Hyperbolic Components of the Complex Exponential Family *

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Abstract

In this paper we describe the structure of the hyperbolic components of the parameter plane of the complex exponential family, as started in [BR]. More precisely, we label each component with a *parameter plane kneading sequence*, and we prove the existence of a hyperbolic component for any given such sequence. We also compare these sequences with the more commonly used *dynamical kneading sequences.*

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1 Introduction

Our goal in this paper is to describe the structure of the hyperbolic components in the parameter plane for the complex exponential family.

Let $E_\lambda(z) = \lambda e^z$ with $\lambda \in \mathbb{C}$. The map E_λ has a unique singular value at 0 (the omitted value). As is well known, the fate of the orbit of 0 determines much of the dynamical behavior of E_λ. For example, if E_λ admits an attracting cycle, then the orbit of 0 must tend to this cycle. As a consequence, E_λ has at most one attracting cycle.

The parameter space of the exponential family was first studied in [BR] and [El1] and later on in [Bo], [D] and [S].

Let Ω_n denote the set of λ-values for which E_λ admits an attracting cycle of period n. The connected components of Ω_n are called *hyperbolic components* and it is conjectured that they are dense in the parameter plane. As shown in [BR] and [El1], any hyperbolic component is simply connected and unbounded, with the exception of Ω_1 which is a cardioid-shaped region containing 0. The region Ω_2 consists of a single component which occupies a large portion of the left half plane. Each Ω_n for $n > 2$ consists of infinitely many distinct components, each of which extends to ∞ in the right half plane.

The arrangement of these hyperbolic components in the λ-plane is quite complicated. A partial description can be found in [BR] where the authors show the existence of infinitely many hyperbolic components of period n in between two hyperbolic components of period $n-1$. Our goal in this paper is to give a more precise description by using the dynamics of the corresponding maps. In particular we shall give a label to each of the components which will describe the dynamical behaviour of the critical orbit for those parameters in the given component. We shall see that this label also determines the position of the component in the right half plane. See Figure 1.

With this goal in mind, there is a choice to be made. Indeed, there are two ways to identify the various hyperbolic components in the λ-plane. Each of these involves the association of a *kneading sequence* to the component. This sequence is a string of $n-2$ integers. For technical reasons we precede the string with a 0 and end the string with a $*$. That is, a kneading sequence assumes the form $0s_1 \ldots s_{n-2}*$ with $s_j \in \mathbb{Z}$. The $*$ denotes a "wild card" that will be described below.

One of the two kneading sequences is a *dynamical kneading sequence* (K-kneading sequence) which is useful mainly in the dynamical plane (see [BD]), since it determines the topological structure of the Julia set of E_λ for any λ in the hyperbolic component. The other kneading sequence is a

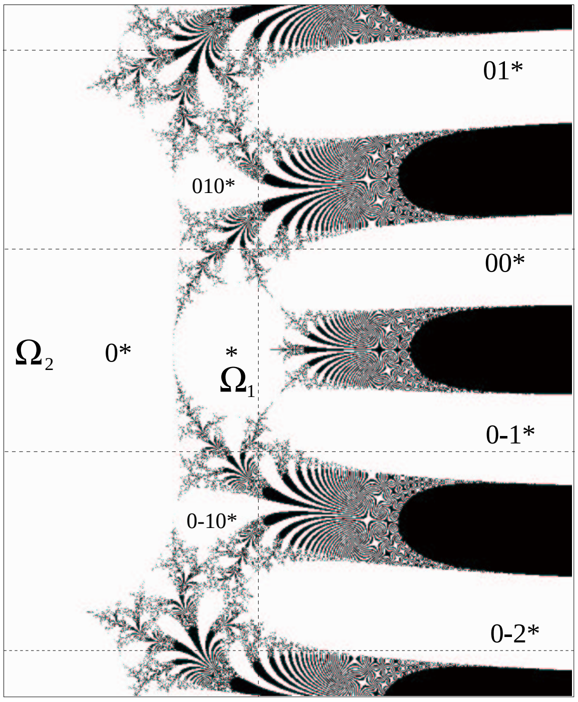

Figure 1: The parameter plane of E_λ. White regions correspond to hyperbolic components. Black smooth regions are due to numerics. Dotted lines have been drawn on the imaginary axis and on the horizontal lines with imaginary parts $-3\pi, -\pi, \pi$ and 3π.

parameter plane kneading sequence (S-kneading sequence) and, as we shall see, is more useful for describing the structure of the λ-plane. The main result in this paper is as follows.

Theorem A. *Fix $n \geq 3$ and let $s_1, \ldots, s_{n-2} \in \mathbb{Z}$. There exists a hyperbolic component $W_{s_1 \ldots s_{n-2}}$ that extends to ∞ in the right half plane and such that if $\lambda \in W_{s_1 \ldots s_{n-2}}$, the map E_λ has an attracting cycle of period n with parameter plane kneading sequence $s = 0 s_1 \ldots s_{n-2} *$. Moreover, the components $W_{s_1 \ldots s_{n-2}}$ are ordered lexicographically.*

From the proof of this theorem one obtains the following corollary (see Figure 2).

Corollary B. *Let $W_{s_1 \ldots s_{n-2}}$ be as in Theorem A. Then, in between this hyperbolic component and $W_{s_1 \ldots s_{n-2}+1}$, there exist hyperbolic components $W_{s_1 \ldots (s_{n-2}+1)\, k}$, for any $k \in \mathbb{Z}$.*

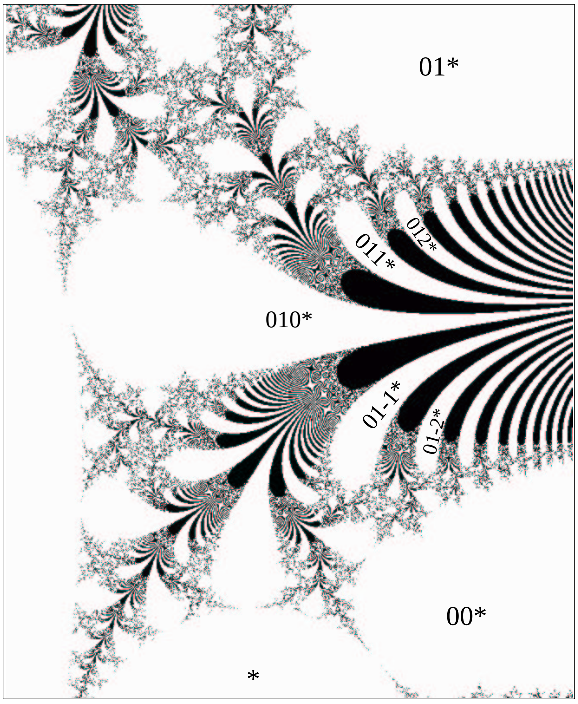

Figure 2: Magnification of Figure 1 showing infinitely many period 4 components in between two period 3 components.

In this statement the words "in between" refer to the ordering given by the imaginary part, since all hyperbolic components extend to infinity in the right half plane.

These results give a description of the ordering of the hyperbolic components in the far right half plane as a function of their kneading sequence. Note that these are existence type results. Although uniqueness is most likely true, this fact does not follow directly from our work in this paper. D. Schleicher [S] has announced some results in this direction using the coding of hairs in parameter space.

In Section 2 below we define each of these kneading sequences and discuss several of their properties. We also derive an algorithm for obtaining one sequence given the other.

In Section 3 we prove Theorem A, that is, we show the existence of hyperbolic components corresponding to any S-kneading sequence.

2 Kneading sequences

Let us consider a hyperbolic component Ω of period $n > 2$. The main goal of this section is to define two different kneading sequences associated to the parameter value $\lambda \in \Omega$. We shall also study the relation between the two sequences and give an algorithm that transforms one into the other.

We start by giving a topological description of the dynamical plane of $E_\lambda(z) = \lambda e^z$ that holds for any parameter λ in the hyperbolic component Ω.

