Arnold Disks and the Moduli of Herman Rings of the Complex Standard Family

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Dedicated to Bodil Branner on her 60th birthday

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Abstract

We consider the Arnold family of analytic diffeomorphisms of the circle $x \mapsto x + t + \frac{a}{2\pi}\sin(2\pi x) \mod (1)$, where $a, t \in [0, 1)$ and its complexification $f_{\lambda,a}(z) = \lambda z e^{\frac{a}{2}(z-\frac{1}{z})}$, with $\lambda = e^{2\pi i t}$a holomorphic self map of $\mathbb{C}^*$. The parameter space contains the well known Arnold tongues $\mathcal{T}_\alpha$ for $\alpha \in [0, 1)$ being the rotation number. We are interested in the parameters that belong to the irrational tongues and in particular in those for which the map has a Herman ring. Our goal in this paper is twofold. First we are interested in studying how the modulus of this Herman ring varies in terms of the parameter a, when a tends to 0 along the curve $\mathcal{T}_\alpha$. We survey the different results that describe this variation including the complexification of part of the Arnold tongues (called *Arnold disks*) which leads to the best estimate. To work with this complex parameter values we use the concept of the *twist coordinate*, a measure of how far from symmetric the Herman rings are. Our second goal is to investigate the slice of parameter space that contains all maps in the family with twist coordinate equal to one half, proving for example that this is a plane in $\mathbb{C}^2$. We show a computer picture of this slice of parameter space and we also present some numerical algorithms that allow us to compute new drawings of non–symmetric Herman rings of various moduli.

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1 Introduction

In this paper we deal with the holomorphic maps of $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ given by

$$f_{\lambda,a}(z) = \lambda z \mathrm{e}^{\frac{a}{2}(z-\frac{1}{z})},$$

for $\lambda = e^{2\pi it} \in \mathbb{S}^1$ and $a \in [0,1)$ (to start with). This family is called *the complex Arnold (or standard) family*, since $f_{\lambda,a}$ restricted to the unit circle, corresponds with the well known Arnold family of circle maps

$$x \mapsto x + t + \frac{a}{2\pi}\sin(2\pi x) \pmod 1.$$

For the given range of parameter values, the maps $f_{\lambda,a}$ are symmetric with respect to the unit circle, and they have two critical points which lie on the negative real line. The points at 0 and ∞ are essential singularities. Since the restriction of these maps to the unit circle is a diffeomorphism of the circle, we may assign a well defined rotation number to each member of the family. In this paper we consider the maps with irrational rotation numbers. (See [F] for a description of the dynamics for rational values of the rotation number.)

We consider the level sets of a given rotation number in the (t,a)–parameter plane. Given $\alpha \in [0,1)$ the set $\mathcal{T}_\alpha = \{(t,a) \in [0,1) \times [0,1) \mid \mathrm{rot}\#(f_{\lambda,a}) = \alpha,\ \lambda = e^{2\pi it}\}$ is called the Arnold tongue of rotation number α. It is well known that $\mathcal{T}_\alpha$ is a set with interior if $\alpha \in \mathbb{Q}$ and, if $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ then $\mathcal{T}_\alpha$ is a Lipschitz curve connecting $(\alpha, 0)$ with $(t', 1)$ for some $t' \in (0,1)$ [A]. Indeed, the curve can be parametrized as $\{(t(a), a) \mid 0 \le t \le 1\}$ where the function $a \mapsto t(a)$ is Lipschitz. See Figure 1.

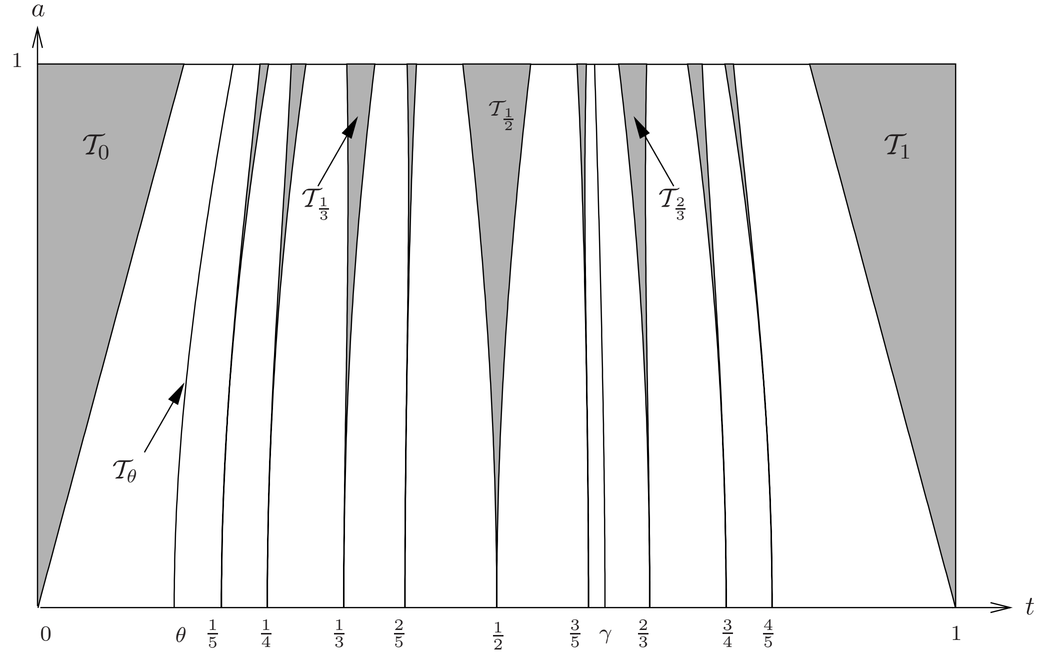

Figure 1: Rational Arnold tongues in the parameter space of the family $f_{\lambda,a}$ for $\lambda = e^{2\pi it}$, $t \in \mathbb{R}/\mathbb{Z}$, up to denominator 5. Irrational tongues for $\gamma = \frac{\sqrt{5}-1}{2}$ and $\theta = \sqrt[5]{2} - 1$. (Picture made by Lukas Geyer.)

Let α be the rotation number of $f_{\lambda,a}$. It follows from theorems of Poincaré and Denjoy (see e.g. [dMvS]) that, if α is irrational, then $f_{\lambda,a}$ is topologically conjugate to the the rigid rotation $\mathcal{R}_\alpha(z) = \mathrm{e}^{2\pi i\alpha}z$. This means that there exists a homemorphism $\phi : \mathbb{S}^1 \to \mathbb{S}^1$ such that $f_{\lambda,a} \circ \phi = \phi \circ \mathcal{R}_\alpha$ on the unit circle. If ϕ can be chosen to be real analytic, we say that $f_{\lambda,a}$ is *analytically linearizable.*

If a map can be analytically linearized on the unit circle, then the conjugacy ϕ extends (also as a conjugacy) to a neighborhood of the unit circle. As a consequence, there exists a maximal domain, H, called a *Herman ring* around the unit circle where the map can be linearized. That is, there exist a number $0 < r < 1$ and a conformal map $\phi : A_r \to H$ which conjugates $\mathcal{R}_\alpha$ to $f_{\lambda,a}$, where $A_r = \{z \in \mathbb{C} \mid r < |z| < 1\}$. See Figure 2.

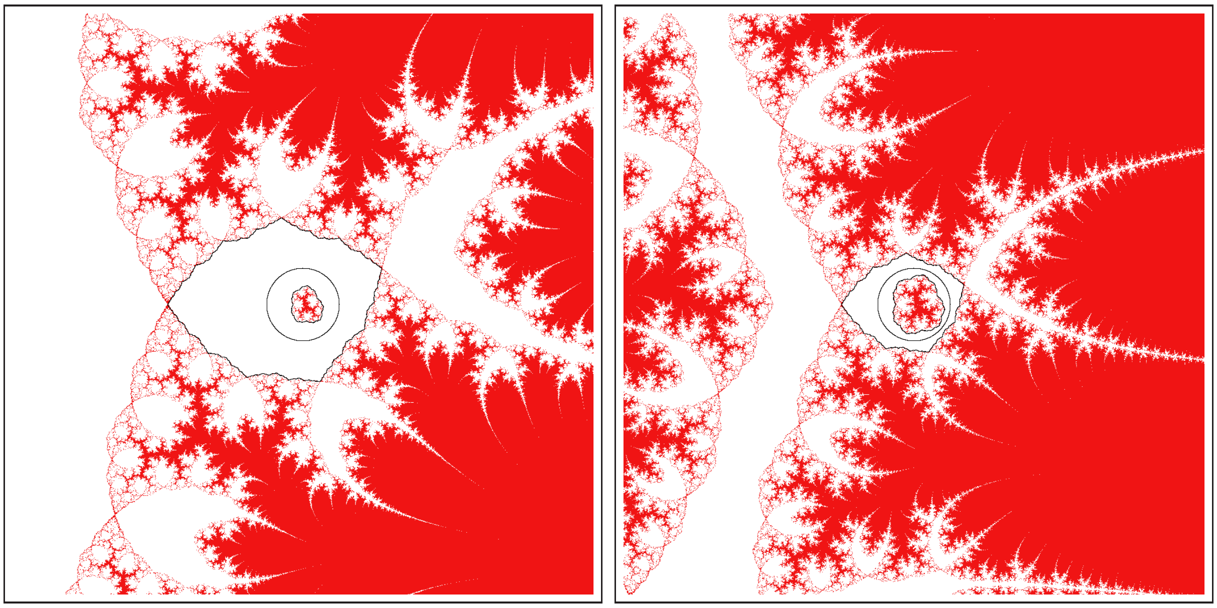

Figure 2: Herman rings for $f_{\lambda,a}$ where we have chosen the parameters $\lambda \in \mathbb{S}^1$ and $a \in (0,1)$ so that the rotation number equals the golden mean. The unit circle is drawn inside each of the rings. Range: $[-8,8] \times [-8,8]$.

The *modulus of* H is $m = \mathrm{mod}(H) = \frac{1}{2\pi}\log\frac{1}{r}$ and we define the *size of* H to be $s = \mathrm{size}(H) = e^{\pi\,\mathrm{mod}(H)}$. Observe that A_r is conformally equivalent to an annulus of the form $\{\frac{1}{s} < |z| < s\}$.

