Deformation of Entire Functions with Baker Domains

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Abstract

We consider entire transcendental functions f with an invariant (or periodic) Baker domain U satisfying a certain condition (which is satisfied always if f restricted to U is proper). First, we classify these domains into three types (hyperbolic, simply parabolic and doubly parabolic) according to the properties of the map they induce in the unit disk, and we give dynamical and geometric criteria to determine the type of a given Baker domain. Second, we study the space of quasiconformal deformations of an entire map with such a Baker domain by studying its Teichmüller space. More precisely, we show that the dimension of this set is infinite if the Baker domain is hyperbolic or simply parabolic, and from this we deduce that the quasiconformal deformation space of f is infinite dimensional. Finally, we prove that the function $f(z) = z + e^{-z}$, which possesses infinitely many invariant Baker domains, is rigid, i.e., any quasiconformal deformation of f is affinely conjugate to f.

1 Introduction

Let $f : S \to S$ be a holomorphic endomorphism of a Riemann surface S. Then f partitions S into two sets: the Fatou set $\Omega(f)$, which is the maximal open set where the iterates $f^n, n = 0, 1, \ldots$ form a normal sequence; and the Julia set $J(f) = S \setminus \Omega(f)$ which is the complement.

If $S = \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, then f is a rational map, and every component of $\Omega(f)$ is eventually periodic by the non-wandering domains theorem in [Sullivan 1982]. There is a classification of the periodic components of the Fatou set: Such a component can either be a cycle of rotation domains or the basin of attraction of an attracting or indifferent periodic point.

If $S = \mathbb{C}$ and f does not extend to $\widehat{\mathbb{C}}$ then f is an entire transcendental mapping (i.e., infinity is an essential singularity) and there are more possibilities. For example a component of $\Omega(f)$ may be wandering, that is, it will never be iterated to a periodic component.

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Like for rational mappings there is a classification of the periodic components of $\Omega(f)$ (see [Bergweiler 1993]) and compared to rational mappings, entire ones allow for one more possibility: A period p periodic component U is called a Baker domain, if for all $z \in U$ we have $f^n(z) \to \infty$, as $n \to \infty$. The first example of an entire function with a Baker domain was given by Fatou in [Fatou 1920], who considered the function $f(z) = z + 1 + \mathrm{e}^{-z}$ and showed that the right half-plane is contained in an invariant Baker domain. Since then, many other examples have been considered, showing various properties that are possible for this type of Fatou components (see for example [Eremenko & Lyubich 1987], [Bergweiler 1995], [Baker & Domínguez 1999], [Rippon & Stallard 1999(1)], [Rippon & Stallard 1999(2)], [König 1999] and [Baranski & Fagella 2000]). It follows from [Baker 1975] that a Baker domain of an entire function is simply connected.

Taking an iterate of the map if necessary we consider only the cases of invariant Baker domains. We remark that in a Baker domain, orbits tend to infinity at a slow rate. More precisely, if γ is an unbounded invariant curve in a Baker domain (and hence all its points tend to infinity under iteration), then there exists a constant $A > 1$ such that $|f(z)| \leq A|z|$ for all $z \in \gamma$ [Bergweiler 1993]. This is in contrast to the fact that points in $\mathbb{C}$ that tend to infinity exponentially fast belong to the Julia set of f and, even more, every point in the Julia set is the limit of such escaping points.

There is another important difference between rational and entire transcendental mappings which concerns the singularities of the inverse map f^{-1} or *singular values*. In the rational case, the points for which some branch of f^{-1} fails to be well defined are precisely the *critical values*, i.e., the images of the zeros of f'. In the entire case, one more possibility is allowed, namely the *asymptotic values*, which are points $a \in \mathbb{C}$ for which there exists a curve $\gamma(t) \to \infty$ as $t \to \infty$ satisfying $f(\gamma(t)) \to a$ as $t \to \infty$. It follows from a theorem of Denjoy, Carleman and Ahlfors that entire functions of finite order may have only a finite number of asymptotic values (see e.g. [Nevanlinna 1970] or [Hua & Yang 1998] Theorem 4.11), but in the other extreme there exists an entire map for which every value is an asymptotic value.

As it is the case with basins of attraction and rotation domains, there is also a relation between Baker domains and the singularities of the inverse map. In particular, it is shown in [Eremenko & Lyubich 1992] that Baker domains do not exist for a map such that the set $\mathrm{Sing}(f^{-1})$ is bounded, where $\mathrm{Sing}(f^{-1})$ denotes the closure in $\mathbb{C}$ of the set of singular values. The actual relationship between this set and a Baker domain U is related to the distance of the singular orbits to the boundary of U (see [Bergweiler 2001] for a precise statement). We remark that it is not necessary, however, that any of the singular values be inside the Baker domain. Indeed, there are examples of Baker domains with an arbitrary number of singular values (including none) inside.

When the map f restricted to the Baker domain U is proper, we call U a *proper Baker domain.* In particular the degree of f restricted to U is finite. In the special case where this degree is one we call the domain U *univalent.*

In [Baranski & Fagella 2000] there is given a classification of univalent Baker domains in terms of the map they induce in the unit disk via the Riemann map. Our first goal in this paper is to extend this classification to accomodate a larger class of Baker domains. More precisely let $\varphi : U \to \mathbb{D}$ denote a Riemann map, mapping U to the unit disk. Such a map conjugates f to a self-mapping of $\mathbb{D}$ that we denote by B_U. The map B_U is called the *inner function* associated to U. If U is proper then this mapping is a (finite) Blaschke product.

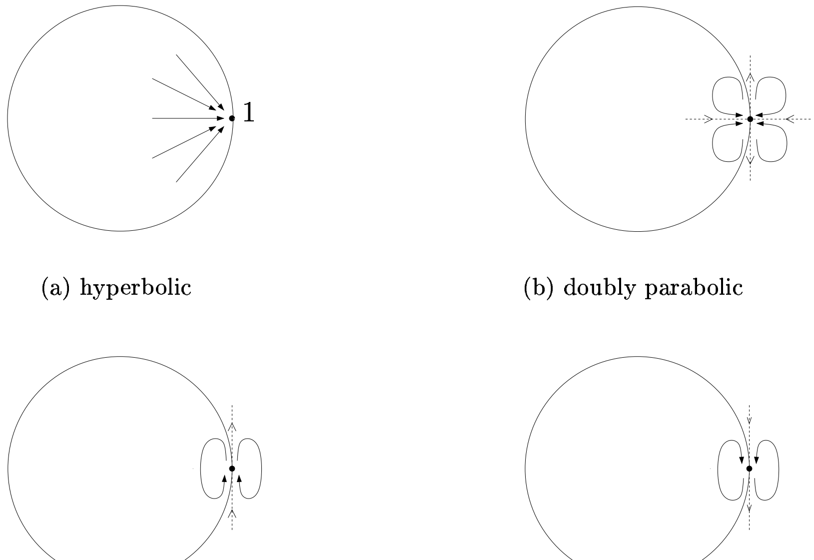
It follows from the Denjoy-Wolff theorem (see e.g. [Milnor 1999], Thm. 5.4), that there exists a point $z_0 \in \partial\mathbb{D}$ such that B_U^n converges towards the constant mapping z_0 locally uniformly in $\mathbb{D}$ as n tends towards infinity. If B_U extends analytically to a neighborhood of z_0 we call U a *regular* Baker domain. In particular, proper Baker domains are a subclass of the regular Baker domains. This class of maps was studied in [Bergweiler 2001]. By invariance we see that $0 < B_U'(z_0) \leq 1$. If we assume that we have normalized the conjugacy φ so that this point is $z_0 = 1$, then B_U is uniquely determined up to conjugation with a Möbius transformation that preserves the unit disk and 1. It follows that $B_U'(1)$ is well defined. In Section 3 we classify regular Baker domains into three different types (see Figure 1).

Proposition 1. *Let f be entire and U a regular Baker domain. Let B_U be the inner function associated to U. Then either*

(1) $0 < B_U'(1) < 1$ *and we call U* hyperbolic, *or*

(2) $B_U(z) = z \pm ia(z-1)^2 + \mathcal{O}((z-1)^3)$, *for some $a > 0$ and we call U* simply parabolic, *or*

(3) $B_U(z) = z - a(z-1)^3 + \mathcal{O}((z-1)^4)$ *for some $a > 0$, and we call U* doubly parabolic. *In this case $f : U \to U$ has degree at least* 2.



(a) hyperbolic

(b) doubly parabolic

(c) simply parabolic $B_U(z) = z + ia(z-1)^2 + \cdots$

(d) simply parabolic $B_U(z) = z - ia(z-1)^2 + \cdots$

Figure 1: The three (or four) possibilities for the dynamics of B_U. By the symmetry of the map, $\mathbb{D}$ and $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ must belong to the basin of attraction of 1 and hence the Julia set must be a subset of the unit circle.

An equivalent classification is given independently in [Bergweiler 2001] together with estimates of the hyperbolic metric in each of the three cases.

Additional to the classification above we present some geometric and dynamical criteria that allow us to determine the class which a given Baker domain belongs to. More precisely we first have the following preliminary definition.

Definition. A domain $\mathcal{G}$ is called *an invariant petal at infinity* if

(a) $\mathcal{G}$ is connected, simply connected and unbounded,

(b) the boundary of $\mathcal{G}$ (as a subset of $\mathbb{C}$) is a simple curve, and

(c) f is a conformal isomorphism of $\mathcal{G}$ onto itself.

In Section 3 we prove the following criteria in terms of the existence of invariant petals at infinity.

Proposition 2. *Let f be entire and U an invariant regular Baker domain. Then,*

(1) *U is hyperbolic, if and only if U does not contain the closure, in $\mathbb{C}$, of an invariant petal at infinity;*

(2) *U is simply parabolic, if and only if U contains the closure of an invariant petal at infinity, $\mathcal{G}$, and any other such petal intersects $\mathcal{G}$;*

(3) *U is doubly parabolic, if and only if U contains the closure of two disjoint invariant petals at infinity.*

It is a natural question to ask wether examples of Baker domains of all three types exist. In Section 4 we give examples of functions with Baker domains of each of the three types. However, our examples for hyperbolic and simply parabolic domains are univalent and, to our knowledge, no examples are known of such maps with degree larger than one.

Our second goal in this paper is to study the possible quasiconformal deformations of entire maps with a Baker domain. We can consider the space of entire mappings with a fixed regular Baker domain as a subset of the space of entire mappings modulo conjugacy with affine mappings. It is natural to ask how this set looks. It is easy to see it cannot be open, since any entire map with a regular Baker domain can be approximated by polynomials, and no polynomial possesses a Baker domain. Lifting maps with Herman rings (see Example 1 in section 4) for different rotation numbers converging to a rational p/q, shows that the set is not closed. Can it have components that are reduced to points? By considering the space of quasiconformal deformations we will see that if such a point exists, the corresponding mapping can only have regular Baker domains which are doubly parabolic.

