Limiting Dynamics for The Complex Standard Family

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Abstract

The complexification of the standard family of circle maps $\mathbf{F}_{\alpha\beta}(\theta) = \theta + \alpha + \beta\sin(\theta) \bmod (2\pi)$ is given by $F_{\alpha\beta}(\omega) = \omega e^{i\alpha} e^{(\beta/2)(\omega - 1/\omega)}$ and its lift $f_{\alpha\beta}(z) = z + \alpha + \beta\sin(z)$. We investigate the 3-dimensional parameter space for $F_{\alpha\beta}$ that results from considering α complex and β real. In particular, we study the 2-dimensional cross sections β =constant as β tends to 0. As the functions tend to the rigid rotation $F_{\alpha,0}$, their dynamics tend to the dynamics of the family $G_\lambda(z) = \lambda z e^z$ where $\lambda = e^{-i\alpha}$. This new family exhibits behavior typical of the exponential family together with characteristic features of quadratic polynomials. For example, we show that that the λ-plane contains infinitely many curves for which the Julia set of the corresponding maps is the whole plane. We also prove the existence of infinitely many sets of λ values homeomorphic to the Mandelbrot set.

1 Introduction

There has been much interest in the two parameter family of maps of the circle given by

$$\mathbf{F}_{\alpha\beta}(\theta) = \theta + \alpha + \beta \sin(\theta) \pmod{2\pi}, \quad \theta \in \mathbf{R} \tag{1}$$

where α and β are real parameters and $0 \leq \theta \leq 2\pi$.

This family is known as the "standard family" and its parameter space contains the well known Arnold Tongues (Arnold [1961]) (see Fig. 1). These tongues correspond to parameters for which there exists a periodic cycle of a given period and a given rate of rotation around the circle.

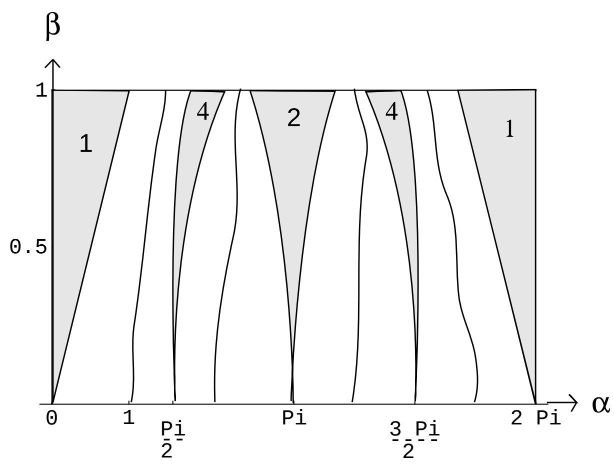

Figure 1: Sketch of the Arnold tongues and their periods: parameter space for $\alpha, \beta \in \mathbf{R}$.

In order to better understand this picture, we will investigate the standard family in the complex plane allowing not only the variable but also the parameters to be complex.

In the complex plane, the standard family is represented by

$$F_{\alpha\beta}(\omega) = \omega\, e^{i\alpha}\, e^{\frac{\beta}{2}(\omega - \frac{1}{\omega})}.$$

where ω is now a complex variable. It is easy to check that for $\alpha, \beta \in \mathbf{R}$, $F_{\alpha\beta}$ preserves the unit circle and that, on this circle, $F_{\alpha\beta}$ coincides with the standard family $\mathbf{F}_{\alpha\beta}$. For each α and β, $F_{\alpha\beta}$ is an entire function of $\mathbf{C}^*$ $(\mathbf{C} - \{0\})$ with two critical points and essential singularities at 0 and ∞. Such maps have been studied by Keen [1988], Kotus [1990] and Kotus [1987].

When considering $\alpha, \beta \in \mathbf{C}$, the parameter space for this family is 4-dimensional and includes the Arnold tongues picture as a 2-dimensional slice. In this paper we will analyze the 3-dimensional slice that corresponds to $\beta \in \mathbf{R}$ and $\alpha \in \mathbf{C}$. More precisely, we will reduce to the case $|\beta| \leq 1$ and describe the changes that occur when β decreases.

When β tends to zero, we encounter a striking phenomenon: while the function is tending to the trivial map $\omega \to \omega\, e^{i\alpha}$, its dynamics are tending to the dynamics of another map definitely different from a linear map. Our results, stated and proved in Sec. 4, concern the parameter space of this new family, $G_\lambda(z) = \lambda\, z\, e^z$. The diffeomorphism analogue of this family, known as the semi-standard family, has been studied by Gelfreich et al. [1992], among others. We show that G_λ combines characteristic features of polynomial families with those typical of entire functions (see Baker & Rippon [1984], Devaney [1991], Devaney & Krych [1984], Devaney & Tangerman [1986], Eremenko [1989], Eremenko & Lyubich [1984], Eremenko & Lyubich [1992], Goldberg & Keen [1986]). Baker [1970] and Jang [1992] proved that for a certain sequence of parameter values, the maps G_λ have Julia sets which coincide with the whole complex plane. We show that, as in the case of the exponential family (see Devaney et al. [1990]), there are infinitely many curves of parameter values for which the Julia set of the corresponding map is the whole plane. On the other hand, we find many regions in parameter space for which the maps behave in a quadratic-like fashion, which is never the case in the exponential family.

2 The Complex Standard Family

We consider the standard family of circle maps from a complex point of view, that is, a family of complex maps whose restriction to the unit circle is described by Eq. (1). In order to find an expression for them we first define their corresponding lifts on the covering space $\mathbf{C}$ by:

$$f_{\alpha\beta}(z) = z + \alpha + \beta \sin(z) \tag{2}$$

with $z \in \mathbf{C}$. Now we conjugate these functions by the projection e^{iz} which sends the real line to the unit circle. We get:

$$F_{\alpha\beta}(\omega) = \omega\, e^{i\alpha}\, e^{\frac{\beta}{2}(\omega - \frac{1}{\omega})}. \tag{3}$$

with $\omega \in \mathbf{C}^*$.

This is a family of entire functions on $\mathbf{C}^*$. They also fall into the category of functions of finite type since the set of singular values consists of two critical values. We also have asymptotic values at 0 and ∞, which coincide with the essential singularities and therefore neither of them are in the domain.

Given a complex function f we will denote its Julia set by $J(f)$. It is defined as the set of points $z \in \mathbf{C}$ for which the family of iterates $\{f^n\}$ is not a normal family in any neighborhood of z. Alternatively, it may also be defined as the closure of the set of repelling periodic points. Fatou [1926] showed that the Julia set of an entire transcendental function is a closed, non-empty, perfect set, that it is completely invariant under f and that periodic points are dense in $J(f)$. The same is true for $J(f) \cup \{0\}$ if f is entire in $\mathbf{C}^*$, as is $F_{\alpha\beta}$. The complement of the Julia set, $\mathbf{C} - J(f)$, is called the stable set and we denote it by $N(f)$. Keen [1988] and Kotus [1990] proved that every component of the stable set of an entire function on $\mathbf{C}^*$ of finite type is periodic or preperiodic. This establishes the nonexistence of wandering domains and gives an alternative definition for the Julia set as the closure of the set of points that tend to 0 or ∞ (in certain fixed asymptotic directions) under iteration. Since each attracting periodic orbit must attract a critical or asymptotic value, there are at most finitely many of these orbits and in the case when all the singular values tend to *zero* or ∞, the Julia set is the whole complex plane.

For each α and $\beta \neq 1$, $F_{\alpha\beta}$ has 2 critical points given by

$$c_1 = e^{i \arccos \frac{-1}{\beta}} \quad ; \quad c_2 = e^{-i \arccos \frac{-1}{\beta}}$$

When $\beta = 1$, $c = -1$ is the only critical point.

In the cases we are considering, β is always a real parameter less or equal than 1. Hence, the critical points for $F_{\alpha\beta}$ always lie on the negative real line . Since they are the projection of the critical points for $f_{\alpha\beta}$, which are complex conjugates , we have that c_1 and c_2 are symmetric with respect to the unit circle. This symmetry holds also for their entire orbits in the following way:

Proposition 2.1 *If $\beta \leq 1$ and α is any complex number, then, for all $n \geq 0$, $F^n_{\alpha\beta}(c_1)$ and $F^n_{\overline{\alpha}\beta}(c_2)$ are symmetric with respect to the unit circle. That is, their arguments are equal and their moduli are inverses of each other.*

Proof : A simple computation shows that for any complex number z,

$$f_{\overline{\alpha}\beta}(\overline{z}) = \overline{f_{\alpha\beta}(z)}.$$

Since the critical points of $f_{\alpha\beta}$ which we denote by c and $\overline{c}$ are complex conjugates, we have that all their iterates are also complex conjugate. The projection by e^{iz} of two complex conjugate points gives two points which are symmetric with respect to the unit circle. Hence, since the projection of the iterates of c and $\overline{c}$ by $f_{\alpha\beta}$ are the iterates of c_1 and c_2 by $F_{\alpha\beta}$, the result follows.

$\square$

We consider two dimensional slices of parameter space that correspond to the α-plane for a a fixed real β-value. The real axis of such a slice reflects the different intervals we encounter when we cross the Arnold tongues horizontally (see Figs. 1, 2). In the computer pictures (see Fig. 3,4) we color black any α-value for which the orbit of the critical point does not "escape" to 0 nor to ∞ in 100 iterations. We consider that the orbit has escaped if an iterate is larger that e^{10} or smaller than e^{-10} in modulus.

Figure 3 shows the result of applying this algorithm for $\beta = 1$ and Fig. 4 is for $\beta = 0.5$. We observe that each tongue-interval in the real axis has an extension in the complex plane. When $\beta = 1$, case for which there is one critical point only, there is a not surprising symmetry with respect to the real axis. For $\beta < 1$ this symmetry disappears. This does not mean that the dynamics of $F_{\alpha\beta}$ are different for α and $\overline{\alpha}$, but only means that the roles of the two critical points are different. When we consider $\overline{\alpha}$, these roles interchange. In other words, if we flip these pictures with respect to the real axis, we obtain the information about the other critical point. This difference in the two critical orbits allows the existence of plenty of bistable regions, i.e. regions of parameter values for which each critical point is attracted to a different periodic orbit. Note that this is never the case when both parameters are real (and $\beta < 1$).

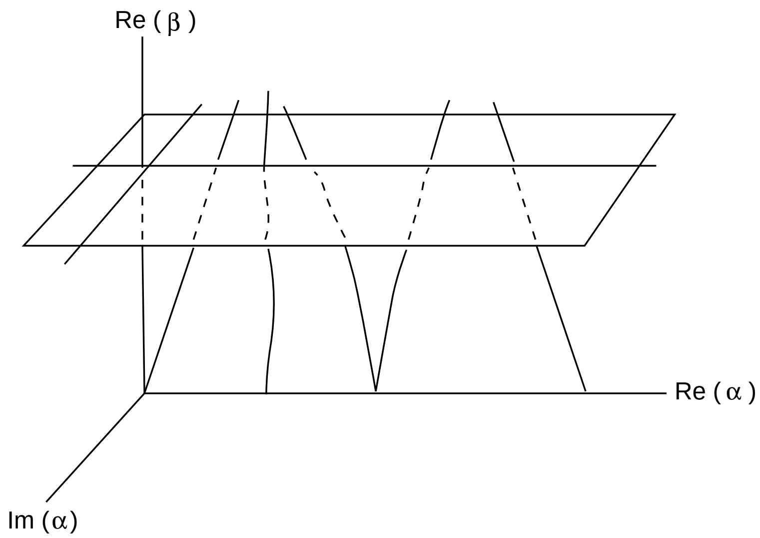

Figure 2: 2-dimensional slices of 3-dimensional parameter space.

