\mu) \longrightarrow L^{p,q}(\nu), \tag{5.9}$$

with constant controlled by $\frac{C}{(p-1)^m}$ for every $1 < p \leqslant p_0$. Before tackling the problem of obtaining endpoint estimates close to $p = 1$, let us recall the definition of a general Lorentz space:

**Definition 5.12.** *Given $q \in (0, \infty)$, a measure $\nu$ and a weight $\omega$, we define*

$$\Lambda_\nu^q(\omega) = \left\{ f\ \nu\text{-measurable} : \|f\|_{\Lambda_\nu^q(\omega)} := \left(\int_0^\infty (f_\nu^*(t))^q \omega(t) dt\right)^{1/q} < \infty \right\}.$$

Notice that this definition includes most of the spaces that we have worked with so far: $L^{p,q}(\nu) = \Lambda_\nu^q(t^{q/p-1})$, $L^p(\nu) = \Lambda_\nu^p(1)$ or $L\log L(\nu) = \Lambda_\nu^1(\log_1(1/s))$. Another notion that we will need to define is the Galb of a quasi-Banach space:

**Definition 5.13.** *Given a quasi-Banach space $X$, we define*

$$\mathrm{Galb}(X) := \left\{ \{c_n\}_n : \sum_{n=0}^\infty c_n f_n \in X \ \text{ whenever } \|f_n\|_X \leqslant 1 \right\},$$

*endowed with the norm* $\|\{c_n\}_n\|_{\mathrm{Galb}(X)} = \sup_{\|f_n\|_X \leqslant 1} \left\| \sum_{n=0}^\infty c_n f_n \right\|_X$.

This concept was introduced in [123] and studied in the context of Lorentz spaces in [22]. Naturally, if $X$ is a Banach space, then $\mathrm{Galb}(X) = \ell^1$, but, for instance, $\mathrm{Galb}(L^{1,\infty}(\mu)) = \ell \log \ell$. This fact is also known as the Stein-Weiss lemma, and can be found in [114]. We will start by proving an estimate for functions in the unit ball of $L^\infty(\mu)$, that is, $\mu$-measurable functions that are essentially bounded by 1.

**Lemma 5.14.** *Let $T$ be a sublinear operator as in* (5.9)*. Then, for every $f \in L^\infty(\mu)$ such that $\|f\|_\infty \leqslant 1$, we have*

$$\|Tf\|_{\Lambda^q_\nu(\omega)} \lesssim \|f\|_{L^1(\mu)} \log_1^m \frac{1}{\|f\|_{L^1(\mu)}},$$

*where $\omega(t) = \min\{t^{q-1}, t^{q/p_0-1}\}$.*

*Proof.* Let $1 < p \leqslant p_0$. On the one hand, by our boundedness hypothesis and the fact that $\|f\|_\infty \leqslant 1$,

$$\|Tf\|_{L^{p,q}(\nu)} \lesssim \frac{\|f\|_{L^p(\mu)}}{(p-1)^m} \leqslant \frac{\|f\|_{L^1(\mu)}^{1/p}}{(p-1)^m}.$$

On the other hand, since $1 < p \leqslant p_0$,

$$\begin{aligned} \|Tf\|_{L^{p,q}(\nu)} &\geqslant \left( \int_0^1 (Tf)^*_\nu(t)^q t^{q-1} dt + \int_1^\infty (Tf)^*_\nu(t)^q t^{q/p_0-1} dt \right)^{1/q} \\ &= \left( \int_0^\infty (Tf)^*_\nu(t)^q \omega(t) dt \right)^{1/q} = \|Tf\|_{\Lambda^q_\nu(\omega)}. \end{aligned}$$

Bringing both estimates together and taking infimum over $p$ on both sides, we get that

$$\|Tf\|_{\Lambda^q_\nu(\omega)} \lesssim \inf_{1<p\leqslant p_0} \frac{\|f\|_{L^1(\mu)}^{1/p}}{(p-1)^m} \lesssim \|f\|_{L^1(\mu)} \log_1^m \frac{1}{\|f\|_{L^1(\mu)}},$$

by Lemma 5.3. $\square$

**Lemma 5.15.** *Given $1 < p_0 \leqslant q < \infty$, we have that*

$$\mathrm{Galb}(\Lambda^q_\nu(\omega)) = \ell(\log \ell)^{1/q'},$$

*where $\omega(t) = \min\{t^{q-1}, t^{q/p_0-1}\}$ and $1/q + 1/q' = 1$.*

*Proof.* This lemma is a direct consequence of [22, Corollary 3.7], by which we only need to check that $W(s)/s^q$ is equivalent to a bounded, decreasing function. In [22], the authors work with $\Lambda^q(\omega)$, taking $\nu$ to be the Lebesgue measure, but when it comes to the Galb, the way functions are rearranged plays no role and we can apply their result. Here, as usual, for a weight $\omega$ we denote $W(t) := \int_0^t \omega(s)ds$. If we make the computations, we get that

$$\frac{W(s)}{s^q} = \frac{1}{q}\chi_{(0,1)}(s) + \left( \frac{1-p_0}{qs^q} + \frac{p_0}{qs^{q-q/p_0}} \right) \chi_{[1,\infty)}(s),$$

which is decreasing and bounded by $1/q$. $\square$

With this, we are ready to prove our main result.

**Theorem 5.16.** *Fix $\sigma$-finite measures $\mu$ and $\nu$, $1 < p_0 \leqslant q < \infty$, $m > 0$, and let*

$$T : L^p(\mu) \longrightarrow L^{p,q}(\nu)$$

*be a bounded sublinear operator with constant controlled by $\frac{C}{(p-1)^m}$ for every $1 < p \leqslant p_0$. Then*

$$T : L(\log L)^m(\log_3 L)^{1/q'}(\mu) \longrightarrow \Lambda^q_\nu(\omega)$$

*is bounded, with $\omega(t) = \min\{t^{q-1}, t^{q/p_0-1}\}$.*

Notice that this is consistent with Theorem 5.7 if we formally take $q = \infty$.

*Proof.* We will follow the general scheme introduced in [27]. Let $f$ be a positive function and write

$$f = \sum_{k=0}^{\infty} 2^{2^k} \widetilde{f}_k, \tag{5.10}$$

with

$$\widetilde{f}_k = \begin{cases} \frac{1}{2} f \chi_{\{0 \leqslant f \leqslant 2\}}, & \text{if } k = 0, \\ \frac{1}{2^{2^k}} f \chi_{\{2^{2^{k-1}} < f \leqslant 2^{2^k}\}}, & \text{if } k \geqslant 1. \end{cases}$$

It holds that $\|\widetilde{f}_k\|_\infty \leqslant 1$ for every $k \geqslant 0$. Assume that $f \in L^1(\mu) \cap L^\infty(\mu)$, and hence, the sum in (5.10) is finite. Now, by sublinearity and the lattice property of $\Lambda^q_\nu(\omega)$,

$$\|Tf\|_{\Lambda^q_\nu(\omega)} \leqslant \left\| \sum_{k=0}^{\infty} 2^{2^k} |T\widetilde{f}_k| \right\|_{\Lambda^q_\nu(\omega)} = \left\| \sum_{k=0}^{\infty} 2^{2^k} D(\|\widetilde{f}_k\|_{L^1(\mu)}) \frac{|T\widetilde{f}_k|}{D(\|\widetilde{f}_k\|_{L^1(\mu)})} \right\|_{\Lambda^q_\nu(\omega)},$$

where $D(t) := t \log_1^m \frac{1}{t}$. But by Lemma 5.14,

$$\left\| \frac{|T\widetilde{f}_k|}{D(\|\widetilde{f}_k\|_{L^1(\mu)})} \right\|_{\Lambda^q_\nu(\omega)} \lesssim 1, \quad k \geqslant 0,$$

and hence, if we denote $A_k := 2^{2^k} D(\|\widetilde{f}_k\|_{L^1(\mu)})$ for every $k \geqslant 0$, we get that

$$\|Tf\|_{\Lambda^q_\nu(\omega)} \lesssim \sup_{\|g_k\|_{\Lambda^q_\nu(\omega)} \leqslant 1} \left\| \sum_{k=0}^{\infty} A_k g_k \right\|_{\Lambda^q_\nu(\omega)} = \|\{A_k\}_k\|_{\mathrm{Galb}(\Lambda^q_\nu(\omega))} = \|\{A_k\}_k\|_{\ell(\log \ell)^{1/q'}},$$

by Lemma 5.15. Now, define the function $\varphi(t) := \log_1^m t \log_3^{1/q'} t$, which is essentially constant on $[0, 2]$ and on the intervals $[2^{2^{k-1}}, 2^{2^k}]$ for every $k \geqslant 0$. The statement for $[0, 2]$ is clear, so let us check the latter:

$$c_k := \varphi(2^{2^k}) \approx 2^{km} \log_1^{1/q'} k \approx 2^m 2^{km} \log_1^{1/q'}(k+1) = c_{k+1},$$

and since $\varphi$ is increasing, we have that $\varphi(t) \approx c_k$ for every $t \in [2^{2^{k-1}}, 2^{2^k}]$. Consequently, if we define

$$a_k = \begin{cases} \int_{0 \leqslant f \leqslant 2} f\varphi(f)d\mu, & \text{if } k = 0, \\ \int_{2^{2^{k-1}} < f \leqslant 2^{2^k}} f\varphi(f)d\mu, & \text{if } k \geqslant 1, \end{cases}$$

we obtain, for $k \geqslant 1$,

$$\begin{aligned} A_k = 2^{2^k} D(\|\widetilde{f}_k\|_{L^1(\mu)}) &= 2^{2^k} D\left(\frac{1}{2^{2^k}} \int_{2^{2^{k-1}} < f \leqslant 2^{2^k}} f d\mu\right) \\ &\approx 2^{2^k} D\left(\frac{1}{2^{2^k} c_k} \int_{2^{2^{k-1}} < f \leqslant 2^{2^k}} f\varphi(f)d\mu\right) = 2^{2^k} D\left(\frac{a_k}{2^{2^k} c_k}\right), \end{aligned}$$

and the analogous for $k = 0$. Therefore, we can write

$$\|Tf\|_{\Lambda^q_\nu(\omega)} \lesssim \left\| \left\{ 2^{2^k} D\left(\frac{a_k}{2^{2^k} c_k}\right) \right\}_k \right\|_{\ell(\log \ell)^{1/q'}}.$$

