Residual Julia sets of rational and transcendental functions

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Abstract

The *residual Julia set*, denoted by $J_r(f)$, is defined to be the subset of those points of the Julia set which do not belong to the boundary of any component of the Fatou set. The points of $J_r(f)$ are called *buried points* of $J(f)$ and a component of $J(f)$ which is contained in $J_r(f)$ is called a *buried component*. In this paper we survey the most important results related with the residual Julia set for several classes of functions. We also give a new criterium to deduce the existence of buried points and, in some cases, of unbounded curves in the residual Julia set (the so called *Devaney hairs*). Some examples are the sine family, certain meromorphic maps constructed by surgery and the exponential family.

1. Introduction

Given a map $f : X \to X$, where X is a topological space, the sequence formed by its iterates will be denoted by $f^0 := \mathrm{Id}$, $f^n := f \circ f^{n-1}$, $n \in \mathbb{N}$. When f is a holomorphic map and X is a Riemann surface the study makes sense and is non-trivial when X is either the Riemann sphere $\widehat{\mathbb{C}}$, the complex plane $\mathbb{C}$ or the complex plane minus one point $\mathbb{C} \setminus \{0\}$. All other interesting cases can be reduced to one of these three.

In this paper we deal with the following classes of maps (partially following [Be]).

$\mathcal{R} = \{f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}} \mid f \text{ is rational of degree at least two}\}$.

$\mathcal{E} = \{f : \mathbb{C} \to \mathbb{C} \mid\ f \text{ is transcendental entire}\}$.

$\mathcal{M} = \{f : \mathbb{C} \to \widehat{\mathbb{C}} \mid f \text{ is transcendental meromorphic with at least one not omitted pole}\}$.

Note that functions in $\mathcal{M}$ have one single essential singularity. This class is usually called the general class of meromorphic functions (see [BKY]).

If f is a map in any of the classes above and we denote by X its domain of definition, the *Fatou set* $F(f)$ (or stable set) consists of all points $z \in X$ such that the sequence of iterates of f is well defined and forms a normal family in a neighborhood of z. The *Julia set* (or chaotic set) is its complement and it is denoted by $J(f) = \widehat{\mathbb{C}} \setminus F(f)$.

The second author was partially supported by MEC's grants BFM2003-09504 and BFM2002-01344 and CIRIT's grant 2001/SGR-70.

2000 *Mathematics Subject Classification*: Primary 37F10, Secondary 30D05.

Key Words: iteration, Fatou set, Julia set, Residual Julia set, Buried points, Buried components.

Classes $\mathcal{R}$ and $\mathcal{E}$ are classical and were initially studied by P. Fatou and G. Julia, and later by many other authors. Introductions to rational functions can be found in the books by Beardon [Bea1], Carleson and Gamelin [CG], Milnor [Mi] and Steinmetz [S] .

Functions in $\mathcal{E}$ and $\mathcal{M}$ have been studied more recently. For a general survey including all the above classes we refer to Bergweiler in [Be] or [XC].

Many properties of $J(f)$ and $F(f)$ are much the same for all classes above but different proofs are needed and some discrepancies arise. For any of these maps we recall some well estabished facts: by definition, the Fatou set $F(f)$ is open and the Julia set $J(f)$ is closed; the Julia set is perfect and non-empty; the sets $J(f)$ and $F(f)$ are completely invariant under f; for z_0 any non exceptional point, $J(f)$ coincides with the closure of the backward orbit of z_0; and finally the repelling periodic points are dense in $J(f)$.

The possible dynamics of a periodic connected component U of the Fatou set of f (i.e. $f^p(U) \subset U$, for some $p \geq 1$), is classified in one of the following possibilities: attracting domain, parabolic domain, rotation domain (Siegel disc or Herman ring) or Baker domain also called parabolic domains at ∞. Herman rings do not exist for $f \in \mathcal{E}$. A Fatou component that is neither periodic nor pre-periodic is called a wandering domain. Neither Baker domains nor wandering domains exist for f in $\mathcal{R}$ or in $\mathcal{E}$ of *finite type* (i.e. such that the inverse function has only finitely many singularities).

We define the *residual Julia set* of f denoted by $J_r(f)$ as the set of those points of $J(f)$ which do not belong to the boundary of any component of the Fatou set $F(f)$. The points of $J_r(f)$ are called *buried points* of $J(f)$ and a component of $J(f)$ that belongs to $J_r(f)$ is called a *buried component*. This concept was first introduced in the context of Kleinian groups. Abikoff in [Ab1, Ab2] defined the residual set $\Lambda_r(\Gamma)$ of a Kleinian group Γ to be the subset of those points of the limit set $\Lambda(\Gamma)$ which do not lie on the boundary of any component of the complement of $\Lambda(\Gamma)$. Abikoff gave examples where $\Lambda_r(\Gamma) \neq \emptyset$. In his well known paper [Su] Sullivan draws attention to the dictionary of correspondences between complex dynamics and Kleinian groups (see [Mo2, Chapter 5] for a first version of what is called a Sullivan's dictionary). Following this idea in 1988 McMullen [M] defined a buried component of a rational function to be a component of the Julia set which does not meet the boundary of any component of the Fatou set. Similarly, for a buried point of the Julia set. McMullen gave an example of a rational function with buried components. Beardon studied this example in his book [Bea1] and he also gave conditions under which the existence of buried components was assured (see [Bea2]). After these results several mathematicians have studied buried components for rational functions (see [BD1] [Mo1],[Mo2] [Q2]). The first discussion about residual Julia sets for functions in class $\mathcal{E}$ of finite type was given by Qiao in 1995 [Q1]. He also gave some conditions for $f \in \mathcal{E}$, of finite type, under which the Julia set contained buried points [Q2]. Different examples of $f \in \mathcal{E}$ with $J_r(f) \neq \emptyset$ and related results were given independently by Domínguez in 1997 [D1]. In [BD1] Baker and Domínguez discused some results of Morosawa and Qiao on conditions for $f \in \mathcal{R}$ to have buried points or buried components. They showed that these results can be extended to functions in class $\mathcal{M}$.

This paper attempts to describe some of the results mentioned above on the residual Julia set for different classes of functions such as rational, transcendental entire and transcendental meromorphic (see Sections 2, 3 and 4).

In Section 5 we prove some new results about the residual Julia set of some classes of entire or meromorphic functions and apply them to the sine family and to some meromorphic functions constructed by surgery. We also give certain conditions under which all points on the so called Devaney hairs (or rays) are buried points. In the particular case of exponential functions, having an attracting periodic orbit of period greater than one we show that all hairs except possibly a countable number of them are buried components. Moreover we characterize these exceptions in terms of the kneading sequence.

2. Basic properties of the residual Julia set

We give here some basic results about the residual Julia set which hold for functions which belong to any of the classes defined in the introduction. In this section, f will denote a function in $\mathcal{R}, \mathcal{E}$ or $\mathcal{M}$.

The first proposition deals with completely invariant components of the Fatou set.

PROPOSITION 2.1.. *If the Fatou set of f has a completely invariant component, then the residual Julia set is empty.*

Proof. If the Fatou set has a completely invariant component U, then $\overline{U}$ is also completely invariant. Hence, by the minimality of $J(f)$, we have $\partial U = J(f)$ so that the residual Julia set $J_r(f) = \emptyset$ by definition. □

In particular for a nonlinear polynomial P the unbounded Fatou component is completely invariant and the residual Julia set is empty. Perhaps this contributed to the relatively late recognition of the possible existence of residual Julia sets.

The following is a trivial observation.

PROPOSITION 2.2.. *If there exists a buried component of $J(f)$, then $J(f)$ is disconnected.*

Proof. Observe that buried components are defined as connected components of the Julia set which are all buried. For such components to exist, $J(f)$ must be disconnected. □

Notice that this is only for buried components, not for buried points. There are examples (see next section) of maps with a connected Julia set and residual julia set nonempty.

Finally, what follows is a result that was proven by Morosawa [Mo1] for rational functions and by Baker and Dominguez [BD1] (in a slightly stronger form) for functions in the remaning classes.