A natural question, not yet solved for general functions, is to know which (optimal) conditions on the map and the rotation number allow us to conclude that the map is analytically linearizable. Works of Rüssmann [Rü], Herman [Her] and Yoccoz [Y] conclude that an analytic circle map sufficiently close to a rigid rotation and whose rotation number is a Brjuno number, is always analytically linearizable. In our case, the condition on the map translates into requiring that the parameter a be small enough. On the other hand, for this particular family, it is known that the Brjuno condition is optimal in the following sense: any member of the Arnold family which is analytically linearizable must have a Brjuno rotation number. This was proven by Geyer in [G], using holomorphic surgery to relate the complex Arnold family to the semistandard map $E_\alpha(z) = e^{2\pi i\alpha}ze^z$, and then establishing the optimality of the Brjuno condition for the maps E_α (see Proposition 2.1).

The semistandard family E_α is in many ways very related to the complex Arnold family. It is often fruitful to rescale the Arnold family to make it a perturbation not of the rigid rotation but of the semistandard map. Indeed, if we change variables by letting $w = \frac{az}{2}$ we obtain a rescaled family

$$g_{\lambda,a}(w) = \lambda w e^{w} e^{-\frac{a^2}{4w}}.$$

Observe that the invariant circle is now that of radius $a/2$. When $a = 0$, the singular limit of this family is the semistandard map. It is often very convenient to work with the rescaled Arnold family, and in fact we shall do so in many parts of the paper. Since both families are conjugate to each other, the linearizability problems are equivalent.

Observe that for all $\alpha \in \mathbb{C}$, the maps $E_\alpha(z)$ are entire transcendental maps which have $z = 0$ as a fixed point of derivative $\lambda = e^{2\pi i\alpha}$. Hence for $\alpha \in \mathbb{R}/\mathbb{Z}$ this is a neutral fixed point. The linearizability problem for fixed points is very related to the one for circle maps. As before, it consists of knowing under which conditions the map is conformally conjugate to the linear map $z \mapsto \lambda z$, although in this case we require the conjugacy to hold in a neighborhood of the fixed point. When the fixed point is linearizable, the maximal neighborhood Δ_α where this is possible is called a *Siegel disk* (See Figure 3). Hence, if E_α has a Siegel disk Δ_α around 0, there exists a conformal map $\phi : \mathbb{D} \to \Delta_\alpha$ mapping 0 to 0, such that $E_\alpha(\phi(z)) = \phi(\lambda z)$. The quantity $r_\alpha = \mid \phi'(0) \mid$ is called *the conformal radius* of Δ_α.

The linearizability problem for the semistandard map is completely solved, in the sense that it is known that E_α is linearizable around $z = 0$ if and only if α is a Brjuno number ([Bru, G]).

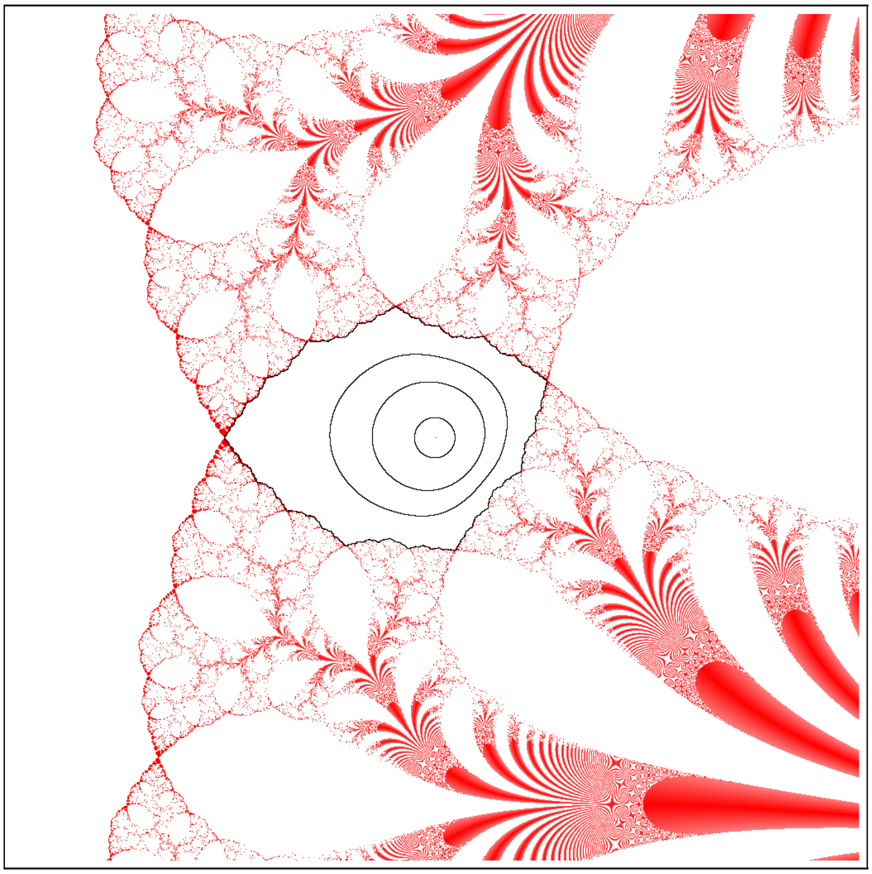

Figure 3: Siegel disk of the function $E_\alpha(z) = e^{2\pi i\alpha} z e^{z}$, with rotation number α, equal to the golden mean. Some orbits have been drawn inside the Siegel disk. Range: $[-2, 2] \times [-2, 2]$.

We now return to the Arnold family. Fix a Brjuno number α. We consider the parameter values for which the rotation number of $f_{\lambda,a}$ is α, and the map is analytically linearizable. That is, the piece (or pieces, a priori) of the Arnold tongue $\mathcal{T}_\alpha$ for which we find a Herman ring in the dynamical plane of $f_{\lambda,a}$. We are interested in understanding how the modulus or the size of the Herman ring varies in terms of the parameter a, precisely when a tends to zero. With this goal in mind, we present a survey of the results that lead to these type of estimates. We do this in two parts: one looking at the "real" parameter space (Section 2) and two, considering its complexification (Section 3), i.e., allowing λ and a to be complex and studying the complex version of (the linearizable part of) the Arnold tongues, called *Arnold disks.* We see how this last point of view leads to the best estimate on the variation of the modulus which is the following.

Theorem 1. *Let α be a fixed Brjuno number and consider the Arnold tongue $\mathcal{T}_\alpha$ of rotation number α. Let $(\lambda(a), a) \in \mathcal{T}_\alpha$ and a be small enough so that $f_{\lambda(a),a}$ has a Herman ring. Let $m(a)$ be its modulus and $s(a)$ the corresponding size. Then, as $a \to 0$,*

$$s(a) = \mathrm{e}^{\pi\, m(a)} = \frac{2r_\alpha}{a} + \mathcal{O}(a),$$

where r_α is the conformal radius of the Siegel disk of the semistandard map E_α.

If we work with the rescaled Arnold family, the moduli of the rings are obviously the same. But changing variables also in the conjugation plane, we see that the scaled Herman ring is conformally equivalent to an annulus of the form $\{\frac{a^2}{4\tilde{s}(a)} < |z| < \tilde{s}(a)\}$ where $\tilde{s}(a) := \frac{a}{2}s(a)$. Observe that this annulus has the circle of radius $a/2$ as the equator, exactly as the ring does. The quantity $\tilde{s}(a)$ is not a conformal invariant.

Using this terminology, Theorem 1 for the rescaled Arnold family reads as follows.

Theorem 2. *Let α be a fixed Brjuno number and consider the Arnold tongue $\mathcal{T}_\alpha$ of rotation number α. Let $(\lambda(a), a) \in \mathcal{T}_\alpha$ and a be small enough so that $f_{\lambda(a),a}$ and hence $g_{\lambda(a),a}$ have a Herman ring whose modulus is $m(a)$ and whose size is $s(a)$. Then, as $a \to 0$, the quantity $\tilde{s}(a)$ has a limit. More precisely,*

$$\tilde{s}(a) = r_\alpha + \mathcal{O}(a^2),$$

where r_α is the conformal radius of the Siegel disk of the semistandard map E_α.

Intuitively, one can say that the limit when $a \to 0$, of the Herman rings of rotation number α of the rescaled Arnold family are the Siegel disk of the semistandard map E_α.

The second part of the paper (see Section 4) is devoted to study a particular slice of the complex parameter space, more precisely the slice containing those maps whose Herman rings have their boundaries rotated half a turn with respect to each other. We first describe the location of this slice in $\mathbb{C}^2$ (see Theorem 4.1) and show a computer drawing of it.

Finally, Section 5 is dedicated to numerics. The computer drawings in this paper needed some new algorithms to be developed, given the difficulties that one encounters when the symmetries of the map are no longer present. In this final part we present these algorithms which are reusable for other types of functions.

2 Real parameter space

In this section we present two results. The first one concerns the parametrization of the linearizable piece of an irrational Arnold tongue and it is the "real" version of Theorem 5 in Section

3. The second result is a first estimate of the size of Herman rings in terms of the parameter a which was obtained in [FSV]. Given a Brjuno number α and its Arnold tongue $\mathcal{T}_\alpha$, we define $\mathcal{T}_\alpha^{\mathrm{AL}}$ as the analytically linearizable part of $\mathcal{T}_\alpha$, i.e., the set of parameter values $(\lambda, a) \in \mathcal{T}_\alpha$ such that $f_{\lambda,a}$ has a Herman ring around $\mathbb{S}^1$.