We claim that the right-hand side of the previous expression is uniformly bounded whenever $\{a_k\}_k \in \ell^1$ with $\|\{a_k\}\|_{\ell^1} = 1$. If we prove this, we would have that $\|Tf\|_{\Lambda^q_\nu(\omega)} \lesssim 1$ for every function $f \in L^\infty(\mu)$ such that

$$1 = \sum_{k=0}^{\infty} a_k = \int_{\mathbb{R}^n} f\varphi(f)d\mu \approx \|f\|_{L(\log L)^m (\log_3 L)^{1/q'}(\mu)},$$

and would get the sought-after boundedness on $L^\infty(\mu) \cap L(\log L)^m(\log_3 L)^{1/q'}(\mu)$. But since this is a dense subspace of $L(\log L)^m(\log_3 L)^{1/q'}(\mu)$, we would have completed the proof. So let us show our claim:

$$\begin{aligned} \left\| \left\{ 2^{2^k} D\left(\frac{a_k}{2^{2^k} c_k}\right) \right\}_k \right\|_{\ell(\log \ell)^{1/q'}} &\lesssim \sum_{k=1}^{\infty} \log_1^{1/q'} k \frac{a_k}{2^{km} \log_1^{1/q'} k} \log_1^m \frac{2^{2^k} 2^{km} \log_1^{1/q'} k}{a_k} \\ &\approx \sum_{k=1}^{\infty} \frac{a_k}{2^{km}} \log_1^m \frac{2^{2^k}}{a_k} \lesssim \sum_{k=1}^{\infty} \frac{a_k}{2^{km}} \log_1^m 2^{2^k} + \sum_{k=1}^{\infty} \frac{a_k}{2^{km}} \log_1^m \frac{1}{a_k} \\ &\lesssim \sum_{k=1}^{\infty} a_k + 1 \approx 1, \end{aligned}$$

where in the first estimate we use the definition of $D$ and $c_k \approx 2^{km} \log_1^{1/q'} k$, and in the last one we use that $a_k$ tends to zero as $k$ tends to infinity in order to conclude that $a_k \log_1^m \frac{1}{a_k}$ is bounded and the second series is finite.

$\square$

### 5.4.2 Extrapolation of $T : L^{p,q} \to L^{p,q}$ near $p = 1$

In this part, we will assume that $(\Omega_1, \mu)$ and $(\Omega_2, \nu)$ are $\sigma$-finite, non-atomic measure spaces. This, for instance, guarantees that every decreasing, right-continuous function on $(0, \infty)$ is the decreasing rearrangement with respect to $\mu$ of some $\mu$-measurable function (see [5, Chapter 2]). In [33], the authors obtain an extrapolation result for operators $T : L^{p,\infty}(\mu) \to L^{p,\infty}(\nu)$. When both $\mu$ and $\nu$ are the Lebesgue measure, they apply Antonov's extrapolation theorem to the composition $TM$, where $M$ denotes the Hardy-Littlewood maximal operator:

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| dy, \quad f \in L^1_{\text{loc}}.$$

It is easily shown that $M$ is bounded from $L^p$ into $L^{p,\infty}$ with a uniform constant independent of $p$ (when $p$ is close to 1), and hence $TM : L^p \to L^{p,\infty}$ is bounded with the same constant $\|T\|_{L^{p,\infty} \to L^{p,\infty}}$ as $T$. The key estimate for this operator is that (see [5, Chapter 3])

$$(Mf)^*(t) \approx f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) ds,$$

and hence, for general $\sigma$-finite, non-atomic measures, it is enough to consider some operator such that its decreasing rearrangement with respect to $\mu$ is equivalent to $f^{**}_\mu$. Let us give a constructive example:

**Definition 5.17.** *Let $(\Omega_1, \mu_1)$ and $(\Omega_2, \mu_2)$ be $\sigma$-finite measure spaces. A map $\rho : \Omega_1 \to \Omega_2$ is said to be a measure-preserving transformation if, whenever $E$ is a $\mu_2$-measurable set, then $\rho^{-1}(E)$ is a $\mu_1$-measurable set and*

$$\mu_1(\rho^{-1}(E)) = \mu_2(E).$$

Now, let $\rho$ be a measure-preserving transformation between $(\Omega_1, \mu)$ and $(0, \infty)$ with the Lebesgue measure[3]. Since these transformations induce equimeasurability (see [5, Chapter 2, Proposition 7.2]), if we define

$$\mathcal{M}_\mu f(x) = f^{**}_\mu(\rho(x)) = \frac{1}{\rho(x)} \int_0^{\rho(x)} f^*_\mu(s) ds,$$

it holds that

$$(\mathcal{M}_\mu f)^*_\mu = f^{**}_\mu. \tag{5.11}$$

Now, let us fix $1 < p_0 \leqslant q < \infty$, $m > 0$, and assume that we have a bounded sublinear operator $T : L^{p,q}(\mu) \to L^{p,q}(\nu)$, with constant less than or equal to $\frac{C}{(p-1)^m}$ for every $1 < p \leqslant p_0$. We will follow the ideas in [33] to obtain an endpoint estimate near $p = 1$. First, however, we will need to study the boundedness of $\mathcal{M}_\mu : L^p(\mu) \to L^{p,q}(\mu)$.

[3]Actually, if $(\Omega_1, \mu)$ is a finite measure space, then the transformation will take values in $(0, \mu(\Omega_1))$ instead of $(0, \infty)$. If this were the case, everything would be identical with the obvious changes.

**Lemma 5.18.** *Let $1 < p \leqslant q < \infty$. Then, the operator $\mathcal{M}_\mu : L^p(\mu) \to L^{p,q}(\mu)$ is bounded with optimal constant equivalent to*

$$\left(\frac{p}{q}\right)^{1/q}\left(\frac{1}{p-1}\right)^{1/q}.$$

*Proof.* This lemma is a particular case of [103, Theorem 2]. Nevertheless, in this article the author is not concerned about the dependence on $p$ of the constants, so we need to go over the proof and check that the equivalence between the given bound for $\mathcal{M}_\mu$ and the real one is stable at least when $p$ tends to $1^+$. This can be done for general weights $\omega_1, \omega_2$ and the result gives us that, for $1 < p \leqslant q < \infty$, $\mathcal{M}_\mu : \Lambda^p_\mu(\omega_1) \to \Lambda^q_\mu(\omega_2)$ if and only if

$$A = \sup_{r>0}\left(\int_0^r \omega_2(x)dx\right)^{1/q}\left(\int_0^r \omega_1(x)dx\right)^{-1/p} < \infty,$$

$$B = \sup_{r>0}\left(\int_r^\infty \frac{\omega_2(x)}{x^q}dx\right)^{1/q}\left(\int_0^r\left(\frac{1}{x}\int_0^x \omega_1(s)ds\right)^{-p'}\omega_1(x)dx\right)^{1/p'} < \infty,$$

with optimal constant equivalent to $A + B$. In our case, $\omega_1(x) = 1$, $\omega_2(x) = x^{q/p-1}$ and making the computations we obtain the desired estimate. We need to mention that in [103] all the rearrangements are with respect to the Lebesgue measure (they work with $f^{**}$), but, since we have (5.11), everything is identical if we work with respect to a general $\sigma$-finite, non-atomic measure $\mu$. □

**Theorem 5.19.** *Fix $\mu$ and $\nu$ two $\sigma$-finite, non-atomic measures, $1 < p_0 \leqslant q < \infty$, $m > 0$ and let*

$$T : L^{p,q}(\mu) \longrightarrow L^{p,q}(\nu)$$

*be a bounded, sublinear operator with constant controlled by $\frac{C}{(p-1)^m}$ for every $1 < p \leqslant p_0$. Then,*

$$T : \left[L(\log L)^{m+1/q-1}(\log_3 L)^{1/q'}(\mu)\right]_1 \longrightarrow \Lambda^q_\nu(\omega)$$

*is bounded with $\omega(t) = \min\{t^{q-1}, t^{q/p_0-1}\}$.*

*Proof.* By Lemma 5.18, we have that $\mathcal{M}_\mu$ is bounded from $L^p(\mu)$ into $L^{p,q}(\mu)$ with constant behaving like $\frac{1}{(p-1)^{1/q}}$ when $p \to 1^+$, and hence, the composition

$$T\mathcal{M}_\mu : L^p(\mu) \to L^{p,q}(\nu)$$

is bounded with constant like $\frac{1}{(p-1)^{m+1/q}}$ when $p$ tends to $1^+$. Now, we apply Theorem 5.16 to conclude that

$$T\mathcal{M}_\mu : L(\log L)^{m+1/q}(\log_3 L)^{1/q'}(\mu) \longrightarrow \Lambda^q_\nu(\omega)$$

is bounded. Therefore,

$$\begin{aligned}\|T(\mathcal{M}_\mu f)\|_{\Lambda^q_\nu(\omega)} \lesssim& \int_0^\infty f^*_\mu(t) \log_1^{m+1/q} \frac{1}{t} \log_3^{1/q'} \frac{1}{t} dt\\ \leqslant& \|f\|_{L^1(\mu)} + \int_0^1 f^*_\mu(t) \log_1^{m+1/q} \frac{1}{t} \log_3^{1/q'} \frac{1}{t} dt\\ \approx& \|\mathcal{M}_\mu f\|_{L^{1,\infty}(\mu)} + \int_0^1 (\mathcal{M}_\mu f)^*_\mu(t) \log_1^{m+1/q-1} \frac{1}{t} \log_3^{1/q'} \frac{1}{t} dt,\end{aligned}$$

where in the last step we need to recall that $(\mathcal{M}_\mu f)^*_\mu(t) = \frac{1}{t}\int_0^t f^*_\mu(s)ds$ and apply Fubini's theorem. With this, we have shown that

$$T : E \cap \underline{L(\log L)^{m+1/q-1}(\log_3 L)^{1/q'}(\mu)} \longrightarrow \Lambda^q_\nu(\omega),$$

where

$$E = \left\{g \in L^{1,\infty}(\mu) : g = \mathcal{M}_\mu f, \text{ for some } f \in L^1_{\mathrm{loc}}(\mu)\right\}.$$