PROPOSITION 2.3.. *If the residual Julia set of f is non-empty, then $J_r(f)$ is completely invariant, dense in $J(f)$ and uncountably infinite.*

Proof. The complete invariance of $J_r(f)$ follows easily from the complete invariance of $F(f)$. Then $\overline{J_r(f)}$ must be completely invariant and has more than three points. By the minimality of $J(f)$ we have that $J(f) = \overline{J_r(f)}$. To prove the second part, Morosawa constructs a Cantor set contained in $J_r(f)$, following the method of Abikoff in [Ab2]. □

In their paper [BD1], Baker and Dominguez prove something slightly stronger, namely that $J_r(f)$ is residual in the sense of category theory. We recall that a residual set is the complement of a countable union of nowhere dense sets (which are in this case the boundaries of the Fatou components). Any residual set in this sense, must contain an intersection of open dense subsets. Since $J(f)$ is a complete metric space, it is a Baire space and therefore any residual subset is dense.

3. The residual Julia set for rational functions

The first example of a rational function with $J_r(f) \neq \emptyset$ was given by McMullen [M]. The idea behind this example is to have a Julia set that consists of a Cantor set of nested Jordan curves (see Figure 1). Such an object necessarily has plenty of buried components, as in the middle thirds Cantor set C, where the extreme points of each interval belong to the set but many other points in C are not extreme points and therefore do not belong to the boundary of any component of C^c.

More precisely let

$$R(z) = z^2 + \frac{\lambda}{z^3}, \lambda > 0.$$

One can see that ∞ is a super-attracting fixed point of R. It can be shown that zero and ∞ lie in different components of the Fatou set, say F_0 and F_∞ respectively. When λ is sufficiently small the following facts can be proved.

(a) F_0 and F_∞ are simply connected, while other components of the Fatou set are doubly connected;

(b) $R^n \to \infty$ on $F(R)$;

(c) ∞ attracts all critical points of $R(f)$;

(d) the Julia set is a Cantor set of nested Jordan curves;

(e) there are components of the Julia set which do not meet the boundary of any component of the Fatou set. Such components are quasicircles.

This example can also be found in [Bea1, Chapter 5, p. 266]. In [Bea2] Beardon proves the first general theorem about buried components for $f \in \mathcal{R}$.

THEOREM 3.4 ([Bea2]). *Suppose that $J(f)$ is disconnected, and that every component of the Fatou set has finite connectivity. Then $J(f)$ has a buried component, so $J_r(f) \neq \emptyset$*

The proof of Theorem 3.4 is based on the following result.

THEOREM 3.5. *If $J(f)$ is disconnected, then it has uncountably many components, and each point of $J(f)$ is an accumulation point of distinct components of $J(f)$.*

Indeed, if each component of $F(f)$ has finite connectivity, and if $J(f)$ is disconnected, then there are only countably many components of $J(f)$ which lie on the boundary of some component of the Fatou set. The theorem follows immediately. Qiao in [Q2] improves Beardon's result by showing the following statement.

THEOREM 3.6 ([Q2]). *Let $f \in \mathcal{R}$ and $J(f) \neq \widehat{\mathbb{C}}$. The Julia set $J(f)$ contains buried components if and only if (i) $J(f)$ is disconnected and (ii) $F(f)$ has no completely invariant component.*

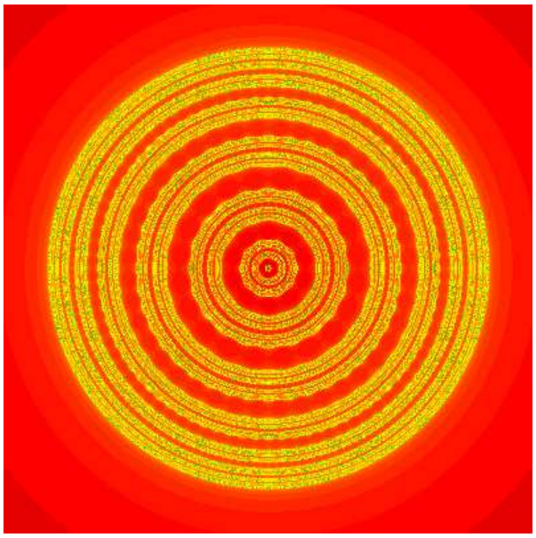

Figure 1: Dynamical plane of $R(z) = z^2 + \frac{\lambda}{z^3}$ where $\lambda = 10^{-8}$. The Julia set is a Cantor set of quasicircles.

Proof. If all Fatou components are finitely connected, the result follows from Beardon's theorem. If not, he proves the existence of a periodic connected component of the Julia set which can be surrounded by a closed curve γ in the Fatou set, such that the bounded component of $\mathbb{C} \setminus \gamma$ contains no periodic Fatou component. Such component of the Julia set is necessarily buried. $\square$

Thus by Qiao's Theorem 3.6, it is not difficult to find examples of rational functions with degree greater than one for which the Julia set contains buried components, so the residual Julia set $J_r(f)$ is not empty. See [Q2] for the complete proof of Theorem 3.6 and examples.

In [BD1] the authors gave a proof of Theorem 3.6, differing from that in [Q2], which can be used also for transcendental entire functions with some changes.

Similar results were proved (independently) by Morosawa (see [Mo1] and [Mo2]).

One can ask if it is possible to have a rational function with connected Julia set and non-empty residual Julia set. Naturally, such a function will have buried points but not buried components. In what follows we describe this example due to Morosawa [Mo1], [Mo2].

A function $f \in \mathcal{R}$ is hyperbolic if each critical point of f has a forward orbit that accumulates at a (super) attracting cycle of f. The following theorem gives a characterization of those hyperbolic functions with non-empty residual Julia set.

Theorem 3.7 ([Mo1]). *Let f be a hyperbolic rational function with degree at least two. Then the residual Julia set is empty if and only if either (i) $F(f)$ has a completely invariant component or (ii) consists of only two components.*

In his proof, Morosawa relies on the fact that the boundary of Fatou components of hyperbolic functions is locally connected. This allows a great control on the union of all these sets.

The example of Morosawa consists of the following hyperbolic rational function.

$$R(z) = \frac{-2z+1}{(z-1)^2}$$

Observe that the set $\{0, 1, \infty\}$ is a super-attracting cycle of $R(z)$, since the critical points of $R(z)$ are 0 and 1. Thus $F(R)$ has countably many components. Moreover, every component of $F(R)$ is eventually absorbed into this super-attracting cycle and R is hyperbolic. Hence the residual Julia set is not empty by Theorem 3.7. It can be shown that each component of the immediate basin on this super-attracting cycle is simply connected. An arbitrary component of $F(R)$ except that of the immediate basin contains no critical point. Thus, every component of $F(R)$ is simply connected. Therefore, the Julia set of R is connected.

This class of examples with locally connected Fatou boundaries was somehow generalized in the following theorem.

THEOREM 3.8 ([Q3]). *Lef $f \in \mathcal{R}$ with degree $d \geq 2$ and $J(f) \neq \mathbb{C}$. Suppose that $F(f)$ has no completely invariant components. Then, either the residual Julia set is nonempty, or the Julia set is not locally connected.*

In fact, this result proves, in the case of locally connected Julia sets, the stronger conjecture of Makienko which reads as follows.

CONJECTURE 3.9 (Makienko, [EL2]). *Lef $f \in \mathcal{R}$. Then, $J(f)$ has buried points if and only if $F(f)$ has no completely invariant components.*

4. Residual Julia set for transcendental entire functions

The results in Section 2 for rational functions can be extended mainly to transcendental meromorphic functions, as we will see in Section 4. However, when we deal with functions in class $\mathcal{E}$, the statements are considerably different. We recall that a component U of the Fatou set that is neither periodic nor preperiodic is called a a wandering component. Sullivan in [Su] proved the non existence of wandering components for a rational function. This result was later extended to functions in class $\mathcal{E}$ of finite type in [EL1] and [GK]. However functions in class $\mathcal{E}$ with infinitely many singularities of f^{-1} may have wandering domains. Examples in class $\mathcal{E}$ can be found in [B] such that $F(f)$ contains some wandering components which roughly speaking form an unbounded sequence of concentric rings U_n. For such functions $J(f)$ is not connected. Thus for functions in class $\mathcal{E}$ it is possible to have both cases of wandering and no wandering components. This turns out to be an important difference when dealing with the residual Julia sets.