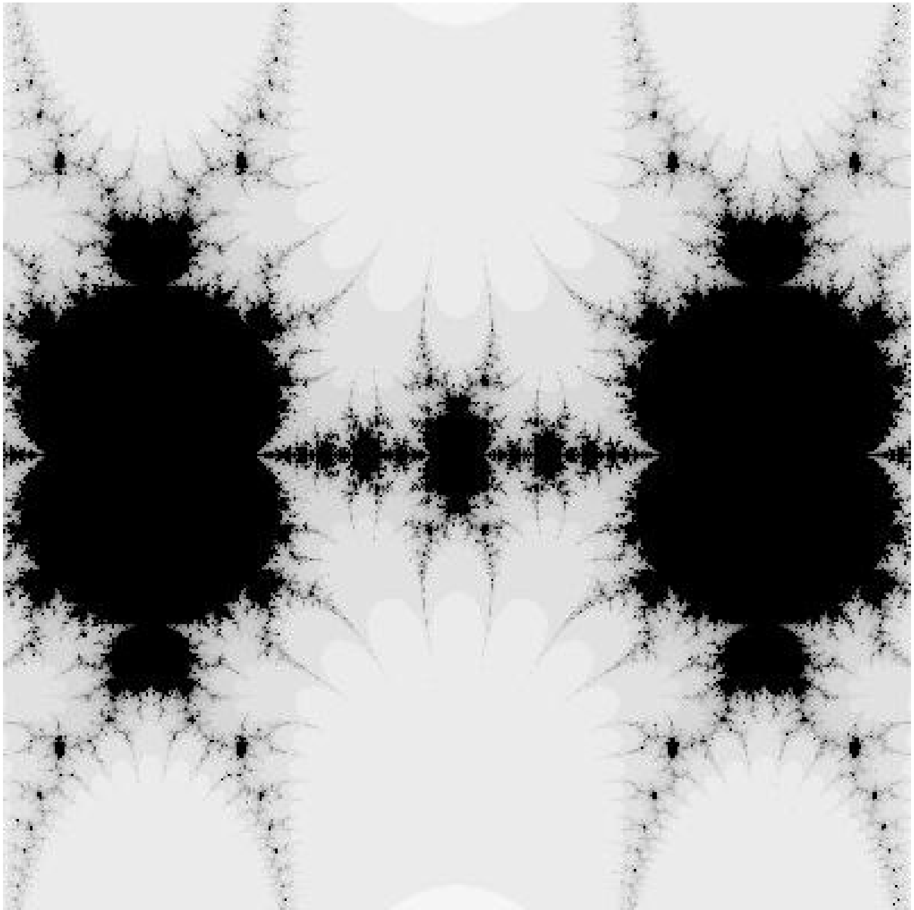

Figure 3: α-plane for $\beta = 1$ fixed. Range: $[-1.4, 7.7] \times [-4.5, 4.5]$.

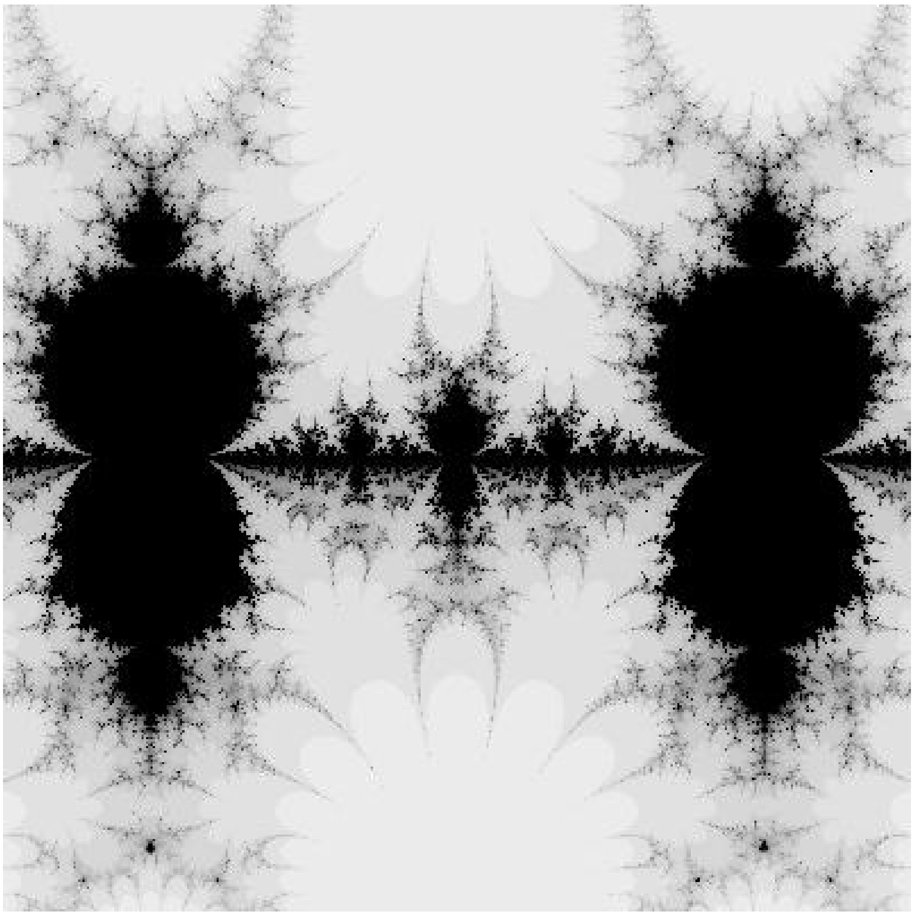

Figure 4: α-plane for $\beta = 0.5$ fixed. Range: $[-1.4, 7.7] \times [-4.5, 4.5]$.

3 Limiting Dynamics

When we look at slices for values of β tending to zero we observe a curious fact: while the maps are tending to the trivial map $F_{\alpha 0}(\omega) = \omega\, e^{i\alpha}$, their dynamics are not. In the upper half plane there seems to be a clear limiting shape. This corresponds to following the behavior of the critical point that tends to 0 as $\beta \to 0$. Symmetrically, the limiting shape appears on the lower half plane when we follow the critical point tending to $-\infty$. It appears that each tongue is tending to a Mandelbrot set! (see figs.5,6).

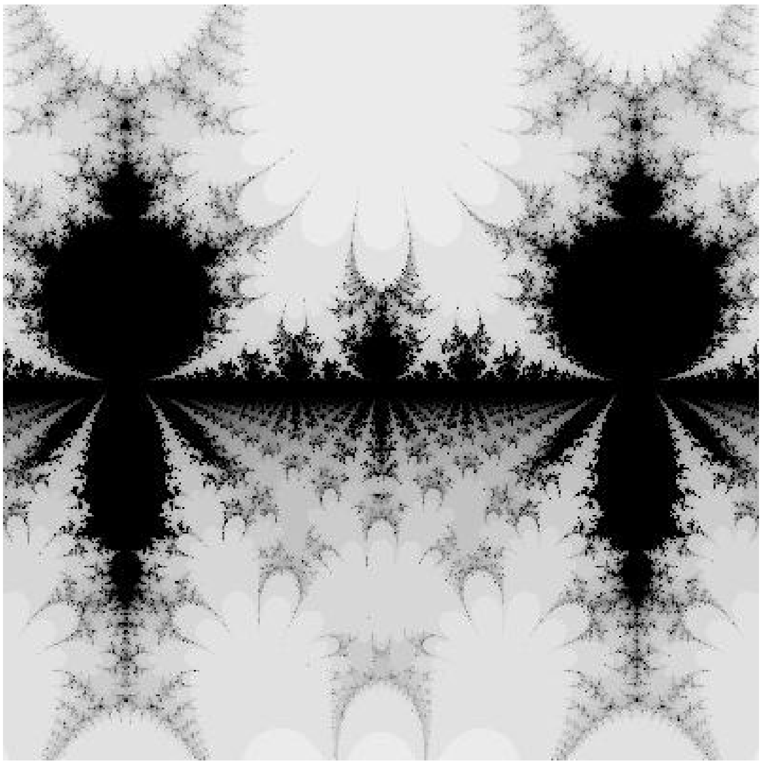

Figure 5: α-plane for $\beta = 0.1$ fixed. Range: $[-1.4, 7.7] \times [-4.5, 4.5]$.

This suggests the possibility of finding a new family of maps conjugate to $F_{\alpha\beta}$ which does not become trivial when β is zero. To find this family we will rescale so that one of the two critical orbits has a limit in the interior of $\mathbf{C}^*$. More precisely, consider the following

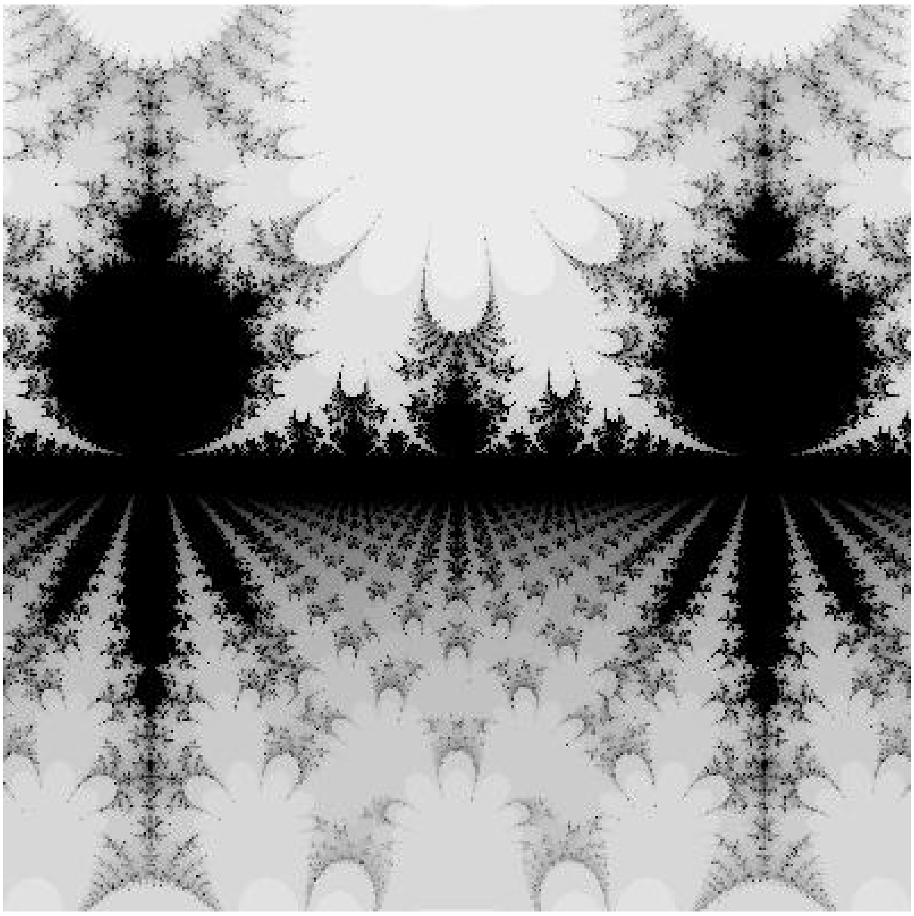

Figure 6: α-plane for $\beta = 0.01$ fixed. Range: $[-1.4, 7.7] \times [-4.5, 4.5]$.

two conjugacies:

Proposition 3.1 *Let* $h_1(\omega) = \frac{\beta}{2}\,\omega$ *and* $h_2(\omega) = \frac{\beta}{2}\,\frac{1}{\omega}$. *Then,*

1. h_1 *conjugates* F *to* $G_1(z) = e^{i\alpha} z\, e^z\, e^{\frac{-\beta^2}{4z}}$
2. h_2 *conjugates* F *to* $G_2(z) = e^{-i\alpha}\, z\, e^z\, e^{\frac{-\beta^2}{4z}}$

Proof : Both computations are essentially the same. We show the second:

$$\begin{aligned} G_2(z) = h_2(F(\omega)) &= \tfrac{\beta}{2}\,\tfrac{1}{F(\omega)} \\ &= \tfrac{\beta}{2}\,\tfrac{1}{\omega}\, e^{-i\alpha}\, e^{-\frac{\beta}{2}\,\omega}\, e^{\frac{\beta}{2}\,\frac{1}{\omega}} \\ &= e^{-i\alpha}\, z\, e^z\, e^{\frac{-\beta^2}{4z}} \end{aligned}$$

where the last step comes from

$$\omega = h_2^{-1}(z) = \frac{\beta}{2z}$$

□

The basic idea is: h_1 contracts the whole complex plane by a factor of $\beta/2$. This guarantees that one of the critical orbits remains bounded as β tends to zero. In the second case we do the same thing but first we interchange the roles of the critical points. Because of the symmetry mentioned above, the resulting function G_2 is the same as G_1 when we substitute α by $-\alpha$. It is therefore equivalent to work with either map. We choose to work with $G(z) = G_2(z)$.

We now set β equal to zero. The resulting map is:

$$G_\alpha(z) = e^{-i\alpha}\, z\, e^z.$$

Proposition 3.2 *G_λ has a fixed point at* 0*, which is also an asymptotic value. There exists a unique critical point at* $z = -1$ *independently of* α*.*

Proof : It is clear that 0 is a fixed point. When $Re(z) \to -\infty$, G_α tends to 0 which makes this point an asymptotic value. The derivative is

$$G'_\alpha(z) = e^{-i\alpha} e^z (1+z)$$

and it has a unique root at $z = -1$.

$\square$

Note that one of the two critical points of $F_{\alpha\beta}$ has collapsed to 0 when setting β equal to zero.

This new family is very interesting by itself. Not only does it reflect the limiting dynamics of the original family $F_{\alpha\beta}$, but it also combines the presence of a critical point at $z = -1$, a fixed asymptotic value at $z = 0$ and an essential singularity at ∞. Throughout the rest of the paper, we will concentrate on the study of this new family.