Theorem 3 ([FG]). *Fix α a Brjuno number and let $f_{\lambda,a}(z) = \lambda z \mathrm{e}^{\frac{a}{2}(z-\frac{1}{z})}$. Then, there exists an $\mathbb{R}-$analytic parametrization*

$$\begin{array}{rccc} \mathcal{F}_\alpha : & (0,1) & \longrightarrow & \mathcal{T}_\alpha^{\mathrm{AL}} \\ & \delta & \longmapsto & \mathcal{F}_\alpha(\delta) = (\lambda(\delta), a(\delta)) \end{array}$$

such that:

(a) *for all $\delta \in (0,1)$, the map $f_{\lambda(\delta),a(\delta)}$ has a Herman ring of modulus $m(\delta) = \frac{1}{\pi} \log \frac{1}{\delta}$ and rotation number α;*

(b) *$\delta \mapsto a(\delta)$ is strictly increasing;*

(c) *$a(\delta) \to 0$ when $\delta \to 0$ and $\lim\limits_{\delta \to 1} a(\delta) = a_* \leq 1$;*

(d) *for all $(\lambda, a) \in \mathcal{T}_\alpha$ such that $a \geq a_*$, the map $f_{\lambda,a}$ has no Herman ring.*

This theorem describes the sets $\mathcal{T}_\alpha^{\mathrm{AL}}$ as connected $\mathbb{R}-$analytic curves that might be the entire Arnold tongue $\mathcal{T}_\alpha$. Moreover, it gives the precise modulus of the Herman ring for each of the parameters δ. On one hand as δ tends to 0, the parameter a tends to 0 and the modulus of the ring tends to infinity (we consider the rigid rotation as having a degenerate Herman ring of infinite modulus). On the other hand, as δ tends to 1, the Herman ring gets thinner and thinner, having in the limit a degenerate Herman ring which contains the unit circle.

Theorem 3 is proven by quasiconformal surgery. We give here an idea of its proof since it illustrates quite well the complex case of the next section.

Proof : Since α is a Brjuno number, for a small enough the map $f_{\lambda,a}$ has a Herman ring. Let us fix a base point, i.e., a pair of parameters (λ_1, a_1) in the Arnold tongue $\mathcal{T}_\alpha$, such that $f_1 := f_{\lambda_1,a_1}$ has a Herman ring H_1 whose modulus we denote by m_1.

Now, given any $s \in (0, \infty)$ the goal is to construct a new map $f_{\lambda(s),a(s)}$ with a Herman ring H_s of modulus $m(s) = s m_1$. Moreover, we want to do this construction in such a way that the map $s \mapsto (\lambda(s), a(s))$ is real analytic and has all the required properties (like monotonicity of $a(s)$). Once this is proven, it is not hard to see that the curve can be reparametrized as desired not depending on a base point.

With this goal in mind, we make a surgery construction which consists only of changing the complex structure of the original map. If H_1 is the Herman ring of f_1, it means that there exists a conformal map $\phi_1 : A_r \to H_1$ where $r = \mathrm{e}^{-2\pi m_1}$, which conjugates $\mathcal{R}_\alpha$ to f_1. We now compose this map with a quasiconformal map $\varphi_s : A_r \to A_{r^s}$ which maps circles to circles. In particular we want it to leave the unit circle invariant and to send the circle of radius r to the circle of radius r^s. Such a map is not hard to compute explicitly, especially if we do so in the covering space of the annulus. It is easy to check that φ_s conjugates $\mathcal{R}_\alpha$ to itself.

We now proceed to change the complex structure on the dynamical plane of f_1. We first change it on the ring H_1 by pulling back the standard complex structure σ_0 on A_{r^s} by the map $\varphi_s \circ \phi_1^{-1}$. This defines a complex structure σ_s on H_1 which has bounded distortion (it is a pull-back by a quasiconformal map) and is invariant under f_1. We then extend σ_s to the whole dynamical plane by using the dynamics of f_1, i.e., pulling back by f_1^n to all the n–th preimages of H_1, and seting $\sigma_s = \sigma_0$ at every point that never falls on H_1 under iteration. This process defines an f_1-invariant complex structure σ_s in all of $\mathbb{C}^*$ with bounded dilatation. By the Measurable Riemann Mapping Theorem, this structure can be integrated, i.e., there exists a quasiconformal homeomorphism $\psi_s : \mathbb{C} \to \mathbb{C}$ that transports σ_s to σ_0. Hence, the map $f_s := \psi_s \circ f_1 \circ \psi_s^{-1}$ is holomorphic and quasiconformally conjugate to f_1. The following diagram commutes.

$$\begin{array}{ccc}
H_s & \xrightarrow{f_s} & H_s \\
{\scriptstyle \phi_s}\uparrow & & \uparrow{\scriptstyle \phi_s} \\
H_1 & \xrightarrow{f_1} & H_1 \\
{\scriptstyle \phi_1}\uparrow & & \uparrow{\scriptstyle \phi_1} \\
A_r & \xrightarrow{\mathcal{R}_\alpha} & A_r \\
{\scriptstyle \varphi_s}\downarrow & & \downarrow{\scriptstyle \varphi_s} \\
A_{r^s} & \xrightarrow{\mathcal{R}_\alpha} & A_{r^s}
\end{array}$$

The set $H_s = \psi_s(H_1)$ is a Herman ring for f_s since the composition $\varphi_s \circ \phi_1^{-1} \circ \psi_s^{-1} : H_s \to A_{r^s}$ is a conformal conjugacy between f_s and $\mathcal{R}_\alpha$. From here one can also see that the modulus of the new ring H_s is equal to $\frac{1}{2\pi} \log \frac{1}{r^s} = sm_1$. Furthermore, f_s must be a member of the complex Arnold family and therefore $f_s = f_{\lambda(s),a(s)}$. This defines the map $s \mapsto (\lambda(s), a(s))$ with the required properties.

□

In view of the theorem above one can ask exactly how the modulus of the Herman ring is tending to infinity, as the parameter a tends to 0. As a first estimate we have the following result, which connects the size of the Herman rings with the conformal radius of the semistandard map of the same rotation number.

Theorem 4 ([FSV]). *Let α be a Brjuno number and r_α the conformal radius of the Siegel disk of the semistandard map $E_\alpha(z) = \mathrm{e}^{2\pi i\alpha} z\mathrm{e}^z$. Let $s(a)$ and $m(a)$ be the size and the modulus of the Herman ring of $f_{\lambda(a),a}$ respectively, with $(\lambda(a), a) \in \mathcal{T}_\alpha^{\mathrm{AL}}$. Then,*

$$s(a) = \mathrm{e}^{\pi m(a)} = \frac{2}{a}(r_\alpha + \mathcal{O}(a \log a)).$$

The proof of Theorem 4 relies on understanding how the maps of the Arnold family are related to the semistandard map E_α. We saw in the introduction how one can relate them by means of a rescaling depending on a, but to really study the limit, it is better to perform a surgery construction that shows why these two families of maps are related. The construction is originally due to Shishikura [S] who used it to construct examples of rational maps with Herman rings starting from polynomials with Siegel disks (and viceversa). Later on, Geyer [G] adapted the proof to the Arnold family and the semistandard map. The result of the construction is summarized in the following proposition.

Proposition 2.1. *Suppose $f = f_{\lambda,a}$ has a fixed Herman ring H with rotation number α. Then the semistandard map $E_\alpha(z) = \mathrm{e}^{2\pi i\alpha} z\mathrm{e}^z$ has a Siegel disk Δ_α and there exists a quasiconformal homeomorphism $\psi : \mathbb{C} \to \mathbb{C}$ and an $E_\alpha-$invariant curve Γ in Δ_α such that*

(a) $\psi(\mathbb{S}^1) = \Gamma$ *and* ψ *maps* $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ *to the unbounded component,* V*, of* $\widehat{\mathbb{C}} \setminus \Gamma$*;*

(b) ψ *conjugates* $f : \widehat{\mathbb{C}} \setminus \mathbb{D} \to \widehat{\mathbb{C}}$ *to* $E_\alpha : \overline{V} \to \widehat{\mathbb{C}}$*;*

(c) $\partial\psi/\partial\bar{z} = 0$ *a.e. on* $\widehat{\mathbb{C}} \setminus \bigcup\limits_{n\geq 0} f^{-n}(\mathbb{D})$ *(in particular* ψ *is conformal in the interior of this set).*

Observe that this proposition relates, at least qualitatively, the members of the Arnold family to the members of the semistandard one. More precisely, it relates all maps in $\mathcal{T}_\alpha^{\mathrm{AL}}$ to the single map E_α. Any Herman ring of rotation number α can be used by this procedure to produce a Siegel disk with the same rotation number.

Remark 2.2. The surgery construction also connects a different unrelated problem for the two families. It is an open problem to find a parameter value α, if it exists, for which E_α has an unbounded Siegel disk (this is a phenomenon which does occur for the exponential family, for example). With this proposition, this becomes equivalent to finding parameter values (λ, a) such that the Herman ring of $f_{\lambda,a}$ contains the essential singularities in its boundary. See [DF] for further discussion.

We proceed to sketch the surgery construction.

Proof of Proposition 2.1: For this proof, let us take the standard annulus normalized in a different way by setting $\mathcal{A}_r = \{z \in \mathbb{C} \mid 1/r < |z| < r\}$ for $r > 1$.

Let $\phi : \mathcal{A}_r \to H$ be a conformal map that conjugates the rigid rotation $\mathcal{R}_\alpha : \mathcal{A}_r \to \mathcal{A}_r$ to $f : H \to H$. Notice that ϕ must be symmetric with respect the unit circle and hence it leaves $\mathbb{S}^1$ invariant.

We now extend ϕ quasiconformally to the unit disk. Denote by $\hat{\phi} : \mathbb{D}_r \to H \cup \mathbb{D}$ a quasiconformal mapping that agrees with ϕ on $\mathbb{D}_r \setminus \mathbb{D}$, maps $\mathbb{D}$ onto $\mathbb{D}$, and fixes 0.

Define a new map $\hat{f} : \mathbb{C} \to \mathbb{C}$ by

$$\hat{f} = \begin{cases} f & \text{on } \mathbb{C} \setminus \mathbb{D}; \\ \hat{\phi} \circ \mathcal{R}_\alpha \circ \hat{\phi}^{-1} & \text{on } \overline{\mathbb{D}}. \end{cases}$$

The map $\hat{f} : \mathbb{C} \to \mathbb{C}$ is a quasiregular mapping with an essential singularity at infinity. It has one critical point (the one of f that is not inside the disk).