More precisely we will consider the Teichmüller space of an entire mapping f with a regular fixed Baker domain, using the general framework given by [McMullen & Sullivan 1998] (see Section 5). We will see that the dimension of this set is infinite if the Baker domain is hyperbolic or simply parabolic, and from this we will deduce that the quasiconformal deformation space of f is infinite dimensional. The precise statement is as follows.

Main Theorem. *Let U be a regular fixed Baker domain of the entire function f and $\mathcal{U}$ its grand orbit. Denote by S the set of singular points of f in $\mathcal{U}$, and by $\widehat{\mathcal{S}}$ the closure of the grand orbit of S taken in $\mathcal{U}$. Then $\mathcal{T}(f,\mathcal{U})$ is infinite dimensional except if U is doubly parabolic and the cardinality of $\widehat{\mathcal{S}}/f$ is finite. In that case the dimension of $\mathcal{T}(f,\mathcal{U})$ equals $\#\widehat{\mathcal{S}}/f - 1$.*

Furthermore we show that the lowest dimension is possible, that is we give an example of a rigid map with a proper Baker domain. Using the Main Theorem we can show the following (see Section 6).

Proposition 3. *The map $f(z) = z + e^{-z}$ is rigid, i.e., if $\widetilde{f}$ is a holomorphic map which is quasiconformally conjugate to f, then $\widetilde{f}$ is affinely conjugate to f.*

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2 Preliminaries

In this section we state some of the results that we use throughout the paper, and mainly in Section 5. We will include the proof of those statements for which we are unable to give a standard reference. The rest can be found, for example, in [Milnor 1999] or [Steinmetz 1993].

2.1 Attracting fixed points and linearizing coordinates

Let V be a domain $\widehat{\mathbb{C}}$ which could be the whole of $\mathbb{C}$ or $\widehat{\mathbb{C}}$, and $f : V \to f(V)$ be holomorphic. Let $z_0 \in V$ be an attracting fixed point, that is with multiplier $\lambda = f'(z_0)$ satisfying $0 < |\lambda| < 1$. Let $\mathcal{A} = \{z \in V \mid f^n(z) \to z_0\}$ denote the basin of attraction of z_0 in V.

We are interested in the changes of variables that conjugate f to a linear map (linearizing coordinates) and, more precisely, in what kind of symmetries are induced by the symmetries of f. The following is a classical result.

Theorem 2.1 (Kœnigs). *Let V be a domain in $\widehat{\mathbb{C}}$ and $f : V \to f(V)$ be holomorphic. Let $z_0 \in V$ be an attracting fixed point and λ be its multiplier. Let $\mathcal{A}$ be the the basin of attraction of z_0 (in V).*

(a) There exists a conformal change of coordinate $w = \phi(z)$, (the linearizing coordinate), defined in a neighborhood V' of z_0, with $\phi(z_0) = 0$, so that $\phi \circ f \circ \phi^{-1}$ is the linear map $w \mapsto \lambda w$, for all w in some neighborhood of the origin. Furthermore, ϕ is unique up to multiplication by a non-zero constant.

(b) The change of coordinate ϕ can be extended to a holomorphic map $\phi : \mathcal{A} \to \mathbb{C}$ so that the diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{f} & \mathcal{A} \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\phi} \\ \phi(\mathcal{A}) & \xrightarrow{\lambda w} & \phi(\mathcal{A}) \end{array}$$

is commutative. If $f : \mathcal{A} \to \mathcal{A}$ is proper then ϕ maps $\mathcal{A}$ onto the whole complex plane.

The extension of ϕ is constructed via the functional equation $\phi(f(z)) = \lambda\phi(z)$ which we require satisfied in the whole basin. For an arbitrary point $z \in \mathcal{A}$, there exists $n \in \mathbb{N}$ such that $f^n(z)$ belongs to V'. Since the original ϕ is well defined in V', the quantity $\phi(f^n(z))$ is well defined and therefore it makes sense to (recursively) define

$$\phi(z) = \frac{1}{\lambda^n}\phi(f^n(z)).$$

Since z was arbitrary, this extends ϕ to the whole $\mathcal{A}$.

Using the construction of this extension we can prove the following proposition. Given a map g we denote by $\mathrm{Fix}(g)$ its fixed points.

Proposition 2.2. *Let $f, V, z_0, \mathcal{A}$ and ϕ be as above and suppose f has a symmetry in V, in the sense that there exists an antiholomorphic involution $\tau : V \to V$ such that $f \circ \tau = \tau \circ f$ (whenever it is well defined) and $\tau(z_0) = z_0$. Then, the linearizing coordinate ϕ can be chosen to be symmetric in the following sense: there exists an antiholomorphic involution $\sigma : \mathbb{C} \to \mathbb{C}$ sending $\phi(\mathcal{A})$ to itself such that*

(1) $\phi \circ \tau = \sigma \circ \phi$ *in* $\mathcal{A}$*;*

(2) $\lambda\sigma(z) = \sigma(\lambda z)$ *for all* $z \in V$*;*

(3) $\phi\left(\mathrm{Fix}(\tau) \cap \mathcal{A}\right) \subseteq \mathrm{Fix}(\sigma)$.

Proof. Since τ is an involution $\tau(V) = V$ and it follows that $\tau(\mathcal{A}) = \mathcal{A}$. Let V' be the neighborhood of z_0 where ϕ is conformal and let us assume w.l.o.g. that V' is symmetric with respect to τ (if this is not the case we can replace V' by the connected component of $V' \cap \tau(V')$ which contains z_0). We then start by defining σ in $\phi(V')$ as

$$\sigma = \phi \circ \tau \circ \phi^{-1}.$$

Hence the following diagram commutes.

$$\begin{tikzcd}
V' \arrow[r, "f"] \arrow[d, "\tau"'] \arrow[ddd, bend right, "\phi"'] & f(V') \arrow[d, "\tau"] \arrow[ddd, bend left, "\phi"] \\
V' \arrow[r, "f"] \arrow[d, "\phi"'] & f(V') \arrow[d, "\phi"] \\
\phi(V') \arrow[r, "z\mapsto\lambda z"] \arrow[d, "\sigma"'] & \phi(V') \arrow[d, "\sigma"] \\
\phi(V') \arrow[r, "z\mapsto\lambda z"] & \phi(V')
\end{tikzcd}$$

Notice that σ is antiholomorphic and it is an involution since $\sigma \circ \sigma = \mathrm{Id}$. We also observe that σ commutes with multiplication by λ. Indeed,

$$\lambda\sigma(z) = \lambda\phi(\tau(\phi^{-1}(z))) = \phi(f(\tau(\phi^{-1}(z)))) = \phi(\tau(f(\phi^{-1}(z)))) = \phi(\tau(\phi^{-1}(\lambda z))) = \sigma(\lambda z).$$

Using this fact, we proceed to extend σ to the whole complex plane, by letting

$$\sigma(w) = \frac{\sigma\left(\lambda^n w\right)}{\lambda^n},$$

where $n \in \mathbb{N}$ is such that $\lambda^n w \in \phi(V')$. Then, by construction, the extension of σ satisfies (2).

To prove (1) we need to show that the extension of σ also satisfies

$$\phi \circ \tau = \sigma \circ \phi$$

in the entire basin of attraction $\mathcal{A}$. Indeed, if $z \in \mathcal{A}$,

$$\sigma(\phi(z)) = \frac{1}{\lambda^n}\sigma(\lambda^n \phi(z)) = \frac{1}{\lambda^n}\sigma(\phi(f^n(z))) = \frac{1}{\lambda^n}\phi(\tau(f^n(z))) = \frac{1}{\lambda^n}\phi(f^n(\tau(z))) = \phi(\tau(z)).$$

Finally we conclude from (1) that if z is a fixed point of τ then $\phi(z)$ is a fixed point of σ and hence (3) is satisfied. $\square$

In the following sections of the paper we will actually be interested in the particular case where the multiplier λ of the fixed is a real number and the symmetry of f is $\tau(z) = 1/\overline{z}$ i.e., reflection with respect to the unit circle. In this case we have the following.

Proposition 2.3. *Let f, V, z_0 and $\mathcal{A}$ be as above and suppose the multiplier $\lambda = f'(z_0)$ satisfies $0 < \lambda < 1$. Suppose $z_0 \in \mathbb{S}^1$ and f is symmetric with respect to the unit circle, i.e., $f \circ \tau = \tau \circ f$ where $\tau(z) = 1/\overline{z}$. Then, for any $\theta \in [0, 2\pi)$, there exists a linearizing coordinate $\phi : \mathcal{A} \to \mathbb{C}$ such that*

(1) *$\phi(\mathcal{A} \cap \mathbb{S}^1) \subseteq L_\theta$ where $L_\theta = \{t e^{i\theta} \mid t \in \mathbb{R}\}$.*

(2) *$\sigma \circ \phi = \phi \circ \tau$ on $\mathcal{A}$ where σ is the reflection with respect to L_θ.*

(3) *If $z_0 = 1$ and L_θ is the imaginary axis then ϕ can be chosen to satisfy $\phi'(z_0) = 1$. In this case we have $\sigma(z) = -\bar{z}$.*

Proof. From proposition 2.2 we know that $\sigma(z)$ is an antiholomorphic involution of the complex plane that commutes with multiplication by λ and such that $\sigma \circ \phi = \phi \circ \tau$ on the basin of attraction $\mathcal{A}$. An antiholomorphic map of degree one of $\mathbb{C}$ must have the form

$$\sigma(z) = a\bar{z} + b$$

for some $a, b \in \mathbb{C}$. Imposing that σ commutes with multiplication by λ one obtains that b must be zero and hence $\sigma(z) = a\bar{z}$. But since σ is an involution, it follows easily that a must be a constant of modulus 1. We observe that the fixed points of σ satisfy $z = 0$ or $z/|z| = \pm\sqrt{a}$. Hence

$$\mathrm{Fix}(\sigma) = L_{\arg(a)/2}$$

which correspond to points on the line going through the origin and $\sqrt{a}$. By Proposition 2.2, $\phi(\mathcal{A} \cap \mathbb{S}^1) \subseteq \mathrm{Fix}(\sigma)$.

We recall that linearizing coordinates are unique up to multiplication by a constant. We may then choose the coordinate $\widetilde{\phi}(z) = e^{i(\theta - \frac{\arg a}{2})}\phi(z)$ which will satisfy

$$\widetilde{\phi}(\mathcal{A} \cap \mathbb{S}^1) \subseteq L_\theta.$$

Finally we observe that if $z_0 = 1$ and we choose a linearizing coordinate satisfying $\phi'(1) = 1$, then ϕ must preserve the tangent vectors at the origin. Hence the piece of unit circle that intersects $\mathcal{A}$ must be mapped under ϕ to part of a straight line with a vertical tangent, i.e., the imaginary axis. $\square$

To end this section we apply the propositions above to the case of Blaschke products. By a Blaschke product we understand a finite product

$$B(z) = \lambda \prod_{i=1}^{d} \frac{z - a_i}{1 - \bar{a}_i z},$$

where each zero $a_i \in \mathbb{D}$ and $|\lambda| = 1$. Every Blasche product is a proper holomorphic of $\mathbb{D}$ onto itself and conversely any proper holomophic self-map of $\mathbb{D}$ is the restriction of a Blaschke product.