4 The Family $z \to \lambda \ z \ \exp(z)$

In order to eliminate the periodicity in α, we define $\lambda = e^{-i\alpha}$ so we will be dealing with the family of maps:

$$G_\lambda(z) = \lambda \, z \, e^z.$$

With this change, we have wrapped the α-plane around the unit circle. The real values of α are now reflected in the unit circle of the λ-plane. The tips of the Arnold tongues that were located at $\alpha = 2\pi p/q$ are now at $\lambda = e^{-2\pi i p/q}$. All the values of α in the lower half plane have been sent inside the circle while the upper half plane is now the complement of the unit disk.

Since the derivative of G_λ at $z = 0$ is precisely λ, this fixed point is an attractor for all values of λ with modulus less than 1. Attached to the unit circle we find all the extensions of the Arnold tongues to the complex plane.

Let us define the set B of parameter values for which the critical value does not escape to ∞ under iteration. A computer picture of this set is shown in Fig. 7 and with a larger range in Fig. 8, where black indicates critical orbits that remain bounded for 100 iterations. Note that plenty of scattered black points are due to the fact that $z = 0$ is an asymptotic value and a fixed point. Points that are sent very close to zero may take a lot of iterations to leave a neighborhood of this point.

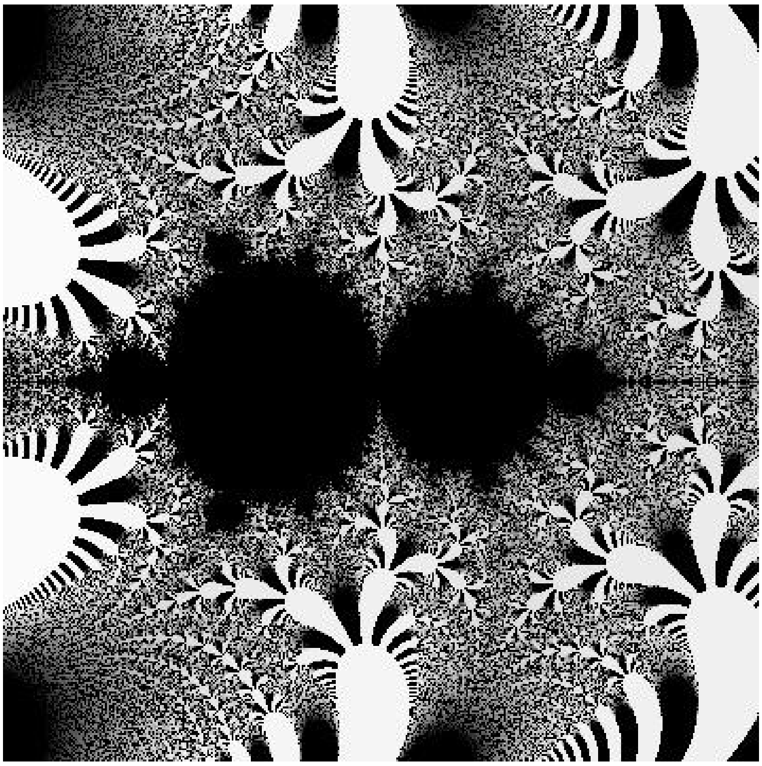

Figure 7: Parameter plane for G_λ. Range: $[-4, 16] \times [-10, 10]$.

When we look at these pictures we notice the combination of two very distinct features: on one hand we encounter structures similar to those one finds for the exponential family (hairs in parameter space, see Devaney et al. [1990]). On the other hand, the obvious component of the set B reminds us of the structure of a quadratic bifurcation set. In this section, we investigate these two different aspects, and we organize it as follows: first we study the dynamical and parameter planes of some families of polynomials $P_{d,\lambda}$ and their

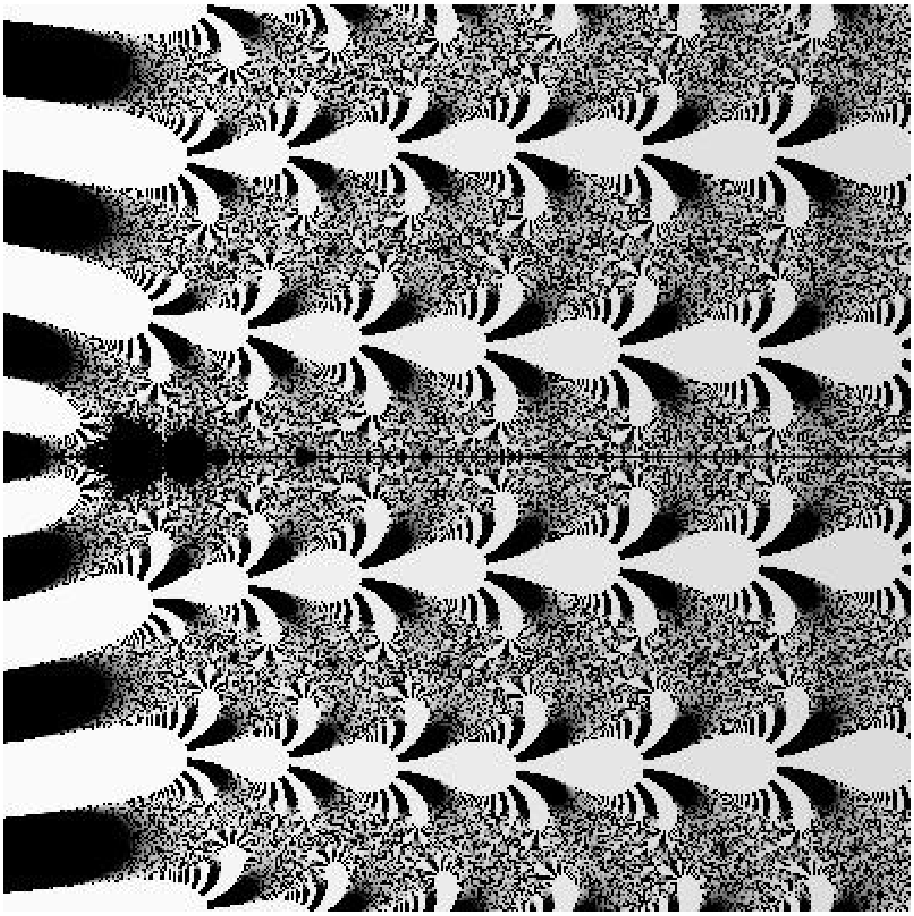

Figure 8: Parameter plane for G_λ. Range: $[-10, 90] \times [-50, 50]$.

relation with the parameter space for G_λ, which will explain the quadratic-like features of the set B. Later on we prove the existence of continuous curves of parameter values for which $J(G_\lambda)$ is the whole plane (i.e. hairs). Finally, we show that there exist infinitely many Mandelbrot sets in the λ-plane, organized by some symbolic dynamics.

4.1 G_λ as a Limit of Polynomials

When we look at the component of B that contains the extensions of the Arnold tongues (Fig. 7) we notice a striking similarity between the structure of this set and that of the Mandelbrot set for the logistic function (as well as many differences). The link between the two is provided by the family of polynomials given by

$$P_{d,\lambda}(z) = \lambda\, z\, (1 + \frac{z}{d})^d.$$

It is clear that $P_{d,\lambda}$ converges to G_λ uniformly in compact sets as $d \to \infty$, but it is also true as we will see, that it converges, in some sense, dynamically.

Unlike entire functions, for a polynomial p of degree k, ∞ is always an attracting fixed point. We define the filled-in Julia set, $K(p)$, as the set of points whose orbit does not tend to infinity under iteration. Then, the Julia set $J(p)$ coincides with the boundary of $K(p)$. It is well known (see Blanchard [1984]) that $K(p)$ is a compact set and it is connected if and only if all the critical points of the polynomial belong to $K(p)$. In general, there exists a neighborhood U of ∞, a real number $r \geq 1$ and a unique analytic isomorphism φ, tangent to the identity at infinity, that conjugates $p\mid_U$ to the map $z \mapsto z^k$ restricted to the exterior of D_R (the disk of radius R). In the case where all the critical orbits are bounded, this conjugacy can be extended to a conjugacy between the exterior of $K(p)$ and the complement of the unit disk. This allows us to define the *external rays* of $K(p)$ in an analogous way to the case of quadratic polynomials (see Douady & Hubbard [1982]). A ray of *external argument* θ is defined by

$$R_p(\theta) = \varphi_p^{-1}(\{re^{2\pi i\theta}\}_{R<r<\infty}).$$

As long as $\lambda \neq 0$, the functions $P_{d,\lambda}$ always have two critical points:

$$c_1^d = \frac{-d}{d+1} \ ; \ c_2^d = -d.$$

The first one appears always with multiplicity one and tends to -1 (the critical point of G_λ) as $d \to \infty$. The second one has multiplicity $d-1$ and tends to $-\infty$ when $d \to \infty$. Since $P_{d,\lambda}(c_2^d) = 0$, and zero is fixed, only c_1^d can escape under iteration. Hence, $K(P_{d,\lambda})$ is connected if and only if the orbit of c_1^d is bounded (see Blanchard [1984]).

We define the connectedness loci as

$$B_d = \{\lambda \in \mathbf{C} \mid \{P_{d,\lambda}^n(c_1^d)\}_n \text{is bounded}\}.$$

These sets are the analogues to the Mandelbrot set for quadratic polynomials. In fact, B_1 is homeomorphic to the logistic Mandelbrot set. Since only one critical point is free, many of the facts proved by Douady & Hubbard [1982] about the Mandelbrot set are also true for the B_d's. In particular:

Theorem 4.1 *For all $d > 0$, B_d is connected.*

Proof : The proof is the same as in Devaney et al. [1990], when they showed that the connectedness loci of the family $\lambda(1+z/d)^d$ are connected. In our case, $P_{d,\lambda}$ is conjugate to the monic family of polynomials $M_{d,\nu}(z) = (z-d\nu)(z+\nu)^d + d\nu$ via the affine map $z \mapsto \frac{d+1}{d}\nu z + d\nu$. If we let $\widetilde{B}_d$ be the analogue of B_d for $M_{d,\nu}$, then one can construct a conformal isomorphism $\psi_d : \mathbf{C} - \widetilde{B}_d \to \mathbf{C} - \overline{D}_1$ (see Douady & Hubbard [1982]). Hence $\widetilde{B}_d$ is conformally equivalent to a disk, and therefore connected for each d. Furthermore this also shows that B_d is connected, since B_d is the image of $\widetilde{B}_d$ under the d-fold covering map $\lambda(\nu) = ((d+1)\nu)^d$.

$\square$

We define the sets

$$H_d = \{\lambda \in \mathbf{C} \mid P_{d,\lambda} \text{ has an attracting cycle}\}.$$

Recall that the attracting cycle is unique since it must attract c_1^d. Following standard notation we call a connected component W of H_d, a *hyperbolic component* of B_d. The hyperbolic components are open and simply connected. For any d, $z = 0$ is a fixed point with eigenvalue λ. Therefore, if λ is in the interior of the unit disk, $P_{d,\lambda}$ has an attracting fixed point. The points $\lambda = e^{2\pi ip/q}$ are period q-doubling bifurcation points. When λ changes from being inside the unit circle to inside the hyperbolic component $W^d_{p/q}$, $z = 0$ changes from attracting to repelling, and a cycle of period q changes from repelling to attracting. We call the point $\lambda = e^{2\pi ip/q}$, the *root* of the hyperbolic component $W^d_{p/q}$. The *limb* $B^d_{p/q}$ of B_d is defined as the connected component of $B_d - D_1$ that contains $W^d_{p/q}$, together with the root point.

We can approximate the sets B_d with the computer as we did with B. The results for $d = 1, 3, 6, 90$ are shown in Figs. 9 to 12. As d gets larger we observe the two following facts: first it appears that parts of the sets B_d remain constant as we change d and only new antennas and ramifications are added. We will refer to this part as the *main body* of B_d. Second, the B_d's appear to be "converging" to the set B. In the rest of the section we will try to make these two statements precise.

We start with the convergence. We show that the hyperbolic components of B_d converge to components of B and viceversa. The same phenomenon was found by Devaney et al. [1990] concerning the polynomial family that approximates the exponential map.