4.1 Functions with no wandering domains

An important class of transcendental entire functions is the class S of functions f of finite type. As mentioned above, functions in class S do not possess wandering domains. The main results related with the residual Julia set for functions in class S were given by Qiao [Q2] and Baker and Domínguez [BD1]. In [Q2], Qiao stated the following theorem.

THEOREM 4.10. *Let $f \in S$ and $J(f) \neq \mathbb{C}$. The Julia set of f contains buried points if and only if $F(f)$ is disconnected.*

The 'only if ' part of Theorem 4.10 is immediate since $F(f)$ being connected would imply the existence of a unique connected component of the Fatou set which has to be completely invariant. In the 'if' part Qiao separates the proof in two cases. If the components of the Fatou set are all bounded, then he shows that there exist continua of buried points, all tending to infinity under iteration (in fact these are the "Devaney hairs", see Section 5). The construction he uses is the same as in [DT]. When some component of the Fatou set is unbounded, a new construction is needed. The author constructs a set of unbounded regions and shows that they all contain points of the Julia set, but at the same time one of them does not contain any periodic component of the Fatou set. Hence all repelling periodic points in this region must be buried.

The last steps in Qiao's proof were not clear to the authors in [BD1] who gave an alternative construction to show the following result. We remark that functions in class $\mathcal{E}$ can have at most one completely invariant component of $F(f)$ but, *a priori*, there could also be other Fatou components. If $f \in S$, any completely invariant component must form the whole Fatou set.

THEOREM 4.11. *Let $f \in \mathcal{E}$ such that $F(f)$ is not connected. Suppose that there are no wandering domains, no completely invariant Fatou component and no Baker domains in which f is univalent.*

(i) If all periodic components of $F(f)$ are bounded, then $J_r(f) \neq \emptyset$.

(ii) If there are unbounded periodic Fatou components and ∞ is an accessible boundary point in one of these components, then there are buried components of $J(f)$ which are unbounded, so $J_r(f) \neq \emptyset$.

Before proving Theorem 4.11, we state the following useful lemma (for a proof see [BD1]).

LEMMA 4.12. *If $f \in \mathcal{E}$ has no wandering domains and $J(f)$ has no buried components, then there is some periodic cycle of Fatou components $G_1, G_2, \ldots G_p$, such that $J(f) = \overline{K(G_j)}$ for all $1 \leq j \leq p$, where the bar denotes closure and $K(G_j)$ denotes the union of all components of $J(f)$ which meet ∂G_j.*

Sketch of the proof of Theorem 4.11 (ii). Suppose that the Fatou set has an unbounded periodic component H such that $f^p(H) = H$ and ∞ is accessible in H. Since H is unbounded it is simply-connected and either the immediate basin of attraction of an attracting, parabolic fixed point of f^p, a Siegel disc or a Baker domain such that f^p is not univalent in H. Further, since H is not completely invariant there is a non-periodic component K such that $f^p(K) \subset H$.

The set $E = \{e^{i\theta}$: radial limit $\Psi(e^{i\theta}) = \infty$, $\Psi : D(0,1) \to H$ is the Riemann map$\}$ contains infinitely many points by a result in [BD2]. Let us take values θ_j, $1 \leq j \leq 4$, such that $0 < \theta_1 < \theta_2 < \theta_3 < \theta_4 < 2\pi$ and radii $\lambda_j : z = re^{i\theta_j}$, $0 < r < 1$. Assume that $e^{i\theta_1}$ and $e^{i\theta_3}$ are in E and that the limits (as $r \to \infty$) on the radii λ_2, λ_4 exist and are finite points α_2, α_4 in ∂H. It can be shown that there is a simple path $\Gamma \subset H$ which runs to ∞ at both ends. If g is an appropiate branch of f^{-p}, then consider the continuation of g along Γ and obtain a simple path Γ' which runs to ∞ at both ends and divides K.

The points α_2, α_4 in ∂H are separated by Γ and the points $p_2 = g(\alpha_2)$, $p_4 = g(\alpha_4)$ in ∂K are separated by Γ'. If we assume that there are no buried components, then there is a periodic component L of the Fatou set such that $J(f) = \overline{T}$,where T is the union of all components of the Julia set which meet ∂L (see Lemma 4.12). Thus there are points q_2, q_4 of T so close to p_2, p_4 that are separated by Γ'. The points q_2, q_4 belong to different components of the Julia set which are also separated by Γ'. These components contain points r_2, r_4 in ∂L. Then there are points in L which are also close to r_2, r_4 but they are separated by Γ'. This is impossible since the Fatou components L and K are distinct. □

REMARK 4.13. Observe that when $f \in S$ and $F(f)$ is not connected there is neither wandering domains, completely invaraint domain nor Baker domains. Thus Theorem 4.11 gives the following corollary.

COROLLARY 4.14. *Let $f \in S$ such that $F(f)$ is not connected. If either all periodic components of $F(f)$ are bounded or there exists an unbounded periodic component in which ∞ is an accessible point, then $J_r(f) \neq \emptyset$. In the latter case there are buried components of $J(f)$.*

Examples of the above results can be given such as the family $\lambda \sin z$ for some values of λ and the family λe^z for any λ such that there is an attracting p-cycle (see also Section 5).

REMARK 4.15. We do not know any examples of an unbounded Fatou component G such that ∞ is not accesible along any path in G. If there are no such examples then Corollary 4.14 would imply Theorem 4.10.

4.2 Functions with wandering domains

It is well known for $f \in \mathcal{E}$ that a multiply connected component of $F(f)$ must be wandering [B]. The Julia set for $f \in \mathcal{E}$ cannot be totally disconnected, as may happen for polynomials. However singleton components in the Julia set can occur. If the Julia set has a singleton component $\{\psi\} \neq \infty$ then $F(f)$ has a multiply-connected component. The converse of this statement is also true and we will see that this yields to the existence of buried points in the Julia set. The main result in [D1] is as follows.

THEOREM 4.16. *If $f \in \mathcal{E}$ and $F(f)$ has a multiply-connected component, then the Julia set has singleton components and such components are buried and dense in $J(f)$.*

Sketch of the proof. The result of Baker [B] mentioned above states that if $F(f)$ has a multiply-connected component U, then U and all its iterates are bounded wandering components and $f^n \to \infty$ in U (as $n \to \infty$). Also there is a simple curve γ in U such that $f^n(\gamma)$ is a curve in $f^n(U) = U_n$ on which $|z|$ is large and winds round zero. In [D1] the following results were shown: (i) $f^{n+1}(\gamma)$ is in $U_{n+1} \neq U_n$, winds round zero and must be outside γ_n, (ii) there is a component N_n of the Fatou set between U_n and U_{n+1} and an integer $m \geq 2$ such that $f(N_n) \subset U_n$ and (iii) the component N_n is a multiply-connected component which does not wind round zero i.e. zero is in the unbounded component of $\overline{N_n^c}$. Thus picking a repelling periodic point η with period p, N_n (as above) for five different values of n and choosing $m \in \mathbb{N}$ so that the spherical derivative of f^{mp} at η satisfies the hypothesis of Ahlfors five island Theorem (see [A]), it can be shown that the disc $D(\eta, R)$, $R > 0$ contains a sequence of different multiply connected components N_k of the Fatou set with diameter tending to zero and N_{k+1} is inside in one of the inner boundary components of N_k. A sequence $\psi_k \in \partial J(f)$ is a Cauchy sequence which converges to a point $\psi \in J(f)$. By construction, ψ is a buried component of the Julia set. Thus there is a dense subset G of $J(f)$ such that each $\alpha \in G$ is a buried singleton component of $J(f)$. □

5. The residual Julia set for transcendental meromorphic functions

In Section 2 we discussed some results of Morosawa [Mo1] and Qiao [Q2] on conditions for a rational function to have buried points or buried components. In this section we will focus on those results that can be extended to transcendental meromorphic functions.