Proposition 2.4. *Let B be a Blaschke product such that $z_0 = 1$ is an attracting fixed point with multiplier $0 < \lambda < 1$. Let $\mathcal{A}$ be the basin of attraction of 1. Then, there exist a linearizing coordinate ϕ that maps $\mathcal{A} \cap \mathbb{S}^1$ onto the imaginary axis and $\mathbb{D}$ onto the left half plane.*

Proof. From the Denjoy-Wolff Theorem we know $\mathbb{D} \subseteq \mathcal{A}$. (In fact $\mathcal{A}$ is the whole sphere except for a point on the unit circle or a cantor subset of the unit circle.) From Proposition 2.3 we know that we can choose ϕ so that $\phi(\mathcal{A} \cap \mathbb{S}^1)$ lies on the imaginary axes and ϕ maps $\mathbb{D}$ into the left half plane. Now observe that the whole imaginary axis must be in the image of ϕ. Indeed, let l be a neighborhood of 1 in $\mathbb{S}^1$ which is mapped under ϕ to a neighborhood L of 0 in the imaginary axis. Then for any $n \in \mathbb{N}$, the set $f^{-n}(l)$ will be mapped by ϕ to $\frac{1}{\lambda^n} L$ and clearly, the union of these segments cover the imaginary axis. Since B maps $\mathbb{D}$ to $\mathbb{D}$, it maps $\widehat{\mathbb{C}} - \overline{\mathbb{D}}$ to itself and $B^{-1}(\mathbb{S}^1) = \mathbb{S}^1$. This proves that ϕ maps $\mathcal{A} \cap \mathbb{S}^1$ to the entire imaginary axis. Then, by symmetry, either ϕ or $-\phi$ map $\mathbb{D}$ to the left half plane. $\square$

Remark 2.5. If a regular Baker domain is not proper, then the inner function has infinite degree. In this case, Proposition 2.4 still holds if we change "onto the half plane" by "into the half plane", since B_U might have omitted values. Still, the image of a neighborhood of $z = 1$ covers a neighborhood of the origin.

2.2 Parabolic fixed points and Fatou coordinates

Let f be a holomorphic map on a domain V of the complex plane and let $z_0 \in V$ be a parabolic fixed point of f which we assume to have multiplier equal to 1. That is, f can be written near z_0 as

$$f(z) = z + a(z - z_0)^{q+1} + \mathcal{O}((z - z_0)^{q+2})$$

for some $q \geq 1$ and some $a \in \mathbb{C} \setminus \{0\}$.

The dynamics in a neighborhood of z_0 is well understood (see for example [Milnor 1999] or [Steinmetz 1993]).

It is well known that there are $2q$ equally spaced invariant directions at z_0 (unit vectors in the tangent space), alternatingly attractive and repelling. Any orbit that converges to the parabolic point must do so in one of the attracting directions, i.e., the ratio $\frac{f^{n+1}(z) - f^n(z)}{|f^{n+1}(z) - f^n(z)|}$ must converge to one of them. Likewise, backward orbits (whenever well defined) which converge to z_0, do so in one of the repelling directions.

Given such an attracting direction v_j in the tangent space, the *parabolic basin of attraction* $\mathcal{A}_j$ is defined as the set of points $z \in V$ that converge to z_0 in the direction of v_j. These basins are disjoint open sets and we denote by $\mathcal{A}_j^0$ the unique component of $\mathcal{A}_j$ that maps to itself under f. Notice that any orbit in $\mathcal{A}_j$ must eventually enter and remain in $\mathcal{A}_j^0$.

Let us choose a neighborhood N of z_0 small enough so that f maps N diffeomorphically onto some neighborhood N' of z_0. Thus f^{-1} is well defined and holomorphic mapping N' to N. Let v be an attracting direction at z_0.

We now define the concept of a parabolic petal, following [Milnor 1999].

Definition 2.6. A simply connected open set $\mathcal{P} \subseteq N \cap N'$ such that $f(\mathcal{P}) \subseteq \mathcal{P}$ is called an *attracting petal* for f at z_0, in the direction of v if

(a) f^n restricted to $\mathcal{P}$ converges uniformly to the constant function $z \mapsto z_0$, and

(b) an orbit is eventually absorbed by $\mathcal{P}$ if and only if it converges to z_0 in the direction of v.

Likewise, $\mathcal{P}'$ is a *repelling petal* for f in the direction v' if it is an attracting petal for f^{-1} in this direction.

The Leau-Fatou Flower Theorem gives the existence of q attracting petals $\mathcal{P}_1, \dots, \mathcal{P}_q$ for f in the q attracting directions at z_0, and q repelling petals $\mathcal{P}'_1, \dots, \mathcal{P}'_q$ for f in the q repelling directions at z_0. These petals may be chosen so that they are bounded by real analytic curves everywhere except at z_0 where they come tangent to the adjacent repelling directions. The attracting and repelling petals alternate cyclically around z_0 so that $\mathcal{P}_j$ intersects exactly $\mathcal{P}'_{j-1}$ and $\mathcal{P}'_j$. See Figure 2. In the special case $q = 1$ the intersection $\mathcal{P} \cap \mathcal{P}'$ has two distinct connected components. Observe that, by definition, each attracting petal $\mathcal{P}_j$ determines a corresponding basin of attraction $\mathcal{A}_j$ as defined above.

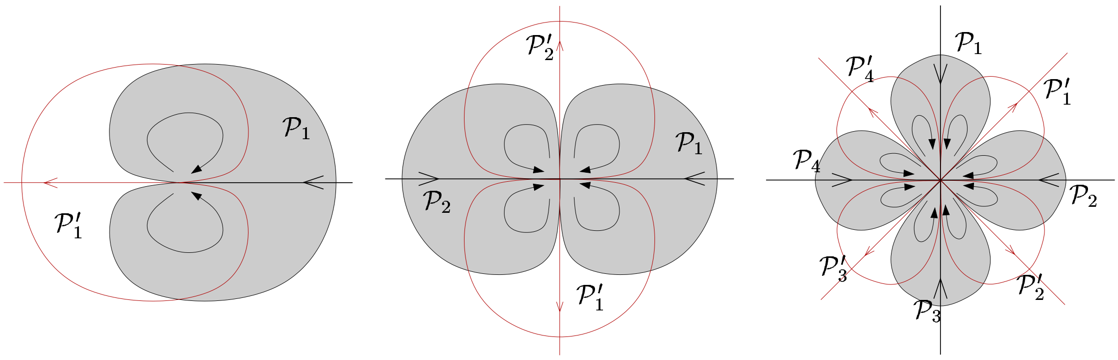

Figure 2: Arrangement of the attracting and repelling petals at a parabolic point with $q = 1, 2$ and 4.

The complete understanding of the dynamics inside each of the petals is given by the following linearization theorem.

Theorem 2.7 (Fatou coordinates). *Let f be a holomorphic map in a domain V and $z_0 \in V$ a parabolic fixed point with multiplier 1. Let $\mathcal{P}$ be one of the attracting (or repelling) petals of f at z_0 and $\mathcal{A}$ be the attracting (or repelling) basin determined by $\mathcal{P}$. Then,*

(a) There exists a conformal embedding $\alpha : \mathcal{P} \to \mathbb{C}$, (unique up to composition with a translation), which conjugates f to translation by 1, i.e., such that

$$\alpha(f(z)) = 1 + \alpha(z)$$

for all $z \in \mathcal{P} \cap f^{-1}(\mathcal{P})$. Moreover, $\alpha \sim -a/(z-z_0)^q$ as $z \to z_0$ in $\mathcal{P}$ and $\mathcal{P}$ may be chosen so that the image of α contains a right half plane (or a left half plane in the repelling case).

(b) *The map α extends uniquely to a holomorphic map $\alpha : \mathcal{A} \to \mathbb{C}$ defined in the entire attracting basin. If $f : \mathcal{A} \to \mathcal{A}$ is proper then ϕ sends $\mathcal{A}$ onto the whole plane.*

The map α is called the Fatou coordinate *of f in $\mathcal{P}$.*

We remark that the extension of the Fatou coordinate in (b) is constructed as usual by iterating $z \in \mathcal{A}$ enough times until it falls in $\mathcal{P}$, then applying the local coordinate and finally going back by $w \mapsto w+1$ the same number of times. Clearly this extension is no longer bijective but its image covers the whole complex plane when $f : \mathcal{A} \to \mathcal{A}$ is proper.

Similarly to the attracting case, we are now interested in the possible symmetries of α. More precisely we have the following proposition.

Proposition 2.8. *Let $f, z_0, \mathcal{P}, \mathcal{A}$ and α be as above and suppose f has a symmetry, i.e., there exists an antiholomorphic involution $\tau : V \to V$ such that $f \circ \tau = \tau \circ f$. Then, the Fatou coordinate α is also symmetric in the following sense: there exists an antiholomorphic involution $\sigma : \mathbb{C} \to \mathbb{C}$ such that*

(1) *$\alpha \circ \tau = \sigma \circ \alpha$ in $\mathcal{A}$;*

(2) *$\sigma(z) + 1 = \sigma(z+1)$ for all $z \in \mathbb{C}$;*

(3) *$\alpha\left(\mathrm{Fix}(\tau) \cap \mathcal{A}\right) \subseteq \mathrm{Fix}(\sigma)$.*

The proof of this proposition mimics that of 2.2 substituting "multiplication by λ" by "translation by one". Hence we shall omit it.

In the particular case where the symmetry of f is $\tau(z) = 1/\overline{z}$ i.e., reflection with respect to the unit circle, we can see exactly what the symmetry induced by α is. In this case we have the following.