Actually, as we said at the beginning of this section, this is also true if we replace $\mathcal{M}_\mu$ by any sublinear operator $S$ satisfying $(Sf)^*_\mu \approx f^{**}_\mu$. Taking this into account, by[4] [33, Remark 3.1], we actually have that, for every $B > 0$,

$$T : E^B \cap \underline{L(\log L)^{m+1/q-1}(\log_3 L)^{1/q'}(\mu)} \longrightarrow \Lambda^q_\nu(\omega),$$

where

$$E^B = \left\{g \in L^{1,\infty}(\mu) : \exists h \text{ with } \frac{1}{B} g^*_\mu(s) \leqslant h^{**}_\mu(s) \leqslant B g^*_\mu(s)\right\},$$

and using the same argument as in [33, Theorem 3.3 and 3.5] we get the result. □

Just as a remark, we see that the extrapolation result obtained in [33] for operators mapping $L^{p,\infty}(\mu)$ into $L^{p,\infty}(\nu)$ (stated in Theorem 5.11) is still true if we weaken the hypotheses to operators with domain $L^{p,p'}(\mu)$ instead of $L^{p,\infty}(\mu)$. The result can be stated as follows:

**Theorem 5.20.** *Fix a couple of $\sigma$-finite, non-atomic measures $\mu$ and $\nu$, $1 < p_0 < \infty$, $m > 0$, and let*

$$T : L^{p,p'}(\mu) \longrightarrow L^{p,\infty}(\nu)$$

*be a bounded sublinear operator with constant controlled by $\frac{C}{(p-1)^m}$ for every $1 < p \leqslant p_0$. Then,*

$$T : \left[L(\log L)^{m-1} \log_3 L(\mu)\right]_1 \longrightarrow R_m(\nu)$$

*is bounded with $R_m(\nu)$ as in Theorem 5.11.*

[4]At this point is where the hypothesis of non-atomic measures is needed.

*Proof.* The key to this improvement is given by the fact that $\mathcal{M}_\mu$ maps $L^p(\mu)$ into the smaller[5] space $L^{p,p'}(\mu)$ also with a uniform constant independent of $p$ as $p \to 1^+$, and hence, we have that

$$T\mathcal{M}_\mu : L^p(\mu) \longrightarrow L^{p,\infty}(\nu)$$

with constant $\frac{1}{(p-1)^m}$ for values of $p$ near 1 and the proof follows exactly as in [33]. To prove this statement we use Lemma 5.18 with $q = p'$ and check that the constant does not blow up as $p$ tends to $1^+$. □

## 5.5 A different behavior of the constant

In view of Theorem 5.5, it is interesting to know if we can apply this theory to operators

$$T : L^{p,\infty}(\mu) \longrightarrow L^{p,\infty}(\nu)$$

whose boundedness constant behaves like

$$\log_1\left(\frac{1}{p-1}\right)\frac{1}{p-1},$$

instead of $\frac{1}{(p-1)^m}$. The following proposition presents a slight modification of Theorem 5.11 so that we can apply it to an operator of this type.

**Theorem 5.21.** *Let $1 < p_0 < \infty$. Fix $\mu$ and $\nu$ two $\sigma$-finite measures and let $T$ be a sublinear operator. If we define*

$$C_p \approx \log_1\left(\frac{1}{p-1}\right)\frac{1}{p-1},$$

*then:*

*(i) Antonov type: If $T : L^p(\mu) \longrightarrow L^{p,\infty}(\nu)$ is bounded for every $1 < p \leqslant p_0$ with constant $C_p$, then*

$$T : L \log L \log_2 L \log_3 L(\mu) \longrightarrow \widehat{R}_1(\nu).$$

*(ii) Carro - Tradacete type: If $T : L^{p,\infty}(\mu) \longrightarrow L^{p,\infty}(\nu)$ is bounded for every $1 < p \leqslant p_0$ with constant $C_p$, then*

$$T : [L \log_2 L \log_3 L(\mu)]_1 \longrightarrow \widehat{R}_1(\nu).$$

*Here, $\widehat{R}_1(\nu)$ is the space of $\nu$-measurable functions such that*

$$\|f\|_{\widehat{R}_1(\nu)} = \sup_{t>0} \frac{t f_\nu^*(t)}{\log_1 t \log_2 t} < \infty.$$

[5] In [33] the authors use that $\mathcal{M}_\mu$ maps $L^p(\mu)$ into $L^{p,\infty}(\mu) \supsetneq L^{p,p'}(\mu)$ uniformly in $p \approx 1$.

*Idea of the proof.* The first step in the proof of Antonov's result (which goes exactly as the proof of Theorem 5.16) is obtaining an estimate for integrable, bounded functions $f$. In this case, we would get

$$(Tf)_\nu^*(t) \leqslant \inf_{1<p\leqslant p_0} \log_1\left(\frac{1}{p-1}\right)\frac{1}{p-1}\left(\frac{\|f\|_{L^1(\mu)}}{t}\right)^{1/p}.$$

Using Remark 5.4, we can compute the infimum and

$$(Tf)_\nu^*(t) \lesssim \frac{\|f\|_{L^1(\mu)}}{t}\log_1\frac{t}{\|f\|_{L^1(\mu)}}\log_2\frac{t}{\|f\|_{L^1(\mu)}}.$$

Hence

$$\|Tf\|_{\widehat{R}_1(\nu)} = \sup_{t>0}\frac{t(Tf)_\nu^*(t)}{\log_1 t\log_2 t} \lesssim D(\|f\|_{L^1(\mu)}),$$

where $D(t) = t\log_1\frac{1}{t}\log_2\frac{1}{t}$. The proof now continues as in the classical case but with this new function $D$ that adds the $\log_2$-factor to the outcome. Concerning $(ii)$, the idea (at least when $\mu=\nu$ are the Lebesgue measure, otherwise replace $M$ by $\mathcal{M}_\mu$ and argue as in Theorem 5.19), is to apply Antonov's result to the composition $TM$, where $M$ is the Hardy-Littlewood maximal operator. In our case, $TM$ maps $L^p$ into $L^{p,\infty}$ with constant $C_p$, so we need $(i)$ instead of Antonov's classical theorem in order to conclude that

$$TM : L\log_1 L\log_2 L\log_3 L \longrightarrow \widehat{R}_1$$

is bounded. If we write what this means, we have

$$\|T(Mf)\|_{\widehat{R}_1} \lesssim \int_0^\infty f^*(t)\log_1\frac{1}{t}\log_2\frac{1}{t}\log_3\frac{1}{t}dt \lesssim \|Mf\|_{1,\infty} + \int_0^1 (Mf)^*(t)\log_2\frac{1}{t}\log_3\frac{1}{t}dt.$$

Therefore, we have shown that if $E$ is the set of functions $g\in L^{1,\infty}$ such that $g = Mf$ for some locally integrable function $f$, then

$$T : E \cap \underline{L\log_2 L\log_3 L} \longrightarrow \widehat{R}_1. \tag{5.12}$$

From this point on, the proof follows exactly as in [33], but now (5.12) translates into boundedness for functions in the space $[L\log_2 L\log_3 L]_1$. □

## 5.6 Back to the Rubio de Francia setting

Recall that, at the beginning of this chapter, we investigated the behavior in $p$ of the boundedness constants for operators related to the extrapolation theory of Rubio de Francia. Let us see what we can deduce from that. We will start with the classical case of the $A_p$ theory. The first natural step is to use Yano's theorem in its original $L^p \to L^p$ version (see Section 5.2) with the boundedness constant (5.1) that comes from Rubio de Francia's extrapolation. The (standard) result is the following:

**Theorem 5.22.** *Let* $1 < p_0 < \infty$, $\beta > 0$ *and let* $T$ *be a sublinear operator such that*

$$T : L^{p_0}(w) \longrightarrow L^{p_0}(w)$$

*is bounded for every* $w \in A_{p_0}$ *with constant* $C_{p_0}\|w\|_{A_{p_0}}^{\beta}$. *Then,*

$$T : L(\log L)^{\beta(p_0-1)}(\mathbb{R}^n) \longrightarrow E_{\beta(p_0-1)}(\mathbb{R}^n)$$

*is also bounded.*

Next, we will start from a restricted weak-type $(p_0, p_0)$ boundedness instead and extrapolate an $L^{p,\infty} \to L^{p,\infty}$ estimate. The class of operators to which these two results apply is the same, since interpolation together with Rubio de Francia's extrapolation and the Reverse Hölder property of $A_p$ weights yield that a sublinear operator of restricted weak-type $(p_0, p_0)$ for every weight in $A_{p_0}$ is also of strong-type $(p_0, p_0)$ for every weight in this class. However, their boundedness constant can improve significantly, as we pointed out in (5.2) and (5.3) with the maximal operator $M$.

**Theorem 5.23.** *Let* $1 < p_0 < \infty$, $\alpha > 0$ *and let* $T$ *be a sublinear operator such that*

$$T : L^{p_0,1}(w) \longrightarrow L^{p_0,\infty}(w)$$

*is bounded for every* $w \in A_{p_0}$ *with constant* $C_{p_0}\|w\|_{A_{p_0}}^{\alpha}$. *Then, for every* $u \in A_1$,

$$T : \left[L(\log L)^{(\alpha+1)(p_0-1)} \log_3 L(u)\right]_1 \longrightarrow R_{(\alpha+1)(p_0-1)}(u)$$

*is also bounded.*

*Proof.* We just need to use Theorem 5.2 to conclude that such an operator maps

$$T : L^{p,\infty}(u) \longrightarrow L^{p,\infty}(u)$$

with constant behaving like $\frac{1}{(p-1)^{\alpha(p_0-1)+p_0}}$, and then use Theorem 5.11 to extrapolate. □

If we had $\alpha = \beta$, then

$$\left[L(\log L)^{(\alpha+1)(p_0-1)} \log_3 L(\mathbb{R}^n)\right]_1 \subsetneq L(\log L)^{\alpha(p_0-1)}(\mathbb{R}^n),$$

but as we mentioned above, for a given operator $T$, the value of $\alpha$ in Theorem 5.23 might be much better (i.e. smaller) than the $\beta$ in Theorem 5.22. Moreover, notice that Theorem 5.23 is valid for every $u \in A_1$, and not just for the Lebesgue measure. Let us give an example.