The map $\hat{f}$ is not holomorphic on $\mathbb{D}$, but there it preserves the complex structure defined by the Beltrami form

$$\mu = \frac{\bar{\partial}\hat{\phi}^{-1}}{\partial\hat{\phi}^{-1}}.$$

Pulling back this Beltrami form via $\hat{f}$, we see that there exists a Beltrami form $\hat{\mu}$ that coincides with μ on $\mathbb{D}$, vanishes on $\mathbb{C} \setminus \bigcup_{n\geq 0} \hat{f}^{-n}(\mathbb{D})$ and that is invariant by $\hat{f}$, in the sense

$$\hat{f}^*\hat{\mu} = \hat{\mu}.$$

By the Measurable Riemann Mapping Theorem, there exists a quasiconformal homeomorphism $\psi : \mathbb{C} \to \mathbb{C}$ which fixes 0, sends ω (the critical point) to -1 and such that

$$\hat{\mu} = \frac{\bar{\partial}\psi}{\partial\psi}.$$

Then, the map $\psi \circ \hat{f} \circ \psi^{-1} : \mathbb{C} \to \mathbb{C}$ is an entire transcendental map with one critical point at -1, which fixes 0 and is conjugate to the rotation $\mathcal{R}_\alpha$ in a neighborhood of 0. One can see with some further argument (see [G] or [FSV]) that such a map must be the semistandard map, i.e.,

$$E_\alpha = \psi \circ \hat{f} \circ \psi^{-1}.$$

The map ψ is the required conjugacy.

□

To prove the estimate in Theorem 4 one needs to make this surgery construction more explicit and quantitative. The idea is to redo the procedure for the rescaled Arnold family depending on the parameter a and make each of the steps explicit; for example, finding a convenient extension $\hat{\phi}_a$ to the disk so that one can compute its coefficient K_a of quasiconformality. The key part of the proof is finding a good estimate for the quantity $(\psi_a \circ \hat{\phi}_a)'(0)$ in terms of the parameter a (where we are using the notation in the proof above).

3 Complex parameter space

In this section we complexify the parameter space, to improve the bounds obtained in the previous section. The analog of the analytically linearizable part of an (irrational) Arnold tongue is an Arnold disk, which we show to be a disk holomorphically embedded in the parameter space.

Consider the family $\{f_{\lambda,a}\}$ for $\lambda \in \mathbb{C}^*$ and $a \in \mathbb{C}$. Even if λ is not contained in the unit circle and a is not real, $f_{\lambda,a}$ may have a fixed Herman ring. If this is the case there is no reason why it should be symmetric with respect to the unit circle and, indeed, this certainly does not seem to be the case in Figures 4 and 5.

When a vanishes $f_{\lambda,a}$ becomes a linear map $z \mapsto \lambda z$ and we consider that map to have a Herman ring of infinite modulus when $\lambda = \mathrm{e}^{2i\pi\alpha}$ and α is a Brjuno number.

Definition 3.1. Suppose $\alpha \in \mathcal{B}$. Let

$$\mathcal{D}_\alpha = \{(\lambda, a) \in \mathbb{C}^* \times \mathbb{C} : f_{\lambda,a} \text{ has a Herman ring of rotation number } \alpha\}.$$

We call $\mathcal{D}_\alpha$ an *Arnold disk.*

The reason for the choice of the name Arnold disk is given by the following theorem that shows that Arnold disks indeed are disks embedded in $\mathbb{C}^2$.

Theorem 5. *Let α be an arbitrary Brjuno number and denote by r_α the conformal radius of the Siegel disk of the semistandard map E_α. The set $\mathcal{D}_\alpha$ is the image of the unit disk $\mathbb{D}$ under an injective holomorphic mapping*

$$\mathcal{F}_\alpha : \mathbb{D} \to \mathcal{D}_\alpha.$$

This mapping can be taken to satisfy the following.

(a) $\mathcal{F}_\alpha(0) = (e^{2i\pi\alpha}, 0)$, *and* $\mathcal{F}'_\alpha(0) = (0, 2r_\alpha)$;

(b) *letting* $\mathcal{F}_\alpha(\delta) = (\lambda(\delta), a(\delta))$, *we have that λ is even and a is odd, i.e. for all $\delta \in \mathbb{D}$*

$$\lambda(-\delta) = \lambda(\delta),\ \ a(-\delta) = -a(\delta);$$

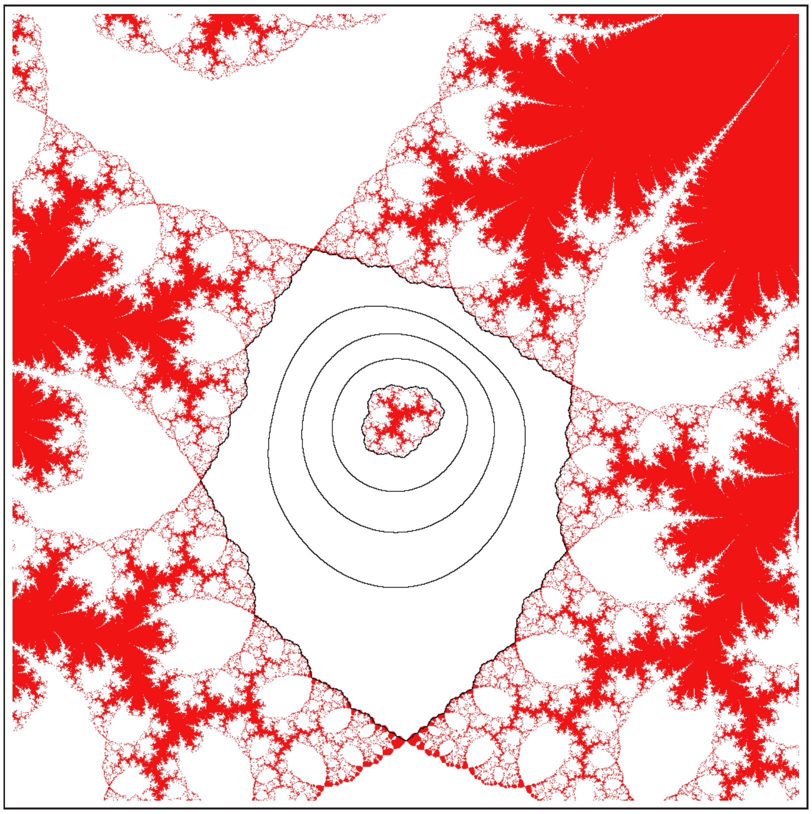

Figure 4: Herman ring of rotation number equal to the golden mean, in the dynamical plane of $f_{\lambda,a}$, where $(\lambda, a) = (\mathrm{e}^{2\pi i 0.622359931841}, 0.5i)$. Range: $[-5,5] \times [-5,5]$.

(c) *for all* $\delta \in \mathbb{D}$, *the modulus* $m(\delta)$ *of the Herman ring of* $f_{\mathcal{F}(\delta)}$ *satisfies*

$$m(\delta) = \frac{1}{\pi} \log \frac{1}{|\delta|};$$

(d) *for some* $\epsilon > 0$, $\mathcal{F}_\alpha(\mathbb{D}_\epsilon)$ *is the graph of a holomorphic map* $a \mapsto \lambda(a)$;

(e) *as* $|a| \to 0$ *the modulus* $m(a)$ *of the Herman ring of* $f_{\lambda(a),a}$ *satifies*

$$e^{\pi m(a)} = \frac{2r_\alpha}{|a|} + \mathcal{O}(a).$$

Part (d) is a corollary of a more general result by Risler (see [Ri]). Notice that part (e) is an improvement of the estimate obtained in the previous section.

We will not prove the properties in the order they are stated. First we see that (d) and (e) follow from the previous three properties. Indeed, part (d) immediately follows from (a), (b) and the implicit function theorem. To see (e) we first note that it follows from (a) that $a(\delta) = 2r_\alpha\delta + \mathcal{O}(\delta^2)$. Since a is an odd function of δ we get $a(\delta) = 2r_\alpha\delta + \mathcal{O}(\delta^3)$, and by the inverse function theorem $\delta(a) = \frac{a}{2r_\alpha} + \mathcal{O}(a^3)$. Combining this fact with (c) we get

$$e^{\pi m(a)} = \frac{1}{|\delta(a)|} = \frac{2r_\alpha}{|a|} + \mathcal{O}(a).$$

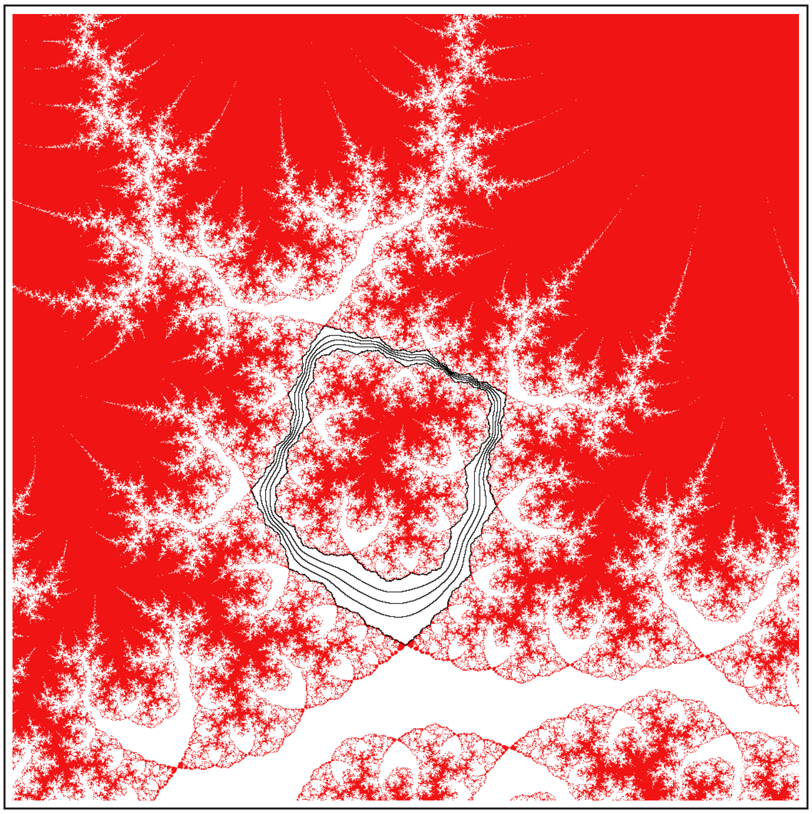

Figure 5: Herman ring of rotation number equal to the golden mean, in the dynamical plane of $f_{\lambda,a}$, where $(\lambda, a) = (\mathrm{e}^{2\pi i 0.642219660059}, i)$. Range: $[-4, 4] \times [-4, 4]$.