Proposition 2.9. *Let f be holomorphic in a domain V and z_0 be a fixed point of f with $f'(z_0) = 1$. Let $\mathcal{P}$ be one of the attracting petals at z_0 and $\mathcal{A}$ the basin of attraction determined by $\mathcal{P}$. Suppose f is symmetric with respect to the unit circle, i.e., $f \circ \tau = \tau \circ f$ where $\tau(z) = 1/\overline{z}$. Then, for any $y \in \mathbb{R}$, there exists a Fatou coordinate $\alpha : \mathcal{A} \to \mathbb{C}$ such that*

(1) *$\alpha(\mathcal{A} \cap \mathbb{S}^1) \subseteq R_y$ where $R_y = \{t + iy \mid t \in \mathbb{R}\}$.*

(2) *$\sigma \circ \alpha = \alpha \circ \tau$ on $\mathcal{A}$ where σ is the reflection with respect to R_y.*

Proof. From proposition 2.8 we know that $\sigma(z)$ is an antiholomorphic involution of the complex plane that commutes with translation by 1, and such that $\sigma \circ \alpha = \alpha \circ \tau$ on the basin of attraction $\mathcal{A}$. An antiholomorphic map of degree one of $\mathbb{C}$ must have the form

$$\sigma(z) = a\bar{z} + b$$

for some $a, b \in \mathbb{C}$. Imposing that σ commutes with translation by 1, one obtains that a must be 1. But since σ is an involution, it follows easily that b must be purely imaginary. Hence $\sigma(z) = \bar{z} + ib'$ for some $b' \in \mathbb{R}$. We observe that the fixed points of σ are

$$\mathrm{Fix}(\sigma) = \{z \in \mathbb{C} \mid \mathrm{Im}(z) = \frac{b'}{2}\}$$

which is a horizontal line. By Proposition 2.8, $\alpha(\mathcal{A}\cap\mathbb{S}^1)\subseteq \mathrm{Fix}(\sigma)$.

We recall that Fatou coordinates are unique up to composition with a translation. We may then choose the coordinate $\widetilde{\alpha}(z)=\alpha(z)+i(y-\frac{b'}{2})$ which will satisfy

$$\mathrm{Im}(\widetilde{\alpha}(\mathcal{A}\cap\mathbb{S}^1)=y.$$

Hence $\widetilde{\sigma}$ is reflection with respect to R_y. □

In the case of a globally defined Blaschke product, the results above apply as follows.

Proposition 2.10. *Let B be a Blaschke product such that $B(1)=B'(1)=1$. Let $\mathcal{P}$ be one of the attracting petals at 1 and $\mathcal{A}$ the basin of attraction determined by $\mathcal{P}$. Then $\mathbb{D}\subseteq\mathcal{A}$ and there exists a Fatou coordinate $\phi:\mathcal{A}\to\mathbb{C}$ that maps $\mathcal{A}\cap\mathbb{S}^1$ onto the real axis and $\mathcal{A}$ onto the upper or lower half plane.*

The proof mimics that of Proposition 2.4 and hence we omit it.

2.3 Quasiconformal mappings

In this section we recall shortly the relevant definitions and results relative to quasiconformal mappings, to be used in Section 5. The standard references are [Ahlfors 1966] and [Lehto & Virtanen 1973]. In this section, $V, V'\subset\mathbb{C}$ are open subsets of the complex plane or more generally, one dimensional complex manifolds.

Definition 2.11. Given a measurable function $\mu: V\to\mathbb{C}$, we say that μ is a $k-$*Beltrami coefficient* of V if $|\mu(z)|\leq k<1$ almost everywhere in V. Two Beltrami coefficients of V are equivalent if they coincide almost everywhere in V.

Definition 2.12. A homeomorphism $\phi: V\to V'$ is said to be $k-$*quasiconformal* if it has locally square integrable weak derivatives and

$$\mu_\phi(z)=\frac{\frac{\partial\phi}{\partial\bar z}(z)}{\frac{\partial\phi}{\partial z}(z)}=\frac{\bar\partial\phi(z)}{\partial\phi(z)}$$

is a $k-$Beltrami coefficient. In this case, we say that μ_ϕ is the *complex dilatation* or the *Beltrami coefficient* of ϕ.

With the same definition, but skipping the hypothesis on ϕ to be a homeomorphism, ϕ is called a $k-$*quasiregular map.*

Definition 2.13. Given a Beltrami coefficient μ of V and a quasiregular map $f: V\to V'$, we define the *pull-back* of μ by f as the Beltrami coefficient of V defined by:

$$f^*\mu=\frac{\frac{\partial f}{\partial\bar z}+(\mu\circ f)\overline{\frac{\partial f}{\partial z}}}{\frac{\partial f}{\partial z}+(\mu\circ f)\overline{\frac{\partial f}{\partial\bar z}}}.$$

We say that μ is f–invariant if $f^*\mu=\mu$. If $\mu=\mu_g$ for some quasiregular map g, then $f^*=\mu_{g\circ f}$.

It follows from Weyl's Lemma that f is holomorphic if and only if $f^*\mu_0=\mu_0$, where $\mu_0\equiv 0$.

Definition 2.14. Given a Beltrami coefficient μ, the partial differential equation

$$\frac{\partial \phi}{\partial \bar{z}} = \mu(z) \frac{\partial \phi}{\partial z} \tag{1}$$

is called *the Beltrami equation.* By *integration* of μ we mean the construction of a quasiconformal map ϕ solving this equation almost everywhere, or equivalently, such that $\mu_\phi = \mu$ almost everywhere.

The famous *Measurable Riemann Mapping Theorem* by Morrey, Bojarski, Ahlfors and Bers states that every k-Beltrami coefficient is integrable.

Theorem 2.15 (Measurable Riemann Mapping Theorem, [Ahlfors 1966]). *Let μ be a Beltrami coefficient of $\mathbb{C}$. Then, there exists a quasiconformal map $\phi : \mathbb{C} \to \mathbb{C}$ such that $\mu_\phi = \mu$. Moreover, ϕ is unique up to post-composition with affine maps.*

We end this section with a lemma that will be important in Section 5. Since we are unable to give a reference, we include its proof here.

Lemma 2.16. *Let $\mathcal{A}$ denote the set of K-quasiconformal homeomorphims $\omega : \mathbb{D} \to \mathbb{D}$ that extend continuously to the boundary as the identity. Then there exists a constant $C = C(K)$ such that for all $\omega \in \mathcal{A}$ and all $z \in \mathbb{D}$ we have that the hyperbolic distance $d_{\mathbb{D}}$ in $\mathbb{D}$ satisfies*

$$d_{\mathbb{D}}(z, \omega(z)) \leq C.$$

Proof. This is a standard compactness argument. Let $\mathcal{B}$ denote the set of K-quasiconformal homeomorphisms of the sphere that fix $-1, 1$ and ∞. We endow $\mathcal{A}$ and $\mathcal{B}$ with the topologies corresponding to uniform convergence. A map $\omega \in \mathcal{A}$ can be extended to the spere, by letting it coincide with the identity outside $\mathbb{D}$. This defines an injection $\mathcal{A} \to \mathcal{B}$ which can be seen to be a homeomorhism onto its image. It is easy to see that the image of $\mathcal{A}$ in $\mathcal{B}$ is closed. Now, it is well-known that $\mathcal{B}$ is sequentially compact (cf. [McMullen 1994]), and it follows that $\mathcal{A}$ is sequentially compact. Then, take a sequence of maps $\omega_n \in \mathcal{A}$ and points $z_n \in \mathbb{D}$ and suppose that $d_{\mathbb{D}}(z_n, \omega_n(z_n)) \to \infty$. Let $\hat{\omega}_n$ be the map we obtain by conjugating ω_n with a Möbius transformation that sends $\mathbb{D}$ to itself and z_n to 0. Now, $\hat{\omega}_n$ is a sequence of maps in $\mathcal{A}$ with $|\omega_n(0)| \to 1$. This is in contradiction with the fact that $\mathcal{A}$ is sequentially compact. □

3 Classification of Baker Domains. Proof of Propositions 1 and 2

Let f be an entire transcendental map and U an invariant regular Baker domain of f. Let B_U be the inner function associated to U as defined in the introduction. Then $0 < B_U'(1) \leq 1$ is well defined. We recall the statement of Proposition 1.

Proposition 1. *Either*

(1) $0 < B_U'(1) < 1$ *and we call U* hyperbolic*, or*

(2) $B_U(z) = z \pm ia(z-1)^2 + \mathcal{O}((z-1)^3)$*, for some $a > 0$ and we call U* simply parabolic*, or*

(3) $B_U(z) = z - a(z-1)^3 + \mathcal{O}((z-1)^4)$ *for some* $a > 0$, *and we call* U doubly parabolic. *In this case* $f : U \to U$ *is of degree at least* 2.

Proof. We need to study the cases when $B_U'(1) = 1$. Then B_U can be written as

$$B_U(z) - 1 = (z-1) + a(z-1)^{q+1} + \mathcal{O}((z-1)^{q+2})$$

for some $q \geq 1$ and $a \neq 0$. By the Fatou Flower Theorem, there exist q immediate attracting basins $\mathcal{A}_1^0, \ldots, \mathcal{A}_q^0$ attached to the point 1 which are disjoint open sets each contained in a sector of angle $2\pi/q$. We claim that the only possible values for q are 1 or 2. Indeed, observe that the whole unit disc is contained in some immediate basin of attraction , say $\mathcal{A}_1^0$, since $\mathbb{D}$ is connected, invariant, and all its points have orbits converging to 1. By symmetry, $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ is also contained in one immediate attracting basin. This implies that there are at most two basins of attraction, and hence q is at most two. In either case, the basins are completely invariant. See Figure 1.

Let us look at the two possibilities separately. For $q = 1$, the change of variable $\zeta = 1/(z-1)$, conjugates B_U on $\mathbb{D}$ to the map

$$\widetilde{B}_U(\zeta) = \zeta - a + \mathcal{O}(1/\zeta)$$

on the left half plane $\{\mathrm{Re} z < -1/2\}$. Observe that the vertical line $\{\mathrm{Re} z = -1/2\}$ is invariant. Taking points on this line close to infinity, we see that a must be purely imaginary. This is equivalent to saying that the repelling direction emanating from $z = 1$ is along the imaginary axis, maybe pointing upwards or maybe downwards.

In the case $q = 2$ there are two repelling directions and since none of them can point into $\mathbb{D}$ (for $z_0 = 1$ attracts all points of the disc) we must have them along the imaginary axis. We can write

$$B_U(z) - 1 = (z-1)(1 + a(z-1)^2 + \mathcal{O}((z-1)^3))$$

and observe that for z on the imaginary axis close to $z = 1$, the quantity $a(z-1)^2$ must be real and positive. Since $z - 1$ is purely imaginary, this can only be acomplished if a is a negative real number. □

Now we would like to find some geometric or dynamical criteria to identify the type of a given Baker domain. Recall that we defined an *invariant petal at infinity* to be a connected, simply connected and unbounded set $\mathcal{G}$ whose boundary (in $\mathbb{C}$) is an invariant simple curve and such that f is a conformal isomorphism of $\mathcal{G}$ onto itself. We recall the statement of Proposition 2.

Proposition 2. *An invariant regular Baker domain* U *is*

(a) *hyperbolic, if and only if* U *does not contain the closure, in* $\mathbb{C}$, *of an invariant petal at infinity;*

(b) *simply parabolic, if and only if* U *contains the closure of an invariant petal at infinity* $\mathcal{G}$ *and any other such petal intersects* $\mathcal{G}$*;*

(c) *doubly parabolic, if and only if* U *contains the closure of two disjoint invariant petals at infinity.*

Proof. Let B_U be the inner function associated to U and f. We first assume that U is simply parabolic. Let $\mathcal{A}$ be the parabolic basin of attraction of 1. We know there exist an attracting parabolic petal $\mathcal{P}$ and a repelling parabolic petal $\mathcal{P}'$ which we choose to be symmetric and tangent to the attracting and the repelling directions at 1 respectively. Hence $\mathcal{P} \cap \mathcal{P}'$ consists of two connected components, one in $\mathbb{D}$ and the other in $\mathbb{C} \setminus \mathbb{D}$. See Figure 3. Let Q denote the connected component inside $\mathbb{D}$.