**Example 5.24.** Consider $M^k = M \circ \overset{k}{\cdots} \circ M$, with $k \geqslant 2$. This is an operator that is under the hypotheses of Rubio de Francia's theorem but is not bounded from $L^1$ to $L^{1,\infty}$, not even in the unweighted case. Therefore, it is interesting to see what we can obtain at the endpoint. We know that, for every $p > 1$ and $w \in A_p$,

(i) $\|M^k\|_{L^p(w)\to L^p(w)} \lesssim \frac{\|w\|_{A_p}^{\frac{k}{p-1}}}{(p-1)^k}$,

(ii) $\|M^k\|_{L^{p,1}(w)\to L^{p,\infty}(w)} \leqslant \|M^k\|_{L^p(w)\to L^{p,\infty}(w)} \lesssim \frac{\|w\|_{A_p}^{\frac{1}{p}+\frac{k-1}{p-1}}}{(p-1)^{k-1}}$.

With (i), we can either apply Yano's theorem directly with $u = 1$, or Theorem 5.22, with $p_0 > 1$ and $\beta = \frac{k}{p_0-1}$. In both cases, we get that

$$M^k : L(\log L)^k(\mathbb{R}^n) \longrightarrow E_k(\mathbb{R}^n).$$

With (ii), we can apply Antonov's theorem only when $u = 1$, and conclude that

$$M^k : L(\log L)^{k-1}\log_3 L(\mathbb{R}^n) \longrightarrow R_{k-1}(\mathbb{R}^n).$$

However, Theorem 5.23 admits any $u \in A_1$, and, for every $\varepsilon > 0$, starting from $p_0 > 1$ such that $\varepsilon = \frac{p_0-1}{p_0} + p_0 - 1$, and $\alpha = \frac{1}{p_0} + \frac{k-1}{p_0-1}$, we get that

$$M^k : \left[L(\log L)^{k-1+\varepsilon}\log_3 L(u)\right]_1 \longrightarrow R_{k-1+\varepsilon}(u).$$

Notice that, for $u = 1$, the best of the three conclusions is the one coming from Antonov's theorem, since the space $L(\log L)^{k-1}\log_3 L(\mathbb{R}^n)$ is larger than both $L(\log L)^k(\mathbb{R}^n)$ and $\left[L(\log L)^{k-1+\varepsilon}\log_3 L(\mathbb{R}^n)\right]_1$. However, by means of Theorem 5.23, we are able to obtain endpoint results when $u \neq 1$.

This idea of finding an optimal relation between the theories of Rubio de Francia and Yano has been gathered and further developed in [25]. The other computation that we carried out at the beginning of this chapter was in Theorem 5.5, where we saw that an operator $T$ that was of restricted weak-type $(p_0, p_0)$ for every weight in $\widehat{A}_{p_0}$ was bounded from $L^{p,\infty}(u)$ into itself for every $u \in A_1$ and $p$ close to 1 with constant behaving like

$$\log_1\left(\frac{1}{p-1}\right)\frac{1}{p-1}.$$

Now, we can use the extrapolation result in Section 5.5 to conclude the following:

**Theorem 5.25.** *Let $1 < p_0 < \infty$, and let $T$ be a sublinear operator such that*

$$T : L^{p_0,1}(w) \longrightarrow L^{p_0,\infty}(w)$$

*is bounded for every $w \in \widehat{A}_{p_0}$ with constant $\varphi(\|w\|_{\widehat{A}_{p_0}})$, where $\varphi$ is an increasing function on $(0,\infty)$. Then, for every $u \in A_1$,*

$$T : [L\log_2 L\log_3 L(u)]_1 \longrightarrow \widehat{R}_1(u).$$

*Proof.* This is a direct consequence of Theorem 5.5 and Theorem 5.21. $\square$

The next result will allow us to obtain a better endpoint estimate for operators under the hypotheses of Theorem 5.25. As a first approach, we will start by considering only monotone operators, and then, we will see that in fact, it can be extended to general sublinear operators.

**Definition 5.26.** *We will say that an operator $T$ is monotone if for every $0 \leqslant f \leqslant g$, it holds that $|Tf| \leqslant |Tg|$.*

**Proposition 5.27.** *Fix an arbitrary weight $u$ on $\mathbb{R}^n$ and let $T$ be a sublinear, monotone operator such that*

*(i)* $\|T\chi_E\|_{L^{1,\infty}(u)} \leqslant u(E)$ *for every measurable set $E$, and*

*(ii)* $T : L^1(u) \cap L^\infty \longrightarrow L^{1,\infty}_{\mathrm{loc}}(u)$ *is bounded.*

*Then,*

$$T : L\log_2 L(u) \longrightarrow L^{1,\infty}_{\mathrm{loc}}(u).$$

*Proof.* Take a non-negative function $f = f_0 + f_1$, with $f_0 = f\chi_{\{f\leqslant 1\}}$ and $f_1 = f\chi_{\{f>1\}}$. By density, we can assume without loss of generality that $f \in L^\infty$. Using that $T$ is sublinear, we have that

$$|Tf| \leqslant |Tf_0| + |Tf_1|,$$

and hence, $\|Tf\|_{L^{1,\infty}_{\mathrm{loc}}(u)} \lesssim \|Tf_0\|_{L^{1,\infty}_{\mathrm{loc}}(u)} + \|Tf_1\|_{L^{1,\infty}_{\mathrm{loc}}(u)}$. Using (ii), we get that for $f_0$,

$$\|Tf_0\|_{L^{1,\infty}_{\mathrm{loc}}(u)} \lesssim \|f_0\|_{L^1(u)} + \|f_0\|_\infty \leqslant \|f\|_{L\log_2 L(u)} + 1.$$

To deal with $f_1$, we write

$$f_1 = \sum_{k=1}^{\infty} f\chi_{E_k} \approx \sum_{k=1}^{\infty} 2^k \chi_{E_k}, \tag{5.13}$$

with $E_k = \{2^{k-1} < f \leqslant 2^k\}$. Since $f$ is bounded, this series is in fact finite. Using that $T$ is monotone and sublinear, we have that

$$|Tf_1| \approx \left| T\left(\sum_{k=1}^{\infty} 2^k \chi_{E_k}\right)\right| \leqslant \sum_{k=1}^{\infty} 2^k |T\chi_{E_k}|,$$

and therefore, using that $\mathrm{Galb}(L^{1,\infty}(u)) = \ell\log\ell$, (i), and $u(E_k) \leqslant \lambda_f^u(2^k)$, we conclude that

$$\begin{aligned}
\|Tf_1\|_{L^{1,\infty}_{\mathrm{loc}}(u)} &\lesssim \left\| \sum_{k=1}^{\infty} 2^k |T\chi_{E_k}| \right\|_{L^{1,\infty}_{\mathrm{loc}}(u)} \lesssim \sum_{k=1}^{\infty} \log_1(k) 2^k \|T\chi_{E_k}\|_{L^{1,\infty}(u)} \\
&\leqslant \sum_{k=1}^{\infty} \log_1(k) 2^k \lambda_f^u(2^k) \lesssim \int_0^\infty \lambda_f^u(s)\log_2(s)ds \lesssim \|f\|_{L\log_2 L(u)}.
\end{aligned}$$

Summing up, we have shown that

$$\|Tf\|_{L^{1,\infty}_{\mathrm{loc}}(u)} \lesssim \|f\|_{L\log_2 L(u)} + 1,$$

and the result follows by linearity (changing $f$ by $\alpha f$ and letting $\alpha$ tend to infinity). $\square$

Notice that the hypothesis that $T$ must be monotone is needed because in (5.13) we have an equivalence instead of an equality. If we want to avoid this monotonicity assumption, we cannot use the standard dyadic decomposition of a function. In [110], however, the author presents the following decomposition of a non-negative function $f$ in an inductive way:

$$f(x) = \sum_{j=1}^{\infty}\sum_{k\in\mathbb{Z}} 2^k \chi_{E_{k,j}}(x) \quad \text{a.e. } x \in \mathbb{R}^n, \tag{5.14}$$

where the sets $E_{k,j}$ depend on $f$ and are defined in such a way that, for every weight $u$,

$$u(E_{k,j}) \leqslant \lambda_f^u(2^{k+j}).$$

We will not give the details of the exact construction, but the idea is very straightforward (see [110, Lemma 4]). With this identity at hand, let us see how we can get rid of the monotonicity assumption in the previous proposition:

**Theorem 5.28.** *Fix an arbitrary weight $u$ on $\mathbb{R}^n$ and let $T$ be a sublinear operator such that*

*(i) $\|T\chi_E\|_{L^{1,\infty}(u)} \leqslant u(E)$ for every measurable set $E$, and*

*(ii) $T : L^1(u) \cap L^\infty \longrightarrow L^{1,\infty}_{\mathrm{loc}}(u)$ is bounded.*

*Then,*

$$T : L\log_2 L(u) \longrightarrow L^{1,\infty}_{\mathrm{loc}}(u).$$

*Proof.* Take a non-negative function $f = f_0 + f_1$, with $f_0 = f\chi_{\{f\leqslant 1\}}$ and $f_1 = f\chi_{\{f>1\}}$. As before, we have that $\|Tf\|_{L^{1,\infty}_{\mathrm{loc}}(u)} \lesssim \|Tf_0\|_{L^{1,\infty}_{\mathrm{loc}}(u)} + \|Tf_1\|_{L^{1,\infty}_{\mathrm{loc}}(u)}$, and for the term with $f_0$, we use $(ii)$ and $\|f_0\|_\infty \leqslant 1$ to get

$$\|Tf_0\|_{L^{1,\infty}_{\mathrm{loc}}(u)} \lesssim \|f\|_{L\log_2 L(u)} + 1.$$

Now, to deal with $f_1$, we make use of (5.14), which states that

$$f_1(x) = \sum_{j=1}^{\infty}\sum_{k\in\mathbb{Z}} 2^k \chi_{E_{k,j}}(x), \quad \text{a.e. } x \in \mathbb{R}^n.$$