5.1 Functions with no wandering domains

For a meromorphic function in $\mathbb{C}$, Baker and Domínguez [BD1] proved the following result for functions with no wandering domains, which is almost a generalization of Theorem 3.6 to meromorphic maps.

THEOREM 5.17. *Let f be meromorphic in $\mathbb{C}$ with no wandering components. Assume that $J(f)$ is not connected and that $F(f)$ has no completely invariant component. Then the residual Julia set $J_r(f)$ is non-empty.*

In order to give the proof of Theorem 5.17 we start with the following lemma.

LEMMA 5.18. *If f is meromorphic in $\mathbb{C}$ with no wandering components and U is a multiply-connected periodic Fatou component such that $\partial U = J(f)$, then U is completely invariant.*

Sketch of the proof. Assume that U is a multiply-connected Fatou component such that for some $p \in \mathbb{N}$ we have $f^p(U) \subset U$ and that $\partial U = J(f)$ but that U is not completely invariant. Observe that any other Fatou component is simply-connected because, otherwise, there would be part of the Julia set bounded away from ∂U. Since there is one component H such that $H \neq U$, $f(H) \subset U$ we obtain a contradiction at once in the case of rational functions, since then $f(H) = U$, $f(\partial H) = \partial U$ and $f(\partial H)$ is connected while ∂U is not. Thus we may assume that f is transcendental so that $J(f)$ and U are unbounded. If g denotes the branch of f^{-1} and if γ is a simple closed path in U which encloses some points of ∂U, it can be proved that the continuation of g maps γ to a simple curve Γ in H which goes to ∞ at both ends. Then the Gross Star Theorem is used to show that U meets both components of $\overline{\Gamma^c}$ which is impossible since $\Gamma \subset H \neq U$. □

Now we are able to prove Theorem 5.17.

Proof of Theorem 5.17. Suppose that f is meromorphic without wandering domains, $J(f)$ is not connected and $J_r(f)$ is empty. Then there is a periodic Fatou component U such that $\partial U = J(f)$ (see [BD1] for a proof of this fact). Since $J(f)$ is not connected, then there exists a component of the Fatou set which is multiply connected. But $\partial U = J(f)$, and hence U is multiply connected. By Lemma 5.18, U is completely invariant which is a contradiction with the hypothesis. □

Note that only meromorphic functions may satisfied the conditions of Theorem 5.17 since, for $f \in \mathcal{E}$, the existence of a multiply-connected component in $F(f)$ implies the existence of wandering domains [B].

For functions $f \in \mathcal{M}$ the following theorem covers almost all cases when $J(f)$ is not connected (see [D2] for the proof).

THEOREM 5.19. *Suppose that $f \in \mathcal{M}$ and either (a) $F(f)$ has a component of connectivity at least three, or (b) $F(f)$ has three doubly-connected components U_i, $1 \leq i \leq 3$, such that one of the following conditions holds.*

(i) Each U_i lies in the unbounded component of the complement of the other two.

(ii) Two components U_1, U_2 lie in the bounded component of U_3^c but U_1 lies in the unbounded component of U_2^c and U_2 lies in the unbounded component of U_1^c.

Then $J(f)$ has singleton components which are buried and dense in $J(f)$.

Another attempt to generalize Theorem 3.6 would be the following.

THEOREM 5.20. *Let $f \in \mathcal{M}$ with no wandering domains. If $F(f)$ has no completely invariant components and if $J(f)$ is disconnected (in such a way that f satisfies the assumptions of Theorem 5.19), then $J_r(f)$ contains singleton components of $J(f)$ which are buried.*

Explicit examples of Theorem 5.20 satisfying hypotesis (a) and (b) of theorem 5.19 can be found in [BD1], such examples were obtained by using Runge's theorem.

In Section 5.2 we provide an example of a meromorphic function obtained by surgery as in [DF] satisfying the hypothesis of Theorem 5.20 and containing unbounded continua of buried points. It is an open problem to find an example with disconnected Julia set not satisfying any of the hypothesis of Theorem 5.19. Likewise it would be interesting to know under which conditions a function with connected Julia set must have non empty residual Julia set.

5.2 Functions with wandering domains

All the theorems mentioned above have the condition that the function has no wandering domains. For functions in class $\mathcal{M}$ it is well know that there are examples with wandering domains of any prescribed connectivity. Thus it is possible to give examples of functions in class $\mathcal{M}$ with wandering domains where the Julia set is disconnected and the Fatou set has no completely invariant domain. The following theorem in [D2] gives an example of this fact. The idea of this example was motivated by a similar example for $f \in \mathcal{E}$ (see [B]).

THEOREM 5.21. *Let $f(z) \in \mathcal{M}$ and suppose that $F(f)$ has multiply-connected components $A_i, i \in \mathbb{N}$ all different, such that each A_i separates $0, \infty$ and $F(A_i) \subset A_{i+1}$ for $i \in \mathbb{N}$. Then $J(f)$ has a dense set of buried singleton components.*

More interesting examples can be constructed by using results of complex approximation theory, see [D2].

6. Hairs in the Residual Julia Set

In this section we deal with some classes of entire or meromorphic transcendental maps that contain unbounded continua of buried points in their Julia sets. These continua are the well known *Devaney hairs*, also called dynamic rays.

If $f : \mathbb{C} \to \mathbb{C}$ is an entire transcendental function, a *hair of f* is defined as a curve $\gamma : (0, \infty) \to \mathbb{C}$ in the Julia set of f, such that $\gamma(t) \underset{t\to\infty}{\longrightarrow} \infty$ and $f^n(\gamma(t)) \underset{n\to\infty}{\longrightarrow} \infty$ for any $t \in (0, \infty)$. If the limit $\lim_{t\to 0} \gamma(t)$ exists (say, it equals z_0) and is finite we say that the hair *lands* at z_0. This point is called the *endpoint* of the hair. In other words, all points on the curve γ have orbits that tend (exponentially fast) towards the essential singularity at infinity, while endpoints might escape or not.

Hairs were initially described for the exponential function by Devaney and Krych in [DK] and then extended to the exponential family, $z \mapsto \lambda \exp(z)$ (for several classes of parameter values) in [DGH] and later in [B*et al*]. More recently, Schleicher and Zimmer [SZ] have extended this description to most parameter values. For more general classes of entire transcendental functions the main reference is by Devaney and Tangerman [DT], where they show that *Cantor Bouquets* i.e., Cantor sets of hairs, appear in any entire transcendental function of finite type that has at least one *hyperbolic asymptotic tract*. A hyperbolic asymptotic tract is an unbounded connected open set where orbits that remain in it behave in an "exponential fashion", that is, f increases the modulus and the derivative exponentially (see [DT] for precise definitions). As an example, the right half plane is a hyperbolic asymptotic tract for the exponential family, while the upper and the lower half plane are hyperbolic asymptotic tracts for any map in the sine family $z \mapsto \lambda \sin(z)$. It seems reasonable to think that the proof of the existence of Cantor bouquets can be easily generalized for entire transcendental maps of *bounded type* (the set of singularities of f^{-1} is contained in a bounded set) with at least one hyperbolic asymptotic tract. In fact, such extension has been announced by Rottenfüsser and Schleicher for an even larger class of functions of bounded type. Similar and other extensions can also be found in [BK].

Transcendental meromorphic functions may also contain Devaney hairs in their Julia sets. Although no general theory is known, many examples can be constructed using, for example, surgery techniques or otherwise adding a pole to a known entire map (see Section 6.2).

We will refer to the class of transcendental functions that possess unbounded continua of escaping points, i.e. Devaney hairs as $\mathcal{H}$.