Hence to prove the theorem we need to construct the mapping $\mathcal{F}_\alpha$, and establish properties (a), (b) and (c). To do so it is convenient to work with the family $g_{\lambda,b}(w) = \lambda w e^w e^{-b/4w}$, where $\lambda \in \mathbb{C}^*$ and $b \in \mathbb{C}$. The map $w = \frac{a}{2}z$ conjugates $f_{\lambda,a}$ to $g_{\lambda,b}$ with $b = a^2$. Another advantage of working with the g family is that we get rid of the symmetry $f_{\lambda,a}(-z) = -f_{\lambda,-a}(z)$. When $b = 0$ the map $g_{\lambda,b}$ is the semistandard map $w \mapsto \lambda w e^w$ and we adopt the convention that the Siegel disk of $g_{\lambda,0}$ is a Herman ring of (one sided) infinite modulus when $\lambda = e^{2i\pi\alpha}$ and α is a Brjuno number.

We define the analogue of the Arnold disk for the $g_{\lambda,b}$ family as follows.

$$\mathcal{D}'_\alpha = \{(\lambda, b) \in \mathbb{C}^* \times \mathbb{C} : g_{\lambda,b} \text{ has a Herman ring with rotation number } \alpha\}$$

Following our convention $\mathcal{D}'_\alpha$ always contains the point $(\mathrm{e}^{2i\pi\alpha}, 0)$.

We state the analog of Theorem 5.

Proposition 3.2. *There exists a holomorphic injection $\mathcal{G}_\alpha : \mathbb{D} \to \mathbb{C}^* \times \mathbb{C}$ that maps the unit disk onto $\mathcal{D}'_\alpha$, and satisfies*

(a') $\mathcal{G}_\alpha(0) = (e^{2i\pi\alpha}, 0)$;

(c') for all $\delta \in \mathbb{D}$, the modulus $m(\delta)$ of the Herman ring of $g_{\mathcal{G}(\delta)}$ satisfies

$$m(\delta) = \frac{1}{2\pi} \log \frac{1}{|\delta|}.$$

Before giving the proof of the proposition we define an invariant called the twist coordinate of the Herman ring. This is most easily done when the two boundary components of the Herman ring H of $g_{\lambda,b}$ are quasicircles, each containing a critical point. (This is the case when α is of bounded type.) Now, there is a conformal isomorphism $\phi : H \to A_r$ where A_r is the round annulus $\{r < |z| < 1\}$. This map extends as a homeomorphism $\overline{H} \to \overline{A_r}$. We can take this isomorphism to map the outer boundary to the outer boundary and the critical point there to 1. The inner critical point is then mapped to a point $re^{2i\pi\Theta}$. We call the number $\Theta \in \mathbb{R}/\mathbb{Z}$ the twist coordinate of $g_{\lambda,b}$.

In general we cannot assume that ∂H consists of two quasicircles each containing a critical point. But since the boundary components are contained in the closure of the forward orbits of the critical points, they are made up of dynamically marked points, and we can still measure to what extend one boundary is twisted with respect to the other one (see [BFGH] for details).

When b is real and positive and $|\lambda| = 1$ then reflection in the circle with center at the origin and radius $\sqrt{b}/2$ conjugates $g_{\lambda,b}$ to itself, and in this case it is easy to check that the twist parameter equals zero.

Figures 6 and 7 show two examples of Herman rings with a twist coordinate of 1/2 and rotation number equal to the golden mean. The drawings are computed in the dynamical plane of the lift of $f_{\lambda,a}$, that is, $z \mapsto z + t + \frac{a}{2\pi}\sin(2\pi z)$, in order to observe the symmetries better. In fact, these two pictures correspond, once projected back, to the two rings in Figures 4 and 5.

Proof of Proposition 3.2: A portion of the proposition can be deduced from [McS]. Indeed, from their results it can be shown that each component of $\mathcal{D}'_\alpha$ is a pointed disk. Here we show that there is only one component and that the puncture corresponds to $g_{\exp(2i\pi\alpha),0}$. This is done by an explicit construction.

We will give a rough sketch of how to construct the mapping $\mathcal{G}_\alpha$. To give an idea of the mapping, we first describe the inverse map $\Pi : \mathcal{D}'_\alpha \to \mathbb{D}$. The modulus of $\Pi(\lambda, b)$ is given in terms of the modulus m of the Herman ring of $g_{\lambda,b}$ and the argument is determined by the twist coordinate. More precisely

$$\Pi(\lambda, b) = \exp(-2\pi m + 2i\pi\Theta).$$

We give an outline of the construction of the map $\mathcal{G}_\alpha$. For the details (and there are quite a few), refer to [BFGH]. First we choose a base point g_{δ_0} with a Herman ring H with the desired rotation number. The mapping is produced by changing the complex structure on H and its preimages, as we did in the proof of Theorem 3. This time, we not only change the modulus of the ring but also introduce a twist of one boundary with respect to the other one. In this way, for each $\delta \in \mathbb{D}^*$ we obtain a new member of the family $g_{\lambda(\delta),a(\delta)}$ with a Herman ring whose modulus and twist coordinate are

$$\begin{aligned} m(\delta) &= \tfrac{1}{2\pi}\log\tfrac{1}{|\delta|} \\ \Theta(\delta) &= \tfrac{1}{2\pi}\arg(\delta). \end{aligned}$$

This defines the mapping $\mathcal{G}_\alpha$ from $\mathbb{D}^*$ to the parameter space, satisfying property (c'). Since the Herman ring separates 0 and one critical point from ∞ and the other critical point we can deduce that when δ tends towards 0, then b tends toward 0 as well. By a surgery construction one can show that if $g_{\lambda,b}$ has a Herman ring and b is small then λ is close to $e^{2i\pi\alpha}$. It follows that the constructed mapping extends past the puncture as required in (a'). Finally one shows that the construction does not depend on the choice of base point and that Π indeed is an inverse.

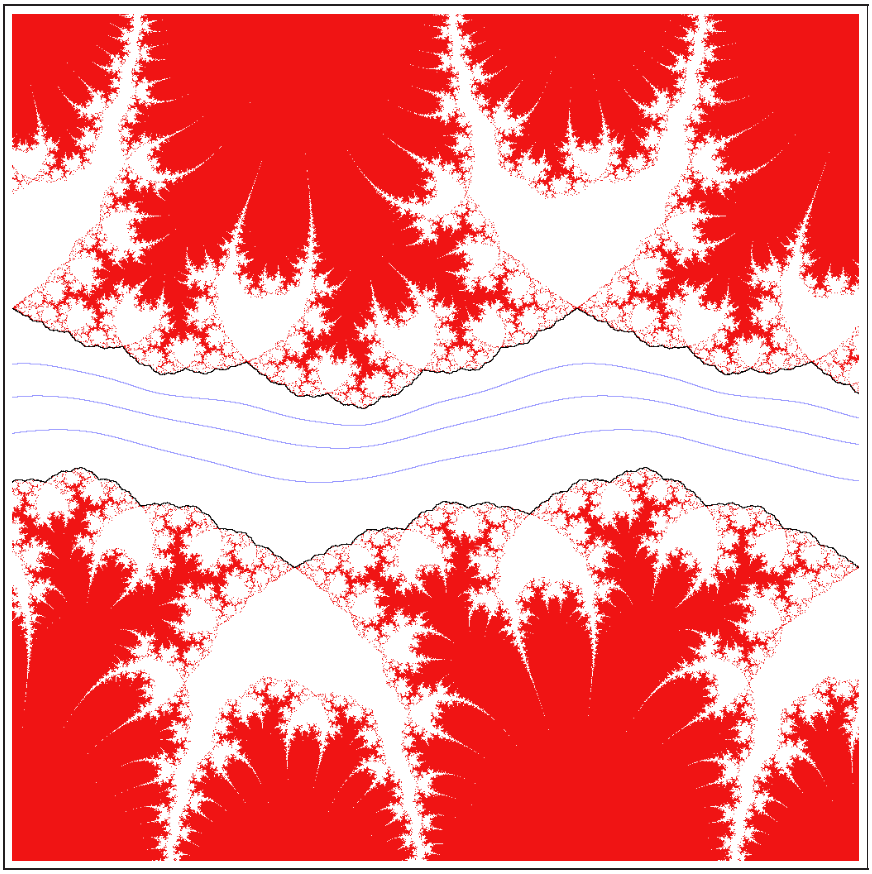

Figure 6: Lift of a Herman ring in the dynamical plane of $z \mapsto z + t + \frac{a}{2\pi}\sin(2\pi z)$. Compare to Figure 4.

$\square$

Let us now finish the proof of Theorem 5. We need to prove properties (a), (b) and (c). Notice that the mapping $(\lambda, a) \mapsto (\lambda, a^2) : \mathcal{D}_\alpha \to \mathcal{D}'_\alpha$ provides a two to one covering map ramified at $(\exp(2i\pi\alpha), 0)$ above $(\exp(2i\pi\alpha), 0)$. Hence, there exists an injective holomorphic map $\mathcal{F}_\alpha : \mathbb{D} \to \mathcal{D}_\alpha$ such that the following diagram commutes:

$$\begin{tikzcd} \mathbb{D} \arrow[r, "\mathcal{F}_\alpha"] \arrow[d, "\delta\mapsto\delta^2"'] & \mathcal{D}_\alpha \arrow[d, "(\lambda,a)\mapsto(\lambda,a^2)"] \\ \mathbb{D} \arrow[r, "\mathcal{G}_\alpha"'] & \mathcal{D}'_\alpha. \end{tikzcd}$$

This mapping is unique if we require that the second coordinate of $\mathcal{F}_\alpha(\delta)$ is real and positive when δ is real and positive. Letting $\mathcal{F}_\alpha(\delta) = (\lambda(\delta), a(\delta))$ we get from the diagram that $(\lambda(\delta), a(\delta)^2) = \mathcal{G}_\alpha(\delta^2)$. It follows that $\lambda(-\delta) = \lambda(\delta)$ so λ is even. It also follows that $a(\delta)^2 = a(-\delta)^2$ so a is either even or odd. Then a has to be odd, because otherwise it would contradict that $\mathcal{F}_\alpha$ is injective. We have proven property (b). Since the Herman ring of $\mathcal{F}_\alpha(\delta)$ is conformally isomorphic to the Herman ring of $\mathcal{G}_\alpha(\delta^2)$ property (c) immediately follows from property (c') in Proposition 3.2.