For every $N > 0$, set $f_1^N$ to be the following truncated series:

$$f_1^N(x) = \sum_{j=1}^{N} \sum_{|k|\leqslant N} 2^k \chi_{E_{k,j}}(x).$$

Since this is an equality, we do not need the monotonicity of $T$ in order to obtain

$$\|Tf_1^N\|_{L^{1,\infty}_{\mathrm{loc}}(u)} \lesssim \sum_{j=1}^{\infty} \log_1(j) \sum_{k\in\mathbb{Z}} 2^k \log_1(|k|)\lambda^u_{f_1}(2^{k+j}). \tag{5.15}$$

Recall that $u(E_{k,j}) \leqslant \lambda^u_{f_1}(2^{k+j})$ and that the logarithmic terms come from the fact that $\mathrm{Galb}(L^{1,\infty}(u)) = \ell \log \ell$. Here we have used that the series defining $f_1^N$ is finite to apply the sublinearity of $T$, and once this is done, we majorize the result by the whole series. Next, fix $j \geqslant 1$ and split the inner sum into three pieces: $I_j^1 + I_j^2 + I_j^3$. The first one will be

$$I_j^1 = \sum_{k<-j} 2^k \log_1(|k|)\lambda^u_{f_1}(2^{k+j}) \leqslant \|f\|_{L\log_2 L(u)} \sum_{k=j+1}^{\infty} 2^{-k} \log_1(k).$$

Here we used that, since $f_1 > 1$, we have that $\lambda^u_{f_1}(2^{k+j}) \leqslant \|f\|_{L\log_2 L(u)}$ whenever $k < -j$, because in this case,

$$\lambda^u_{f_1}(2^{k+j}) = \lambda_{f_1}(1) \leqslant \|f\|_{L\log_2 L(u)}.$$

The second term we need to consider is

$$\begin{aligned} I_j^2 &= \sum_{k=-j}^{0} 2^k \log_1(|k|)\lambda^u_{f_1}(2^{k+j}) = 2^{-j} \sum_{k=-j}^{0} 2^{k+j} \log_1(|k|)\lambda^u_{f_1}(2^{k+j}) \\ &\leqslant 2^{-j}\|f\|_{L\log_2 L(u)} \sum_{k=0}^{j} \log_1(k). \end{aligned}$$

Here we just used that $t\lambda^u_f(t) \leqslant \|f\|_{L\log_2 L(u)}$, for every $t > 0$. Finally,

$$\begin{aligned} I_j^3 &= \sum_{k=1}^{\infty} 2^k \log_1(k)\lambda^u_{f_1}(2^{k+j}) \leqslant 2^{-j} \sum_{k=1}^{\infty} 2^{k+j} \log_1(k+j)\lambda^u_{f_1}(2^{k+j}) \\ &\lesssim 2^{-j} \int_0^\infty \lambda^u_f(s)\log_2(s)ds \lesssim 2^{-j}\|f\|_{L\log_2 L(u)}. \end{aligned}$$

Now we go back to (5.15) and using the bounds for $I_j^m$, $m = 1, 2, 3$, we conclude that

$$\|Tf_1^N\|_{L^{1,\infty}_{\mathrm{loc}}(u)} \lesssim \|f\|_{L\log_2 L(u)} \sum_{j=1}^{\infty} \log_1(j) \left( \sum_{k=j+1}^{\infty} 2^{-k}\log_1(k) + 2^{-j}\sum_{k=0}^{j} \log_1(k) + 2^{-j} \right).$$

The second and third terms in parentheses, together with the $\log_1(j)$ term outside, are obviously convergent when $j \geqslant 1$. For the first one, a simple rearrangement of the sums shows that

$$\sum_{j=1}^{\infty} \log_1(j) \sum_{k=j+1}^{\infty} 2^{-k} \log_1(k) = \sum_{k=2}^{\infty} 2^{-k} \log_1(k) \sum_{j=1}^{k-1} \log_1(j) \leqslant \sum_{k=2}^{\infty} 2^{-k} k \log_1(k)^2 < \infty,$$

which is exactly the second one but with the indices $k, j$ interchanged. Therefore, we have that $\|Tf_1^N\|_{L^{1,\infty}_{\mathrm{loc}}(u)} \lesssim \|f\|_{L\log_2 L(u)}$. If we show that $f_1^N$ converges to $f_1$ in $L\log_2 L(u)$, then we can conclude $\|Tf_1\|_{L^{1,\infty}_{\mathrm{loc}}(u)} \lesssim \|f\|_{L\log_2 L(u)}$ and hence,

$$\|Tf\|_{L^{1,\infty}_{\mathrm{loc}}(u)} \lesssim \|f\|_{L\log_2 L(u)} + 1.$$

We finish the proof by linearity as in the previous proposition. To show that $f_1^N \to f_1$ in $L\log_2 L(u)$, we observe that the difference $f_1(x) - f_1^N(x)$ decreases to zero for almost every $x \in \mathbb{R}^n$, since $f_1^N$ is a partial sum of a convergent series of positive terms, and this coincides with $f_1$ almost everywhere. In particular, its decreasing rearrangement with respect to $u$ satisfies that

$$(f_1 - f_1^N)^*_u(t) \longrightarrow 0, \quad \text{a.e. } t \in (0, \infty).$$

On the other hand, $|f_1 - f_1^N|$ can be pointwise controlled by $f_1 \in L\log_2 L(u)$, so

$$\left|(f_1 - f_1^N)^*_u(t) \log_2 \frac{1}{t}\right| \leqslant (f_1)^*_u(t) \log_2 \frac{1}{t} \in L^1(0, \infty).$$

Therefore, by the dominated convergence theorem,

$$\|f_1 - f_1^N\|_{L\log_2 L(u)} = \int_0^\infty (f_1 - f_1^N)^*_u(t) \log_2 \frac{1}{t} dt \longrightarrow 0,$$

as $N \to \infty$, so we finish the proof. $\square$

**Corollary 5.29.** *Let $T$ be a sublinear operator such that, for some $1 < p_0 < \infty$ and every $w \in \widehat{A}_{p_0}$,*

$$T : L^{p_0,1}(w) \longrightarrow L^{p_0,\infty}(w)$$

*is bounded, with constant controlled by $\varphi_{p_0}(\|w\|_{\widehat{A}_{p_0}})$ and $\varphi_{p_0}$ an increasing function on $(0, \infty)$. Then, for every $u \in A_1$,*

$$T : L\log_2 L(u) \longrightarrow L^{1,\infty}_{\mathrm{loc}}(u)$$

*is also bounded.*

*Proof.* In [28, Theorem 2.11 and Corollary 2.16], the authors prove that such an operator satisfies, for every $u \in A_1$:

(i) $\|T\chi_E\|_{L^{1,\infty}(u)} \lesssim u(E)$ for every measurable set $E$, and

(ii) $T : L(\log L)^{\varepsilon}(u) \longrightarrow L^{1,\infty}_{\text{loc}}(u)$ is bounded for every $\varepsilon > 0$.

We know that $L^1(u) \cap L^{\infty}$ is continuously embedded in any rearrangement invariant space with respect to the measure given by $u$ (see [5, Chapter 2, Theorem 6.6]). In particular, since $L(\log L)^{\varepsilon}(u)$ is rearrangement invariant, we have that

$$T : L^1(u) \cap L^{\infty} \hookrightarrow L(\log L)^{\varepsilon}(u) \to L^{1,\infty}_{\text{loc}}(u),$$

so we can apply Theorem 5.28 to deduce the desired boundedness. Notice that we would have enough with (ii) for some $\varepsilon > 0$. $\square$

**Remark 5.30.** *This result can be seen as a self-improvement of [28, Corollary 2.16]. In fact, Corollary 5.29 improves Theorem 5.25, that was already stronger than [28, Corollary 2.16], since*

$$L(\log L)^{\varepsilon}(u) \subsetneq [L \log_2 L \log_3 L(u)]_1 \subsetneq L \log_2 L(u).$$

*Obviously, all these endpoint results close to $L^1$ make sense when the operator $T$ is not $(\varepsilon, \delta)$-atomic approximable, because otherwise, we already have that*

$$T : L^1(u) \longrightarrow L^{1,\infty}(u).$$

# Chapter 6

# Pointwise Bounds via Yano's Theory

## 6.1 Introduction

In this last part of the thesis we will present another application of the ideas behind Yano's extrapolation. However, we will no longer deal with weighted estimates in the context of $A_p$ weights. Our goal now is to take advantage of extrapolation techniques to obtain pointwise bounds for integral operators. We will motivate this chapter[1] with an example. In 1917, J. Radon [101] introduced a transformation that reconstructed a function from its projections. Later, in 1972, G. Hounsfield was able to build the first $x$-ray computed tomography scanner using the Radon transform to recover an object from its projection data [74]. The special case in which all projections are identical and hence, a single projection is enough for an exact object reconstruction, was already solved by N. H. Abel [1] in 1826. He used the following integral operator, called the Abel transform:

$$Af(x) = \int_x^\infty \frac{f(t)t}{\sqrt{t^2 - x^2}} dt. \tag{6.1}$$

In many papers dealing with the Abel transform, the starting condition on the function $f$ is that "it decays at infinity faster than $1/t$". Obviously, if the information that we have on the function $f$ is just that $f(t) \lesssim \frac{1}{t}$, then we cannot say anything about $Af$ since $A\left(\frac{1}{t}\right) \equiv \infty$. However, if we assume that the decay of $f$ at infinity is a little faster, namely, that there exists $p_0 > 1$ such that, for every $1 < p \leqslant p_0$ and every $t > 0$,

$$f(t) \leqslant \frac{C}{t^{2-\frac{1}{p}}}, \tag{6.2}$$

then $Af(x) < \infty$ for every $x > 0$ and

$$Af(x) \lesssim \int_x^\infty \frac{1}{\sqrt{t^2 - x^2}\, t^{1-\frac{1}{p}}} dt \lesssim \frac{x^{\frac{1}{p}-1}}{p-1}.$$

[1]The results that we present are gathered in [23].