By considering the Fatou coordinates it is easy to check that there exists a curve Γ (as small as we want) emanating from the parabolic point tangentially to the repelling direction, and converging to the same point tangentially to the attracting direction. Thus, this curve Γ is the boundary of an invariant petal at infinity.

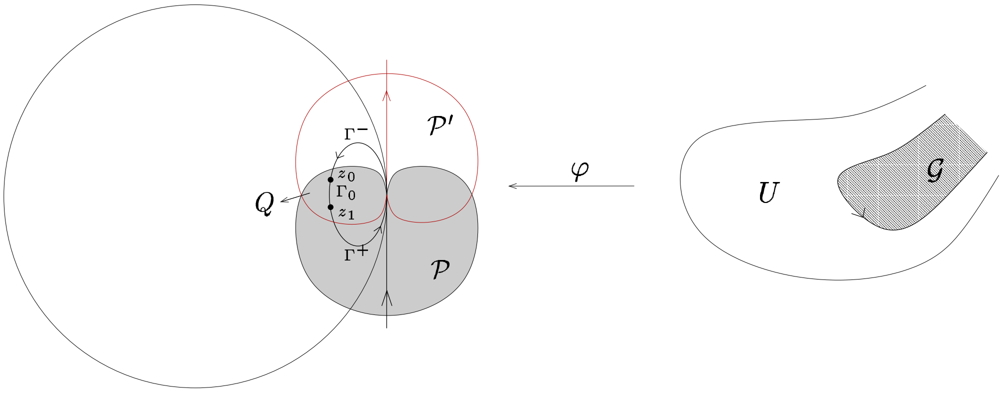

Figure 3: Construction of an invariant petal at infinity in the simply parabolic case.

For the doubly parabolic case we leave to the reader to check the existence of two invariant petals at infinity (see Figure 4).

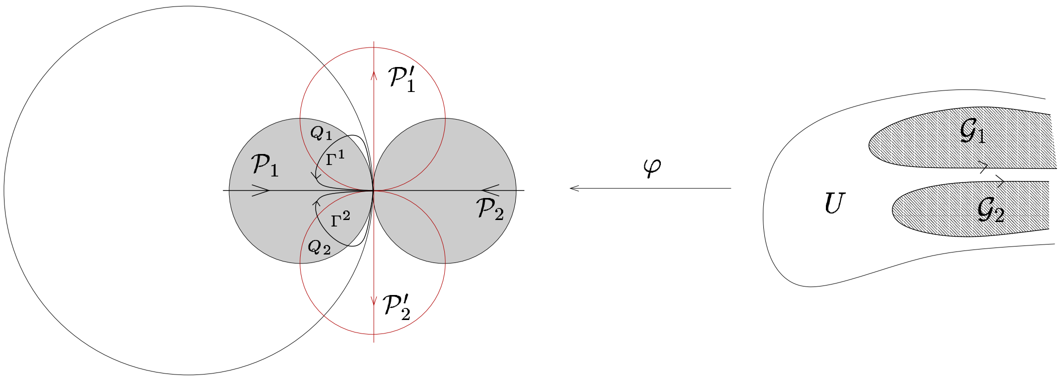

Figure 4: Construction of two invariant petals at infinity in the doubly parabolic case.

Hence, if U is parabolic then U contains an invariant petal at infinity. So if U does not contain such a petal it is necessarily hyperbolic.

To see the other implication, we suppose that U is hyperbolic and assume $\mathcal{G}$ is an invariant petal at infinity whose closure is contained in U. First observe that $\mathcal{G}$ corresponds to an open domain $\widetilde{\mathcal{G}}$ in the unit disk which is mapped univalently onto itself by B_U. Since B_U converges to the constant function 1, this point is contained in the boundary of $\widetilde{\mathcal{G}}$. By assumption the boundary of $\widetilde{\mathcal{G}}$ does not meet the unit circle in any other point.

Let ϕ denote the linearizing coordinates defined on the basin of attraction of $z_0 = 1$, which conjugate B_U to multiplication by $\lambda = B_U'(1)$ on the complex plane. Observe that no bounded domain can be invariant by this linear map, since any point must enter, after a finite number of iterations, a disk of any prescribed size around 0. But now, $\phi(\widetilde{\mathcal{G}})$ is a bounded domain which must be invariant under multiplication by λ. This concludes the proof of (a).

To finish the proof, it will suffice to show that if U is simply parabolic and $\mathcal{G}_1, \mathcal{G}_2 \subset U$ are invariant petals at infinity then the intersection $\mathcal{G}_1 \cap \mathcal{G}_2$ is non-empty.

The two petals $\mathcal{G}_1$ and $\mathcal{G}_2$ correspond to two domains $\widetilde{\mathcal{G}}_1$ and $\widetilde{\mathcal{G}}_2$ in the unit disk, whose boundaries in $\mathbb{D}$ are connected curves. As above, these curves must have both their ends at $z = 1$ and, because f is degree one on each of the domains, the dynamics on the boundaries have a definite direction.

Now, we are assuming that U is simply parabolic, which means that B_U has one attracting and one repelling direction, both on the imaginary axis. By the Fatou Flower Theorem, any orbit that tends to $z = 1$ must do so tangent to the attracting direction. Likewise, any orbit escaping from $z = 1$ must exit tangent to the repelling axis. This leaves only one possibility for the boundaries of $\tilde{\mathcal{G}}_1$ and $\tilde{\mathcal{G}}_2$, namely that each of their ends is tangent to the imaginary axis. But two such domains must necessarily intersect each other.

□

An immediate corollary of these topological criteria is the following.

Corollary 3.1. *The type of a regular Baker domain is preserved under conjugacy by homeomorphisms.*

4 Examples

Examples of hyperbolic and simply parabolic univalent Baker domains were already given in [Baranski & Fagella 2000], but we include them here for completeness. Additionally we present an example of a degree two doubly parabolic domain.

Up to this date, we do not know of any example of a hyperbolic or simply parabolic regular Baker domain with degree larger than one or any doubly parabolic domain with degree larger than two. We summarize this in the following table.

	Univalent	$1 <$ degree
Hyperbolic	Example 1	?
Simply parabolic	Example 2	?
Doubly parabolic	$\times\times\times$	Example 3

Example 1. (univalent, hyperbolic)

Let $f(z) = z + \alpha + \beta\sin(z)$ for $0 < \alpha < 2\pi$ and $0 < \beta < 1$. Projecting f by $w = e^{iz}$, we obtain the map

$$F(w) = e^{i\alpha} w e^{\frac{\beta}{2}(w - 1/w)}$$

which is a holomorphic self-map of $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. It is easy to check that F restricted to the unit circle $\mathbb{S}^1$ is the well-known standard family of circle maps.

For appropriately chosen values of the parameters α and β, the map F has a Herman ring V symmetric with respect to $\mathbb{S}^1$. It is easy to check that lifting V by e^{iz} we obtain a Fatou component U of f, which is an invariant Baker domain, symmetric with respect to the real axis. See Figure 5. Since V is a rotation domain, the map F is univalent in V. Using the fact $f(z + 2k\pi) = f(z) + 2k\pi$ one can easily show that $f|_U$ must also be univalent.

One can check that U is conformally equivalent to a horizontal band B of finite height and that f in U is conjugate to a horizontal translation in B. It follows easily that U is hyperbolic.

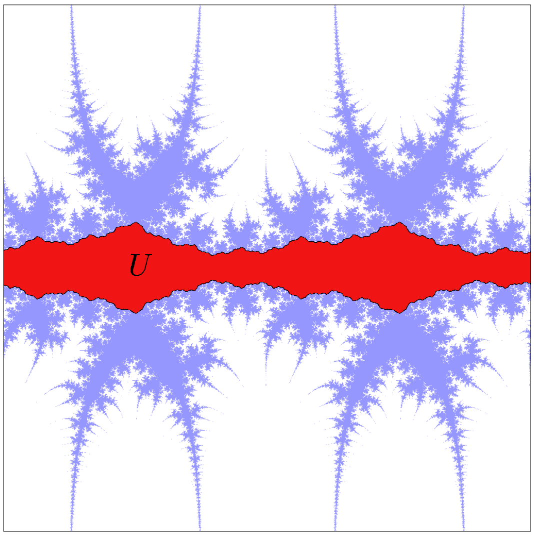

Figure 5: The dynamical plane of $f(z) = z + \alpha + \beta \sin(z)$ for certain values of α and β such that f has a univalent hyperbolic Baker domain.

Example 2. (univalent, simply parabolic) Let $g(w) = \lambda w \exp w$. Then 0 is a fixed point of g with multiplier λ. The map g has only one critical point at $z = -1$. Observe that g is semiconjugate to the map $f(z) = z + \log(\lambda) + e^z$ by $w = e^z$.

Let $\lambda = e^{2\pi i\theta}$ where θ is chosen so that g has a Siegel disk Δ around 0 (we can choose θ to be any Brjuno number). Then Δ lifts under e^z to a domain U which contains a left half plane. See Figure 6. The invariant closed curves in Δ lift to invariant almost vertical curves in U, the points of which move upwards towards infinity. Hence U is a Baker domain which is easily seen to be univalent. Observe that each of these curves is the boundary of an invariant petal at infinity, hence this is a parabolic example and, since it is univalent, it is simply parabolic.

Example 3. (degree 2, doubly parabolic)

In this section we study the example

$$f(z) = z + e^{-z},$$

which was also investigated in [Baker & Domínguez 1999], showing the existence of infinitely many invariant Baker domains for f. We start by proving the same fact using different arguments and then proceed to show that the domains are doubly parabolic.

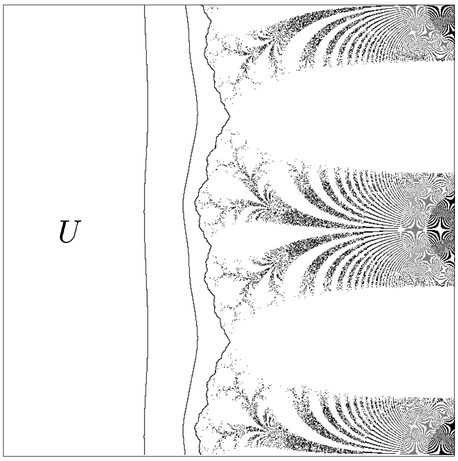

Figure 6: The dynamical plane of $f(z) = z + \alpha + e^z$ with $\alpha = \frac{\sqrt{5}-1}{2}$, which contains a univalent simply parabolic Baker domain.