2.1 The fingers and the glove

If $\lambda \in \Omega$, the map $E_\lambda(z) = \lambda \exp(z)$ has an attracting periodic orbit of period $n > 1$. This orbit varies analytically with λ as long as λ lies in the hyperbolic component. Let $z_0(\lambda), z_1(\lambda) = E_\lambda(z_0), \ldots z_{n-1}(\lambda) = E_\lambda(z_{n-2})$ be the points of the periodic orbit. To simplify notation we will omit the dependence on λ whenever it is understood.

Let A^* denote the immediate basin of attraction of the periodic orbit and, for $0 \leq i \leq n-1$, define $A^*(z_i)$ to be the connected component of A^* which contains z_i. We name the points in the orbit so that the asymptotic value 0 belongs to $A^*(z_0)$.

We now construct geometrically and define what we call *fingers*. More details can be found in [BD]. For $\nu \in \mathbb{R}$, let $H_\nu = \{z \mid \mathrm{Re}(z) > \nu\}$.

Definition. An unbounded simply connected $F \in \mathbb{C}$ is called a *finger* of width c if

a) F is bounded by a single simple curve $\gamma \in \mathbb{C}$

b) There exists ν such that $F \cap H_\nu$ is simply connected, extends to infinity and satisfies

$$F \cap H_\nu \subset \{z \mid \mathrm{Im}(z) \in [\psi - \frac{c}{2}, \psi + \frac{c}{2}]\} \text{ for some } \psi \in \mathbb{R}.$$

Observe that the preimage of any finger which does not contain 0 consists of infinitely many fingers of width smaller than 2π which are $2\pi i$–translates of each other.

We begin the construction by choosing $B = B(\lambda)$ to be a disk in $A^*(z_0)$ that contains both 0 and z_0, and having the property that B is mapped strictly inside itself under E_λ^n.

We now take successive preimages of the disk B. More precisely, let B_{n-1} be the open set in $\mathbb{C}$ which is mapped to B. Note that, since $0 \in B$, it follows that B_{n-1} has a single connected component which contains a left half-plane, and whose image under E_λ wraps infinitely many times over $B \setminus \{0\}$. Clearly the point z_{n-1} belongs to the set B_{n-1}, which lies inside $A^*(z_{n-1})$.

We now consider the preimage of B_{n-1}. It is easy to check (by looking at the image of vertical lines with increasing real part) that this preimage consists of infinitely many disjoint fingers of width smaller than 2π which are $2\pi i$-translates of each other. We define $B_{n-2} \subset A^*(z_{n-2})$ to be the connected component such that $z_{n-2} \in B_{n-2}$. The map E_λ takes B_{n-2} conformally onto B_{n-1}.

Similarly, we define the sets $B_{n-3}, \ldots, B_0$, by setting B_i to be the connected component of $E_\lambda^{-1}(B_{i+1})$ that contains the point z_i. These inverses are all well defined and the map E_λ sends B_{i+1} conformally onto B_i. Each B_i belongs to the immediate basin $A^*(z_i)$. The following characterization of the sets $B_i, i = 0, \ldots, n-2$ is proved in [BD].

Proposition 2.1. *Let $n \geq 2$. For $i = 0, \ldots, n-2$, B_i is a finger of width $b_i \leq 2\pi$.*

It follows from the above construction that the width of the finger B_{n-2} that is mapped by E_λ conformally onto B_{n-1} (essentially B_{n-1} is the left half plane) is much larger than the width of the other fingers B_i, $i = 0, \ldots, n-3$, that map conformally onto B_{i+1}. So we will refer to B_{n-2} as the *big finger.*

We proceed to the final step, by defining the set

$$G = \{z \in \mathbb{C} \mid E_\lambda(z) \in B_0\}$$

which we call the *glove*. We observe from the above construction that G is a connected set and $B_{n-1} \subset G \subset A^*(z_{n-1})$. See Figure 3. Moreover, the complement of G consists of infinitely many fingers, each of which are $2\pi i$ translates of each other. We index these infinitely many connected components by V_j, $j \in \mathbb{Z}$, so that $2\pi ij \in V_j$.

In fact, these V_j form a set of fundamental domains for the Julia set of E_λ in the following sense:

- $J(E_\lambda) \subset \bigcup_{j \in \mathbb{Z}} V_j$.
- E_λ maps each V_j conformally onto $\mathbb{C} \setminus B_0$, and so $E_\lambda(V_j) \supset J(E_\lambda)$.

Hence, for each $j \in \mathbb{Z}$ we have a well defined inverse branch of E_λ:

$$L_j = L_{\lambda,j} : \mathbb{C} \setminus B_0 \longrightarrow V_j.$$

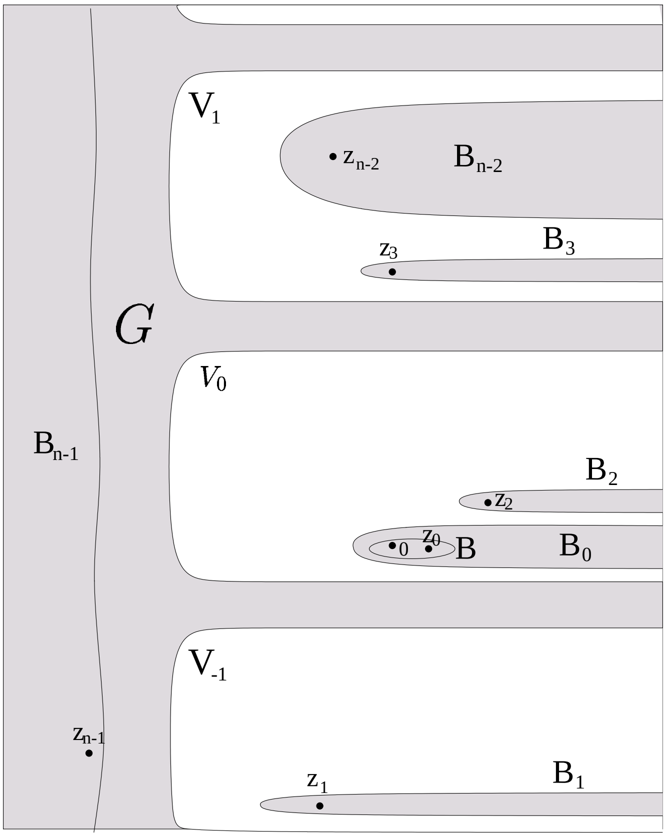

Figure 3: Sketch of the sets B_0 to B_{n-1}, G and V_j for $j \in \mathbb{Z}$. Points in grey belong to the basin of attraction of the periodic orbit.

Note that B_0 lies inside V_0 since $0 \in B_0$. The other fingers $B_1, \dots, B_{n-2}$ may lie inside any of the fundamental domains V_j, depending on the value of λ. In particular, several B_i may lie in the same V_j.

2.2 K–Kneading sequences and S-Kneading sequences

We first introduce the kneading sequence given by the fundamental domains V_j. We define the *K–kneading sequence* of a value $\lambda \in \Omega$ as

$$K(\lambda) = 0\, k_1\, k_2\, k_3\, \dots\, k_{n-2} *$$

where $B_j \subset V_{k_j}$ for all $1 \leq j \leq n-2$. We use $*$ for the position of the point z_{n-1}, since this point does not belong to any of the V_j. Since all the boundaries of the B_i move analytically with λ, it follows that this kneading sequence is constant throughout the entire hyperbolic component Ω.

We define the *K–itinerary* of any point $z \in J(E_\lambda)$ to be

$$K(z) = k_0\, k_1\, k_2\, k_3\, \ldots$$

where $E_\lambda^j(z) \in V_{k_j}$ for any $j \geq 0$.

One can then use these itineraries together with the kneading sequence to give a complete description of the structure of the Julia set for E_λ in terms of symbolic dynamics. See [BD].