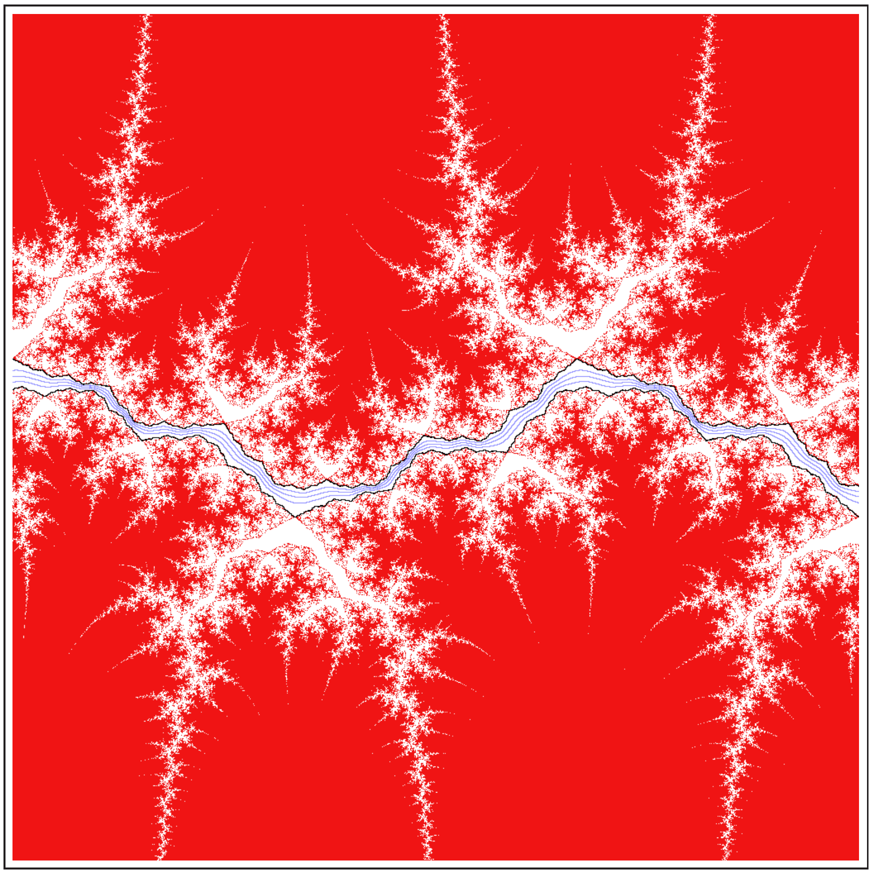

Figure 7: Lift of a Herman ring in the dynamical plane of $z \mapsto z + t + \frac{a}{2\pi}\sin(2\pi z)$. Compare to Figure 5.

So to conclude we need only show that $\mathcal{F}'_\alpha(0) = (\lambda'(0), a'(0)) = (0, 2r_\alpha)$. That $\lambda'(0) = 0$ follows immediately from the fact that λ is even. On one hand, we know from Theorem 1 that

$$\frac{2}{a}(r_\alpha + o(1)) = \mathrm{e}^{\pi m(a)},$$

for $a > 0$. On the other hand, we have from (c) that

$$\mathrm{e}^{\pi m(a(\delta))} = \frac{1}{\delta}.$$

Combining these two facts we get $a(\delta) = \frac{1}{2}r_\alpha\delta + o(\delta)$. This proves (a) and finishes the proof of Theorem 5.

4 The slice of twist coordinate equal to $\frac{1}{2}$.

As it was already mentioned, in general it is not easy to locate complex parameters $(\lambda, a) \in \mathbb{C}^2$ for which the Arnold map $f_{\lambda,a}$ has a Herman ring. The main reason is that, as we saw, these parameters live in surfaces in $\mathbb{C}^2$ isomorphic to disks, one for each fixed rotation number.

There are two exceptional cases where it is not so difficult to locate these parameter values. The first one is the "real" or symmetric case, i.e, when the Herman rings are symmetric with respect to the unit circle or, equivalently, the case where the twist coordinate is equal to 0. Two facts make the computation easier: first, we know that the unit circle is always an invariant curve in the ring, which allows us to compute the rotation number of the map; and second and most important, all these parameters lie on the plane (or cylinder) $\{(\lambda, a) \in \mathbb{R}/\mathbb{Z} \times [0,1)\}$. Consequently we can apply, for example, bissection methods to locate parameter values for which the ring exists and has a given rotation number (see Section 5 for details).

The second exceptional case turns out to be the slice for which the twist parameter is equal to 1/2. That is, the two boundary components of the Herman rings are rotated half a turn with respect to each other (see Section 3). Although the symmetry is broken in this case (there is another kind of symmetry which we will describe later) we still have the important property of having these parameter values located on a plane of $\mathbb{C}^2$, namely $\{(\lambda, a) \in \mathbb{R}/\mathbb{Z} \times i\mathbb{R}\}$. This is exactly what we show in the following proposition.

Proposition 4.1. *Suppose $f_{\lambda,a}$ has a Herman ring. Then, the twist coordinate equals* 1/2 *if and only if $\lambda \in \mathbb{R}/\mathbb{Z}$ and $a = i\tilde{a}$ with $\tilde{a} \in \mathbb{R}$.*

Proof : For the proof we shall use again the rescaled Arnold family $g_{\lambda,b}$. Fix a rotation number $\alpha \in \mathcal{B}$. Recall from Theorem 3.2 that $\mathcal{G}_\alpha$ defines a holomorphic bijection between $\mathbb{D}$ and $\mathcal{D}'_\alpha$, such that $g_{\lambda(\delta),b(\delta)}$ has a Herman ring with twist coordinate given by the argument of δ. More precisely,

$$\Theta(\delta) = \frac{1}{2\pi} \arg(\delta).$$

The image by $\mathcal{G}_\alpha$ of the interval $[0,1)$ is exactly the piece of Arnold tongue $\mathcal{T}_\alpha^{\mathrm{AL}}$, since it follows from Theorem 3 and the injectivity of $\mathcal{G}_\alpha$ that those are the only maps with Herman rings having twist coordinate equal to 0. But now let us look at the map $\mathcal{G}_\alpha$ restricted to the interval $(-1,1)$. By holomorphy, both components, $\lambda(\delta)$ and $a(\delta)$, must be real analytic. The first one, $\delta \mapsto \lambda(\delta)$ maps $(0,1)$ into S^1 or equivalently, $\delta \mapsto t(\delta)$ maps $(0,1)$ into the reals (where $\lambda = \mathrm{e}^{2\pi i t}$). It follows that the Taylor series of $t(\delta)$ must have real coefficients and hence, the whole image $t(-1,1)$ must be real. The same argument shows that $b(-1,1)$ must also be real.

We conclude that parameters (λ, b) for which the Herman of $g_{\lambda,b}$ has twist parameter one half (i.e., the image of $(-1,0)$ under $\mathcal{G}_\alpha$) lie in $\mathbb{S}^1 \times \mathbb{R}^-$.

To return to the non rescaled Arnold family, recall that $\mathcal{F}_\alpha(\delta) = (\lambda(\delta), a(\delta))$ where $(\lambda(\delta), a(\delta)^2) = \mathcal{G}_\alpha(\delta^2)$. By lifting we deduce that $a(-1,0) \in i\mathbb{R}$ and $\lambda(-1,0) \in \mathbb{S}^1$.

To see the other implication, suppose that $f_{\lambda,a}$, $\lambda \in \mathbb{S}^1$, $a \in i\mathbb{R}$ has a fixed Herman ring H. The map $f_{\lambda,a}$ has a symmetry $f_{\lambda,a}(-\frac{1}{\bar{z}}) = -\frac{1}{\overline{f_{\lambda,a}(z)}}$. So H is symmetric with respect to $\tau(z) = -\frac{1}{\bar{z}}$.

We claim that a linearizing map $\psi : H \to \{\frac{1}{r} < |z| < r\}$ will have this symmetry as well. Indeed $\mathcal{R}_\alpha$ and $f_{\lambda,\alpha}$ commute with τ, and hence the map $\tilde{\psi} := \tau\psi\tau : H \to \{\frac{1}{r} < |z| < r\}$ is another linearizing map of H. With this normalization, such maps are unique up to post composition with a rigid rotation, thus $\tilde{\psi} = \mathcal{R}_\theta\psi$ for some $\theta \in [0, 2\pi)$. Now

$$\psi\tau = \tau\mathcal{R}_\theta\psi = \mathcal{R}_\theta\tau\psi.$$

Hence $\psi = \mathcal{R}_\theta\tau\psi\tau = \mathcal{R}_{2\theta}\psi$. It follows that $2\theta = 0 \bmod 1$ or equivalently, that $\theta = 1/2$ or $\theta = 0$. But the first option is not possible because in such a case, $\mathcal{R}_\theta = \tau$ on the unit circle and therefore

$\psi\tau = \psi$ on the equator of H, i.e, on $\psi^{-1}(\mathbb{S}^1)$. This would contradict with the injectivity of ψ and hence $\theta = 0$. We have then proved that $\psi\tau = \tau\psi$.

Observe that, as a consequence, every marked point in the boundary of H will have the same property, from which we conclude that the twist coordinate must be 1/2.