Theorem 4.2

1. *If G_λ has an attracting cycle of period k, then there exists d^* such that for all $d \geq d^*$, $P_{d,\lambda}$ has an attractive periodic point of period k.*

2. *If $P_{d,\lambda}$ has an attracting cycle of period k for infinitely many d's, then G_λ has a periodic point of period k which is either attracting or indifferent.*

Proof :

1. This is clear from uniform convergence of $P^k_{d,\lambda}$ and the Schwarz lemma.

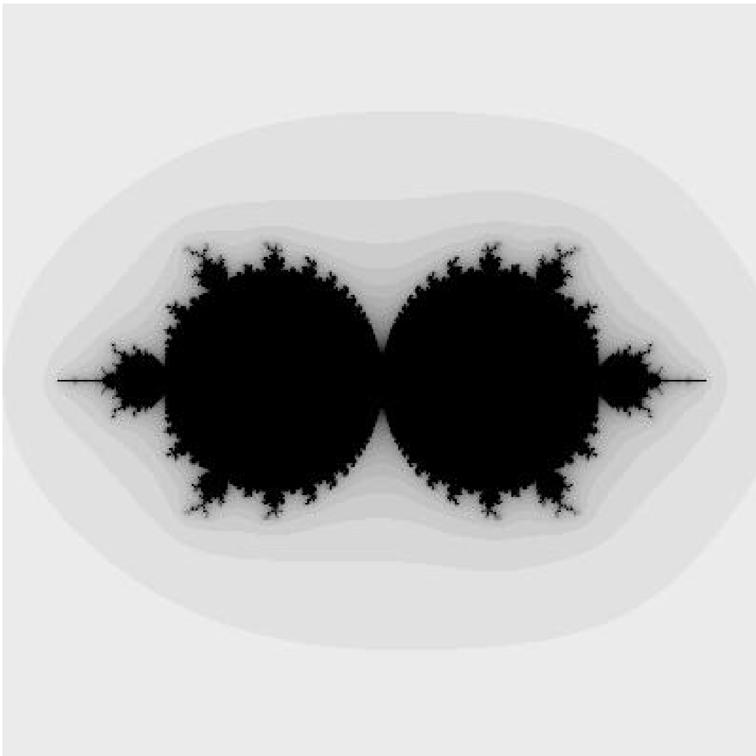

Figure 9: Parameter plane for $P_{\lambda,1}$. Range: $[-2.5, 4.5] \times [-3.5, 3.5]$.

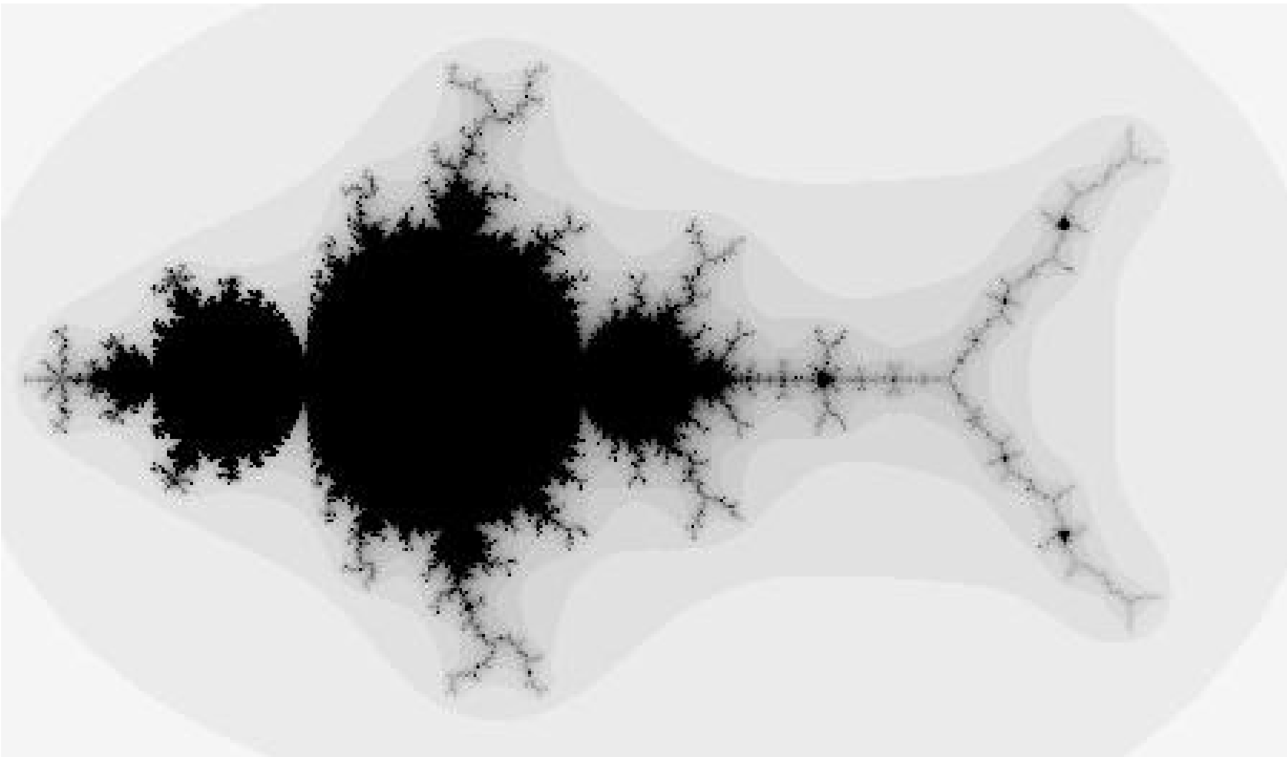

Figure 10: Parameter plane for $P_{\lambda,3}$. Range: $[-3, 14] \times [-5, 5]$.

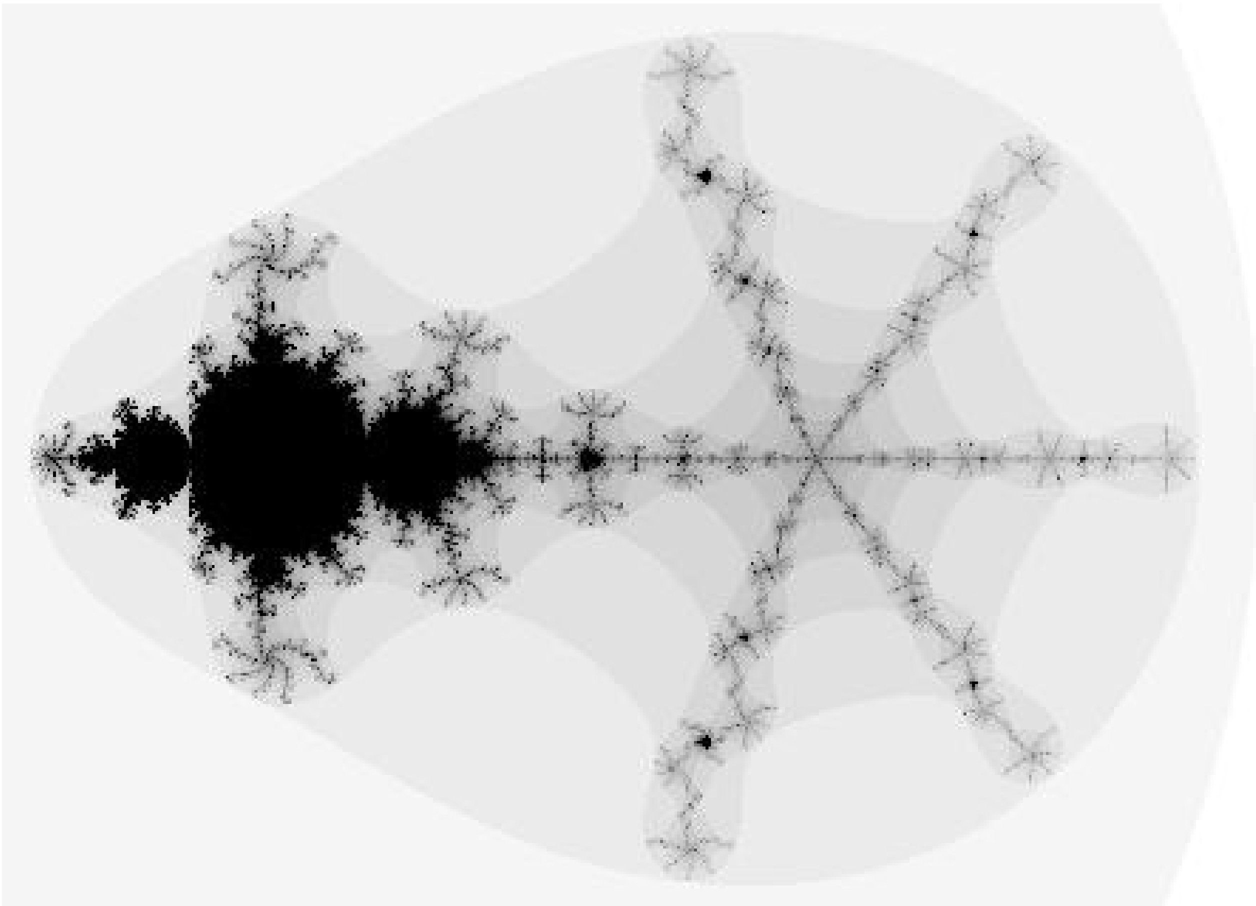

Figure 11: Parameter plane for $P_{\lambda,6}$. Range: $[-4, 30] \times [-12, 12]$.

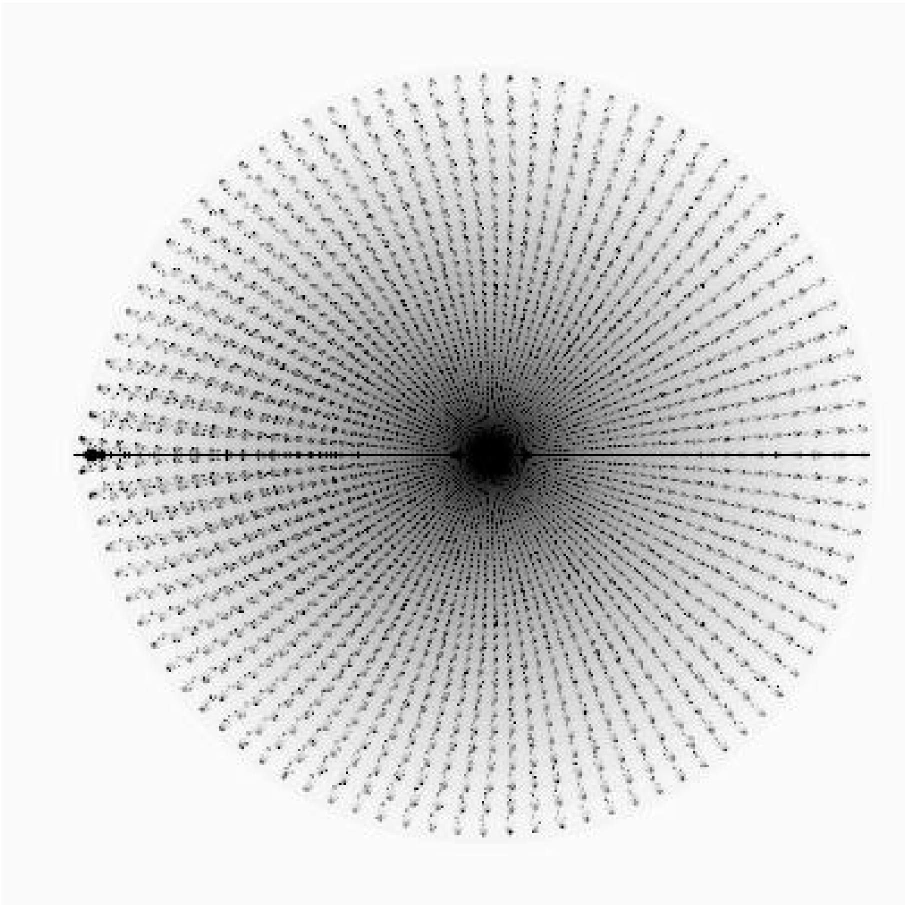

Figure 12: Parameter plane for $P_{\lambda,90}$. Range: $[-50, 500] \times [-275, 275]$

2. We need the following lemma:

Lemma 4.1 *If $z_1, \dots, z_n$ is an attracting periodic orbit for $P_{d,\lambda}$, then for some i*

$$|z_i| < \frac{2d}{d-2}.$$

Proof : The derivative of $P_{d,\lambda}$ can be expressed as

$$P'_{d,\lambda}(z_k) = \frac{P_{d,\lambda}(z)(1+z+z/d)}{z(1+z/d)}.$$

Hence, for any $k = 1, 2, \dots, n$,

$$|(P^n_{d,\lambda})'(z)| = \left|\prod_{i=1}^{n} P'_{d,\lambda}(z_i)\right| = \left|\frac{\prod z_{i+1} \prod(1+z_i+z_i/d)}{\prod z_i \prod(1+z_i/d)}\right| = \prod \left|1 + \frac{z_i}{1+z_i/d}\right|.$$

Since the cycle is attracting, this derivative must be less than 1. Thus for some i,

$$|z_i| < 2|1 + \frac{z_i}{d}| < 2 + 2\frac{|z_i|}{d}$$

and the result follows.