We now define the *S-kneading sequence* of a value $\lambda \in \Omega$. This sequence was introduced in [S]. If we look at the dynamical plane very far to the right, we see that any finger is basically a straight horizontal band; therefore it makes sense to define the order of fingers in terms of their imaginary part. In this fashion, we can speak about fingers sitting *above* or *below* each other. Likewise, we can talk about the *upper boundary* and the *lower boundary* of a finger, as long as we look in the far right half plane.

Consider the half plane $H_\mu = \{z \in \mathbb{C} \mid \mathrm{Re}(z) > \mu\}$ for a fixed μ large enough. Define the family of fingers $F_j,\ j \in \mathbb{Z}$ to be the infinitely many connected components of the preimage of B_{n-1}. We observe that the fingers F_j are the $2k\pi i$-translations of the big finger for any $k \in \mathbb{Z}$. We index these sets consecutively so that F_0 is the one immediately above B_0. For any $j \in \mathbb{Z}$, let T_j be the region in H_μ that lies between the upper boundaries of F_{j-1} and F_j (so, we have $F_j \cap H_\mu \subset T_j$). See Figure 4.

Finally, we define the *S-kneading sequence* of a value $\lambda \in \Omega$ as

$$S(\lambda) = 0\, s_1\, s_2\, s_3\, \ldots\, s_{n-2} *$$

where $B_j \cap H_\mu \subset T_{s_j}$ for all $1 \leq j \leq n-2$. It is easy to check that this definition does not depend on the choice of μ, as long as μ is chosen to be large enough so that the boundary of the fingers $B_j,\ j = 0, \ldots n-2$ and $F_j,\ j \in \mathbb{Z}$ crosses the boundary of H_μ exactly twice. See Figure 4.

We observe that the regions T_i do not define a family of fundamental domains in the sense explained above. Consequently, the S–itinerary (defined in the obvious way) is not well defined for all points in the Julia set, but only for those whose orbits have sufficiently large real part. Although this shows that S-kneading sequences and itineraries are not suitable for use in the dynamical plane, we shall see that they are very convenient when the parameter plane is considered. Therefore, it is of interest to be able to use both of these kneading sequences

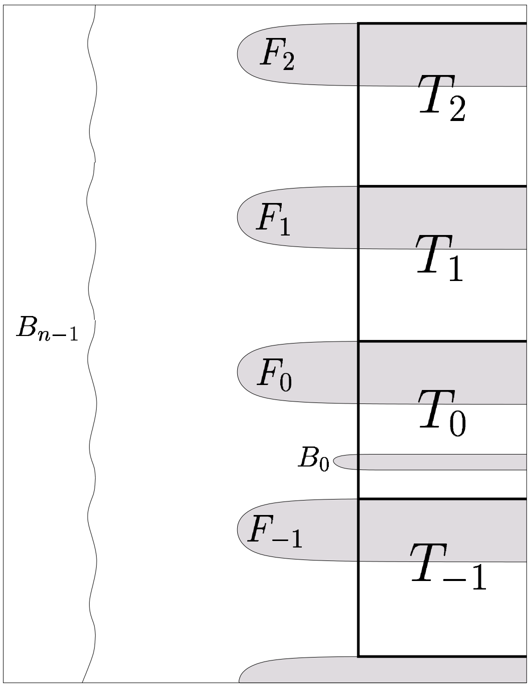

Figure 4: The families F_i and T_i.

2.3 Translation Algorithm

In this section we describe an algorithm that relates the K– and S–kneading sequences. Let us denote the S–kneading sequence of E_λ by

$$S = 0 s_1 s_2 \ldots s_{n-2} * .$$

We will show how to compute the K–kneading sequence

$$K = 0 k_1 k_2 \ldots k_{n-2} * .$$

associated to λ.

The algorithm is composed of two steps. The first step is to attach a sign ($+$ or $-$) to each of the zero entries of S (with the exception of the first entry of the sequence that will remain as 0). This sign indicates that the corresponding B_i is above (0^+) or below (0^-) B_0, at least far to the right. The second step will determine each of the k_i based on s_i and s_{i+1}, except for the last entry k_{n-2} which will be determined by s_{n-2} and s_1.

Step 1: deciding on 0^+ or 0^-

Let $s_i = 0$. Then $B_i \subset V_0$ and so either B_i lies above or below B_0 in the far right half plane. We attach the superscript $+$ or $-$ to 0 depending on whether B_i is above (0^+) or below (0^-) B_0. We write $* = \infty$ for ordering purposes.

Consider the words $s_1 s_2 \dots$ and $s_{i+1} s_{i+2} \dots$. Compare these two words until finding the minimal $j \geq 1$ such that $s_j \neq s_{i+j}$. Then we set

$$s_i = \begin{cases} 0^+ & \text{if } s_j < s_{i+j} \\ 0^- & \text{if } s_j > s_{i+j} \end{cases}$$

We show that this rule gives the correct superscript. Since $s_i = 0$, B_i meets T_0 as well as B_0. We must decide if B_i is above or below B_0.

If $s_1 > s_{i+1}$ (resp. $s_1 < s_{i+1}$) then B_{i+1} is below (resp. above) B_1. Since the order is preserved inside one fundamental domain we can deduce that B_i is below (resp. above) B_0. Hence $s_i = 0^-$ (resp. 0^+). Observe that having defined $* = \infty$ takes care of the case $s_{i+1} = *$, i.e., the case of the big finger.

We end by induction. Let us assume $s_j = s_{i+j}$ for $j = 1, \dots, k$ but $s_{k+1} \neq s_{i+k+1}$. Then B_j and B_{i+j} live in T_{s_j}, $j = 1, \dots, k$, and hence, their relative order can be decided by looking at their respective images B_{k+1} and B_{i+k+1}. There are two cases.

If $s_{k+1} > s_{i+k+1}$ then B_{i+k+1} is below B_{k+1}, and consequently, B_{i+j} is below B_i for all $j = 1 \dots, k$. So, B_i is below B_0 and $s_i = 0^-$.

If $s_{k+1} < s_{i+k+1}$ we substitute "above" for "below" in the previous paragraph and conclude that $s_i = 0^+$.

In particular we remark that there are two cases that do not occur: (a) $s_i = 0^+$ and $s_{i+1} \leq 0^-$ in the case $s_1 \geq 0^+$, and (b) $s_i = 0^-$ and $s_{i+1} \geq 0^+$ in the case $s_1 \leq 0^-$.

Step 2: obtaining k_i

Let S be a modified S-kneading sequence by adding the corresponding 0^+ and 0^- symbols. There are two completely symmetric cases: $s_1 \geq 0^+$ and $s_1 \leq 0^-$. Before starting we set $1 - 1 = 0^+$ and $-1 + 1 = 0^-$. Now, for any i with $1 \leq i \leq n - 2$,

(a) If $s_1 \geq 0^+$ then $$k_i = \begin{cases} s_i & \text{if } i = n-2 \text{ or } s_{i+1} \geq 0^+ \\ s_i - 1 & \text{if } s_{i+1} \leq 0^- \end{cases}$$

(b) If $s_1 \leq 0^-$ then $\quad k_i = \begin{cases} s_i + 1 & \text{if } i = n-2 \text{ or } s_{i+1} \geq 0^+ \\ s_i & \text{if } s_{i+1} \leq 0^- \end{cases}$

We now prove that for a given $\lambda \in \Omega$ the above rule translates any S to a unique K. We consider the case $s_1 \geq 0^+$, the other case is being symmetric.

We denote by g_i the piece of the glove G that falls into the region T_i. Since $s_1 \geq 0^+$, B_1 is above B_0 and hence, the glove g_0 must be below B_0.

This implies that V_0 is the fundamental domain between the gloves g_0 and g_1 and, in general, each V_i lies between g_i and g_{i+1}, in particular including F_i. This last remark implies that the last digit of the sequence will not change. That is, $k_{n-2} = s_{n-2}$.