Therefore, taking infimum over $1 < p \leqslant p_0$ as in Lemma 5.3, we get that, for every $x > 0$,

$$Af(x) \lesssim \log_1 \frac{1}{x}.$$

In this chapter we will prove that we can obtain the same upper bound for $Af(x)$ under a condition on the decay of $f$ at infinity weaker than (6.2). It is clear from the setting that the underlying idea (and hence, the techniques we will use) is the same as in Yano's extrapolation theory.

This problem seems to be of interest even when we are dealing with integral operators of the form

$$T_K f(x) = \int_0^\infty K(x,t) f(t)dt, \tag{6.3}$$

with $K$ a positive kernel. This class of operators includes

$$S_a f(t) = \int_0^\infty a(s) f(st) ds, \tag{6.4}$$

with $a$ being a weight. These operators were first introduced by Braverman [8] and Lai [81] and also studied by Andersen in [2]. In particular, they cover the cases of Hardy operators, Riemann-Liouville, Calderón operator, Laplace and Abel transforms, among many others.

The general setting will be the following: let $w$ be a weight and, as usual, we write $W(t) = \int_0^t w(s)ds$. This weight will be fixed and hence, the constants $C$ (explicit or implicit) appearing in the inequalities of this chapter may depend on it. We will assume that $W(t) > 0$, for every $t > 0$. Moreover, since $W$ is increasing, it is equivalent to a strictly increasing function and hence, we can assume without loss of generality that $W$ has an inverse, that we will denote[2] by:

$$W^{(-1)} : (0, W(\infty)) \longrightarrow (0, \infty).$$

Let us consider positive, measurable functions $f$ satisfying

$$f(t) \lesssim \frac{1}{W(t)}, \quad t \in (0, \infty),$$

and an operator $T_K$ as in (6.3). Obviously, for such an $f$, it holds that

$$T_K f(x) \lesssim \int_0^\infty \frac{K(x,t)}{W(t)} dt = M(x),$$

and hence the function $M$ is an upper pointwise bound for $T_K$ on that set of functions. However, on many occasions, $M \equiv \infty$ and no interesting information can be obtained

[2] We use the notation $W^{(-1)}$ for the inverse function because we will keep $W^{-1}$ to denote $\frac{1}{W}$.

without assuming some extra condition. As in the example of the Abel transform, we will assume that $M \equiv \infty$ but, for every $1 < p \leqslant p_0$,

$$\int_0^\infty \frac{K(x,t)}{W(t)^{1/p}} dt < \infty.$$

In fact, we will need to have some control on how this quantity blows up when $p$ is close to 1 (as it happened in Yano's theory with the boundedness constants), so to be precise, we will assume that it can be controlled by $\frac{1}{(p-1)^m}$. That is, there exists $m > 0$ such that, for every $x$,

$$U(x) := \sup_{1<p\leqslant p_0} (p-1)^{mp} \left( \int_0^\infty \frac{K(x,t)}{W(t)^{1/p}} dt \right)^p < \infty. \tag{6.5}$$

In Section 6.3, we will see that this is the case of many other interesting examples. Since our goal is to find pointwise upper bounds, we will work with the following normed spaces:

**Definition 6.1.** *We say that a measurable function $f \in B(W)$ if and only if $W^{-1}$ is a pointwise upper bound for $f$, that is*

$$B(W) := \left\{ f \ \textit{measurable} \ : \|f\|_{B(W)} = \sup_{t>0} f(t)W(t) < \infty \right\}.$$

We observe that if (6.5) is satisfied, then clearly

$$\int_1^\infty \frac{K(x,t)}{W(t)} dt \lesssim \inf_{1<p\leqslant p_0} \frac{U(x)^{1/p}}{(p-1)^m} \lesssim U(x) \Big( \log_1 \frac{1}{U(x)} \Big)^m,$$

but this computation fails completely whenever we are dealing with values of the variable $t$ close to zero. Hence, we want to find conditions on the functions $f \in B(W)$ so that the above bound remains true for the whole operator, that is

$$T_K f(x) \lesssim U(x) \Big( \log_1 \frac{1}{U(x)} \Big)^m.$$

## 6.2 Main Results

In order to give the proof of our main theorem, we need the following result. This can be regarded as a variant of Antonov's theorem (see Theorem 5.7), since the spaces $B(U^{-1/p})$ are closely related to $L^{p,\infty}$ spaces. However, since we do not use decreasing rearrangements to define them, here the limiting space $U^{-1}$ as $p$ tends to 1 (at least formally) is still normed, so we can avoid the use of the Galb of quasi-normed spaces that made the extra $\log_3$-term appear. In fact, the proof is more similar to Yano's theorem.

**Proposition 6.2.** *If $T$ is a sublinear operator such that*

$$T : L^p(w) \longrightarrow B(U^{-1/p})$$

*is bounded, for every $1 < p \leqslant p_0$, with constant less than or equal to $\frac{C}{(p-1)^m}$, then*

$$T : L(\log L)^m(w) \longrightarrow B(U_m^{-1})$$

*is bounded with*

$$U_m(t) = U(t)\Big(\log_1 \frac{1}{U(t)}\Big)^m. \tag{6.6}$$

*Proof.* The proof follows the standard scheme of Yano's extrapolation theorem in its modern version (see [19, 30, 127]) but we include it for the sake of completeness. Let $f$ be a positive function satisfying $\|f\|_\infty \leqslant 1$. Then,

$$\sup_{t>0} Tf(t)U^{-1/p}(t) \lesssim \frac{\|f\|_{L^p(w)}}{(p-1)^m} \leqslant \frac{\|f\|_{L^1(w)}^{1/p}}{(p-1)^m},$$

and hence

$$\begin{aligned} Tf(t) &\lesssim \inf_{1<p\leqslant p_0} \frac{1}{(p-1)^m} \left(\|f\|_{L^1(w)}U(t)\right)^{1/p} \lesssim \|f\|_{L^1(w)}U(t)\left(\log_1 \frac{1}{\|f\|_{L^1(w)}U(t)}\right)^m \\ &\lesssim \|f\|_{L^1(w)}\Big(\log_1 \frac{1}{\|f\|_{L^1(w)}}\Big)^m U_m(t). \end{aligned}$$

From here, it follows that, if $\|f\|_\infty \leqslant 1$, then

$$\|Tf\|_{B(U_m^{-1})} \lesssim D_m(\|f\|_{L^1(w)}), \tag{6.7}$$

where $D_m(s) = s\left(\log_1 \frac{1}{s}\right)^m$. Now, for a bounded function with $|f| \geqslant 1$, whenever $f \neq 0$, we can decompose

$$f = \sum_{n\geqslant 0} 2^{n+1} f_n,$$

where $f_n = 2^{-(n+1)} f\chi_{E_n}$ and $E_n = \{2^n < f \leqslant 2^{n+1}\}$. Clearly $\|f_n\|_{L^1(w)} \leqslant \lambda_f^w(2^n)$, and together with the fact that $\|f_n\|_\infty \leqslant 1$ and $B(U_m^{-1})$ is a normed space, we can use (6.7) on every $f_n$ to conclude that

$$\begin{aligned} \|Tf\|_{B(U_m^{-1})} &\lesssim \sum_{n=0}^\infty 2^n D_m(\|f_n\|_{L^1(w)}) \lesssim \sum_{n=0}^\infty 2^n D_m(\lambda_f^w(2^n)) \\ &\lesssim \int_0^\infty D_m(\lambda_f^w(y))dy \approx \|f\|_{L(\log L)^m(w)}, \end{aligned}$$

as we wanted to see. We extend this estimate to a general function (not necessarily bounded) by a density argument. $\square$

**Lemma 6.3.** *If $f$ is a decreasing function, then*

$$\|f\|_{L(\log L)^m(w)} = \int_0^\infty f(t)\Big(\log_1 \frac{1}{W(t)}\Big)^m w(t)dt.$$

*Proof.* We have that, for every $s > 0$,

$$\lambda_f^w(s) = \int_{\{x>0: f(x)>s\}} w(x)dx = \int_0^{\lambda_f(s)} w(x)dx = W(\lambda_f(s)).$$

Therefore,

$$\begin{aligned} f_w^*(t) &= \inf\{s > 0 : \lambda_f^w(s) \leqslant t\} = \inf\{s > 0 : W(\lambda_f(s)) \leqslant t\} \\ &= \inf\{s > 0 : \lambda_f(s) \leqslant W^{(-1)}(t)\} = f^*(W^{(-1)}(t)) = f(W^{(-1)}(t)), \end{aligned}$$

and hence,

$$\begin{aligned} \|f\|_{L(\log L)^m(w)} &= \int_0^\infty f_w^*(t) \left(\log_1 \frac{1}{t}\right)^m dt = \int_0^\infty f(W^{(-1)}(t)) \left(\log_1 \frac{1}{t}\right)^m dt \\ &= \int_0^\infty f(t)\Big(\log_1 \frac{1}{W(t)}\Big)^m w(t)dt. \end{aligned}$$

□

The following result follows immediately by Hölder's inequality:

**Lemma 6.4.** *Let $w$ be a weight on $(0,\infty)$ and let $P_w$ be the generalized Hardy operator*

$$P_w f(x) = \frac{1}{W(x)} \int_0^x f(s)w(s)ds.$$

*Then,*

$$P_w : L^p(w) \longrightarrow B(W^{1/p})$$

*is bounded with constant* 1.