□

Now assume that for infinitely many d's, $P_{d,\lambda}$ has an attracting cycle of period k, and let z^d be the points on those cycles satisfying the lemma above. Then, the sequence $\{z^d\}_d$ has an accumulation point z^*. Since $P^k_{d,\lambda} \to G^k_\lambda$ as $d \to \infty$, z^* is a fixed point for G^k_λ. Given that the convergence is uniform in compact sets, we have that this fixed point is either attracting or indifferent.

□

We now discuss the structure of the main body of B_d. We have the following conjecture:

Conjecture 1 *For any $d \geq 0$ and for any p/q, the limb $B^d_{p/q}$ contains a set homeomorphic to the Mandelbrot set, whose "cardioid" is $W^d_{p/q}$.*

We provide the evidence supporting this conjecture. The case $d = 1$ is clear since B_1 is a logistic Mandelbrot set. The function $c = \lambda(2-\lambda)/4$ maps the 1-limb of B_1 in a $1-1$ fashion to the Mandelbrot set, while maps each p/q-limb of B_1 to the p/q-limb of M. The technique of tuning, introduced by Douady & Hubbard [1985], shows that each of these limbs contains a set homeomorphic to M.

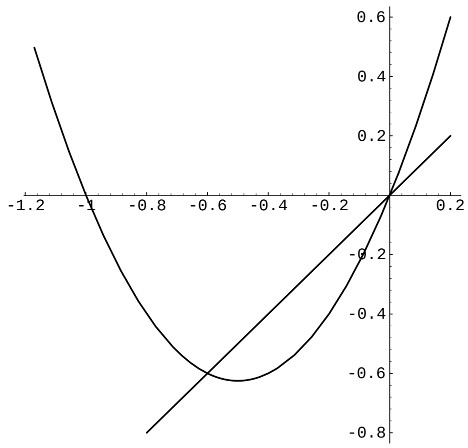

Figure 13: Graph of $P_{\lambda,1}$, for $\lambda \in \mathbf{R}$.

For the cases $d > 1$ we need to look at the Julia sets of $P_{d,\lambda}$ to see why our conjecture might be true. We first concentrate in the 1-limb B_1^d, that is, the connected component of B_d attached to the unit circle at $\lambda = 1$. Hence it contains the interval $I_d = [1, \lambda_1^d]$, where λ_1^d is the parameter value for which $c_1^d \mapsto -d \mapsto 0$. Figs. 13, 14 show the graphs of $P_{\lambda,1}$ and $P_{d,\lambda}$ in the cases of d even and odd respectively, for $\lambda \in I_d$. Note that, in each case, the interval $[-d, 0]$ is invariant and $P_{d,\lambda}$ has a unique critical point in this interval. Moreover as λ decreases from 1 to λ_1^d, the critical value decreases monotonically to $-d$. Hence we expect the dynamics of this family for $\lambda \in I_d$ to be similar to those of the quadratic family.

For any $\lambda \in B_1^d$, not necessarily real, the Julia set has the following characteristics:

1. The point $z = 0$ is a repelling fixed point. There is exactly one ray attached to 0. This follows since the bifurcation that occurred at $\lambda = 1$ was simply a change in

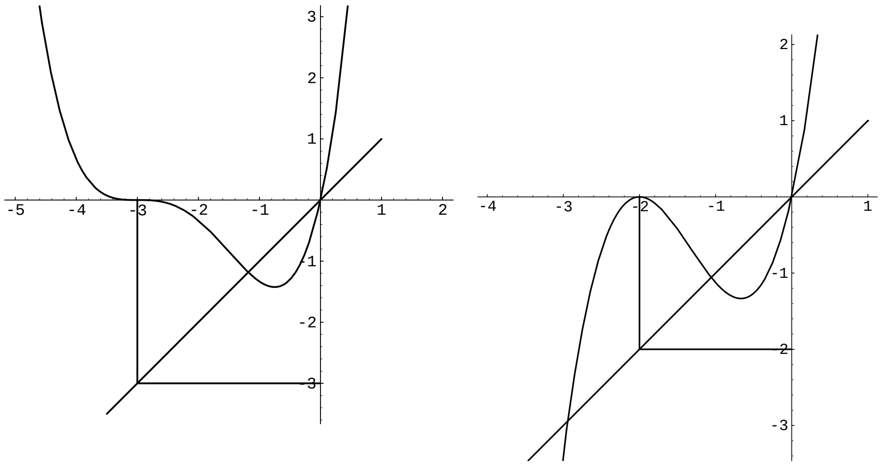

Figure 14: Graph of $P_{d,\lambda}$for d odd ($d = 3$) and d even ($d = 2$).

stability type at 0, from attracting to repelling.

2. The critical point $z = -d$ is the only nonzero preimage of zero, and it maps to 0 with multiplicity d. Hence there are d rays that land at $z = -d$, with arguments $\frac{1}{d+1}, \frac{2}{d+1}, \ldots, \frac{d}{d+1}$. See Fig. 15. These rays divide the complex plane into d different open sectors $S_1, S_2 \ldots S_d$ where S_1 is the sector that contains zero. Note that in the quadratic case, $d = 1$, only S_1 is present. S_1 always contains the interval $(-d, \infty]$ and hence the free critical point.

3. With the exception of S_1, all the other sectors map onto the whole plane minus the 0-ray in a $1 - 1$ fashion. S_1 does it in a $2 - 1$ fashion.

The idea is the following: the λ-values in B_1^d for which the critical orbit is entirely contained in S_1 correspond to maps $P_{d,\lambda}$ which are conjugate to $P_{1,\lambda}$ when restricted to the set of points that never leave S_1 under iteration. Hence, these λ values form a set in B_1^d homeomorphic to the Mandelbrot set. This explains why, when we increase d, we only see new antennas or ramifications appearing on the B_d's: the Mandelbrot-like set remains while λ-values for which the critical orbit leaves S_1 in the new possible ways are added.

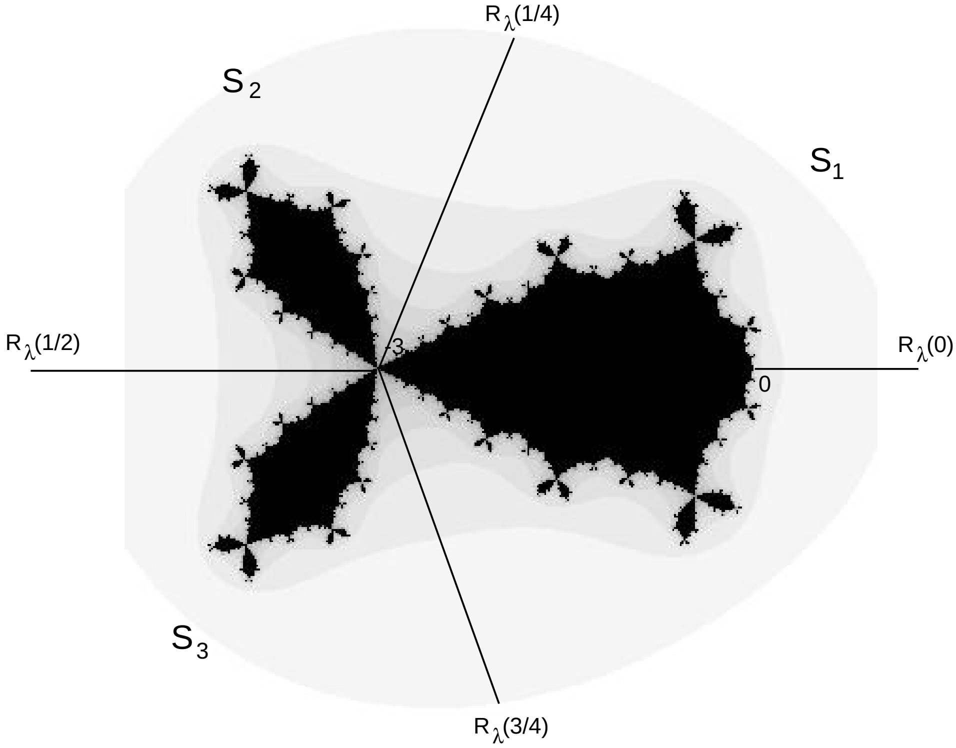

Figure 15: The filled in Julia set for $P_{\lambda,3}$ when $\lambda \in B_1^3$.

In order to prove the statement above rigorously, we would need to use the technique of surgery on complex polynomials as it was done by Branner & Douady [1986]. Using surgery, one can construct a quadratic polynomial on the $1/d$ limb of the Mandelbrot set for every polynomial in B_1^d, and viceversa. This process constructs a homeomorphism between these two limbs. The statement then, follows by tuning (see Douady & Hubbard [1985]). The complete proof will be included in a later paper.

A similar situation occurs for all the other limbs $B_{p/q}^d$. The Julia sets in these cases have a different structure than the one described above but they all share the following characteristics: when $\lambda = e^{ip/q}$, $z = 0$ ceases to be attracting and a cycle of period q becomes attracting. Hence the point $z = 0$ has q different rays attached to it, whose arguments are q-periodic under the function $\theta \mapsto (d+1)\theta$. These rays define q sectors in $\mathbf{C}$ which we denote by $S_1^0, \ldots, S_q^0$, where S_1^0 is the sector that contains the critical point. The only preimage of 0, $z = -d$, has $d\,q$ rays that attach to it which correspond to the preimages of the rays at 0 (see Fig. 16). As before, these rays define $d\,q$ sectors $S_1, \ldots, S_d q$, where S_1 denotes the sector containing the critical point. Consider now the set $S = S_1^0 \cap S_1$. One can easily check that, under $P_{d,\lambda}^q$, S maps to the whole complex plane minus the complement of S_1^0 with degree 2. As above, the λ-values for which the entire orbit of the critical point stays inside $S \cup S_{d+1} \cup S_{d+2}$ form a set homeomorphic to the Mandelbrot set.

We now describe a phenomenon that occurs as $d \to \infty$. We have already shown that the hyperbolic components of B_d tend to hyperbolic components of B. Hence one might think that the Mandelbrot-like structure would also be preserved in the limit. In fact, we conjecture that this may be the case but with a major difference: among others, all the Misiurewicz points for which the critical point is eventually mapped to 0 are missing (they tend to ∞ as $d \to \infty$). Let us illustrate this fact in the case of real values of λ. When λ is positive, the end of the Mandelbrot-like set corresponds to the λ-value for which the critical value is $-d$, and hence $P_{d,\lambda}^2(c_1^d) = 0$ (see Fig. 17). Since $-d$ is tending to $-\infty$, this endpoint must be tending to ∞. In the case of λ negative, the Misiurewicz point that is tending to $-\infty$ is the λ-value for which $P_{d,\lambda}^2(c_1^d) = -d$ (see Fig. 18). This must also be

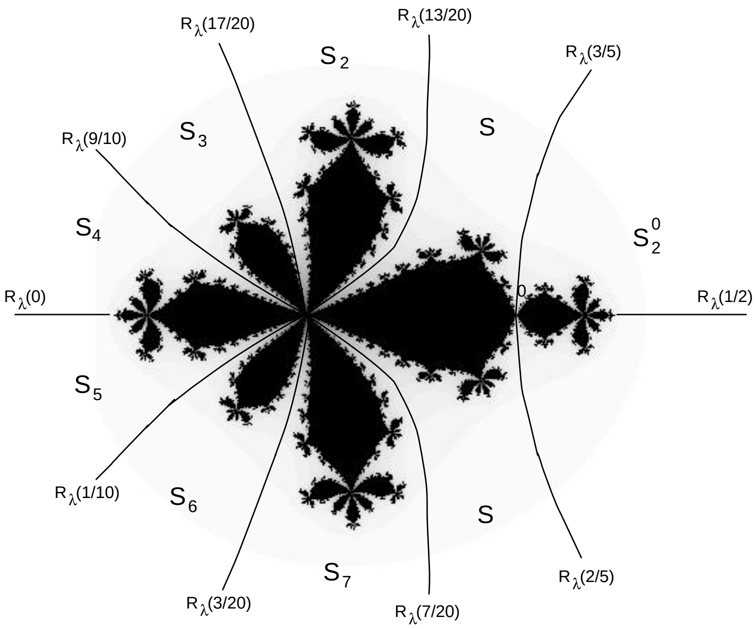

Figure 16: The filled in Julia set for $P_{\lambda,3}$ when $\lambda \in B^3_{1/2}$.

the case for any Misiurewicz points of the type described above since the only preimage of $z = 0$ by $P_{d,\lambda}$ has gone to $-\infty$, making 0 into an omitted point for G_λ. Hence the limit of the $\mathcal{M}_\lambda$'s cannot be really homeomorphic to a Mandelbrot set. However, we believe it is homeomorphic to a Mandelbrot set with these particular arms stretched to reach infinity. A more precise study of this peculiar set will be given in a later paper.