□

Although the symmetry with respect to the unit circle (or to the real line in the lift) is lost for maps in this slice (where $|\lambda| = 1$ and $a = i\tilde{a}$, $\tilde{a} \in \mathbb{R}$), we just saw that another symmetry appears. Indeed, it is easy to check that

$$f_{\lambda,i\tilde{a}}(-\frac{1}{\bar{z}}) = -\frac{1}{\bar{f}_{\lambda,i\tilde{a}}(c)}$$

and that the lift $F_{t,i\tilde{a}}(z) = z + t + i\frac{\tilde{a}}{2\pi}\sin(2\pi z)$ satisfies

$$F_{t,i\tilde{a}}(\bar{z} + \pi) = \overline{F_{t,i\tilde{a}}(z)} + \pi.$$

As a consequence, the two critical points of the lift (looked in the cylinder) which are located at

$$\omega_1 = \frac{\pi}{2} - i\,\mathrm{arcsinh}(\frac{1}{\tilde{a}}) \quad \text{and} \quad \omega_2 = \frac{3\pi}{2} + i\,\mathrm{arcsinh}(\frac{1}{\tilde{a}})$$

do not have independent dynamics (as in the general case). It then makes sense to compute a $(t, \tilde{a})$–plane picture where we check if the orbit of the critical point escapes to infinity or otherwise remains bounded. The result is shown in Figure 8, where we have also superposed the rational tongues of rotation number $1/4, 1/2$ and $3/4$ and the irrational curve corresponding to rotation number equal to the golden mean (see Section 5.4 for the algorithms).

Figures 4 and 5 show the dynamical plane for two of the parameters in the irrational curve while Figures 6 and 7 correspond to the lifts of these.

We observe from Figure 8 that many of the rational tongues do not seem to cross this slice. For example, it is easy to check that none of the maps in this parameter plane can have an attracting fixed point, and therefore, there is no zero – tongue emanating from the point $(0, 0)$. Similarly, there does not seem to be any rational tongue of odd denominator attached to the bottom line $\tilde{a} = 0$.

5 Numerical algorithms

In this section we describe the numerical algorithms used to create the pictures in the paper. The methods are quite general and may be used to compute the same type of pictures for other families possessing a cycle of Herman rings (or Siegel disks) as long as it is the only existing periodic Fatou cycle.

We start by assuming we already know the parameter values for which the map has a Herman ring H. Later on we shall see how to compute them, but first we see how to draw a dynamical plane picture with these given parameter values.

Escaping algorithms usually work poorly for holomorphic maps of $\mathbb{C}^*$ (i.e., those with essential singularities at 0 and ∞). It is common for their Julia set to have positive measure and it therefore appears very thick. Moreover, exponentiating repeatedly makes overflows and underflows appear too often and too soon.

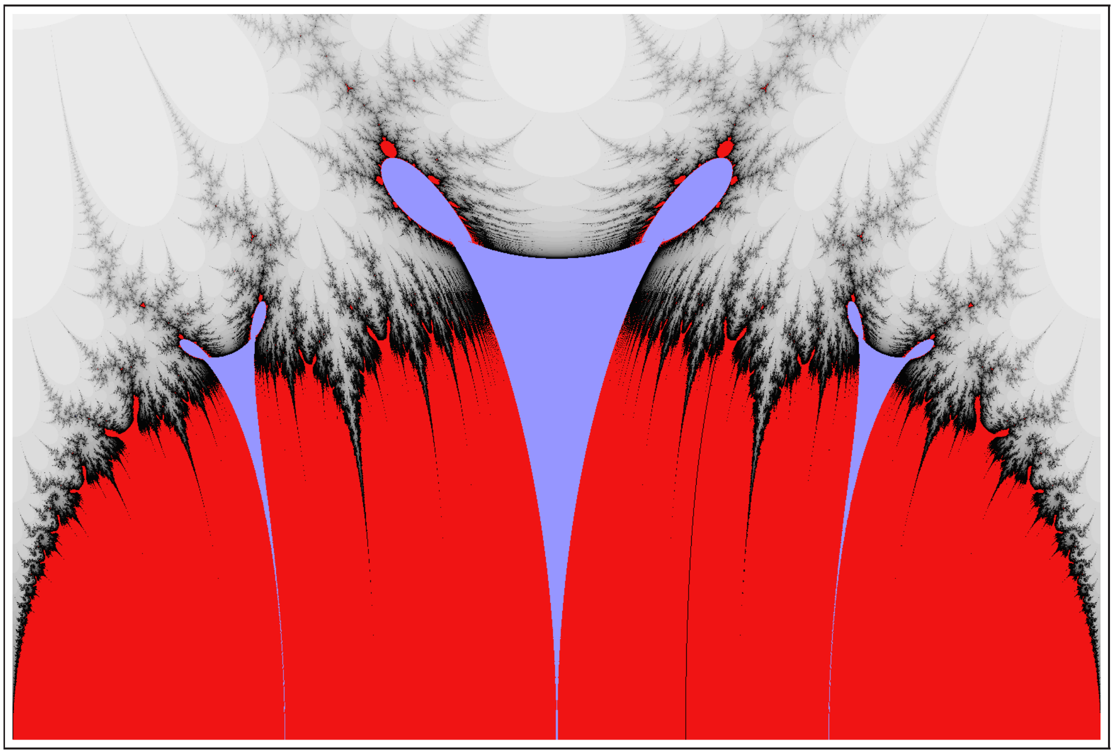

Figure 8: $(t, \tilde{a})-$parameter plane where the map $f_{\lambda,i\tilde{a}}$ is iterated to check if the critical orbits seem to remain bounded (dark grey). Range: $[0,1] \times [0,2]$. All Herman rings of maps in this slice have twist parameter one half. Superposed, we find the rational tongues of rotation numbers $1/4, 1/2$ and $3/4$ and the irrational curve corresponding to rotation number equal to the golden mean. See Section 5.4.

The algorithms used here to draw the dynamical planes are of a different nature. Given a pixel, we ask wether the corresponding center point eventually falls inside the Herman ring, in which case it is painted in white. Pixels which do not satisfy this property are painted in color.

To be able to answer this key question we must first find what we call a *base domain* of the Herman ring, i.e., a set A inside the ring satisfying the following: every point in the Herman ring has an orbit which eventually intersects A. These base domains are of a different shape depending on the map we work with.

In all cases, we use the following important fact: the orbit of the critical points accumulates on the boundary of the Herman ring. Hence, we always can compute two lists of points that correspond to the critical orbits. These points are drawn in the picture so the boundary of the ring is outlined.

5.1 Symmetric Herman rings (Figure 2)

In this case we look for a base domain in the form of a true annulus around the unit circle, since we know that the unit circle is always completely contained in the ring. To find the width of the annulus we use the symmetry of the ring. Indeed, we find the point on the outer boundary which has the smallest modulus, say $r > 1$. Necessarily, the reflexion of this point with respect to the unit circle is the point on the inner boundary with the largest modulus, $\frac{1}{r}$. Then every orbit of H meets the annulus $A = \{\frac{1}{r} < |z| < r\}$ and therefore A is a base domain. See Figure

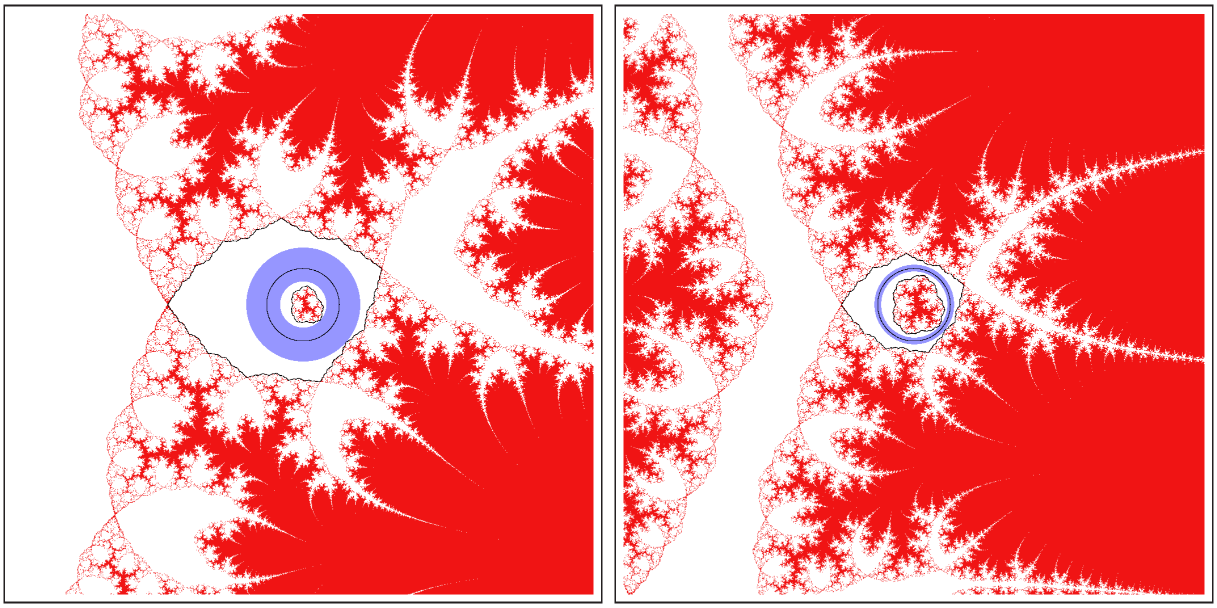

Figure 9: Herman rings with rotation number equal to the golden mean, for the map $f_{\lambda,a}$ for chosen parameters $(t, a) = (0.614526385907, 0.5)$ (left) and $(t, a) = (0.610074404161, 0.8)$ (right) where $\lambda = \exp(2\pi i t)$. The base annulus is drawn inside each of the rings. Range: $[-8, 8] \times [-8, 8]$. Compare to Figure 2.

9.

5.2 Siegel disk (Figure 3)

If the invariant Fatou component is a Siegel disk with center p, we may look for a base domain in the form of a disk, centered also at p. To find its radius, we choose from all the points on the boundary of the disk (i.e., the critical iterates) the one that is closest to the center of the disk, say at distance r. The disk $D(p, r)$ is a base domain, and all invariant circles must cross the radial segment that joints p with the closest point on the boundary. See Figure 10.