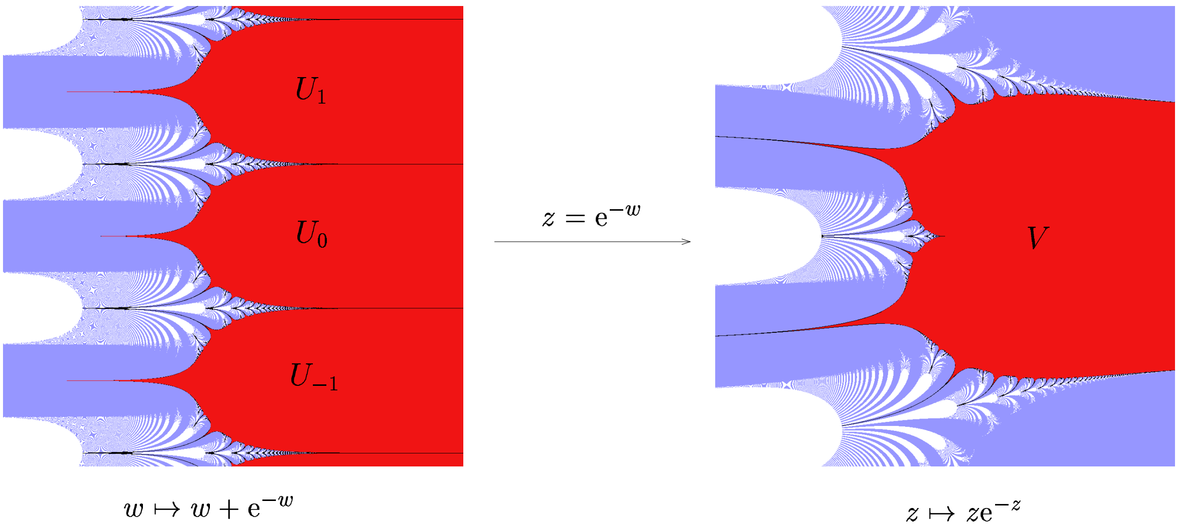

Figure 7: To the left is an illustration of the dynamics of $f : w \mapsto w + \mathrm{e}^{-w}$. This map possesses a sequence of fixed doubly parabolic proper Baker domains $\ldots, U_{-1}, U_0, U_1, \ldots$. The map f is semiconjugate to $g : z \mapsto z\mathrm{e}^{-z}$ by $z = \mathrm{e}^{-w}$. The Baker domains of f correspond to the immediate parabolic basin of attraction of the parabolic fixed point 0 of g.

To study the dynamics of f it is convenient to work with the map $g(w) = we^{-w}$ that is semiconjugate to f by $w = e^{-z}$. Observe that $w = 0$ is a fixed point of g of multiplier 1, and $g(w) = w - w^2 + \mathcal{O}(w^3)$ near 0. From the results in Section 2 we know that there are exactly one attracting and one repelling direction of g at 0; they are the positive and negative real axes respectively. There exists an attracting petal $\mathcal{P}$ of f at 0 which determines a basin of attraction $\mathcal{A}$. Let $\mathcal{A}^0$ denote the immediate basin of attraction, i.e., the connected component of $\mathcal{A}$ that contains $\mathcal{P}$. Then, $\mathcal{A}^0$ also contains the unique critical point $w = 1$. Clearly, all orbits that tend to $w = 0$ must do so tangentially to the positive real axis, while the backward orbits that tend to 0 do so tangentially to the negative real axis.

We now lift this picture back to the dynamical plane of f (see Figures 7 and 8). Observe that the preimages of $\mathbb{R}^-$ under e^{-z} are the horizontal lines $\{\text{Im}z = (2k+1)\pi, k \in \mathbb{Z}\}$. Hence all of them are invariant by f and their points have orbits whose real part tends to $-\infty$ exponentially fast. This implies that all of them lie in the Julia set of f.

The horizontal strips that lie in between these preimages are mapped to the whole dynamical plane of g in a one-to-one fashion and, therefore, they each contain a preimage of $\mathcal{A}^0$. Let us denote these preimage by $\ldots, U_{-1}, U_0, U_1, \ldots$, and observe that each U_k contains the invariant horizontal line $\text{Im}z = 2k\pi$, since these are mapped to the positive real axis by e^{-z}. Hence, for all $k \in \mathbb{Z}$, the set U_k is invariant and its points tend to infinity under iteration of f (since this is the preimage of 0 under the conjugation). Therefore each of these sets is an invariant Baker domain.

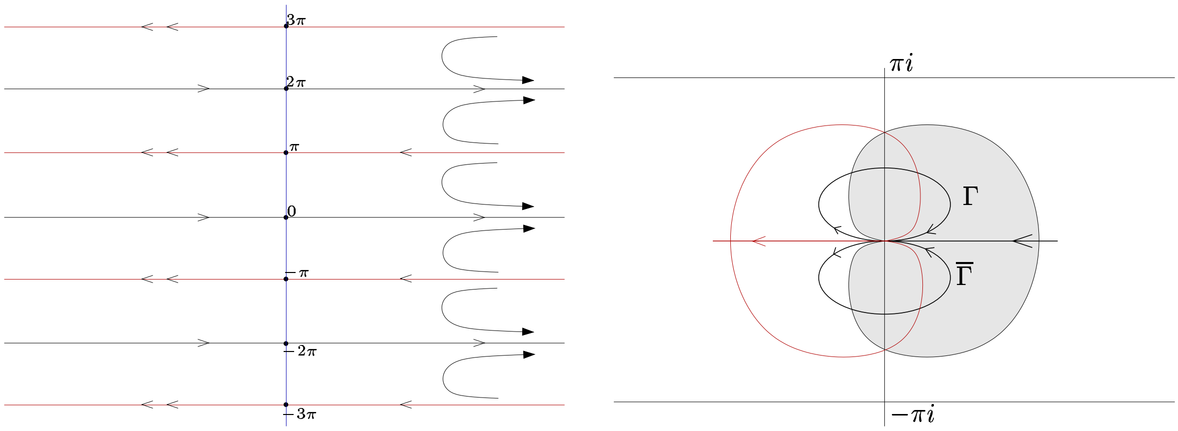

Figure 8: Left: Sketch of the dynamical plane of f. There is an invariant Baker domain in every strip $\text{Im}z \in ((2k+1)\pi, (2k+3)\pi)$, $k \in \mathbb{Z}$. Right: Construction of invariant petals in the dynamical plane of g. Compare to Figure 7.

We now proceed to check that each U_k is doubly parabolic by constructing two disjoint invariant petals at infinity inside U_k. By considering Fatou coordinates in the dynamical plane of g, one can check that there exist two invariant curves, Γ and $\overline{\Gamma}$ emanating from 0 in the direction of $\mathbb{R}_-$ and tending to 0 tangentially to $\mathbb{R}_+$, see Figure 8 to the right. The domains bounded by these curves lift to two invariant petals at infinity in each U_k.

5 Deformations. Proof of the Main Theorem.

In this section we consider the Teichmüller space of an entire mapping f with a regular fixed Baker domain, using the general framework given by [McMullen & Sullivan 1998]. We will see that the dimension of this set is infinite if the Baker domain is hyperbolic or simply parabolic, and from this we will deduce that the quasiconformal deformation space of f is infinite dimensional. For some preliminaries on quasiconformal mappings see Section 2.3.

Let V be an open subset of the complex plane or more generally a one dimensional complex manifold and f a holomorphic endomorphism of V. Define an equivalence relation $\sim$ on the set of quasiconformal homeomorphisms on V by identifying $\phi : V \to V'$ with $\psi : V \to V''$ if there exists a conformal isomorphism $c : V' \to V''$ such that $c \circ \phi = \psi$, i.e. the following diagram commutes.

$$\begin{tikzcd} V \arrow[r, "\phi"] \arrow[dr, "\psi"'] & V' \arrow[d, "c"] \\ & V'' \end{tikzcd}$$

It then follows that $\phi \circ f \circ \phi^{-1}$ and $\psi \circ f \circ \psi^{-1}$ are conformally conjugate (although the converse is not true in general). Then the deformation space of f on V is

$$\mathrm{Def}(f, V) = \{\phi : V \to V' \text{ quasi conformal} \mid \mu_\phi \text{ is } f\text{-invariant } \}/ \sim .$$

As a consequence of the Measurable Riemann Mapping Theorem (see [Ahlfors 1966] or Theorem 2.15) one obtains a bijection between $\mathrm{Def}(f, V)$ and

$$\mathcal{B}_1(f, V) = \{f - \text{invariant Beltrami forms } \mu \in L^\infty \text{ with } ||\mu||_\infty < 1\},$$

and this is used to endow $\mathrm{Def}(f, V)$ with the structure of a complex manifold. Indeed, $\mathcal{B}_1(f, V)$ is the unit ball in the Banach space of f-invariant Beltrami forms equipped with the infinity norm.

We denote by $\mathrm{QC}(f, V)$ the set of quasiconformal automorphisms of V that commute with f. A family of q.c. mappings is called uniformly K-q.c. if each element of the family is K-q.c.

A hyperbolic Riemann surface V is covered by the unit disk; in fact V is isomorphic to $\mathbb{D}/\Gamma$ where Γ is a Fuchsian group. Let $\Omega \subseteq \mathbb{S}^1$ denote the complement of the limit set of Γ. Then $(\mathbb{D} \cup \Omega)/\Gamma$ is a bordered surface and Ω/Γ is called the ideal boundary of V. A homotopy $\omega_t : V \to V, 0 \leq t \leq 1$ is called *rel ideal boundary* if there exists a lift $\hat{\omega}_t : \mathbb{D} \to \mathbb{D}$ that extends continously to Ω as the identity. If V is not hyperbolic then the ideal boundary is defined to be the empty set.

We denote by $\mathrm{QC}_0(f, V) \subseteq \mathrm{QC}(f, V)$ the subgroup of automorphisms which are homotopic to the identity rel the ideal boundary of V through a uniformly K-q.c. subset of $\mathrm{QC}(f, V)$.

Earle and McMullen [Earle & McMullen 1988] prove the following result for hyperbolic subdomains of the Riemann sphere.

Theorem 5.1. *Suppose $V \subseteq \widehat{\mathbb{C}}$ is a hyperbolic subdomain of the Riemann sphere. Then a uniformly quasiconformal homotopy $\omega_t : V \to V, 0 \leq t \leq 1$ can be extended to a uniformly* *quasiconformal homotopy of $\widehat{\mathbb{C}}$ by letting $\omega_t = \mathrm{Id}$ on the complement of V. Conversely, a uniformly quasiconformal homotopy $\omega_t : V \to V$ such that each ω_t extends continuously as the identity to the topological boundary $\partial V \subseteq \widehat{\mathbb{C}}$ is a homotopy rel the ideal boundary.*

Proof. The proof can be found in [Earle & McMullen 1988]: Proposition 2.3 and the proof of Corollary 2.4 imply the first statement. Theorem 2.2 implies the second. □

The group $\mathrm{QC}(f, V)$ acts on $\mathrm{Def}(f, V)$ by $\omega_* \phi = \phi \circ \omega^{-1}$. Indeed if ϕ and ψ represent the same element in $\mathrm{Def}(f, V)$ then $\omega_* \phi = \omega_* \psi$ as elements of $\mathrm{Def}(f, V)$.