Consider s_i for $1 \leq i < n-2$. So B_i lies in T_{s_i}. By the observations above, either (see Figure 5)

1. B_i lies in V_{s_i} because the piece of the glove g_{s_i} is below B_i (case $k_i = s_i$), or

2. B_i lies in $V_{s_i - 1}$ because the piece of the glove g_{s_i} is above B_i (case $k_i = s_i - 1$).

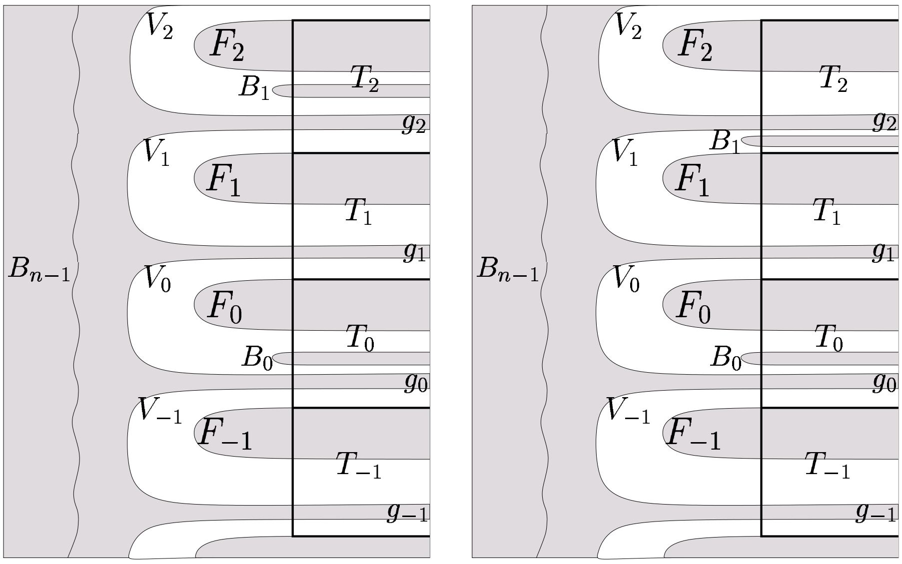

Figure 5: Example of the two possibilities: the S-kneading sequence 02∗ translating either into 02∗ or into 01∗.

It is straightforward to check that the first case occurs if and only if B_{i+1}

is above B_0, i.e., $s_{i+1} \geq 0^+$. The second case occurs if and only if B_{i+1} is below B_0, i.e., $s_{i+1} \leq 0^-$.

As an example, consider the S–kneading sequence

$$S = 0 - 2\ 0\ 0 - 1\ 2\ 3\ 0\ 0 - 1\ 2\ 0\ 0\ \ * .$$

After the first step we have

$$S = 0 - 2\ 0^+\ 0^+ - 1\ 2\ 3\ 0^+\ 0^+ - 1\ 2\ 0^+\ 0^+\ *,$$

and after the second step the corresponding K–kneading sequence is

$$K = 0 - 1\ 1\ 0^+\ 0^-\ 3\ 4\ 1\ 0^+\ 0^-\ 3\ 1\ 1\ \ * .$$

We finally observe that the above 2-step algorithm can also be used in the reverse direction, that is, for a given K with the symbols 0^+ and 0^- we obtain, via the inverse algorithm, a unique S. Next section will refer to this point taking into account the admissibility.

2.4 Properties

Why are we working with two distinct kneading sequences? The answer to this question is based on the fact that the two sequences have different properties and consequently, they are suitable depending on the problem under consideration.

More precisely, the K-kneading sequences work well to study the dynamical plane because they are defined by using fundamental domains. These domains work for all points of the Julia set and give rise to good symbolic dynamics and consequently to a complete description of the Julia set (see [BD]). In contrast, when working in parameter plane, one can find many different hyperbolic components sharing the same K-kneading sequence. For instance, for any $n \in \mathbb{N}$, all hyperbolic components of period n bifurcating from the main cardioid have their K-kneading sequence given by $K = 0000\dots0$. To fix this uniqueness problem we might consider the symbols 0^+ and 0^- as before. But then, an admissibility problem arises, without an obvious way to decide if a sequence is admissible or not (except, of course, going through the inverse algorithm to check if the resulting sequence is possible).

The S-kneading sequences do not involve fundamental domains (in the complete sense) and hence they are not as useful as the K-kneading sequences to describe the dynamical plane. However, we prove in the next section that all sequences are admissible, that is, we can find a hyperbolic component Ω

corresponding to any given sequence of integers. Moreover, these sequences give plenty of information about the location of the periodic orbit.

The uniqueness of hyperbolic components having a given S-kneading sequence seems a natural result but it is not straightforward from the construction below.

Finally, we remark that the method of finding those hyperbolic components in the next section makes it possible to provide a global picture of their distribution in the plane.

3 Hyperbolic Components. Proof of the main result.

Our goal in this section is to construct a parameter value λ, for which E_λ has an attracting cycle with any given S-kneading sequence. We first consider the special case where the S-kneading sequence consists of a single digit; the proof in this case makes use of many of the ideas of the general case, but in a simpler setting.

3.1 The case 0k*

The result follows from the next two propositions.

Proposition 3.1. *Fix $k \in \mathbb{Z}$. For $a \in \mathbb{R}$, let $\lambda_a = a + (2k+1)\pi i$. Then, for sufficiently large values of a, the map E_{λ_a} has an attracting cycle of period 3.*

Proof. We assume throughout that $a \geq |2k+1|\pi$, so that $|\operatorname{Arg}(\lambda_a)| \leq \pi/4$, where Arg denotes the principal branch of the argument. Then $\lambda_a = E_{\lambda_a}(0)$ lies in the right half plane, but $E^2_{\lambda_a}(0) = \lambda_a \exp(\lambda_a)$ lies in the left half plane since $E^2_{\lambda_a}(0) = -e^a \lambda_a$. Choosing a large, we may assume that $a < |\lambda_a| \leq a+1$. Since

$$\frac{3\pi}{4} \leq |\operatorname{Arg}(E^2_{\lambda_a}(0))| \leq \pi$$

it follows that

$$\begin{aligned} \text{Re } (E^2_{\lambda_a}(0)) &= |\lambda_a| e^a \cos(\text{Arg } (E^2_{\lambda_a}(0))) \\ &\leq -\frac{|\lambda_a|}{\sqrt{2}} e^a \\ &< -ae^a/\sqrt{2}. \end{aligned}$$

Let U_2 be the ball of radius 1 about $E^2_{\lambda_a}(0)$. The preimage of U_2 containing λ_a is an open set U_1 which is mapped univalently onto U_2 by E_{λ_a}, and the preimage of U_1 containing 0 is another open set, say U_0, which is mapped univalently onto U_2 by $E^2_{\lambda_0}$. We claim that there is an attracting cycle of period 3 whose orbit under E_{λ_a} lies in U_0, U_1, and U_2. Let F denote the appropriate branch of the inverse of $E^2_{\lambda_a}$ that takes U_2 univalently onto U_0. See Figure 6. By the Koebe 1/4 Theorem, we have

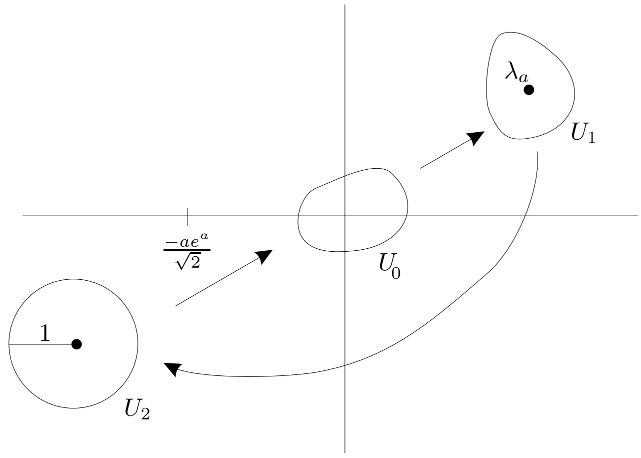

Figure 6: The sets U_0, U_1 and U_2 in the proof og Proposition 3.1

$$\begin{aligned}
\operatorname{dist}(0, \partial U_0) &\geq \frac{1}{4}|F'(E^2_{\lambda_a}(0))| \\
&= \frac{1}{4} \cdot \left|\frac{1}{\lambda_a}\right| \cdot \left|\frac{1}{\lambda_a e^{\lambda_a}}\right| \\
&= \frac{e^{-a}}{4|\lambda_a|^2} \\
&\geq \frac{e^{-a}}{4(a+1)^2}.
\end{aligned}$$