Now, we are ready to prove the main result of this chapter, following the ideas introduced in [33]. Again, this can be regarded as an extrapolation similar to that in Theorem 5.11 or Theorem 5.19, where first we make a composition with a suitable maximal operator (in this case, $P_w$), and then we use an Antonov-like result (now, Proposition 6.2):

**Theorem 6.5.** *Let $T_K$ be defined as in* (6.3) *and satisfying*

$$U(x) := \sup_{1<p\leqslant p_0} (p-1)^{mp} \left(\int_0^\infty \frac{K(x,t)}{W(t)^{1/p}} dt\right)^p < \infty.$$

*The function $U_m$ will stand for the expression in* (6.6). *Then, for every $x$,*

$$T_K f(x) \lesssim \|f\|_{D_m(W)} U_m(x),$$

*where*

$$\|f\|_{D_m(W)} = \|f\|_{B(W)} + \int_0^1 \sup_{s>0} \left( \min\left( \frac{W(s)}{W(t)}, 1 \right) f(s) \right) \left( \log_1 \frac{1}{W(t)} \right)^{m-1} w(t)dt.$$

*Proof.* By (6.5), we have that

$$T_K : B(W^{1/p}) \longrightarrow B(U^{-1/p})$$

with constant less than or equal to $(p-1)^{-m}$ and hence, by the previous lemma,

$$T_K \circ P_w : L^p(w) \longrightarrow B(U^{-1/p})$$

is bounded with the same behavior of the constant. Then, applying Proposition 6.2, we obtain that

$$T_K \circ P_w : L(\log L)^m(w) \longrightarrow B(U_m^{-1})$$

is bounded. Now, since for $t$ small enough, say $t \leqslant \delta < 1$, it is easy to see that

$$\left( \log_1 \frac{1}{W(t)} \right)^m \approx \int_t^1 \left( \log_1 \frac{1}{W(s)} \right)^{m-1} \frac{w(s)}{W(s)} ds,$$

we have that

$$\int_0^\delta g(t) \left( \log_1 \frac{1}{W(t)} \right)^m w(t)dt \lesssim \int_0^1 P_w g(t) \left( \log_1 \frac{1}{W(t)} \right)^{m-1} w(t)dt.$$

Therefore, by Lemma 6.3, if $g$ is decreasing,

$$\begin{aligned} \sup_{t>0} \frac{T_K(P_w g)(t)}{U_m(t)} &\lesssim \int_0^\infty g(t) \left( \log_1 \frac{1}{W(t)} \right)^m w(t)dt \\ &\lesssim \|g\|_{L^1(w)} + \int_0^\delta g(t) \left( \log_1 \frac{1}{W(t)} \right)^m w(t)dt \\ &\lesssim \|P_w g\|_{B(W)} + \int_0^1 P_w g(t) \left( \log_1 \frac{1}{W(t)} \right)^{m-1} w(t)dt. \end{aligned} \tag{6.8}$$

Let us now assume that $f \in B(W)$ is a decreasing function satisfying

$$\int_0^1 \frac{\sup_{s \leqslant t} W(s) f(s)}{W(t)} \left( \log_1 \frac{1}{W(t)} \right)^{m-1} w(t)dt < \infty.$$

Set $H(t) = \sup_{s\leqslant t} W(s)f(s)$. With this definition, it is clear that $H$ is an increasing function such that $H(0) = 0$ and $\frac{H(t)}{W(t)}$ is decreasing, so we have that $H\left(W^{(-1)}(t)\right)$ is quasi-concave on $(0, W(\infty))$. It is known (see [5, Chapter 2]) that in this case, there exists $h$ decreasing such that $H\left(W^{(-1)}(t)\right) \approx \int_0^t h(s)ds$, with equivalence constant 2, so by a change of variables, there exists $g$ decreasing such that

$$\frac{1}{2}H(t) \leqslant \int_0^t g(s)w(s)ds \leqslant 2H(t).$$

On the other hand,

$$f(t) \leqslant \frac{H(t)}{W(t)} \approx \frac{\int_0^t g(s)w(s)ds}{W(t)} = P_w g(t),$$

and thus $T_K f(t) \lesssim T_K(P_w g)(t)$. Therefore, using (6.8)

$$\sup_{t>0} \frac{T_K f(t)}{U_m(t)} \lesssim \sup_{t>0} \frac{T_K(P_w g)(t)}{U_m(t)} \lesssim \|P_w g\|_{B(W)} + \int_0^1 P_w g(t)\Big(\log_1 \frac{1}{W(t)}\Big)^{m-1} w(t)dt.$$

Since

$$\|P_w g\|_{B(W)} = \sup_{t>0} W(t)\frac{\int_0^t g(s)w(s)ds}{W(t)} \approx \sup_{t>0} H(t) = \|f\|_{B(W)},$$

and

$$P_w g(t) \approx \frac{\sup_{s\leqslant t} W(s)f(s)}{W(t)},$$

we obtain that, for every decreasing function $f \in B(W)$,

$$\sup_{t>0} \frac{T_K f(t)}{U_m(t)} \lesssim \|f\|_{B(W)} + \int_0^1 \frac{\sup_{s\leqslant t} W(s)f(s)}{W(t)}\Big(\log_1 \frac{1}{W(t)}\Big)^{m-1} w(t)dt. \tag{6.9}$$

Finally, if we take a general function $f \in B(W)$, we can consider its least decreasing majorant

$$F(t) = \sup_{r\geqslant t} f(r).$$

We have that $F \in B(W)$ is decreasing and $f \leqslant F$. Hence, $T_K f(x) \leqslant T_K F(x)$ and the result follows immediately applying (6.9) to the function $F$, since we have the equality of norms

$$\|F\|_{B(W)} = \sup_{t>0} F(t)W(t) = \sup_{t>0}\sup_{r\geqslant t} f(r)W(t) = \sup_{t>0} f(t)W(t) = \|f\|_{B(W)},$$

and

$$\begin{aligned}
\frac{\sup_{s\leqslant t} W(s)F(s)}{W(t)} &= \frac{\sup_{s\leqslant t} W(s)\sup_{r\geqslant s} f(r)}{W(t)} = \frac{\sup_{r>0} f(r)W(\min(t,r))}{W(t)} \\
&= \frac{\max\left(\sup_{s\leqslant t} f(s)W(s), W(t)\sup_{s\geqslant t} f(t)\right)}{W(t)} \\
&= \sup_{s>0}\left(\min\left(\frac{W(s)}{W(t)},1\right) f(s)\right).
\end{aligned}$$

□

Notice that the natural setting for Theorem 6.5 is that of decreasing functions, and we just extend it to general functions by considering their least decreasing majorants. In fact, if $f$ is itself decreasing, the expression for $\|f\|_{D_m(W)}$ can be written in a simpler way. The next corollary is just the result that we get in this setting and corresponds to the estimate in (6.9):

**Corollary 6.6.** *Under the hypotheses of Theorem 6.5 we have that, for every decreasing function $f$,*

$$T_K f(x) \lesssim \|f\|_{D_m(W)} U_m(x),$$

*where*

$$\|f\|_{D_m(W)} = \|f\|_{B(W)} + \int_0^1 \frac{\sup_{s\leqslant t} W(s)f(s)}{W(t)}\Big(\log_1 \frac{1}{W(t)}\Big)^{m-1} w(t)dt.$$

Extrapolation results (such as the analogous to Antonov's theorem) for operators that are only defined on the cone of decreasing functions can be found in [29]. Finally, the following corollary gives a bound for the iterative operator of order $n \in \mathbb{N}$, $T_K^n f = T_K(T_K^{n-1} f)$:

**Corollary 6.7.** *Assume that $T_K$ satisfies* (6.5)*, with $U \approx W^{-1}$. Then, for every $n \in \mathbb{N}$, we have that*

$$T_K^n f(x) \lesssim \|f\|_{D_{nm}(W)} \frac{1}{W(x)} \left(\log_1 W(x)\right)^{nm}.$$

*Proof.* Since $T_K$ satisfies (6.5), with $U \approx W^{-1}$, we have that

$$T_K : B(W^{1/p}) \longrightarrow B(W^{1/p}),$$

with constant less than or equal to $(p-1)^{-m}$, so we can iterate to conclude that the same holds for $T_K^n$, with constant controlled by $(p-1)^{-nm}$. The proof now follows as in Theorem 6.5. □

## 6.3 Examples and applications

In this section, we will use Theorem 6.5 on some interesting examples. Obviously, if one is only interested in decreasing functions, all the conditions can be written as in Corollary 6.6 instead.

### 6.3.1 The Abel transform

Let us start by solving the initial question about the Abel transform.

**Corollary 6.8.** *If a positive measurable function $f(t) \lesssim 1/t$ satisfies that*

$$\int_1^\infty \sup_y \left(f(y)y\min(y,t)\right)\frac{dt}{t^2} < \infty \tag{6.10}$$

*then, for every $x > 0$,*

$$Af(x) \lesssim \log_1 \frac{1}{x}.$$

Before giving the proof, we should emphasize the fact that it is very easy to verify that condition (6.10) is weaker than (6.2).

*Proof.* First of all, making a change of variables, it is immediate to see that, if $g(s) = f(\frac{1}{s})\frac{1}{s^2}$ and

$$T_K g(x) = \int_0^x \frac{g(s)}{\sqrt{x^2 - s^2}} ds,$$

then, for every $x > 0$,

$$Af(x) = \frac{1}{x} T_K g\Big(\frac{1}{x}\Big). \tag{6.11}$$

On the other hand, we have that

$$\sup_{1<p\leqslant 2} (p-1)^p \left(\int_0^x \frac{1}{\sqrt{x^2 - s^2} s^{1/p}} ds\right)^p \approx \frac{1}{x} < \infty,$$

and therefore, applying Theorem 6.5, we get

$$T_K g(x) \lesssim \frac{\log_1 x}{x},$$

whenever $g \in B(W)$ with $W(t) = t$ and

$$\int_0^1 \sup_{s>0} \left(g(s)\min\left(\frac{s}{t}, 1\right)\right) dt < \infty.$$

The result now follows rewriting this condition in terms of $f$ and using (6.11). $\square$

### 6.3.2 The Riemann-Liouville operator

Given $\alpha > 0$, let us consider the Riemann-Liouville operator

$$R_\alpha f(x) = \int_0^x f(t)(x-t)^{\alpha-1}dt.$$

This operator, and in particular its boundedness in the context of weighted $L^p$ spaces, has been studied in many papers such as [29], [93] or [100]. Our contribution is the following:

**Corollary 6.9.** *Fix $\alpha > 0$. If a positive measurable function $f(t) \lesssim 1/t$ satisfies that*

$$\int_0^1 \sup_{s>0} \left( \min\left(\frac{s}{t}, 1\right) f(s) \right) dt < \infty,$$