In this section we want to give a new criterium to assure the existence of buried points in the Julia set, and also some conditions for a transcendental map under which we can assure that the Devaney hairs are in the residual Julia set, i.e., unbounded curves of buried points. If the endpoints are also buried then the hair and its endpoint form a buried component. The precise statement is as follows.

PROPOSITION 6.22. *Let $f \in (\mathcal{R} \cup \mathcal{E} \cup \mathcal{M})$, and $A \subsetneq \mathbb{C}$ a closed set with nonempty interior. Suppose the following conditions are satisfied:*

a) $(\mathbb{C} \setminus A) \cap J(f) \neq \emptyset$, *and*

b) all Fatou components of f eventually iterate inside A and never leave again. That is, if Ω is a Fatou component, $f^n(\Omega) \subset A$ for all $n > N$, where N depends on Ω.

Then the residual Julia set is nonempty. More precisely, the residual Julia set contains the set

$$\{z \in J(f) \mid f^n(z) \notin A \text{ for infinitely many values of } n\}.$$

In particular if A is bounded, any point of the Julia set with an unbounded forward orbit belongs to the residual Julia set.

Proof. Since any point on the boundary of a Fatou component maps to another point with the same property, we have that all these points must eventually fall into A and never leave again. Thus no point of the Julia set which leaves A infinitely often can be in the boundary of a Fatou component.

Now, the complementary of A is an open set which contains points of $J(f)$. Let $z \in (J(f) \cap \mathbb{C} \setminus A)$, and let U be a neighborhood of z entirely contained in $\mathbb{C} \setminus A$. Since periodic points are dense in $J(f)$, it follows that U must contain a periodic point of the Julia set. This point has to come back to itself infinitely often. Since it lies in the complementary of A, this must be a point in the residual Julia set. □

Remark 6.23. 1. The hypothesis inmediately rule out the case of a completely invariant component. Indeed if U is a completely invariant component, then $J(f) = \partial U$. But U must be in A because it is invariant, which implies that the whole Julia set is in A, contradicting that some Julia set must be in the complement.

2. The hypothesis allow for wandering domains (as long as they eventually wander inside A) or Baker domains of any kind (as long as they are contained in A)

3. If A is bounded and we assume $f \in \mathcal{H}$, then it follows that all points in the Devaney hairs are buried points. Clearly no point on a hair (except may be the endpoint) can be in the boundary of any Fatou component, since such points escape exponentially fast to infinity.

4. If A is bounded and the map is meromorphic, it follows that all poles and its preimages are buried points.

In what follows we will analyze three examples which fall under the hypothesis of Proposition 6.22. The two first examples – the sine family and an example of a meromorphic surgery constructed by surgery – have bounded Fatou components while the third example is the exponential family that has unbounded Fatou components, and we treat it apart.

6.1 The Sine family

An important example is given by the sine family $S_\lambda(z) = \lambda \sin(z)$. This family has two critical values (with symmetric orbits) and no asymptotic values, hence $S_\lambda \in S$ for all λ. Since both, the far upper and the far lower half plane are hyperbolic asymptotic tracts, it follows from the standard arguments that $S_\lambda \in \mathcal{H}$ and all functions in this family have a pair of Cantor Bouquets [DT], one in the upper half plane and another one in the lower half plane. See Figure 2. How the landing of these hairs occurs is a fact that depends on the parameter value.

For $|\lambda| < 1$, the fixed point $z = 0$ is attracting and its basin is completely invariant and therefore unbounded [DS]. For $\lambda = 1$, the Fatou set consists of the parabolic basin of 0, all whose connected components are bounded (see [Bha]). For $\lambda = e^{2\pi i\theta}$ with θ and irrational number of bounded type, the map S_λ has an invariant Siegel disk around $z = 0$ and all other Fatou componets are its preimages. It has been announced in [Z] that this Siegel disk must be bounded (and therefore all its preimages because S_λ has no asymptotic values). For other λ parameter values, it is not known to our knowledge wether the Fatou components of S_λ are bounded or not.

At the end of this section we prove the following.

Proposition 6.24. *Let $\lambda \in \mathbb{C}$ such $|\mathrm{Re}(\lambda)| \geq \frac{\pi}{2}$. Then, all Fatou components of S_λ are bounded.*

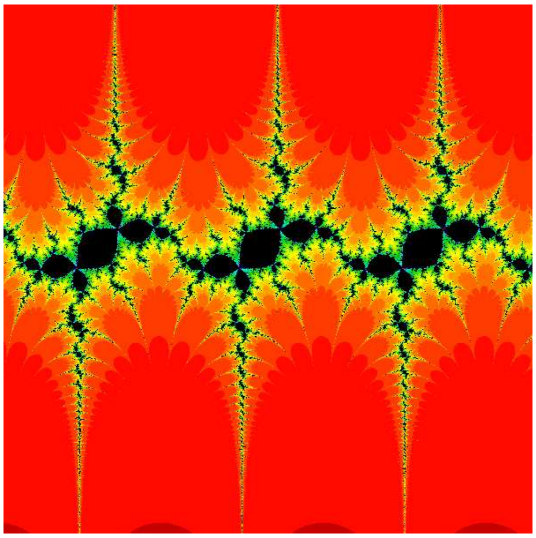

Figure 2: Dynamical plane of $S_\lambda(z) = \lambda \sin(z)$ where $\lambda = 1.88853 + i0.673125$. There are two attracting orbits of period three. Their inmediate basins of attraction are bounded and all other Fatou components are preimages of those. All Devaney hairs consist of buried points.

Remark 6.25. In fact, we conjecture that this is true for all $|\lambda| \geq 1$, but our construction only works for this smaller set (see Subsection 6.1.1).

Since the sine family cannot have neither Baker domains nor wandering domains, it follows that we are always under the hypothesis of Proposition 6.22. Hence we have the following corollary.

Corollary 6.26. *Let $\lambda \in \mathbb{C}$ such $|\mathrm{Re}(\lambda)| \geq \frac{\pi}{2}$. Then, all Devaney hairs of S_λ consist of buried points.*

Notice that if the Julia set is the whole plane then, trivially, all its points are buried.

6.1.1 Boundedness of Fatou components (Proof of Proposition 6.24)

The main point in the proof is the following Proposition.

Proposition 6.27. *Let $\lambda \in \mathbb{C}$ be such that $|\mathrm{Re}(\lambda)| \geq \frac{\pi}{2}$. Then there is a fixed hair in the upper half plane whose endpoint is the repelling fixed point $z = 0$. More precisely, there exists an invariant curve $\{\gamma(t)\}_{0 \leq t < \infty}$ such that*

1. $|\mathrm{Re}\gamma(t)| \leq \pi/2$ *for all* $t \geq 0$.
2. $\lim_{t\to\infty} \gamma(t) = \infty$ *and* $\lim_{t\to 0} \gamma(t) = 0$.
3. *For all* $t > 0$, $\lim_{n\to\infty} S_\lambda^n(\gamma(t)) = \infty$.