Definition 5.2. The Teichmüller space $\mathcal{T}(f, V)$ is the deformation space $\mathrm{Def}(f, V)$ modulo the action of $\mathrm{QC}_0(f, V)$, i.e. $\mathcal{T}(f, V) = \mathrm{Def}(f, V)/\mathrm{QC}_0(f, V)$. If V is a one dimensional complex manifold we denote by $\mathcal{T}(V)$ the Teichmüller space $\mathcal{T}(\mathrm{Id}, V)$.

Teichmüller space can be equipped with the structure of a complex manifold and a (pre)-metric (we refer to [McMullen & Sullivan 1998]).

Let us give a rough idea of Teichmüller space and the motivation for studying it. In holomorpic dynamics one is often interested in studying the set $\mathcal{F}$ of holomorphic mappings that are quasiconformally conjugate to a given holomorhic map $f : V \to V$ modulo conjugacy by conformal isomorphisms. Such a mapping can be written as $\phi \circ f \circ \phi^{-1}$ for a $\phi \in \mathrm{Def}(f, V)$. Now $\phi \circ f \circ \phi^{-1}$ and $\psi \circ f \circ \psi^{-1}$ are conformally conjugate exactly when they represent the same element in $\mathrm{Def}(f, V)/\mathrm{QC}(f, V)$. So we can study $\mathcal{F}$ by looking at $\mathrm{Def}(f, V)/\mathrm{QC}(f, V)$. Clearly the Teichmüller space is related to this space, and it can be shown to be, at least morally, a covering of it. Because of the nice properties of Teichmüller space, this space is often more convenient to study than $\mathcal{F}$.

For an endomorphism f of the space V, the *grand orbit* of $y \in V$ is the set $\{x \in V \mid f^n(x) = f^m(y)$ for some $n, m > 0\}$. The grand orbit of a set is the union of the grand orbits of its elements. The *grand orbit relation* is the equivalence relation such that $x \sim y$ if and only if they have the same grand orbit. We denote by V/f the quotient space obtained from V by identifying points under the grand orbit relation of f.

Sullivan and McMullen prove stronger versions of the following two theorems.

Theorem 5.3. *Let f be an entire mapping, and suppose that U_α is a family of pairwise disjoint completely invariant open subsets of $\mathbb{C}$. Then*

$$\mathcal{T}(f, \cup U_\alpha) \simeq \prod \mathcal{T}(f, U_\alpha).$$

Proof. This follows from Theorem 5.5 in [McMullen & Sullivan 1998]. □

Theorem 5.4. *Suppose every component of the one-dimensional manifold V is hyperbolic, $f : V \to V$ is a holomorphic covering map, and the grand orbit relation of f is discrete. If V/f is connected then V/f is a Riemann surface and*

$$\mathcal{T}(f, V) \simeq \mathcal{T}(V/f).$$

Proof. This is a consequence of Theorem 6.1 in [McMullen & Sullivan 1998]. □

After these general definitions, we return to the case where f is an entire mapping with a fixed regular Baker domain U. By definition $f : U \to U$ is conjugate to its inner function $B_U : \mathbb{D} \to \mathbb{D}$, with a non-repelling fixed point at 1.

Proposition 5.5. *Let f be an entire mapping with a regular fixed Baker domain U. Let $\mathcal{U}$ denote the grand orbit of U and set $W = \mathcal{U}/f$. Then we have the following three mutually exclusive possibilities.*

1. *U is hyperbolic and W is an annulus of finite modulus;*
2. *U is simply parabolic and W is an annulus of one-sided infinite modulus i.e. conformally isomorphic to $\{x + iy \mid y > 0\}/\mathbb{Z}$;*
3. *U is doubly parabolic and W is an annulus of two-sided infinite modulus i.e. conformally equivalent to $\mathbb{C}/\mathbb{Z}$.*

Proof. Let B_U be the inner function associated to U (see Section 1). Extend B_U to $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ by reflection and to the unit circle where possible.

In the hyperbolic case B_U has an attracting fixed point at 1 and every point in $\mathbb{D}$ converges towards 1 under iteration. By Proposition 2.4 and Remark 2.5, the local conjugacy between B_U and multiplication by $0 < \lambda < 1$ can be extended to a (semi)conjugacy mapping $\mathbb{D}$ into the left halfplane, covering a neighborhood of zero. It follows that the conjugacy induces a conformal isomorphism from $\mathbb{D}/B_U$ to the left half plane modulo the group generated by $z \mapsto \lambda z$. So $\mathbb{D}/B_U$ is an annulus of modulus $(-\log \lambda)/\pi$.

In the simply parabolic case B_U has a simple parabolic fixed point at one. The unit disk is contained in the immediate basin of attraction and by symmetry so too is the complement of the closure of the unit disk. There exists a holomorhic mapping, sending the immediate attracting basin of 1 into $\mathbb{C}$, covering a half plane and conjugating B_U to translation by 1, (see Proposition 2.7). By symmetry (see Proposition 2.10) this mapping can be taken to map $\mathbb{D}$ into, say, the upper half plane. It induces a conformal isomorphism betwen $\mathbb{D}/B_U$ and the upper half plane modulo the group generated by $z \mapsto z + 1$.

In the doubly parabolic case the unit disk coincides with a basin of attraction of the parabolic fixed 1. As before we have a linearizing mapping, conjugating the dynamics of B_U on $\mathbb{D}$ to $z \mapsto z + 1$ on $\mathbb{C}$, (see 2.10). This mapping induces a conformal isomorphism sending $\mathbb{D}/B_U$ to $\mathbb{C}/\mathbb{Z}$.

Finally notice that $\mathbb{D}/B_U$ is conformally equivalent to U/f that again is conformally isomorphic to $\mathcal{U}/f$. □

We now move on to show that the grand orbit of the set of singular values is formed by dynamically distinguished points. More precisely we have the following proposition.

Proposition 5.6. *Let f be an entire mapping and $\mathcal{U}$ a totally invariant open set whose connected components are simply connected and hyperbolic. Denote by S the set of singular values of f in $\mathcal{U}$. Then any $\omega \in \mathrm{QC}_0(f, \mathcal{U})$ restricts to the identity on the closure of the grand orbit of S in $\mathcal{U}$.*

To prove the proposition we need the following lemma.

Lemma 5.7. *Let V be a simply connected hyperbolic subset of $\mathbb{C}$, and $f : V \to \mathbb{C} \setminus \{a, b\}$ be a holomorphic map into the thrice punctured sphere. Suppose $\gamma : [0, +\infty] \to \widehat{\mathbb{C}}$ is a curve such that*

1. *$\gamma([0, +\infty)) \subset V$,*

2. $\gamma(+\infty) \in \partial V \cup \{\infty\}$, *and*

3. $\lim_{\tau \to +\infty} f \circ \gamma(\tau) = x_0 \in \widehat{\mathbb{C}}$.

Let $(z_n) \subset V$ be a sequence converging to the boundary of V in $\widehat{\mathbb{C}}$ and satisfying that $d_V(z_n, \gamma) \leq C$ for some C. Then, if $f(z_n)$ converges to a point in $\widehat{\mathbb{C}}$ this point must be x_0.

Proof. Set $x_1 = \lim f(z_n)$. We must show that $x_1 = x_0$. Let $\phi : \mathbb{H} \to V$ be a Riemann mapping that sends the upper half plane $\mathbb{H}$ conformally onto V. By [Pommerenke 1991] (Proposition 2.14) the curve $\tilde{\gamma} = \phi^{-1} \circ \gamma|_{[0,+\infty)}$ extends continuously to a curve $\tilde{\gamma} : [0, +\infty] \to \overline{\mathbb{H}} \cup \{\infty\}$, with $\tilde{\gamma}(+\infty) \in \partial H \cup \{\infty\}$. By replacing ϕ with another Riemann mapping we can suppose $\tilde{\gamma}(\infty) = 0$. Let $w_n = \phi^{-1}(z_n)$ and let $\tau_n \geq 0$ be a sequence such that $d_{\mathbb{D}}(\tilde{\gamma}(\tau_n), w_n) \leq C$; we must have $t_n \to +\infty$. Let L_n denote the affine mapping that maps $\mathbb{H}$ onto itself and sends $\gamma(\tau_n)$ to i. Set $g_n = f \circ \phi \circ L_n^{-1} : \mathbb{H} \to \mathbb{C} \setminus \{a, b\}$. By Montel's theorem g_n is a normal sequence and by passing to a subsequence we suppose that g_n converges to a map $g_\infty : \mathbb{H} \to \widehat{\mathbb{C}}$, locally uniformly in $\mathbb{H}$. Clearly $g_\infty(i) = x_0$.

We claim that g_∞ is the constant mapping. Let $r \in (0, 1)$ be arbitrary. Take $\tau_n' > \tau_n$ such that

$$|L_n \circ \tilde{\gamma}(\tau_n') - L_n \circ \tilde{\gamma}(\tau_n)| = |L_n \circ \tilde{\gamma}(\tau_n') - i| = r.$$

Such τ_n' exists since $\mathrm{Im}(L_n \circ \tilde{\gamma}(\tau)) \to 0$ as $\tau \to +\infty$. Now, $g_n \circ L_n(\tilde{\gamma}(\tau_n')) = \gamma(\tau_n') \to x_0$ and it follows that there exists a point on the circle with center i and radius r that g_∞ maps to x_0 (any accumulation point of $\{L_n(\tilde{\gamma}(\tau_n'))\}$ will do). Since r was arbitrary, the Identity Theorem implies that g_∞ is the constant map $w \mapsto x_0$.

Note that $L_n(w_n)$ is contained in the closure of the hyperbolic disk $\{d_{\mathbb{H}}(\zeta, i) < C\}$. On the one hand $g_n(L_n(w_n)) \to x_0$. On the other hand $g_n(L_n(w_n)) = f(z_n) \to x_1$. We conclude that $x_0 = x_1$. ☐

Proof of Proposition 5.6. We first show that ω restricts to the identity on the set of critical values, then that it restricts to the identity on the set of asymptotic values, and finally to the closure of the grand orbit of these two sets.

Let ω_t be a path in $\mathrm{QC}(f, \mathcal{U})$ that connects $\omega_0 = \mathrm{Id}$ to $\omega_1 = \omega$. Since ω_t commutes with f, the set of critical points is ω_t invariant. So, if $c \in \mathcal{U}$ is a critical point, the path $t \mapsto \omega_t(c)$ is a subset of the critical points. Since this set is discrete $\omega_t(c) = c$ for all t. Since ω_t commutes with f we immediately get that every ω_t fixes the critical values.