5.3 Non-symmetric Herman rings (Figures 4 and 5)

This is the hardest case since we do not know a priori the location of the Herman ring in the dynamical plane, nor any orbit (or any point, for that matter) contained inside the ring. We do have, however, the lists of points that correspond to the approximated boundaries of H. We will choose a base domain in the form of a round disk with the condition of being entirely contained in H and touching both boundaries. To do this, we need to find the minimum (at least a local minimum) of the distance from points on the outer boundary to points on the inner boundary. One can do this, for example, by picking a point in the first list (outer), say p_1; then finding the closest point to p_1 in the other list (inner), say q_1; then the closest point to q_1 in the first list, say p_2, etc. We stop once the points do not change any more, and hence we have a pair (p_n, q_n) whose distance between each other is at least a local minimum in the following sense: no point in the inner boundary is closer to p_n than q_n; and viceversa, no point in the outer boundary is closer to q_n than p_n. This assures that the disk of radius $|p_n - q_n|/2$ centered at the middle point

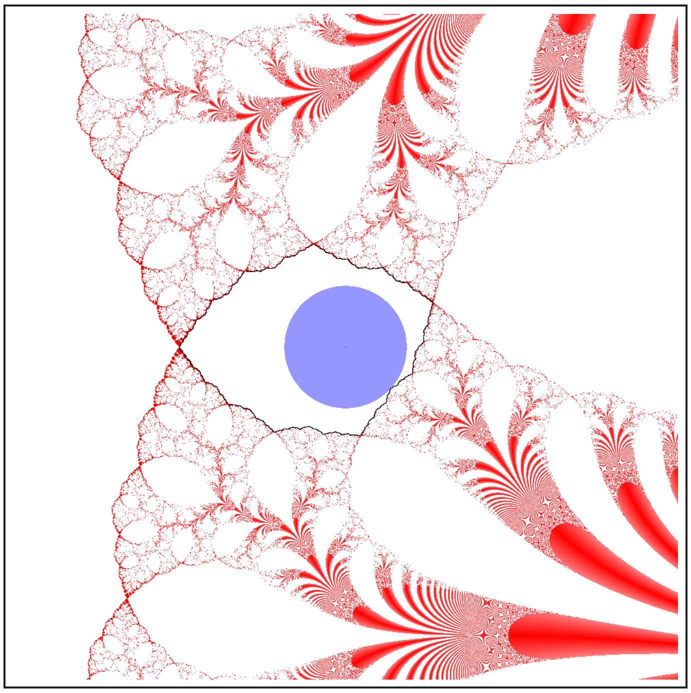

Figure 10: Siegel disk of the function $E_\alpha(z) = e^{2\pi i\alpha} z \mathrm{e}^z$, with rotation number α, equal to the golden mean. The base disk has been drawn inside the disk. Range: $[-2, 2] \times [-2, 2]$.

between p_n and q_n is completely contained in H and touches the two boundaries. See Figure

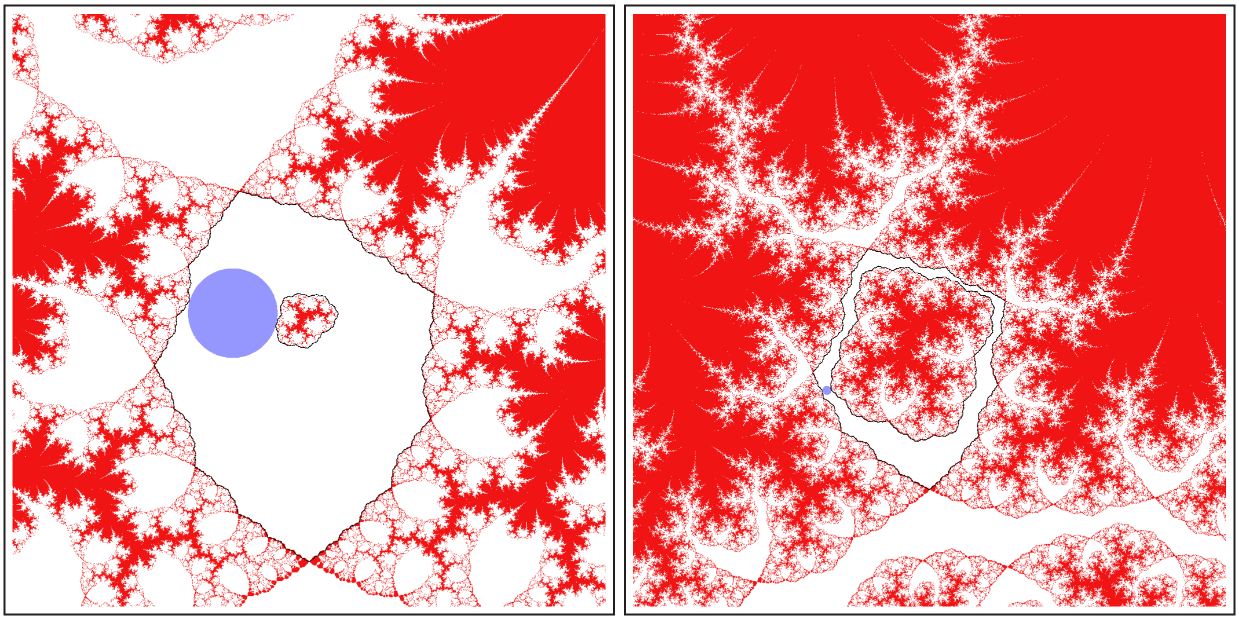

Figure 11: Herman rings of rotation number equal to the golden mean, in the dynamical plane of $f_{\lambda,a}$, where $(\lambda, a) = (\mathrm{e}^{2\pi i 0.622359931841}, 0.5i)$ (left) and $(\lambda, a) = (\mathrm{e}^{2\pi i 0.642219660059}, i)$ (right). The base disks has been drawn inside the ring.

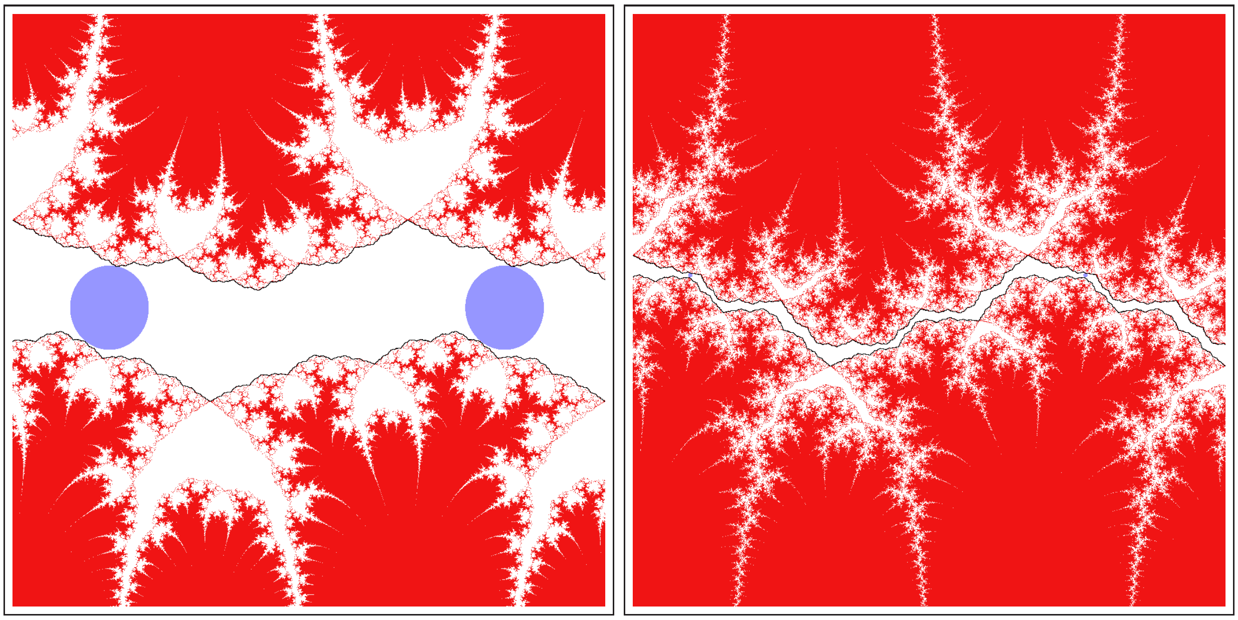

Figure 12: Lifts of the Herman rings in Figure 11, in the dynamical plane of $z \mapsto z + t + \frac{a}{2\pi}\sin(2\pi z)$. The base rings have been drawn inside the Baker domain.

11. A similar computation can be made for the lift of these rings as it is shown in Figure 12.

5.4 Parameter space: finding parameters for a given rotation number.

To draw the images in this paper, we needed to locate parameters in a given parameter slice for which $f_{\lambda,a}$ has a Herman ring with a given rotation number, say α. We observe that if f_{λ_0,a_0} has a Herman ring with rotation number α of bounded type, then the quantity

$$\rho^n_{t_0,a_0}(\omega_{t_0,a_0}) = \frac{F^n_{t_0,a_0}(\omega_{t_0,a_0}) - \omega_{t_0,a_0}}{n}$$

has limit α, where $\omega_{t_0,a_0} \in \partial H$ is a critical point and $\lambda_0 = \mathrm{e}^{2\pi i t_0}$. Also, if $f_{t,a}$ has an attracting periodic cycle, then $\rho^n_{t,a}(\omega_{t,a})$ tends to a rational number. Hence we expect $\rho^N_{t,a}$ for a given N large, to be close to a real number for a substantial part of parameter space.

Suppose we can find $(t_0, a_0), (t_1, a_1)$ such that $\rho^N_{t_0,a_0}$ and $\rho^N_{t_1,a_1}$ are close to being real and

$$\mathrm{Re}(\rho^N_{t_0,a_0}) < \alpha < \mathrm{Re}(\rho^N_{t_1,a_1}).$$

We then pick points randomly on the segment between (t_0, a_0) and (t_1, a_1) until we find (t_2, a_2) such that $\rho^N_{t_2,a_2}$ is close to being real, and replace one of the previous pairs by this pair, as in the classical bisection procedure. If we manage to find parameters for which $\rho^N_{t,a}$ is almost real, the continuity of $\rho^N_{t,a}$ guarantees the convergence of $\rho^N_{t_n,a_n}$ towards α, if the length of the segments decreases to 0.

We do not claim this to be a fullproof method; it is a heuristic one that seems to work reasonably, especially for initial values (t_0, a_0), (t_1, a_1) with $\mathrm{Re}(a_0)$ and $\mathrm{Re}(a_1)$ small.

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