Proof. Let B consist of the vertical strip

$$B = \{z \in \mathbb{C} \mid -\frac{\pi}{2} < \mathrm{Re}(z) < \frac{\pi}{2}\}.$$

See Figure 3. The right (resp. left) vertical boundary of B is mapped to the ray segment starting at λ (resp. $-\lambda$) and going to infinity, given that

$$\lambda \sin(\pm\frac{\pi}{2} + yi) = \pm\cosh(y).$$

Any horizontal segment of the form $\{x + y_0 i\}_{|x|<\frac{\pi}{2}}$ joining the two boundaries is mapped

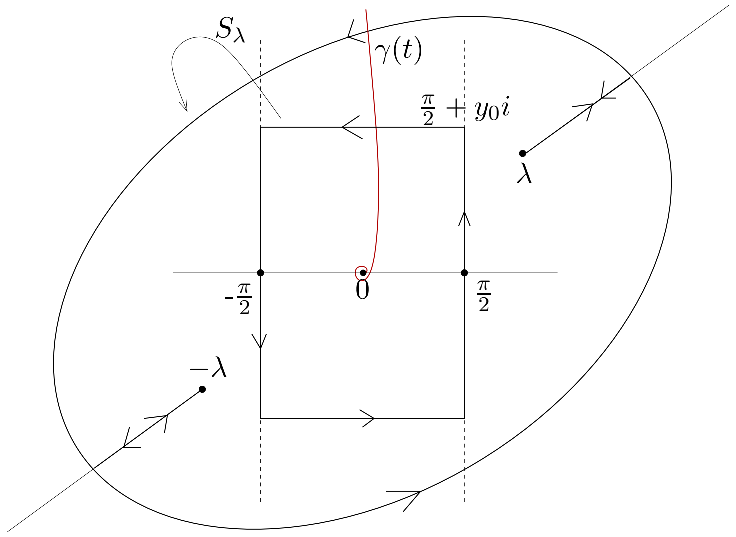

Figure 3: Setup of the proof of Proposition 6.27 for $|\mathrm{Re}(\lambda)| \geq \frac{\pi}{2}$.

under S_λ to half an ellipse of radii $\cosh(y_0)$ and $\sinh(|y_0|)$, rotated by λ. The simmetric segment is mapped to the other half.

It follows from these observations that B is mapped one to one to the whole plane except the 2 ray segments. Hence the standard construcions apply (see e.g. [DT]) to prove, in particular, that the "tail" of the hair $\gamma(t)$ (i.e. for t large enough) exists in B.

In order to see that it lands at $z = 0$, observe that a box like the one in Figure 3 (in fact, independently of its height y_0) is mapped one to one to a set that completely covers it. Hence a well defined branch of the inverse exists inside the box and satisfies the hypothesis of the Schwarz lemma. It follows easily by iterating the inverse that $z = 0$ is the unique fixed point in the box and that $\gamma(t)$ must converge to zero as $t \to 0$. □

We proceed to see how the boundness of the Fatou components follows from Proposition 6.27.

Having a fixed hair $\gamma(t)$ landing at $z = 0$ gives, by symmetry, another invariant hair $\widetilde{\gamma}(t) = -\gamma(t)$ coming from below landing at the same point. These two hairs together with $\{0\}$ form an invariant curve Γ in the Julia set, contained in the strip

$$B = \{z \in \mathbb{C} \mid -\frac{\pi}{2} < \mathrm{Re}(z) < \frac{\pi}{2}\}.$$

Since the function S_λ is 2π-periodic, all the 2π-translations of Γ are vertical curves in the Julia set all mapped to Γ under one iteration.

Hence the Julia set divides the plane in infinitely many "vertical strips" and, as a consequence, no Fatou component can have unbounded real part, or else it would have to intersect the Julia set. Let us name these strips by $\{R_k\}_{k\in\mathbb{Z}}$, where R_K intersects $\mathbb{R}$ at the interval $[2k\pi, 2(k+1)\pi]$.

Now, let us suppose that a periodic Fatou component U, (S_λ has no wandering domains) has unbounded imaginary part. Since U is periodic, all its images must be contained in a finite set of strips, say $R_{-N}, \ldots, R_0, \ldots, R_N$, for some $N \in \mathbb{N}$. We will show that no open set can remain forever in these strips under iteration, unless its imaginary part is bounded.

To that end, choose a point $z_0 = x_0 + iy_0 \in U$ with high enough imaginary part so that

$$\sinh(y_0 - 2\pi) > \frac{4(N+1)\pi}{|\lambda|}. \tag{1}$$

Since U is open, there exists $\delta_0 = \delta_0(y_0)$ such that the round disk $D_0 = D(z_0, \delta_0)$ is contained in U, see Figure 4.

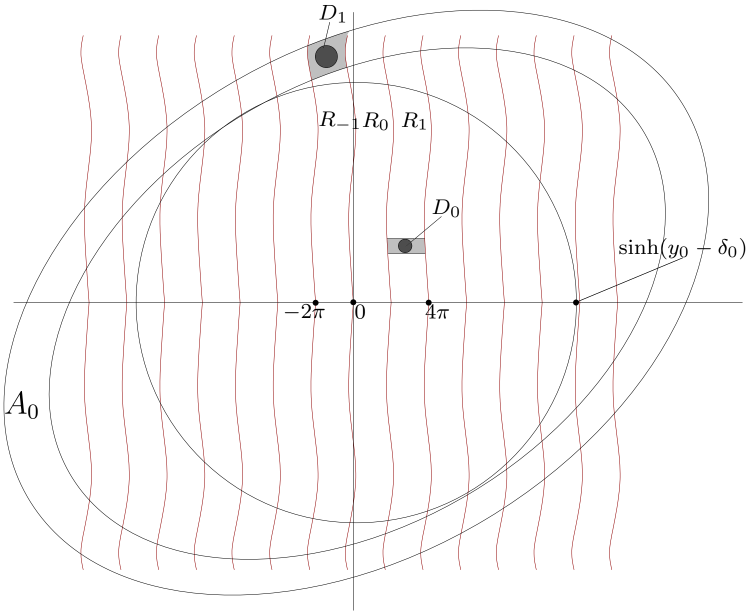

Figure 4: Sketch of the disks D_0 and D_1 in the case $N = 1$.

Assuming U is in R_{k_0}, with $|k_0| \leq N$, the "rectangle" $R_{k_0} \cap \{y_0 - \delta_0 < \mathrm{Im}(z) < y_0 + \delta_0\}$ is mapped to a huge annulus A_0 formed by two concentric ellipses, both transversal to R_k for all $|k| \leq N$, since they contain a disk of radius R where

$$R = |\lambda| \sinh(y_0 - \delta) > |\lambda| \sinh(y_0 - 2\pi) > 4(N+1)\pi.$$

See Figure 4. Since $S_\lambda(D_0)$ belongs to the Fatou component $S_\lambda(U)$ which is constrained to lie in a strip R_{k_1} with $|k_1| \leq N$, it follows that D_0 must be mapped under S_λ to one of the two connected components of $A_0 \cap R_{k_1}$. W.l.o.g. we suppose that $S_\lambda(D)$ is in the component that lies in the upper half plane.

Let $z_1 = x_1 + y_1 = S_\lambda(z_0)$. Since S_λ is univalent in D_0, it follows from the Koebe distortion theorem that

$$\mathrm{dist}(z_1, \partial S_\lambda(D_0)) \geq \frac{1}{4}|S'_\lambda(z_0)| \mathrm{dist}(z_0, \partial D_0).$$

Hence, $S_\lambda(D_0)$ contains a round disk $D_1 = D(z_1, \delta_1)$ where

$$\delta_1 \geq \frac{1}{4}|\lambda||\cos(z_0)|\delta_0 \geq \frac{1}{4}|\lambda|\sinh(y_0)\delta_0,$$

given that $|\cos(x+iy)| \geq \sinh(y)$ for $y \geq 0$.

Now we can apply the same argument to D_1 and conclude that $S_\lambda^2(U)$ must contain a round disk around $z_2 = S_\lambda(z_1)$ of radius δ_2 with

$$\delta_2 \geq \frac{1}{4}|\lambda||\cos(z_1)|\delta_1 \geq \frac{1}{4^2}|\lambda|^2 \sinh(y_1)\sinh(y_0)\delta_0.$$

Using that $\sinh y_0 > \sinh(y_0 - 2\pi)$ and Equation (1), we have that $y_1 >> y_0$ and hence

$$\delta_2 \geq \left(\frac{|\lambda|\sinh(y_0)}{4}\right)^2 \delta_0.$$

Applying the same argument n times we obtain that $S_\lambda^n(U)$ contains a round disk of radius

$$\delta_n \geq \left(\frac{|\lambda|\sinh(y_0)}{4}\right)^n \delta_0,$$

which tends to ∞ as $n \to \infty$. This contradicts the fact that $S_\lambda^n(U)$ is constrained to lie for all n inside one of the strips R_k with $|k| \leq N$.