Now

$$\begin{aligned}
|E^3_{\lambda_a}(0)| &= |\lambda_a| \exp(\operatorname{Re} E^2_{\lambda_a}(0)) \\
&\leq (a+1)\exp(-ae^a/\sqrt{2}) \\
&<< \frac{e^{-a}}{4(a+1)^2}
\end{aligned}$$

for large a. Hence $E^3_{\lambda_a}(0)$ is contained in U_0. Moreover, if $w \in U_2$, then

$$\begin{aligned}
|E_{\lambda_a}(w) - E^3_{\lambda_a}(0)| &\leq \max_{z \in U_2} |E'_{\lambda_a}(z)| \\
&\leq |\lambda_a \exp(\mathrm{Re}\, E^2_{\lambda_a}(0) + 1)| \\
&\leq (a+1)e \exp(-ae^a/\sqrt{2}) \\
&<< \frac{e^{-a}}{4(a+1)^2}
\end{aligned}$$

as before. Hence,

$$\mathrm{dist}(0, \partial E^3_{\lambda_a}(U_0)) \leq (a+1)(e+1)\exp(-ae^a/\sqrt{2}) << \frac{e^{-a}}{4(a+1)^2},$$

and it follows that $E^3_{\lambda_a}(U_0)$ is properly contained in U_0. Thus we have an attracting cycle whose orbit visits U_0, U_1 and U_2. This completes the proof of the proposition. □

Before proceeding, we observe that the above estimates guarantee that the entire half plane Re $z \leq$ Re $E^2_{\lambda_a}(0) + 1$ is contained in the basin of the cycle.

We now claim that the S-kneading sequence of λ_a is $0k*$.

Proposition 3.2. *Let $k \in \mathbb{Z}$ and set $\lambda_a = a + (2k+1)\pi i$. Then for values of a sufficiently large, E_{λ_a} has an attracting 3 cycle with $S(\lambda_a) = 0k*$.*

Proof. Let $\gamma(t) = t + (2k+1)\pi i$ for $t \geq a$. $E_{\lambda_a}(\gamma(t))$ is a straight line which lies to the left of $E^2_{\lambda_a}(0)$. By the above observation, $E_{\lambda_a}(\gamma(t))$ lies in the connected component of the immediate basin of attraction which contains $E^2_{\lambda_a}(0)$. Hence $\gamma(t)$ lies in the component of the immediate basin which contains λ_a.

Let S be the strip $\{z | \, |\mathrm{Im}\, z| \leq \pi\}$. There is a preimage of $\gamma(t)$ contained in the interior of S, at least for t large. We claim that the entire preimage of $\gamma(t)$ lies in S. The preimage of $\gamma(t)$ can never meet the boundary of S, for E_{λ_a} maps the boundary of S into the left half plane, far from $\gamma(t)$. Hence the preimage of $\gamma(t)$ lying in S must be the preimage that contains 0.

We then consider the set B as above so that B contains $E^3_{\lambda_a}(0)$. It then follows that B_2 contains $E^2_{\lambda_a}(0)$ and $E_{\lambda_a}(\gamma(t))$. By taking one more preimage, the big finger B_1 contains λ_a and $\gamma(t)$ and its translations contain the semilines $\{t + (2j+1)\pi \mid t \geq a\}$. Moreover, the finger B_0 contains 0 and the preimage of $\gamma(t)$ in S. It follows then that the fingers are indexed so that $B_1 = F_k$ and hence $S(\lambda_a) = 0k*$. □

3.2 The general case

Now we proceed to the general case. For the remainder of this section we fix a kneading sequence $s = 0s_1s_2\dots s_{n-2}*$. Let $\widehat{s} = \max|s_i|$ and define $M = (2\widehat{s}+1)\pi$. We assume throughout that $a > M$. Let $H(a)$ denote the closed half strip

$$H(a) = \{z|\text{Re } z \geq a, \ |\text{Im } (z)| \leq M\}.$$

We let $L(a)$ denote the left boundary of $H(a)$. We will prove:

Theorem 3.3. *For each sufficiently large a, there is $\lambda_a \in L(a)$ for which E_{λ_a} has an attracting n-cycle with $S(\lambda_a) = s$.*

We will divide the proof in three parts, stated in Propositions 3.5, 3.6 and 3.7. Afterwards we will see how Theorem A (see Section 1) follows.

We denote the first n points on the orbit of 0 by w_i, so $w_0 = 0$, $w_1 = \lambda_a$, $\dots, w_n = E^n_{\lambda_a}(0)$. As in the previous special case, we will construct λ_a so that the orbit of 0 under E_{λ_a} has the following properties:

1. $w_i \in H(a)$ for $i = 1,\dots,n-2$ and $\text{Re}\, w_{i+1} >> \text{Re}\, w_i$ for $i = 0,\dots,n-3$.
2. w_{n-1} lies in the left half plane and
$$|\text{Re } w_{n-1}| >> \text{Re } w_{n-2}$$
3. w_n lies close to 0 and, as in the period 3 case, there is an attracting cycle of period n lying close to $w_0,\dots,w_{n-1}$.

Let $\nu = \nu(a) = |a+(2\widehat{s}+1)\pi i| = \max_{z\in L(a)}|z|$, and note that $\nu(a)-a \to 0$ as $a \to \infty$.

For $-k \leq i \leq k$, let $H_i(a)$ be the substrip of $H(a)$ given by

$$H_i(a) = \{z \in H(a)|\text{Re } z \geq a, (2i-1)\pi \leq \text{Im} z \leq (2i+1)\pi\}.$$

See Figure 7.

For $j = 1,\dots,n-2$, define the functions $w_j(\lambda) = E^j_\lambda(0)$. Note that each w_j is a function of the parameter λ and is analytic. For example, $w_1(\lambda) = \lambda$ and $w_2(\lambda) = \lambda e^\lambda$.

For $j = 1,\dots,n-2$, define

$$I_{s_1\dots s_j}(a) = \{\lambda \in L(a)|w_i(\lambda) \in H_{s_i}(a) \quad \text{for} \quad i = 1,\dots,j\}.$$

Note that $I_{s_1}(a) = L(a) \cap H_{s_1}(a)$ and that the $I_{s_1\dots s_j}$ are nested, assuming they are nonempty. The following Proposition shows that each of the $I_{s_1\dots s_j}$ consists of a single vertical segment.

We say that a smooth curve $\mu(t)$ in $H_{s_i}(a)$ is a *vertical curve* if the curve connects the upper and lower boundaries of $H_{s_i}(a)$.