Now let $x_0 \in \mathcal{U}$ be an asymptotic value, and $\gamma : [0, +\infty) \to \mathcal{U}$ a corresponding asymptotic path; i.e. a curve with the property that $\lim_{\tau \to \infty} \gamma(\tau) = \infty$ and $\lim_{\tau \to \infty} f \circ \gamma(\tau) = x_0$. Let V denote the component of $\mathcal{U}$ that contains γ. Then V is simply connected and hyperbolic, [Eremenko & Lyubich 1992]. By assumption $\omega_t|_V : V \to V$ fixes the ideal boundary of V. Set

$$x_t = \omega_t(x_0) = \lim_{\tau \to \infty} f \circ \omega_t \circ \gamma(\tau)$$

We must show that $x_t = x_0$. Let $\tau_n > 0$ be a sequence tending towards $+\infty$ and set $z_n = \omega_t \circ \gamma(\tau_n)$ By Lemma 2.16 there exists a constant C such that the hyperbolic distance in V satisfies $d_V(\gamma(\tau_n), z_n) \leq C$. Since $f(V)$ is contained in a component of $\mathcal{U}$ which is hyperbolic, we can apply Lemma 5.7 and we get that $x_t = \lim_{n \to +\infty} f(z_n) = x_0$. So ω_t fixes the asymptotic values of f in $\mathcal{U}$.

Since every singular value is in the closure of the set of asymptotic and critical values, we get by continuity that ω_t fixes the singular values of f in $\mathcal{U}$. Since ω_t commutes with f we get that ω_t restricts to the identity on the forward orbit of this set. Now suppose $\omega_t(y) = y$ for all t and that $f^n(x) = y$. Then $\omega_t(x)$ must map into $f^{-n}\{y\}$. Since this set is discrete we get that $\omega_t(x) = x$ for all x. It follows that ω_t restricts to the identity on the grand orbit of S for all t and by continuity this is also true on the closure. $\square$

We can now prove our main theorem whose statement we recall here.

Main Theorem. *Let U be a proper fixed Baker domain of the entire function f and $\mathcal{U}$ its grand orbit. Denote by S the set of singular values of f in $\mathcal{U}$, and by $\widehat{\mathcal{S}}$ the closure of the grand orbit of S taken in $\mathcal{U}$. Then $\mathcal{T}(f,\mathcal{U})$ is infinite dimensional except if U is doubly parabolic and the cardinality of $\widehat{\mathcal{S}}/f$ is finite. In that case the dimension of $\mathcal{T}(f,\mathcal{U})$ equals $\#\widehat{\mathcal{S}}/f - 1$.*

Proof. By Lemma 5.6 every element of $\mathrm{QC}_0(f,\mathcal{U})$ restricts to the identity on $\widehat{\mathcal{S}}$. Hence

$$\begin{aligned}\mathcal{T}(f,\mathcal{U}) &\simeq \mathcal{B}_1(f,\mathcal{U})/\mathrm{QC}_0(f,\mathcal{U}) \simeq \left(\mathcal{B}_1(f,\widehat{\mathcal{S}}) \times \mathcal{B}_1(f,\mathcal{U}-\widehat{\mathcal{S}})\right)/\mathrm{QC}_0(f,\mathcal{U}) \\ &\simeq \mathcal{B}_1(f,\widehat{\mathcal{S}}) \times \left(\mathcal{B}_1(f,\mathcal{U}-\widehat{\mathcal{S}})/\mathrm{QC}_0'(f,\mathcal{U})\right),\end{aligned}$$

where we denote by $\mathrm{QC}_0'(f,\mathcal{U})$ the group formed by the restriction of each element in $\mathrm{QC}_0(f,\mathcal{U})$ to $\mathcal{U} - \widehat{\mathcal{S}}$. Since the elements in $\mathrm{QC}_0(f,\mathcal{U})$ are the identity on $\widehat{\mathcal{S}}$, it follows from Theorem 5.1 that

$$\mathrm{QC}_0'(f,\mathcal{U}) = \mathrm{QC}_0(f,\mathcal{U}-\widehat{\mathcal{S}}).$$

Therefore,

$$\mathcal{T}(f,\mathcal{U}) \simeq \mathcal{T}(f,\mathcal{U}-\widehat{\mathcal{S}}) \times \mathcal{B}_1(f,\widehat{\mathcal{S}}).$$

By Proposition 5.5, $W = \mathcal{U}/f$ is an annulus of finite modulus when U is hyperbolic, one-sided infinite modulus when U is simply parabolic and two-sided infinite modulus when U is doubly parabolic. The subset $T = \widehat{\mathcal{S}}/f \subset W$ is relatively closed in W, so $W - T$ is an open set. We denote the components of W by V_i. Then each $V_i = \mathcal{V}_i/f$ for a completely invariant open subset $\mathcal{V}_i$ of $\mathbb{C}$ and $\cup\mathcal{V}_i = \mathcal{U} - \widehat{\mathcal{S}}$. By Theorem 5.3 we have

$$\mathcal{T}(f,\mathcal{U}-\widehat{\mathcal{S}}) \simeq \prod_i \mathcal{T}(f,\mathcal{V}_i),$$

and by Theorem 5.4 we have

$$\prod_i \mathcal{T}(f,\mathcal{V}_i) \simeq \prod_i \mathcal{T}(V_i).$$

If T contains interior points, then $\mathcal{B}_1(f,\widehat{\mathcal{S}})$ is infinite dimensional, so we can suppose it does not. Then T is a proper subset of W. If T has infinitely many components then a component V_i of $W - T$ is either of infinite connectivity or has ideal boundary (or both). In both cases the Teichmüller space is infinite (see [Gardiner 1987]). So we can assume that T has only finitely many components. If one of these components is not a point then the presence of ideal boundary forces the dimension of the Teichmüller space to be infinite. Consequently we can assume that T is a finite set. Then $W-T$ has only one component; it is an annulus with finitely many punctures. If W is of finite or one-sided infinite modulus, again the presence of ideal boundary will force the dimension to be infinite. So we can suppose that W is an annulus of doubly infinite modulus and U is a doubly parabolic Baker domain.

Since T is finite $\mathcal{B}_1(f, \widehat{\mathcal{S}})$ is trivial and

$$\mathcal{T}(f, \mathcal{U}) \simeq \mathcal{T}(W - T).$$

Finally $W - T$ is conformally equivalent to the sphere with $2 + \#T$ punctures. It is well known that the dimension of Teichmüller space of the sphere with n punctures is $n - 3$. So the dimension of $\mathcal{T}(f, \mathcal{U})$ equals $2 + \#T - 3 = \#T - 1$. The proof is finished recalling that $\#T = \#\widehat{\mathcal{S}}/f$. $\square$

We conclude this section by remarking that the dimension of the Teichmüller space of f on the grand orbit of a Baker domain gives a lower bound of the Teichmüller space of f. Indeed, with $\mathcal{U}$ denoting the grand orbit of a Baker domain, $J(f)$ the boundary of $\mathcal{U}$ and $\mathcal{V}$ the complement of $\mathcal{U} \cup J(f)$ we get

$$\mathcal{T}(f, \mathbb{C}) \simeq \mathcal{T}(f, \mathcal{U}) \times \mathcal{B}_1(f, J(f)) \times \mathcal{T}(f, \mathcal{V}).$$

So in general we expect $\mathcal{T}(f, \mathbb{C})$ to be high dimensional. It may then come as a surprise, that we can give an example of an entire function f with fixed proper Baker domains which is rigid, in the sense that the Teichmüller space $\mathcal{T}(f, \mathbb{C})$ is trivial. We will exhibit such an example in the next section.

6 A rigid example; proof of Proposition 3

In this section we shall show that the doubly parabolic example

$$f(z) = z + e^{-z},$$

is rigid. More precisely, we show the following.

Proposition 3. *The map $f(z) = z + e^{-z}$ is rigid, i.e., if $\widetilde{f}$ is a holomorphic map which is quasiconformally conjugate to f, then $\widetilde{f}$ is conjugate to f by an affine map.*

We need the following preliminary lemma which follows easily from work by Eremenko and Lyubich.

Lemma 6.1. *Let $f(z) = z + e^{-z}$. The Julia set $J(f)$ has measure zero.*

Proof. The mapping $z \mapsto e^{-z}$ semi conjugates f to $g(z) = ze^{-z}$, and $J(f)$ is the preimage of $J(g)$ under $z \mapsto e^{-z}$ (see [Bergweiler 1999]). So to show $J(f)$ has measure zero, it will suffice to show that $J(g)$ has measure zero. The entire function g has exactly one critical point $\omega = 1$ and exactly one asymptotic value $a = 0$. Since the asymptotic value is absorbed by the parabolic fixed point at the origin, and the critical point is being attracted by the parabolic fixed point, Proposition 4 and Theorem 8 in [Eremenko & Lyubich 1992] imply that $J(g)$ has zero measure. $\square$

Proof of Proposition 3. Let U_j, $j \in \mathbb{Z}$, denote the Baker domains of f, and $\mathcal{U}_j$ their grand orbit. The boundary of each open set $\mathcal{U}_j$ coincides with with the Julia set $J(f)$. Since the Julia set is contained in the closure of dynamically distinguished points (periodic points for example), and since f has no other Fatou components we get:

$$\mathcal{T}(f, \mathbb{C}) \simeq \mathcal{T}(f, \cup \mathcal{U}_j) \times \mathcal{B}_1(f, J(f)).$$

By Theorem 5.3

$$\mathcal{T}(f, \cup \mathcal{U}_j) = \prod_j \mathcal{T}(f, \mathcal{U}_j).$$

Since f has no asymptotic values, and each doubly parabolic Baker domain $\mathcal{U}_j$ contains exactly one critical point, we get from Proposition 1 that each $\mathcal{T}(f, \mathcal{U}_j)$ is trivial. So

$$\mathcal{T}(f, \mathbb{C}) \simeq \mathcal{B}_1(f, J(f)).$$

In view of Lemma 6.1 $J(f)$ has measure 0 and so $\mathcal{B}_1(J(f))$ is trivial. In other words $\mathcal{T}(f, \mathbb{C}) \simeq \mathrm{Def}(f, \mathbb{C})/\mathrm{QC}_0(f, \mathbb{C})$ is formed by one point. Since $\mathrm{QC}_0(f, \mathbb{C})$ is a subgroup of $\mathrm{QC}(f, \mathbb{C})$ also $\mathrm{Def}(f, \mathbb{C})/\mathrm{QC}(f, \mathbb{C})$ has cardinality one. Finally this set is in one to one correspondance with the set of entire mappings quasiconformally conjugate to f modulo conjugacy by affine isomorphisms, and the proposition follows. □

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