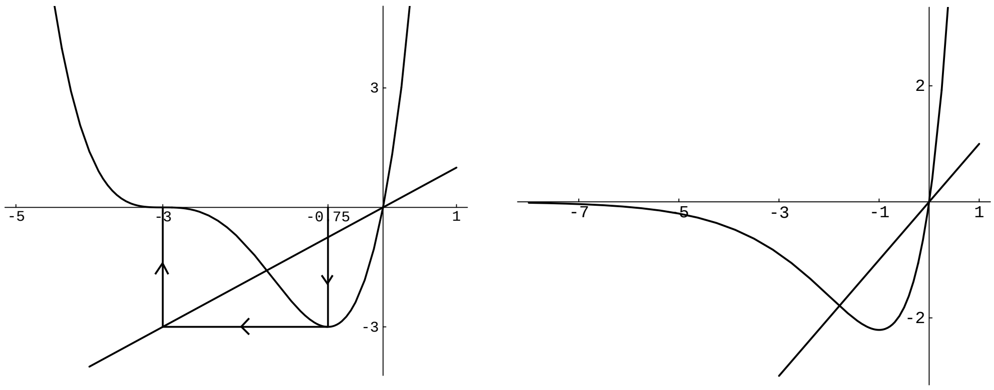

Figure 17: Graphs of $P_{d,\lambda}(d = 3)$ and $G_\lambda$for λpositive.

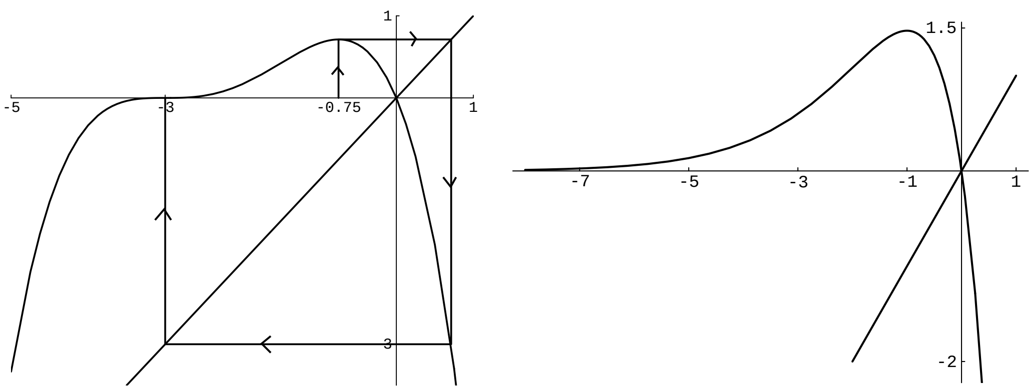

Figure 18: Graphs of $P_{d,\lambda}(d = 3)$ and $G_\lambda$for λnegative.

4.2 Hairs

As mentioned above, when we look at Fig. 8 we can observe structures similar to those found in the exponential family: the "hairs". In this section we will establish the existence of some hairs in parameter space, that is, continuous curves of parameter values for which the critical point escapes to infinity and therefore, the Julia set for these values is the whole complex plane. As in the case of the exponential family (see Devaney et al. [1990]), these hairs probably correspond to the limit of the parameter space rays of $P_{d,\lambda}$. However, we will not use the the polynomial family in this description.

We first need to study the dynamical plane for the functions G_λ.

4.2.1 Dynamical Plane

The goal in this section is to prove that for values of λ in some set Λ of parameters, the repelling fixed point at $z = 0$ comes equipped with a continuous invariant curve of points that tend to infinity under iteration. We call this curve the "fixed hair" and itself and its preimages will be crucial in constructing the hairs in parameter space.

To fix ideas let us give an easy example where the presence of the fixed hair is obvious. Consider the case $\lambda \in \mathbf{R}, \lambda \geq 1$. For these values of λ, $z = 0$ is repelling and the real line is invariant. In these cases, the positive real line is the fixed hair attached to $z = 0$ (see Fig. 19), since all points escape to infinity under iteration.

First we set up some fundamental domains. For any λ in the upper half plane (not including the real line) consider all the preimages of the negative real line. We denote them by σ_k^λ. Each σ_k^λ is determined by the equation:

$$Im(z) = (2k+1)\pi - \arg(z) - \arg(\lambda), \quad k = 0, \pm 1, \pm 2, \ldots \tag{4}$$

where $\arg(z) \in (-\pi, \pi)$ and $\arg(\lambda) \in (0, \pi)$ are well defined because σ_k^λ cannot intersect $\mathbf{R}^-$ for any k. From the equation, it can easily be checked that these curves look as shown in Fig. 20.

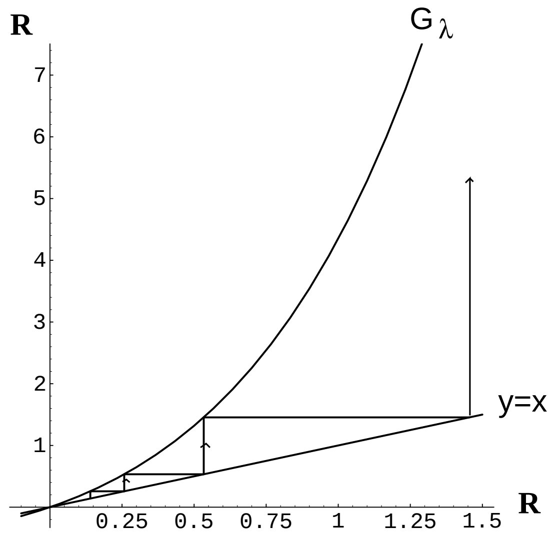

Figure 19: Graph of $G_\lambda$for $\lambda \in \mathbf{R}$

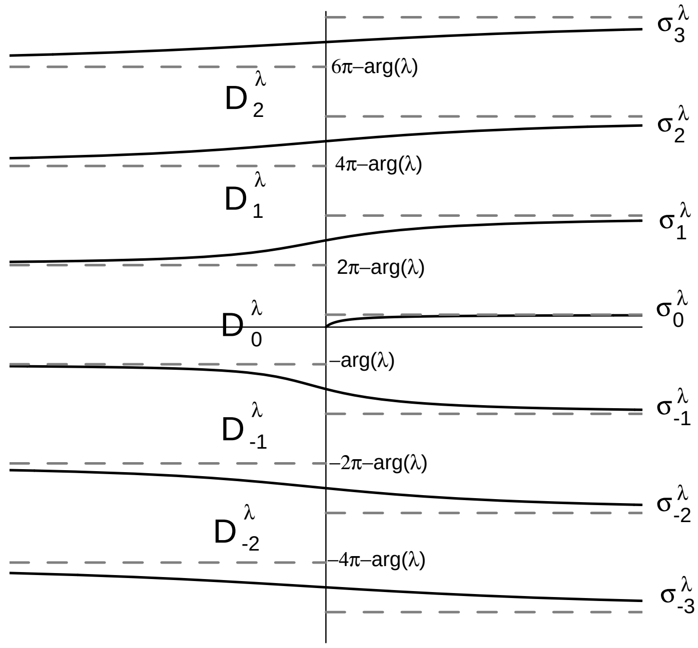

Figure 20: Dynamical plane: preimages of the negative real line and fundamental domains.

As their real parts tend to ∞, the σ_k^λ's are asymptotic to the lines $Imz = -\arg(\lambda) + (2k+1)\pi$. As their real part tends to $-\infty$ they are asymptotic to the lines $Imz = -\arg(\lambda) + 2(k+1)\pi$ when $k < 0$ or $Imz = -\arg(\lambda) + 2k\pi$ when $k > 0$. σ_0^λ is the preimage that contains $z = 0$ and therefore it connects this point with ∞.

If λ belongs to the lower half plane, the preimages of the negative real line are the complex conjugates of the curves we just described. For any λ in the lower half plane, we define

$$\sigma_k^\lambda = \overline{\sigma_{-k}^\lambda}.$$

For every λ, these preimages define fundamental domains. Denote by D_0^λ the strip between σ_1^λ and σ_{-1}^λ. This is a double fundamental domain in the sense that any point in $\mathbf{C}$ (except zero and the critical value), has two preimages in D_0^λ. Denote by D_k^λ the rest of the strips in the natural way (see Fig. 20). For k nonzero, each D_k^λ is a simple fundamental domain. This strip maybe divided in two parts, D_{k+}^λ and D_{k-}^λ separated by the preimage of the positive real axis. For λ in the upper half plane, this preimage is determined by the equation:

$$Im(z) = 2k\pi - \arg(z) - \arg(\lambda).$$

The semistrip D_{k+}^λ maps to the upper half plane by G_λ, while D_{k-}^λ maps to the lower half plane.

We now start the construction of the fixed hair. Consider the ray from zero to infinity that goes through the critical value $-\lambda/e$. On that ray, let μ_λ be the segment from $-\lambda/e$ to infinity. Let l be the preimage of μ_λ in D_0^λ. Since D_0^λ is a double fundamental domain, l consists of two continuous curves joined at the critical point $z = -1$ (see Fig. 21), precisely the two curves that map to $(-\infty, -1]$ by the function $G_1(z) = z\,e^z$. They do not depend on λ and they are represented by the equations:

$$Im(z) = \pm\pi - \arg(z). \tag{5}$$

Let L be the region bounded by l. L is a fundamental domain. It is totally contained in D_0^λ and it contains D_{0+}^λ which implies that the positive real line and the fixed point $z = 0$

are always in its interior. L is the region where the fixed hair will lie. For later purposes, we need the intersection of L with μ_λ to be empty. In order to have this property, we need the following proposition:

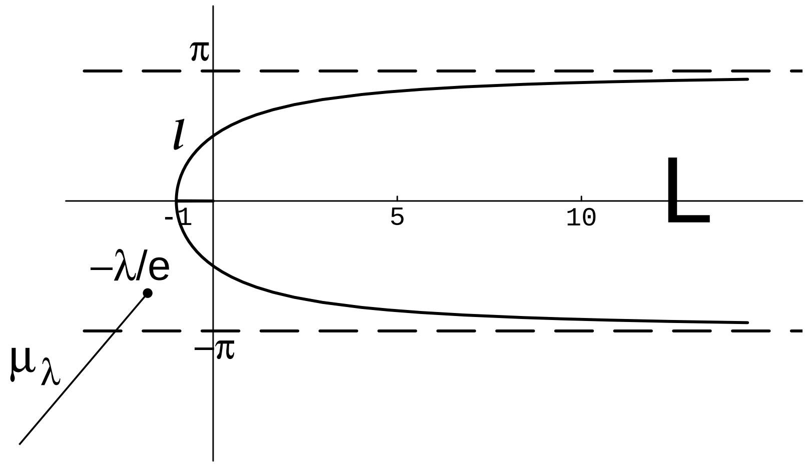

Figure 21: Dynamical plane: the set L.

Let Λ be the set of λ-values satisfying:

$$\Lambda = \{\lambda \in \mathbf{C} \mid |Im(\lambda)| \geq e \arg(\lambda)\}.$$

(See Fig. 22)

Proposition 4.1

1. *Λ contains the sets $\{\lambda \mid |Im(\lambda)| \geq \pi e\}$ and $\{\lambda \in \mathbf{R} \mid \lambda \geq e\}$.*

2. *If $\lambda \in \partial\Lambda$ and $Im(\lambda) > 0$, then the critical value $-\lambda/e$ lies on l.*

Proof : If we write $\arg(\lambda)$ as a parameter $\theta \in (-\pi, \pi)$, the boundary of Λ can be parametrized by:

$$x(\theta) = \frac{e\,\theta}{\tan(\theta)} \;\; , \;\; y(\theta) = e\,\theta.$$

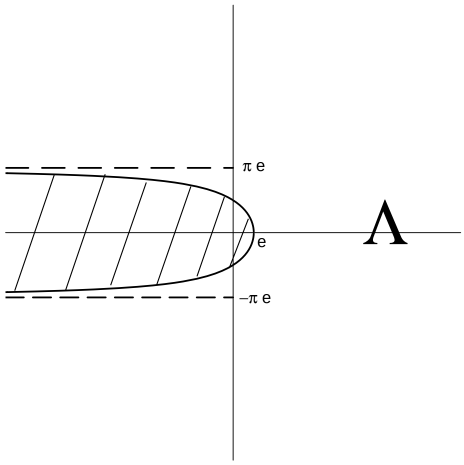

Figure 22: Parameter plane: the set Λ.

From these equations, it is clear that

$$\lim_{\theta \to 0} x(\theta) = e,$$

while

$$\lim_{\theta \to \pi} y(\theta) = \pi\, e$$

and

$$\lim_{\theta \to -\pi} y(\theta) = -\pi\, e$$

These limits, plus the fact that the imaginary part of $\partial\Lambda$ increases when $\arg(\lambda)$ increases, prove the first statement.