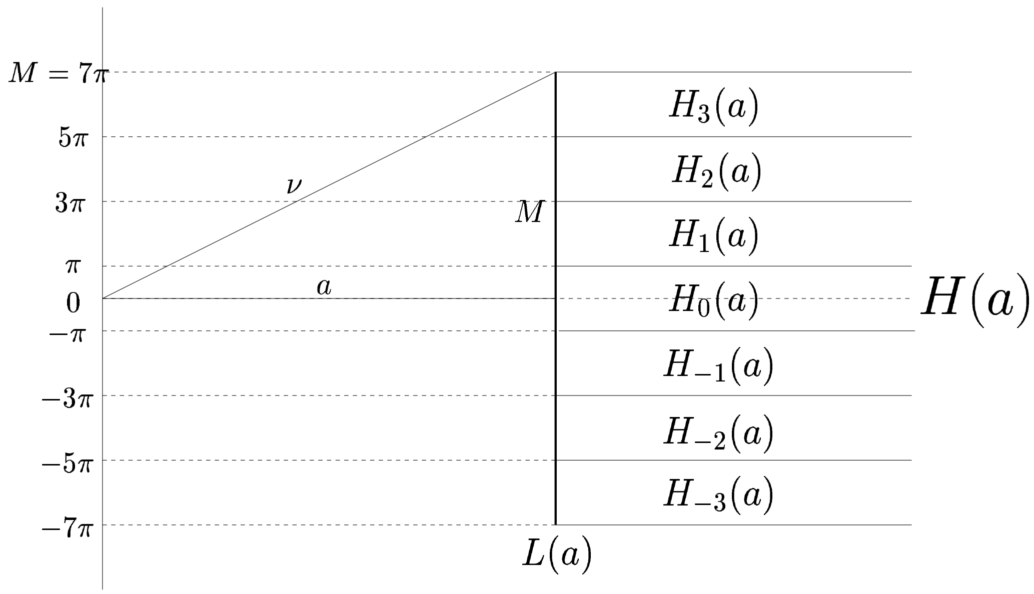

Figure 7: The sets $H(a)$, $L(a)$ and the substrips $H_i(a)$ for the case $\widehat{s} = 3$.

Proposition 3.4. *There exists $a_0 > M$ such that if $a > a_0$ and $1 \leq j \leq n-2$, the set $\{w_j(\lambda) \mid \lambda \in I_{s_1 \ldots s_j}(a)\}$ consists of a single vertical curve in $H_{s_j}(a)$. Hence $I_{s_1 \ldots s_j}$ is a single vertical segment.*

Proof. We parametrize the segment $I_{s_1}(a)$ as $\lambda(t) = a + (2s_1\pi + t)i$ for $t \in (-\pi, \pi)$ and consider the set

$$J_{s_1 s_2}(a) = \{\lambda \in I_{s_1}(a) \mid w_2(t) \subset H(a)\},$$

where $w_2(t) := w_2(\lambda(t)) = \lambda(t)e^{\lambda(t)}$. We will show that given any $\varepsilon > 0$ and taking a large enough,

$$|\operatorname{Arg} w_2'(t) - \frac{\pi}{2}| < \varepsilon, \tag{1}$$

for any t such that $\lambda(t) \in J_{s_1 s_2}(a)$. This implies that, when t runs from $-\pi$ to π, every time that the curve $w_2(t)$ crosses the strip $H(a)$, its tangent vector points upwards and it is almost vertical. It follows that at those instances, the imaginary part of $w_2(t)$ is an increasing function of t and hence, the curve can cross the strip only once. We proceed now to show (1).

Set a_0 large enough so that

$$|\operatorname{Arg} \lambda(t)| < \frac{\varepsilon}{3(n-3)} := \varepsilon'$$

for all $t \in (-\pi, \pi)$.

The tangent vector to $w_2(t)$ is

$$w_2'(t) = \lambda'(t)e^{\lambda(t)}(1 + \lambda(t)) = i\, e^{\lambda(t)}(1 + \lambda(t))$$

and thus

$$|\operatorname{Arg} w_2'(t) - \frac{\pi}{2}| = |\operatorname{Arg} e^{\lambda(t)} + \operatorname{Arg}(1 + \lambda(t))| \leq |\operatorname{Arg} e^{\lambda(t)}| + \varepsilon'.$$

If $\lambda(t) \in J_{s_1 s_2}$, it is clear that $|w_2(t)| > |\lambda(t)|$. Since both are inside the strip $H(a)$, we have that $\varepsilon' > |\operatorname{Arg} w_2(t)| = |\operatorname{Arg} \lambda(t) + \operatorname{Arg} e^{\lambda(t)}|$. It is then easy to see that $|\operatorname{Arg} e^{\lambda(t)}| < 2\varepsilon'$. Plugging this in the expression above, we obtain.

$$|\operatorname{Arg} w_2'(t) - \frac{\pi}{2}| < 3\varepsilon' = \frac{\varepsilon}{n-3} < \varepsilon,$$

as required.

We now proceed to look at $I_{s_1 s_2 s_3}$ which will ilustrate the general case. As above, consider

$$J_{s_1 s_2 s_3}(a) = \{\lambda \in I_{s_1 s_2}(a) \mid w_3(t) \subset H(a),\}$$

where $w_3(t) = w_3(\lambda(t)) = \lambda(t)e^{w_2(t)}$ and $w_2(t) \in H_{s_2}(a)$. For these values of t, we will show

$$|\operatorname{Arg} w_3'(t) - \frac{\pi}{2}| < \varepsilon, \tag{2}$$

The tangent vector to $w_3(t)$ is

$$w_3'(t) = e^{w_2(t)}(\lambda'(t) + \lambda(t)w_2'(t)) = e^{w_2(t)}(i + \lambda(t)w_2'(t))$$

and thus

$$\operatorname{Arg} w_3'(t) = \operatorname{Arg} e^{w_2(t)} + \operatorname{Arg}(i + \lambda(t)w_2'(t))$$

We claim that

$$\frac{\pi}{2} - 4\varepsilon' < \operatorname{Arg}(i + \lambda(t)w_2'(t)) < \frac{\pi}{2} + 4\varepsilon'$$

Indeed, we showed above that

$$\frac{\pi}{2} - 3\varepsilon' < \operatorname{Arg} w_2'(t) < \frac{\pi}{2} + 3\varepsilon'.$$

Moreover, since $|\operatorname{Arg}\lambda(t)| < \varepsilon'$, we obtain

$$\frac{\pi}{2} - 4\varepsilon' < \operatorname{Arg}(\lambda(t)w_2'(t)) < \frac{\pi}{2} + 4\varepsilon'.$$

Finally, it remains to add the vector i to this expression, which makes the argument even closer to $\pi/2$.

To finsih the proof of (2) observe that, by the same argument as in the first case, $\operatorname{Arg} w_3(t) = \operatorname{Arg}(\lambda(t)e^{w_2(t)}) < \varepsilon'$ and hence $|\operatorname{Arg} e^{w_2(t)}| < 2\varepsilon'$. Putting all this together we have

$$\frac{\pi}{2} - 6\varepsilon' < \operatorname{Arg} w_3'(t) < \frac{\pi}{2} + 6\varepsilon'$$

as we wanted to prove.

It is easy to check that we may iterate this procedure and obtain that, for $j = 2, \ldots, n-2$ and for all t such that $\lambda(t) \in J_{s_1 \ldots s_j}(a)$,

$$|\operatorname{Arg} w_j'(t) - \frac{\pi}{2}| < 3(j-1)\varepsilon' = (j-1)\frac{\varepsilon}{n-3} \leq \varepsilon,$$

which concludes the proof of the proposition. □

Proposition 3.5. *Let $\varepsilon > 0$. There exists $a_0 > M$ such that if $a > a_0$, then there is $\lambda_a \in L(a)$ satisfying*

1. *$w_i(\lambda_a) \in H_{s_i}(a)$ for $i = 1, \ldots, n-2$.*
2. *$\operatorname{Im}(w_{n-2}(\lambda_a)) = (2s_{n-2}+1)\pi$.*
3. *$E^{j-1}_{(a-\varepsilon)}(a-\varepsilon) \leq \operatorname{Re} w_j(\lambda_a) \leq |w_j(\lambda_a)| \leq E^{j-1}_{(a+\varepsilon)}(a+\varepsilon)$ for $j = 1, \ldots, n-2$, where E_b is the real exponential $E_b(x) = be^x$.*

Proof. By the proposition above if $\lambda \in I_{s_1 \ldots s_j}(a)$, then the curve $\lambda \to w_j(\lambda)$ is a vertical curve in $H_{s_j}(a)$. We will show that, moreover,

$$E^{j-1}_{(a-\varepsilon)}(a-\varepsilon) \leq \operatorname{Re} w_j(\lambda) \leq E^{j-1}_{(a+\varepsilon)}(a+\varepsilon)$$

for each j. Then λ_a will be defined as the upper endpoint of $I_{s_1 \ldots s_{n-2}}(a)$.