*then, for every $x > 0$,*

$$R_\alpha f(x) \lesssim x^{\alpha-1}(\log_1 x).$$

*Proof.* Making the change of variables $y = \frac{t}{x}$, we have that

$$R_\alpha f(x) = x^\alpha \int_0^1 (1-y)^{\alpha-1} f(yx)dy := x^\alpha I_\alpha f(x),$$

and hence

$$\sup_{1<p\leqslant 2} (p-1)^p \left( I_\alpha \Big(\frac{1}{y^{1/p}}\Big)(x) \right)^p \lesssim \frac{1}{x}.$$

Consequently, if we take $W(t) = t$ and $U(t) = \frac{1}{t}$, we have that $I_\alpha$ is under the hypotheses of Theorem 6.5 and therefore

$$I_\alpha f(x) \lesssim \frac{\log_1 x}{x},$$

whenever $f(t) \lesssim 1/t$ and it satisfies that

$$\int_0^1 \sup_{s>0} \left( \min\left(\frac{s}{t}, 1\right) f(s) \right) dt < \infty.$$

Hence, under these conditions on $f$, it holds that, for every $x > 0$,

$$R_\alpha f(x) \lesssim x^{\alpha-1}(\log_1 x).$$

□

### 6.3.3 Iterative operators

Observe that in the two previous examples, the function $U$ coincides with $W^{-1}$, and hence we can apply Corollary 6.7 to obtain the following:

**Corollary 6.10.** *Let $n \in \mathbb{N}$ and $f$ be positive measurable function with $f(t) \lesssim 1/t$. It holds that:*

- *If $f$ satisfies*
$$\int_1^\infty \sup_y \left(f(y) y \min(y,t)\right) (\log_1 t)^{n-1} \frac{dt}{t^2} < \infty,$$
*then, for every $x > 0$,*
$$A^n f(x) \lesssim \left(\log_1 \frac{1}{x}\right)^n.$$

- *If $f$ satisfies*
$$\int_0^1 \sup_{s>0} \left(\min\left(\frac{s}{t}, 1\right) f(s)\right) \left(\log_1 \frac{1}{t}\right)^{n-1} dt < \infty,$$
*then, for every $x > 0$,*
$$R_\alpha^n f(x) \lesssim x^{\alpha-1} (\log_1 x)^n.$$

### 6.3.4 Braverman-Lai's operators

Let us now consider the operator $S_a$ defined in (6.4) and let us assume the following: there exist an increasing function $D \geqslant 0$, with $D(t) = 0$ if and only if $t = 0$, and a function $E$ so that, for some $m > 0$ and every $1 < p \leqslant p_0$,

$$\int_0^\infty \left(\sup_{t>0} \frac{E(t)}{D(st)}\right)^{1/p} a(s) ds \lesssim \frac{1}{(p-1)^m}. \tag{6.12}$$

Then, one can immediately see that, for every $t > 0$,

$$\int_0^\infty \frac{a(s)}{D(st)^{1/p}} ds \lesssim \frac{1}{(p-1)^m E(t)^{1/p}},$$

and hence, (6.5) holds with $W = D$ and $U \lesssim E^{-1}$. A direct consequence of Theorem 6.5 is the following:

**Corollary 6.11.** *If* (6.12) *holds, then, for every $f \in B(D)$ satisfying*

$$\int_0^1 \sup_{s>0} \left(\min\left(\frac{D(s)}{D(t)}, 1\right) f(s)\right) \left(\log_1 \frac{1}{D(t)}\right)^{m-1} dD(t) < \infty,$$

*we have that*

$$S_a f(x) \lesssim \frac{(\log_1 E(x))^m}{E(x)}.$$

Notice that in the simplest case, when $a(s) = \chi_{(0,1)}(s)$, the operator $S_a f(t) = Sf(t) = \frac{1}{t}\int_0^t f(s)ds$ is the Hardy operator. What we obtain is that, if $D(t) = E(t) = t$, we can take $m = 1$ to conclude that

$$S_a f(t) \lesssim \frac{\log_1 t}{t},$$

whenever $f(t) \lesssim 1/t$ and

$$\int_0^1 \sup_{s>0} \left(\min\left(\frac{s}{t}, 1\right) f(s)\right) dt < \infty.$$

By taking a function $f$ such that $f(t) = \frac{1}{t}$, whenever $t > 1$, we see that the pointwise bound cannot be improved. However, in this particular example, in order to get that pointwise bound, it is possible to weaken the condition on the function near 0 by simply assuming that $f \in L^1(0,1)$.

### 6.3.5 Other applications

In Theorem 6.5, the condition that we require on $f$ is basically that its least decreasing majorant $F$ satisfies $\|F\|_{D_m(W)} < \infty$. To finish this section, we will present two more versions of our main result in which the role of $F$ is played by the decreasing rearrangement $f^*$ and the level function $f^\circ$, respectively.

Assume that $K(x,t)$ is decreasing in $t$. Then, by Hardy's inequality [5, Theorem 2.2], we have that, for every function $f$,

$$T_K f(x) = \int_0^\infty K(x,t) f(t) dt \leqslant \int_0^\infty K(x,t) f^*(t) dt = T_K(f^*)(x),$$

so we can apply Corollary 6.6 to $f^*$ and write the following result:

**Corollary 6.12.** *Under the hypotheses of Theorem 6.5 if, for every $x > 0$, $K(x,t)$ is decreasing in $t \in (0,\infty)$, then*

$$T_K f(x) \lesssim \|f^*\|_{D_m(W)} U_m(x).$$

Similarly, assume now that we have a Volterra operator

$$V_K f(x) = \int_0^x K(x,t) f(t) dt,$$

with $K(x,t)$ decreasing in $t \in (0,x)$. In [99], the authors show that, for every bounded function $f \geqslant 0$ with compact support in $(0,\infty)$, it holds that

$$V_K f(x) \leqslant V_K(f^\circ)(x),$$

where $f^\circ$ is a decreasing function associated with $f$ called the Halperin level function (see [65, 108]). Therefore, this estimate together with Corollary 6.6 and Fatou's lemma yield:

**Corollary 6.13.** *Under the hypotheses of Theorem 6.5, if $K(x,t)$ is decreasing in $t \in (0,x)$ for every $x > 0$, then*

$$V_K f(x) \lesssim \|f^\circ\|_{D_m(W)} U_m(x).$$

## 6.4 Generalization to sublinear operators

Although our motivation has been to study integral operators with positive kernels, our main result can be extended to more general operators as follows:

**Theorem 6.14.** *Let $T$ be a sublinear operator such that, for every $x$*

$$U(x) := \sup_{1<p\leqslant p_0} \sup_{\|f\|_{B(W^{1/p})}\leqslant 1} (p-1)^{pm} Tf(x)^p < \infty.$$

*Then, we have that*

$$Tf(x) \lesssim \|f\|_{D_m(W)} U_m(x).$$

In the proof of Theorem 6.5, we make use of the fact that the operators $T_K$ are monotone. Since now we do not have this property on $T$, we will need to introduce auxiliary functions $\kappa$ and $\rho$ to get around this problem.

*Proof.* We will follow the proof of Theorem 6.5. Let $\kappa$ be an arbitrary function with $\|\kappa\|_\infty \leqslant 2$. Define

$$T_\kappa f := T(\kappa f).$$

By our assumption, it is easy to check that, for every $1 < p \leqslant p_0$,

$$T_\kappa : B(W^{1/p}) \longrightarrow B(U^{-1/p}),$$

with constant controlled by $(p-1)^{-m}$. As before, we get that, for every function $\|\kappa\|_\infty \leqslant 2$ and every $g$ decreasing,

$$\sup_{t>0} \frac{T_\kappa(P_w g)(t)}{U_m(t)} \lesssim \|P_w g\|_{B(W)} + \int_0^1 P_w g(t) \Big( \log_1 \frac{1}{W(t)} \Big)^{m-1} w(t)dt. \tag{6.13}$$

Let us now assume that $f \in B(W)$ is a decreasing function satisfying that

$$\int_0^1 \frac{\sup_{s\leqslant t} W(s)f(s)}{W(t)} \Big( \log_1 \frac{1}{W(t)} \Big)^{m-1} w(t)dt < \infty.$$

If $H(t) = \sup_{s\leqslant t} W(s)f(s)$, we have the existence of a decreasing function $g$ such that

$$\frac{1}{2} H(t) \leqslant \int_0^t g(s)w(s)ds \leqslant 2H(t).$$

With this,

$$f(t) \leqslant \frac{H(t)}{W(t)} \leqslant \frac{2\int_0^t g(s)w(s)ds}{W(t)} = 2P_w g(t),$$

so we can write, for some $\|\kappa\|_\infty \leqslant 2$,

$$f(t) = \kappa(t)P_w g(t).$$

Therefore, for every function $\rho$ with $\|\rho\|_\infty \leqslant 1$, we can use (6.13) with $\|\kappa\rho\|_\infty \leqslant 2$ to show that

$$\begin{aligned}\sup_{t>0} \frac{T(\rho f)(t)}{U_m(t)} = \sup_{t>0} \frac{T_{\kappa\rho}(P_w g)(t)}{U_m(t)} &\lesssim \|P_w g\|_{B(W)} + \int_0^1 P_w g(t)\Big(\log_1 \frac{1}{W(t)}\Big)^{m-1} w(t)dt \\ &\approx \|f\|_{B(W)} + \int_0^1 \frac{\sup_{s\leqslant t} W(s)f(s)}{W(t)}\Big(\log_1 \frac{1}{W(t)}\Big)^{m-1} w(t)dt.\end{aligned} \tag{6.14}$$

Choosing $\rho \equiv 1$, we finish the proof in the decreasing case. For a general function $f \in B(W)$, we consider its least decreasing majorant $F(t) = \sup_{r\geqslant t} f(r)$, which lies in $B(W)$ and satisfies $f \leqslant F$. Hence, we write $Tf(x) = T(\rho F)(x)$ for some $\|\rho\|_\infty \leqslant 1$, and the result follows immediately applying (6.14) together with

$$\|F\|_{B(W)} = \|f\|_{B(W)}$$

and

$$\frac{\sup_{s\leqslant t} W(s)F(s)}{W(t)} = \sup_{s>0}\left(\min\Big(\frac{W(s)}{W(t)}, 1\Big) f(s)\right).$$

$\square$

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**S**



**U**



**V**



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