By symmetry, it is enough to prove the second statement for values of λ in the upper half plane. Hence we must show that if $\lambda \in \partial\Lambda$, then the critical value satisfies Eq. (5) for $-\pi$. By definition of Λ:

$$\begin{aligned} Im(-\lambda/e) &= \tfrac{-1}{e}\, e \arg(\lambda) \\ &= -\pi - (\arg(\lambda) - \pi) \\ &= -\pi - \arg(-\lambda/e). \end{aligned}$$

Hence, Eq. (5) is satisfied.

$\square$

Corollary 4.1 *For all $\lambda \in \Lambda$, $L \cap \mu_\lambda = \emptyset$.*

Proof : By construction, if $\lambda \in \Lambda$ then the critical value is always in the exterior of M and hence, the same holds for μ_λ.

$\square$

Corollary 4.2.1 assures us that the relative position of L and μ_λ is topologically as shown in Fig. 21. Therefore a branch of the inverse of G_λ taking values in the interior of L is well defined. That is,

$$G_\lambda^{-1} : \mathbf{C} - \mu_\lambda \longrightarrow \overset{\circ}{L} .$$

We now define a sequence of functions from parameter plane to dynamical plane. Fix $t \geq 0$ and let $H_n^t : \Lambda \longrightarrow \overset{\circ}{L}$ be

$$H_n^t(\lambda) = G_\lambda^{-n} \circ G_1^n(t).$$

For each n, $H_n^t(\lambda)$ is an analytic function. It iterates t n times forward by the function $G_1(t) = t\, e^t$, and afterwards, n times backwards by G_λ^{-1}. Since the positive real line is entirely inside L, $G_1^n(t)$ can never lie on μ_λ and hence these functions are well defined.

Lemma 4.2 *$\{H_n\}$ is a normal family on Λ. Hence, there exists a subsequence that converges uniformly to an analytic function H^t, which is not identically equal to infinity.*

Proof : Since l is included in D_0^λ, $\bigcup_{n=1}^\infty H_n(\Lambda)$ omits many more than three points on $\mathbf{C}$. Therefore by Montel's theorem, $\{H_n\}$ is a normal family on Λ. This gives the uniform convergence of subsequences to an analytic function H^t. We prove that H^t cannot be identically equal to ∞ by showing that it is finite for all the real values of λ in Λ. If $\lambda \in \Lambda \cap \mathbf{R}$, then $\lambda > e$ and therefore $\lambda = G'_\lambda(0) > G'_1(0) = 1$. This means that we always go back much faster by G_λ^{-1} than we go forward by G_1. Hence for all n and for all t,

$$0 \leq H_n^t(\lambda) < t. \qquad (6)$$

Consequently , no subsequence of $\{H_n\}$ can tend to ∞.

$\square$

Lemma 4.3 *If two subsequences of H_n^t converge uniformly, then they converge to the same limit.*

Proof : Fix $t > 0$ (we drop the superscripts for this proof) and suppose two subsequences tend uniformly to two different functions:

$$\{H_{n_i}\} \implies H^1$$

$$\{H_{m_i}\} \implies H^2.$$

We prove that $H^1 \equiv H^2$ by showing that the two functions agree on the real line. Since they are analytic functions of λ, this implies that they have to agree on all their domain of definition.

Fix $\lambda_0 \in \mathbf{R} \cap \Lambda$. Then,

1. $\{H_n(\lambda_0)\}$ is bounded because of inequality (6):

$$0 \leq H_n^t(\lambda_0) < t$$

   for all n.

2. $\{H_n(\lambda_0)\}$ is decreasing since

$$\begin{aligned} H_{n+1}(\lambda_0) &= G_{\lambda_0}^{-n-1} G_1^{n+1}(t) \\ &= G_{\lambda_0}^{-n} G_{\lambda_0}^{-1} G_1^{n+1}(t) \\ &< G_{\lambda_0}^{-n} G_1^{-1} G_1^{n+1}(t) \\ &= H_n(\lambda_0). \end{aligned}$$

Hence, $\{H_n(\lambda_0)\}$ is convergent. This implies that $\{H_n\}$ converges pointwise to a function $H(\lambda)$ for real values of λ. Therefore, H^1 and H^2 have to be equal to H for all $\lambda \in \mathbf{R}$.

$\square$

Proposition 4.2 *For all $t \geq 0$, the sequence $\{H_n^t\}$ converges uniformly on compact sets of Λ.*

Proof : Once we fix $t > 0$, we deal with a sequence $\{H_n\}$ of elements of a metric space such that every subsequence has a subsubsequence converging uniformly to a unique element H. We must show that all subsequences converge themselves uniformly to H. Suppose a subsequence $\{H_{n_i}\}$ does not converge uniformly (if it does, by lemma 4.3 it converges to H and we are done). Then, we can find $\varepsilon > 0$ and infinitely many H_{n_i}'s such that $|H_{n_i} - H| > \varepsilon$. These H_{n_i}'s form a subsequence which does not have any subsubsequence converging uniformly to H, which contradicts the assumption.

□

Now that we know that the sequence $\{H_n^t\}$ itself converges uniformly, we may define the analytic function

$$H^t(\lambda) = \lim_{n\to\infty} G_\lambda^{-n}(G_1^n(t))$$

which, if we fix λ, is a function from dynamical plane to dynamical plane:

$$H^\lambda(t) = H^t(\lambda) \quad , \quad t \in [0, \infty).$$

(Note that $H^0(\lambda) \equiv 0$.)

We will see that for every λ, the function $H^\lambda(t)$ parametrizes the fixed hair attached to $z = 0$.

Lemma 4.4 *The function $H^\lambda(t)$ conjugates G_1 to G_λ, i.e., the following diagram commutes:*

$$\begin{array}{ccc} \mathbf{R} & \xrightarrow{G_1} & \mathbf{R} \\ {\scriptstyle H^\lambda}\downarrow & & \downarrow{\scriptstyle H^\lambda} \\ \overset{\circ}{M}_\lambda & \xrightarrow{G_\lambda} & \overset{\circ}{M}_\lambda \end{array}$$

Proof : For all $n > 0$ we have:

$$\begin{aligned} H_n^\lambda(G_1(t)) &= G_\lambda^{-k}(G_1^{k+1}(t)) \\ &= G_\lambda(G_\lambda^{-k-1}(G_1^{k+1}(t)) \\ &= G_\lambda(H_{n+1}^\lambda(t)). \end{aligned}$$

We know

$$\{H_n^\lambda(G_1(t))\} \longrightarrow \{H^\lambda(G_1(t))\}$$

while

$$\{G_\lambda(H_{n+1}^\lambda(t))\} \longrightarrow \{G_\lambda(H^\lambda(t))\}.$$

Hence $H^\lambda(G_1(t)) = G_\lambda(H^\lambda(t))$ for all t.

$\square$

Theorem 4.3 *Given any $\lambda \in \Lambda$, the function $H^\lambda(t)$ is continuous with respect to t.*

To prove this result we need several lemmas.

Lemma 4.5 *For all $\lambda \in \Lambda$ and for all $z \in L$,*

$$|G_\lambda^{-1}(z)| \geq G_{|\lambda|}^{-1}(|z|).$$

Hence, for all $c \in \mathbf{R}^+$,

$$|z| \geq G_{|\lambda|}(c) \Longrightarrow |G_\lambda^{-1}(z)| \geq c.$$

Proof : We know

$$|G_\lambda(z)| = |\lambda||z|e^{Re(z)}.$$

Since $Re(z) \leq |z|$, we have:

$$|G_\lambda(z)| \leq |\lambda||z|e^{|z|} = G_{|\lambda|}(|z|).$$

Applying this inequality to $G_\lambda^{-1}(z)$ we get,

$$|z| \leq G_{|\lambda|}(|G_\lambda^{-1}(z)|)$$

which, since $G_{|\lambda|}(|z|)$ is an increasing function on $|z|$, implies:

$$G_{|\lambda|}^{-1}(|z|) \leq |G_\lambda^{-1}(z)|.$$

For the same reason,

$$\begin{aligned} & |z| \geq G_{|\lambda|}(c) \\ \Longrightarrow\ & G_{|\lambda|}^{-1}(|z|) \geq c \\ \Longrightarrow\ & |G_\lambda^{-1}(z)| \geq c. \end{aligned}$$

$\square$

Lemma 4.6 *For all $\lambda \in \Lambda$ and for all $z \in L$ such that $Re(z) > 0$ and $Re(G_\lambda^{-1}(z)) \geq 0$,*

$$Re(G_\lambda^{-1}(z)) \leq G_{|\lambda|}^{-1}(Re(z) + 2\pi).$$

Hence, for all $c \in \mathbf{R}^+$ and for all $z \in M_\lambda$ such that $Re(z) > 0$,

$$Re(z) + 2\pi \leq G_{|\lambda|}(c) \Longrightarrow Re(G_\lambda^{-1}(z) \leq c.$$

Proof : Let $\omega = G_\lambda^{-1}(z)$. Since L is bounded by the horizontal lines $Im(z) = \pm\pi$, the imaginary part of any point in L is bounded by π. That is,

$$Re(G_\lambda(\omega)) \geq |G_\lambda(\omega)| - \pi$$

and

$$|\omega| \geq Re(\omega).$$

because of the assumptions on ω and $z = G_\lambda(\omega)$. Hence,

$$\begin{aligned} Re(G_\lambda(\omega)) + \pi &\geq |G_\lambda(\omega)| \\ &= |\lambda||\omega|e^{Re(\omega)} \\ &\geq |\lambda| Re(\omega) e^{Re(\omega)} \\ &= G_{|\lambda|}(Re(\omega)). \end{aligned}$$

Going back to z,

$$Re(z) + \pi \geq G_{|\lambda|}(Re(G_\lambda^{-1}(z)))$$

which, since $G_{|\lambda|}(Re(z))$ is an increasing function on $Re(z)$ says:

$$G_{|\lambda|}^{-1}(Re(z) + \pi) \geq Re(G_\lambda^{-1}(z)).$$

Now assume $Re(z) > 0$ and $Re(z) + \pi \leq G_{|\lambda|}(c)$. If $Re(G_\lambda^{-1}(z) \leq 0$ it is always less than c and we are done. If not,

$$\begin{aligned}
& Re(z) + \pi \leq G_{|\lambda|}(c) \\
\Longrightarrow\ & G_{|\lambda|}^{-1}(Re(z) + \pi) \leq c \\
\Longrightarrow\ & Re(G_\lambda^{-1}(z)) \leq c.
\end{aligned}$$

$\square$

Proposition 4.3 *Let $M, N \in \mathbf{R}$ be such that $e^M \geq |\lambda|$ and $e^N \geq (1 + \frac{\pi}{N})/|\lambda|$. Then, if $t > 0$, for all $n > 0$,*

1. *$|H_n^\lambda(t)| \geq t - M$ and*
2. *$Re(H_n^\lambda(t)) \leq t + N$.*
3. *if $Re(H^\lambda(t)) > 0$,*
$$t - M - \pi \leq Re(H^\lambda(t)) \leq t + N.$$
4. *if $Re(H^\lambda(t)) > 0$,*
$$t - M \leq |H^\lambda(t)| \leq t + N + \pi.$$

Proof : We prove the first two statements by induction on n.

1. For $n = 1$ we want to show:
$$|G_\lambda^{-1}(G_1(t))| \geq t - M.$$
By lemma 4.5, setting $c = t - M$, this is implied by:
$$|G_1(t)| \geq G_{|\lambda|}(t - M)$$
or,
$$te^t \geq |\lambda|(t - M)e^t e^{-M}.$$

This inequality is satisfied if

$$e^M \geq |\lambda|(1 - \frac{M}{t})$$

which, since $M > 0$, is true by assumption.

Assume it is true for n. We must show:

$$|H_{n+1}^{\lambda}(t)| \geq t - M,$$

or equivalently, using lemma 4.4:

$$|G_{\lambda}^{-1}(H_n^{\lambda}(G_1(t)))| \geq t - M.$$

Applying again lemma 4.5, this is implied by:

$$|H_n^{\lambda}(G_1(t))| \geq G_{|\lambda|}(t - M).$$

By the induction hypothesis for $G_1(t)$, it reduces to show:

$$G_1(t) - M \geq G_{|\lambda|}(t - M)$$

or,

$$te^t - M \geq |\lambda|(t - M)e^t e^{-M}.$$

Dividing both sides by $te^t > 0$, the inequality is equivalent to:

$$e^M(1 - \frac{M}{te^t}) \geq |\lambda|(1 - \frac{M}{t}).$$

Since $e^t > 1$ we have $(1 - \frac{M}{te^t}) \geq (1 - \frac{M}{t})$, and hence, the inequality is satisfied if $e^M \geq |\lambda|$ which is true by assumption.