If $\lambda \in V_{s_1}(a)$, then $\exp(\lambda)$ lies on a circle of radius e^a centered at 0. Hence $\lambda \to w_2(\lambda) = \lambda e^{\lambda}$ is a nearly circular arc contained in the annulus

$$E_a(a) \leq |z| \leq E_\nu(\nu) \qquad (3)$$

where we recall that $\nu = \max_{z \in L(a)} |z|$. This arc crosses $H_{s_2}(a)$ in a single vertical curve η_2, provided a is sufficiently large.

Given $\varepsilon > 0$, we claim we may choose a large enough so that, if $\lambda \in I_{s_1 s_2}(a)$ then

$$E_{a-\varepsilon}(a-\varepsilon) \leq \text{Re } w_2(\lambda) \leq |w_2(\lambda)| \leq E_{a+\varepsilon}(a+\varepsilon). \tag{4}$$

Indeed, both estimates are deduced from Equation (3). The lower estimate holds since the circle of radius $E_a(a)$ meets $H(a)$ in a nearly vertical arc. The upper estimate follows since $\nu(a) - a \to 0$ as $a \to \infty$ and hence we may choose a so that $\nu < a + \varepsilon$.

Now we exponentiate points on η_2. The result is a curve whose endpoints lie in $\mathbf{R}^-$. Multiplication of this curve by the appropriate $\lambda \in I_{s_1 s_2}(a)$ expands this curve, but the image must cross $H_{s_3}(a)$ in a single vertical curve wich we denote by η_3.

As above, we claim that by choosing a large enough we have that, for $\lambda \in I_{s_1 s_2 s_3}(a)$,

$$E^2_{a-\varepsilon}(a-\varepsilon) \leq \text{Re } w_3(\lambda) \leq |w_3(\lambda)| \leq E^2_{a+\varepsilon}(a+\varepsilon). \tag{5}$$

The upper estimate holds since

$$|w_3(\lambda)| = |\lambda| \exp(\text{Re}(w_2(\lambda))) \leq \nu \exp(E_{a+\varepsilon}(a+\varepsilon)) \leq E^2_{a+\varepsilon}(a+\varepsilon).$$

To obtain the lower estimate, first set $R_{a,\varepsilon} = a \exp(E_{a-\varepsilon}(a-\varepsilon))$ and observe that, by Equation (4),

$$|w_3(\lambda)| = |\lambda| e^{\text{Re}(w_2(\lambda))} \geq R_{a,\varepsilon}.$$

By a simple trigonometric argument (see Figure 8) one can see that

$$\text{Re}(w_3(\lambda)) \geq \sqrt{R_{a,\varepsilon} - M^2}. \tag{6}$$

We then have, on one hand

$$R_{a,\varepsilon} - \sqrt{R_{a,\varepsilon} - M^2} \underset{a\to\infty}{\longrightarrow} 0$$

and, on the other hand

$$R_{a,\varepsilon} - E^2_{a-\varepsilon}(a-\varepsilon) = \varepsilon \exp(E_{a-\varepsilon}(a-\varepsilon)) \underset{a\to\infty}{\longrightarrow} \infty.$$

Putting everything together, we obtain the lower estimate in Equation (5).

It is now clear that, continuing in the same fashion we obtain the required $I_{s_1 s_2 \ldots s_j}(a)$. Note that, by construction, if λ is the upper endpoint of $I_{s_1 s_2 \ldots s_j}(a)$, then $z_j(\lambda) \in \partial H_{s_j}(a)$. Hence, pick λ to be the upper endpoint of $I_{s_1 s_2 \ldots s_{n-2}}(a)$ and then $\text{Im}(w_{n-2}(\lambda_a)) = (2s_{n-2}+1)\pi$. This completes the proof of the Proposition. $\square$

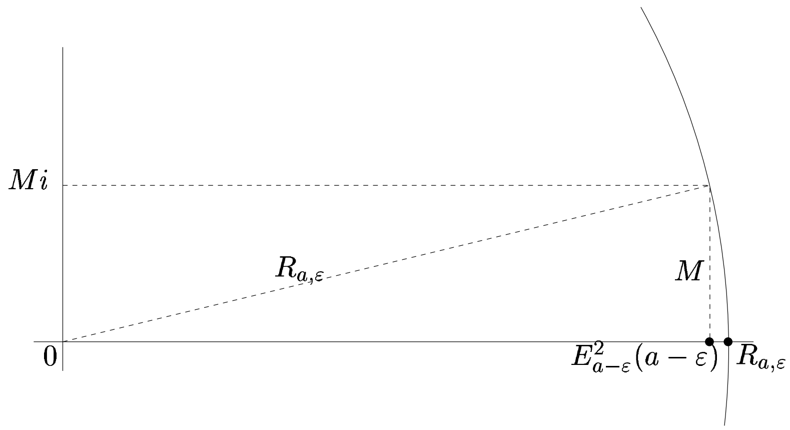

Figure 8: The construction in Equation (6)

Proposition 3.6. *Choose λ_a as in the proposition above. Then E_{λ_a} has an attracting cycle of period n.*

Proof. By the same arguments as in Proposition 3.5, it is clear that

$$E^{n-2}_{a-\varepsilon}(a-\varepsilon) \le |w_{n-1}(\lambda)| \le E^{n-2}_{a+\varepsilon}(a+\varepsilon).$$

We know that $\operatorname{Im} w_{n-2}(\lambda_a) = (2s_{n-2}+1)\pi$, and hence it follows that

$$\operatorname{Re}\, w_{n-1}(\lambda_a) \le -E^{n-2}_{a-\varepsilon}(a-\varepsilon)\cos(\operatorname{Arg}\lambda_a)$$

since $\operatorname{Arg}\, w_{n-1}(\lambda_a) = \operatorname{Arg}\,(\lambda_a)+\pi$. Now $|\operatorname{Arg}\lambda_z| \le \pi/4$ so that

$$\operatorname{Re}\, w_{n-1}(\lambda_a) \le -(E^{n-2}_a(a)-1)/\sqrt{2}.$$

Let B be an open ball of radius 1 about $w_{n-1}(\lambda_a)$. The preimages of B containing $w_j(\lambda_a)$ for $j=1,\dots,n-2$ are open sets, and $E^{n-1-j}_{\lambda_a}$ maps them univalently onto B. Let U be the preimage of B containing 0. Then $E^{n-1}_{\lambda_a}$ maps U univalently onto B.