Once established that all periodic components of the Fatou set are bounded, it follows easily that the other ones are too. Indeed, any preimage of a bounded component must be bounded since S_λ has no asymptotic values. Moreover, S_λ has no wandering domains. This concludes the proof of Proposition 6.24.

6.2 A meromorphic function constructed by surgery

Using surgery methods exactly as in [DF] one can construct meromorphic functions with buried Devaney hairs.

Let us consider as a first map for example the function $f(z) = \lambda \sin(z)$ where λ is chosen so that f has a bounded invariant Siegel disk. This can be acomplished for instance by either

1. taking an appropiate λ with $\mathrm{Re}(\lambda) \geq \pi/2$ on the boundary of the main hyperbolic component attached to the unic disk at $\lambda = 1$ (which has period one), or
2. taking $\lambda = e^{2\pi i\theta}$ with θ an irrational number of bounded type.

In the first case, as λ runs along the boundary of the component, the multiplier of one of the fixed points runs along the unit disk, so there will be invariant Siegel disks for infinitely many values of λ. Moreover, it follows from Proposition 6.24 that they must be bounded. In the second case, it has been proven in [Z] that Siegel disks for these class of parameters are bounded.

We assume in either case that that the Siegel disk of f is centered at $z = 0$ (making an affine change of coordinates if necessary) and its rotation number is a certain number θ.

As a second map consider the quadratic polynomial $\widetilde{f}(z) = \rho z(1-z)$ where $\rho = e^{-2\pi i\theta}$. It is well known that $\widetilde{f}$ has an invariant Siegels disk at $z = 0$ of rotation number $-\theta$.

The surgery construction consists of "gluing" the two dynamical planes (or spheres) where f and $\widetilde{f}$ act, along one of the invariant curves in each of the Siegel disks, γ and $\widetilde{\gamma}$. The result, after performing the details of the surgery, is a new function F which reflects the dynamics

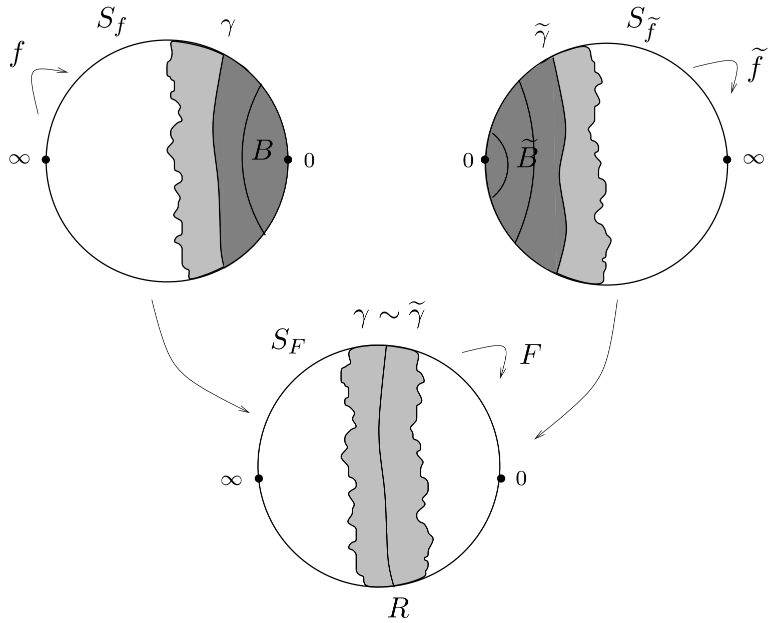

Figure 5: The surgery procedure to construct a meromorphic function with buried Devaney hairs.

of f on the unbounded component of the complement of γ and the dynamics of $\widetilde{f}$ in the bounded one. See Figure 5 and [DF] for the complete details.

The new map F (or an affine conjugate) has still a transcendental singularity at ∞ and no other. It has a bounded Herman ring around γ (since both Siegel disks were bounded), and poles where $\widetilde{f}$ had zeros (i.e. one single pole). The point $z = 0$ is now a superattracting point whose inmediate basin is in the bounded component of the complement of the Herman ring. The function F has also two Cantor bouquets since the dynamics around ∞ have not changed.

It follows easily that all periodic Fatou components of F are bounded and there are no wandering components. Hence we are under the hypothesis of Proposition 6.22 and we conclude that all points which tend to infinity under iteration are buried points. In particular the pole and all its preimages are buried and all Devaney hairs are unbounded continua of buried points.

6.3 The exponential family

The discusions above deal with the cases where all Fatou components are bounded. Notice that this excludes from the discussion an important family like the exponential, since all hyperbolic members of the exponential family have unbounded inmediate basins of attraction. See Figure 6.

With some work, we could see that hyperbolic exponential functions also fall under the hypothesis of Proposition 6.22, with a set A which is unbounded and which contains the inmediate basin of attraction. Hence we could conclude that the residual Julia set is nonempty. However, it is simpler and more profitable to closely study the functions to conclude something stronger, namely that most of the hairs in the Julia set are buried. More precisely we prove the following.

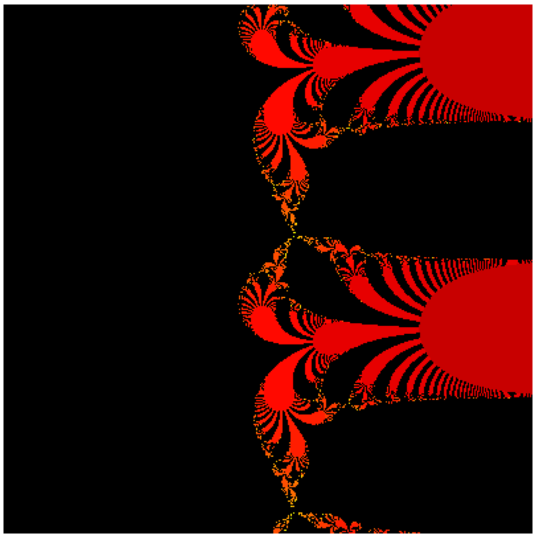

Figure 6: Dynamical plane of $E_\lambda(z) = \lambda \exp(z)$, where λ is chosen so that there is an attracting periodic orbit of period 3.

Proposition 6.28. *Let $\lambda \in \mathbb{C}$ such that $E_\lambda(z) = \lambda e^z$ has an attracting periodic orbit. Then all hairs, except possibly a countable number of them, are buried components.*

To prove this proposition we will associate to the parameter λ an infinite integer sequence called the *kneading sequence* $K(\lambda) = 0s_1 \dots s_{n-1}$, and likewise to each point z in the Julia set another infinite sequence called the *itinerary of* z and denoted by $K(z)$. After seeing the precise construction, it will be clear that all points on a hair must share the same itinerary.

With this notation, Proposition 6.28 is a consequence of the following.

Proposition 6.29. *Let $\lambda \in \mathbb{C}$ such that $E_\lambda(z) = \lambda e^z$ has an attracting periodic orbit of period n. Supose $K(\lambda) =: K = 0k_1 \dots k_{n-1}$ is the kneading sequence. Then, all hairs in the Julia set are buried components except those with itinerary $T\,K\,u_1\,K\,u_2\,K\,u_3\,K \dots$, where T is any finite sequence, and $u_1, u_2, \dots$ are arbitrary entries. Hence only countably many hairs can have points which are not buried.*

The rest of this section is dedicated to prove Proposition 6.29. We start by sketching the construction that allows us to define the kneading sequence of a map. We refer the reader to [DFJ] and [BhDe] for all missing details.

6.3.1 The fingers and the glove

Let $z_0, z_1 = E_\lambda(z_0), \dots z_{n-1} = E_\lambda(z_{n-2})$ be the points of the attracting periodic orbit.

Let A^* denote the immediate basin of attraction of the periodic orbit and, for $0 \le i \le n-1$, define $A^*(z_i)$ to be the connected component of A^* which contains z_i. We name the points in the orbit so that the asymptotic value 0 belongs to $A^*(z_0)$.

We now construct geometrically and define what we call *fingers*. For $\nu \in \mathbb{R}$, let $H_\nu = \{z \mid \mathrm{Re}(z) > \nu\}$.