Let $F : B \to U$ denote the appropriate branch of the inverse of $E^{n-1}_{\lambda_a}$ taking $w_{n-1}(\lambda_a)$ to 0. We have

$$\begin{aligned} |F'(w_{n-1}(\lambda_a))| &= \left| \frac{1}{\prod_{j=0}^{n-2} E'_{\lambda_a}(w_j(\lambda_a))} \right| \\ &= \frac{1}{\prod_{j=1}^{n-1} |w_j(\lambda_a)|} \\ &\ge \frac{1}{\prod_{j=1}^{n-1} (E^{j-1}_{a+\varepsilon}(a+\varepsilon))} \end{aligned}$$

by Proposition 3.5. By the Koebe 1/4 Theorem we have:

$$\text{dist } (0, \partial U) \geq \frac{1}{4} |F'(w_{n-1}(\lambda_a))|$$

$$\geq \frac{1}{4} \frac{1}{\prod_{j=1}^{n-1}(E_{a+\varepsilon}^{j-1}(a+\varepsilon))}.$$

Now consider $w_n(\lambda_a)$. We have

$$\begin{aligned}
|w_n(\lambda_a)| &= |E_{\lambda_a}(w_{n-1}(\lambda_a))| = |\lambda_a| \exp(\text{Re}(w_{n-1}(\lambda_a))) \\
&\leq (a+\varepsilon)\exp(-\frac{1}{\sqrt{2}} E_{a-\varepsilon}^{n-2}(a-\varepsilon)) \\
&<< \frac{1}{4} \frac{1}{\prod_{j=0}^{n-1}(E_{a+\varepsilon}^{j-1}(a+\varepsilon))}.
\end{aligned}$$

The last inequality follows (for a large enough and for ε small enough) since the expression for $|E_{\lambda_a}(w_{n-1}(\lambda_a))|$ contains one higher iterate of E_a. Hence $w_n(\lambda_a)$ lies well within U. We claim that $E_{\lambda_a}(B) \subset U$ as well. Indeed, for $w \in B$, we have

$$\begin{aligned}
|E'_{\lambda_a}(w)| &\leq |E'_{\lambda_a}(w_{n-1}(\lambda_a)+1)| \\
&= |\lambda_a| \exp(\text{Re } w_{n-1}(\lambda_a)+1) \\
&\leq (a+\varepsilon)\exp(-\frac{1}{\sqrt{2}} E_{a-\varepsilon}^{n-2}(a-\varepsilon)+1) \\
&<< \frac{1}{4} \frac{1}{\prod_{j=1}^{n-1}(E_{a+\varepsilon}^{j-1}(a+\varepsilon))}
\end{aligned}$$

as above. This shows that $E_{\lambda_a}(B)$ lies well within U since

$$|E_{\lambda_a}(w) - E_{\lambda_a}(w_{n-1}(\lambda_a))| \leq \max_{w \in B} |E'_{\lambda}(w)|.$$

It follows that E_{λ_a} has an attracting cycle of period n that lies close to $w_j(\lambda_a)$ for $j = 0, \ldots, n-1$. □

The following proposition completes the proof of Theorem 3.2.

Proposition 3.7. *For λ_a as in Proposition 3.6 and a large enough, $S(\lambda_a) = 0 s_1 s_2 \ldots s_{n-2}*$.*

Proof. Let $\gamma(t) = t + (2s_{n-2}+1)\pi i$ with $t \geq \text{Re } w_{n-2}(\lambda_a)$ so $w_{n-2}(\lambda_a)$ is the left hand endpoint of this horizontal line. We claim that $\gamma(t)$ belongs to the basin of attraction of the attracting cycle. Indeed, $E_{\lambda_a}(\gamma(t))$ is a straight line lying to the left of $w_{n-1}(\lambda_a)$. Hence $|E^2_{\lambda_a}(\gamma(t))| \leq |w_n(\lambda_a)|$ and it follows that this line lies in the immediate basin containing $w_{n-1}(\lambda_a)$.

For any $\varepsilon > 0$ we let $\tau = \varepsilon/n$. Then for a sufficiently large we have $|\text{Arg } w_j(\lambda_a)| \leq \tau$ for $j = 1, \dots, n-2$. This follows since $|\text{Arg } w_j(\lambda_a)| \leq |\text{Arg } (a + (2\widehat{s}+1)\pi i)|$ which may be made arbitrarily small as a increases.

Now let $\mu_j(t)$ denote the curve that contains $w_{n-2-j}(\lambda_a)$ and satisfies $E^j_{\lambda_a}(\mu_j(t)) = \gamma(t)$ for $t \geq \text{Re}\, w_{n-2}(\lambda_a)$ and $j = 1, \dots, n-2$. So $\mu_1(t)$ contains $w_{n-3}(\lambda_a)$ while $\mu_{n-2}(t)$ contains 0. By construction, each μ_j is in a different component of the immediate basin of the attracting cycle. To prove the result, we will show that $\mu_j(t) \subset H_{s_{n-2-j}}(a)$ for each $j \leq n-3$ and $|\,\text{Im}(\mu_{n-2}(t))| < \pi$.

Consider $\mu_1(t)$. We have $E_{\lambda_a}(\mu_1(t)) = \gamma(t)$ so that

$$E'_{\lambda_a}(\mu_1(t)) \cdot \mu'_1(t) = \gamma'(t).$$

Therefore

$$\text{Arg } E'_{\lambda_a}(\mu_1(t)) + \text{Arg } \mu'_1(t) = \text{Arg } \gamma'(t) = 0$$

and consequently

$$\begin{aligned} |\text{Arg } \mu'_1(t)| &= |\text{Arg } E'_{\lambda_a}(\mu_1(t))| \\ &= |\text{Arg } E_{\lambda_a}(\mu_1(t))| \\ &= |\text{Arg } \gamma(t)| \\ &\leq \tau. \end{aligned}$$

In particular, this implies that $\mu_1(t)$ lies to the right of its endpoint, $w_{n-3}(\lambda_a)$, for $t > \text{Re}\, w_{n-2}(\lambda_a)$.

Continuing inductively, we find that

$$|\text{Arg } \mu'_j(t)| \leq \tau j$$

so that $|\text{Arg } \mu'_j(t)| \leq \varepsilon$ for all j, and that each $\mu_j(t)$ lies to the right of its endpoint, $w_{n-z-j}(\lambda_a)$.

Now suppose that Im $\mu_j(t_0) = (2k+1)\pi$ for some $k \in \mathbf{Z}$. It follows that $E_{\lambda_a}(\mu_j(t_0))$ lies in the left half plane. But $E_{\lambda_a}(\mu_j(t)) = \mu_{j-1}(t)$ if $j > 1$ and $E_{\lambda_a}(\mu_1(t)) = \gamma(t)$. This contradicts the fact that $\mu_{j-1}(t_0)$ lies to the right of the endpoint of μ_{j-1}. Hence each μ_j must lie in a horizontal strip of width at most 2π and contained between the translates of $\gamma(t)$. This implies that $\mu_j(t) \subset H_{s_{n-2-j}}(a)$, and the result follows. $\square$

This concludes the proof of Theorem 3.3. To end the proof of Theorem A, observe that the result holds for any a larger than a certain value a_0. Following the construction, we then see that we have constructed a curve of λ_a values, one for each sufficiently large $a \in \mathbf{R}$, having the property that Re $\lambda_a = a$ and $S(\lambda) = s$. Note that λ_a lies in the intervals $I_{s_1 \dots s_{n-2}}(a)$ and, by construction, we have Im $(I_{s_1 \dots s_{n-3}\alpha}(a)) <$ Im $(I_{s_1 \dots s_{n-3}\beta}(a))$ if and only if $\alpha < \beta$. Thus, the hyperbolic components *of the same period* are ordered lexicographically. The following Corollary shows how the components of period $n+1$ insert in between the components of period n.

Corollary 3.8. *Suppose* λ_a *and* $\widetilde{\lambda}_a$ *have kneading sequences* $0s_1 \dots s_{n-2}*$ *and* $0s_1 \dots (s_{n-2}+1)*$ *for a sufficiently large. Then, given any* $k \in \mathbb{Z}$*, there is* $\lambda_a(k)$ *with Re* $\lambda_a(k) = a$ *and* $S(\lambda_a(k)) = 0s_1 \dots s_{n-2}+1\, k*$.

Proof. By construction, the λ values in the vertical segment in between λ_a and $\widetilde{\lambda}_a$, are exactly those belonging to $I_{s_1 \dots s_{n-2}+1}(a)$. Hence, if we iterate the process one step further to obtain $\lambda_a(k)$ with $S(\lambda_a(k)) = 0s_1 \dots s_{n-2}+1\, k*$, we must iterate once more for values of λ in this segment. Hence, each of the $\lambda(k)$ belongs to $I_{s_1 \dots s_{n-2}+1}(a)$. □

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