Definition. An unbounded simply connected $F \in \mathbb{C}$ is called a *finger* of *width* c if

a) F is bounded by a single simple curve $\gamma \subset \mathbb{C}$.

b) There exists ν such that $F \cap H_\nu$ is simply connected, extends to infinity, and satisfies

$$F \cap H_\nu \subset \left\{ z \mid \mathrm{Im}(z) \in \left[a - \frac{d}{2}, a + \frac{d}{2} \right] \right\} \text{ for some } a \in \mathbb{R},$$

and c is the infimum value for d.

Observe that the preimage of any finger which does not contain 0 consists of infinitely many fingers of width smaller than 2π which are $2\pi i$–translates of each other.

We begin the construction by choosing B to be a topological disk in $A^*(z_0)$ that contains both 0 and z_0, and having the property that B is mapped strictly inside itself under E_λ^n. This set can be defined precisely using linearizing coordinates.

We now take successive preimages of the disk B (see Figure 7). More precisely, let B_{n-1} be the open set in $\mathbb{C}$ which is mapped to B. Note that, since $0 \in B$, it follows that B_{n-1} has a single connected component which contains a left half plane, and whose image under E_λ wraps infinitely many times over $B \setminus \{0\}$. Note that the point z_{n-1} belongs to the set B_{n-1}, which lies inside $A^*(z_{n-1})$.

We now consider the preimage of B_{n-1}. It is easy to check (by looking at the image of vertical lines with increasing real part) that this preimage consists of infinitely many disjoint fingers of width less than 2π which are $2\pi i$-translates of each other. We define $B_{n-2} \subset A^*(z_{n-2})$ to be the connected component for which $z_{n-2} \in B_{n-2}$. The map E_λ takes B_{n-2} conformally onto B_{n-1}.

Similarly, we define the sets $B_{n-3}, \ldots, B_0$, by setting B_i to be the connected component of $E_\lambda^{-1}(B_{i+1})$ that contains the point z_i. These inverses are all well defined and the map E_λ sends B_i conformally onto B_{i+1}. Each B_i belongs to the immediate basin $A^*(z_i)$. The following characterization of the sets $B_i, i = 0, \ldots, n-2$ is proved in [BhDe].

Proposition 6.30. *Let $n > 2$. For $i = 0, \ldots, n-2$, B_i is a finger of width $c_i < 2\pi$.*

It follows immediately from the above construction that the width of the finger B_{n-2} that is mapped by E_λ conformally onto B_{n-1} is π, while the widths of the other fingers is 0. So we will refer to B_{n-2} as the *big finger*.

We proceed to the final step, by defining the set

$$G = \{z \in \mathbb{C} \mid E_\lambda(z) \in B_0\}$$

which we call the *glove*. We observe from the above construction that G is a connected set and $B_{n-1} \subset G \subset A^*(z_{n-1})$. See Figure 7. Moreover, the complement of G consists of infinitely many fingers, each of which are $2\pi i$ translates of each other. We index these infinitely many connected components by $V_j,\ j \in \mathbb{Z}$, so that $2\pi i j \in V_j$.

In fact, these V_j form a set of fundamental domains for the Julia set of E_λ in the following sense:

- $J(E_\lambda) \subset \bigcup_{j \in \mathbb{Z}} V_j$.
- E_λ maps each V_j conformally onto $\mathbb{C} \setminus B_0$, and so $E_\lambda(V_j) \supset J(E_\lambda)$.

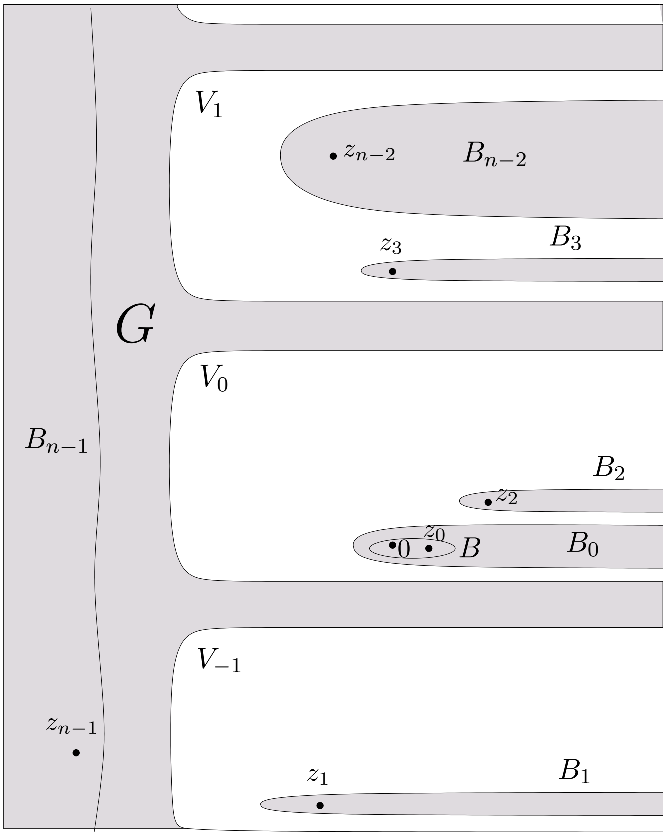

Figure 7: Sketch of the sets B_0 to B_{n-1}, G and V_j for $j \in \mathbb{Z}$. Points in grey belong to the basin of attraction of the periodic orbit.

Hence, for each $j \in \mathbb{Z}$ we have a well defined inverse branch of E_λ:

$$L_j = L_{\lambda,j} : \mathbb{C} \setminus B_0 \longrightarrow V_j.$$

Note that B_0 lies inside V_0 since $0 \in B_0$. The other fingers $B_1, \ldots, B_{n-2}$ may lie inside any of the fundamental domains V_j, depending on the value of λ. In particular, several B_i may lie in the same V_j.

6.3.2 Kneading sequence and itineraries

We first introduce the kneading sequence given by the fundamental domains V_j. We define the *kneading sequence* of λ to be

$$K(\lambda) = 0\, k_1\, k_2\, k_3\, \ldots\, k_{n-2}\, *$$

where $B_j \subset V_{k_j}$ for all $1 \leq j \leq n-2$. We use $*$ for the position of the point z_{n-1}, since this point does not belong to any of the V_j.

We define the *K–itinerary* of any point $z \in J(E_\lambda)$ to be

$$K(z) = k_0\, k_1\, k_2\, k_3\, \ldots$$

where $E_\lambda^j(z) \in V_{k_j}$ for any $j \geq 0$.

Notice that if γ is a hair of the Julia set, all points in γ must share the same itinerary, since the curve cannot cross the boundaries of the fundamental domains V_j given that these belong to the Fatou set.

One can then use these itineraries together with the kneading sequence to give a complete description of the structure of the Julia set for E_λ in terms of symbolic dynamics. See [BhDe].

6.3.3 Conclusion of the proof of Proposition 6.29

We first observe that the connected components of the inmediate basin of attraction satisfy that $\overline{A*(z_i)} \subset V_j$ where $0 \leq i \leq n-2$ and j is such that $B_i \subset V_j$. In other words, the boundary of each of these components is entirely contained in one and only one of the fundamental domains, more precisely the one whose index provides the corresponding entry in the kneading sequence. The only exception is the connected component $A*(z_{n-1}$, whose boundary intersects all the fundamentla domains V_j for all $j \in \mathbb{Z}$. This means that any point z in the boundary of the inmediate basin of attraction, say on the boundary of $A*(z_0)$ for example, must have an itinerary equal to

$$K(z) = \overline{k_0\, k_1\, k_2\, , \ldots\, k_{n-2}\, *}$$

where $*$ stands for any integer (which may be different at every period).

On the other hand, any connected component of the Fatou set will eventually map to the inmediate basin of attraction. Hence any point on the boundary of the Fatou set must have an itinerary which, after a finite number of entries, ends up exactly as $K(z)$. This concludes the proof of Proposition 6.29.

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