2. For $n = 1$ we must show:

$$Re(G_{\lambda}^{-1}G_1(t)) \leq t + N.$$

If the left hand side is negative or 0 then we are done. If not, we can apply lemma 4.6 setting $c = t + N$, and then the inequality is equivalent to :

$$Re(G_1(t) \leq G_{|\lambda|}(t + N) - \pi$$

i.e.

$$te^t \leq |\lambda|(t+N)e^te^N - \pi,$$

or,

$$\frac{te^t+\pi}{te^t+Ne^t} \leq |\lambda|e^N.$$

Since $t, N > 0$, we have:

$$\begin{aligned} \frac{te^t+\pi}{te^t+Ne^t} &\leq \frac{te^t+\pi}{te^t+N} \\ &\leq 1+\frac{\pi}{N}. \end{aligned}$$

Therefore the inequality is satisfied if $e^N \geq (1+\frac{\pi}{N})/|\lambda|$, which is true by assumption.

Assume it is true for n. For $n+1$ we must prove:

$$Re(G_\lambda^{-1}(H_n^\lambda(G_1(t)))) \leq t+N.$$

As before, we use lemma 4.6 to reduce it to:

$$Re(H_n^\lambda(G_1(t))) \leq G_|\lambda|(t+N) - \pi,$$

and the induction hypothesis to get

$$te^t + N \leq |\lambda|(t+N)e^te^N - \pi.$$

This is implied by

$$\frac{te^t+\pi+N}{te^t+Ne^t} \leq |\lambda|e^N.$$

Since $t, N > 0$ we have:

$$\begin{aligned} \frac{te^t+\pi+N}{te^t+Ne^t} &\leq \frac{te^t+\pi+N}{te^t+N} \\ &\leq \frac{te^t+N}{te^t+N} + \frac{\pi}{te^t+N} \\ &\leq 1+\frac{\pi}{N}. \end{aligned}$$

Hence, by assumption the result follows.

3. The right hand inequality comes from taking the limit in statement 2. Taking the limit on statement 1, we have

$$|H^{\lambda}(t)| \geq t - M.$$

Since $H^{\lambda}(t) \in L$, the imaginary part is bounded by π. Therefore, since its real part is positive,

$$Re(H^{\lambda}(t)) \geq |H^{\lambda}(t)| - \pi.$$

Combining both inequalities, the result follows.

4. The left hand comes from taking the limit in statement 1. For the right hand, we have

$$|H^{\lambda}(t)| \leq Re(H^{\lambda}(t)) + \pi \leq t + N + \pi$$

by statement 3.

□

Lemma 4.7 *For all $\lambda \in \Lambda$ and for all $z, \omega \in L$ such that for $i = 1, 2 \ldots, n$ $Re(G_{\lambda}^{-i}(z)) > 0$ and $Re(G_{\lambda}^{-i}(\omega)) > 0$,*

$$|G_{\lambda}^{-n}(z) - G_{\lambda}^{-n}(\omega)| < |z - \omega|/|\lambda|^{n}.$$

Proof : In general, for any $z \in L$ with positive real part, we have

$$|G'_{\lambda}(z)| = |\lambda| e^{Re(z)} |1 + z| \geq |\lambda|$$

and by the inverse function theorem, this implies

$$|(G_{\lambda}^{-1})'(z)| < \frac{1}{|\lambda|}.$$

If $Re(G_{\lambda}^{-i}(z)) > 0$ for $i = 1, 2 \ldots, n$, we have

$$|(G_{\lambda}^{-n})'(z)| < (\frac{1}{|\lambda|})^{n}$$

and therefore, using the mean value theorem :

$$|G_\lambda^{-n}(z) - G_\lambda^{-n}(\omega)| < (\frac{1}{|\lambda|})^n |z - \omega|.$$

$\square$

We will now put together all these estimates to prove that $H_\lambda(t)$ is continuous with respect to t.

Proof of theorem 4.3: We first assume $t_0 > M = \log(|\lambda|) + \pi$. Fix $\varepsilon > 0$ and pick $n > 0$ such that $(1/|\lambda|)^n(3M + 2\pi) < \varepsilon$. Using continuity of G_1^n, choose $\delta > 0$ such that if $|t_0 - t| < \delta$ then $|G_1^n(t_0) - G_1^n(t)| < M$. By proposition 4.3,

$$G_1^n(t_0) - M \leq Re(H^\lambda(G_1^n(t_0))) \leq G_1^n(t_0) + M \ \ and$$

$$G_1^n(t) - M \leq Re(H^\lambda(G_1^n(t))) \leq G_1^n(t) + M.$$

Subtracting these two inequalities,

$$|Re(H^\lambda(G_1^n(t_0))) - Re(H^\lambda(G_1^n(t)))| \leq 3M$$

and since the imaginary parts are bounded,

$$|H^\lambda(G_1^n(t_0)) - H^\lambda(G_1^n(t))| \leq 3M + 2\pi.$$

Using the conjugacy in lemma 4.4 and lemma 4.7, we have

$$\begin{aligned} |H^\lambda(t_0) - H^\lambda(t)| &= |G_\lambda^{-n} H^\lambda G_1^n(t_0) - G_\lambda^{-n} H^\lambda G_1^n(t)| \\ &\leq (1/|\lambda|)^n(3M + 2\pi) \\ &< \varepsilon \end{aligned}$$

which proves continuity at any $t_0 > M$. Note that the hypotheses of lemma 4.7 are satisfied by the results on proposition 4.3.

Now suppose $t_0 < M$. Pick $n > 0$ large enough so that $G_1^n(t_0) > M$. By continuity of G_λ^{-n}, there is a δ' s.t.for any $z, \omega \in M_\lambda$

$$|z - \omega| < \delta' \Longrightarrow |G_\lambda^{-n}(z) - G_\lambda^{-n}(\omega)| < \varepsilon.$$

By continuity of H^λ at $G_1^n(t_0)$, we can find $\delta'' > 0$ such that

$$|T - G_1^n(t_0)| < \delta'' \Longrightarrow |H^\lambda(T) - H^\lambda(G_1^n(t_0))| < \delta'.$$

Now choose $\delta > 0$ so that if $|t - t_0| < \delta$ then $|G_1^n(t_0) - G_1^n(t)| < \delta''$. Then,

$$|H^\lambda(G_1^n(t)) - H^\lambda(G_1^n(t_0))| < \delta'$$

and hence,

$$|G_\lambda^{-n}(H^\lambda(G_1^n(t))) - G_\lambda^{-n}(H^\lambda(G_1^n(t_0)))| < \varepsilon.$$

Using the conjugacy in lemma 4.4, this says

$$|H^\lambda(t) - H^\lambda(t_0)| < \varepsilon.$$

□

Remarks :

1. Because of the estimates in proposition 4.3, $H^\lambda(t)$ escapes to infinity under iteration of G_λ for all $t \in (0, \infty)$. Therefore, $H^\lambda(t) \in J(G_\lambda)$ for all $\lambda \in \Lambda$. We call this continuous curve the fixed hair attached to $z = 0$.

2. Recall that this construction works only for $\lambda \in \Lambda$. However, the same construction would prove that, independently of λ, the tail of this fixed hair is always present. We just have to pick $t > T(\lambda)$, $T(\lambda)$ large enough, in order to assure that we are far from the critical value. From there, the same estimates hold and the inverses are all well defined. One can show that, for values of λ such that the critical value lies inside D_{0+}^λ, this fixed hair always attaches to the repelling fixed point $z = -\log(\lambda)$. This region of parameter values which we denote by Λ' is depicted in Fig. 23. For values of λ neither in λ nor in Λ', we conjecture that the fixed hair must always attach either to $z = 0$ or to $z = -\log(\lambda)$, except for those λ's for which the critical values lies exactly on the fixed hair, and hence, the Julia set of G_λ is the whole plane.

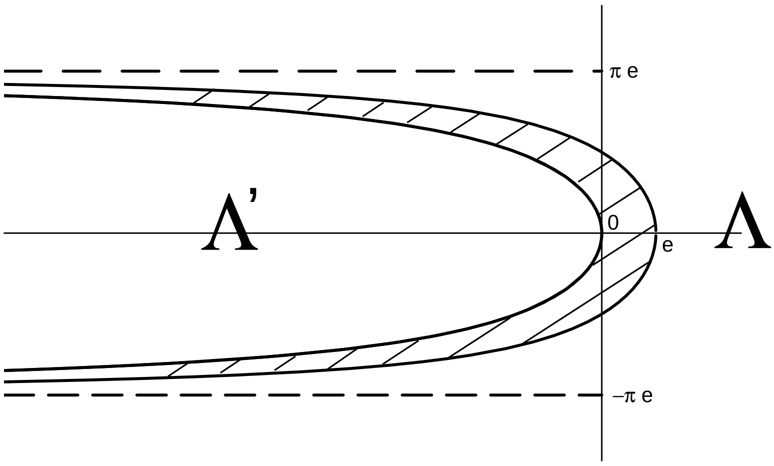

Figure 23: Parameter plane: the set Λ'.

3. The fixed hair is not the only one in dynamical plane. As in other entire functions (see Devaney [1991],Devaney & Krych [1984],Devaney & Tangerman [1986]), these types of curves come attached to any repelling periodic cycle, as well as to other points with bounded itinerary but belonging to the Julia set.

The preimages by G_λ of the fixed hair we just constructed will be important in our construction of some hairs in the parameter plane. These preimages consist of a set of continuous curves which we denote as follows: (see Fig. (24))

Definition For each $\lambda \in \Lambda$, $t > M = \log(|\lambda|) + \pi$, $k \in \mathbf{Z}$, we define $\gamma_k^\lambda(t)$ as the preimage of $H^\lambda(t)$ that belongs to D_k^λ. We define γ_k^λ as the continuous preimage of H^λ that contains the points $\gamma_k^\lambda(t)$ for $t > M$. This defines $\gamma_k^\lambda(t)$ for $t \leq M$ as the preimage of $H^\lambda(t)$ that belongs to γ_k^λ.

The fact that all the γ_k^λ's are preimages of the same line, make the regions bounded by them fundamental domains. For each $k = 1, 2, \ldots$, we denote by S_k^λ the fundamental domain bounded by the curves γ_k^λ and γ_{k+1}^λ (γ_k^λ and γ_{k-1}^λ if $k < 0$). The region S_0, bounded by γ_{-1}^λ and γ_1^λ is a double fundamental domain.

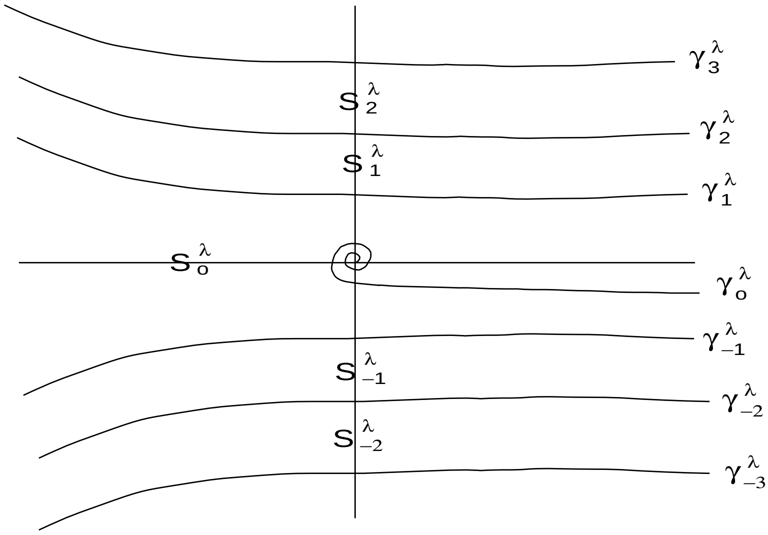

Figure 24: Dynamical plane: sketch of the fixed hair and its preimages.

4.2.2 Parameter Plane

Our goal in this section is to prove that there are continuous curves of parameter values, for which the critical value escapes to infinity under iteration of G_λ, in a particular fashion. Hence, the Julia set for these parameter values is the whole plane. More precisely:

Theorem 4.4 *For any $k \neq 0$, the set of parameter values*

$$\Gamma_k = \{\lambda \in \Lambda \mid \frac{-\lambda}{e} \in \gamma^\lambda_{-k}\}$$

form a continuous curve in parameter plane.

By symmetry of G_λ with respect to the real axis, it is enough to consider parameter values in the upper half plane. Therefore, from now on we fix $k > 0$ and we will work on the region

$$\Lambda^+ = \{\lambda \in \Lambda \mid Im(\lambda) \geq 0\}.$$

Without loss of generality we reduce to the case $k = 1$. The other cases can be proved with totally analogous arguments.

Fix $t > 0$. For these parameter values we define the following functions:

$$\begin{array}{rccc} P^t : & \Lambda^+ & \longrightarrow & \Lambda^+ \\ & \lambda & \longmapsto & -e\,\gamma_{-1}^{\lambda}(t) \end{array}$$

We are interested in finding fixed points of P^t, since those are λ's for which $-\lambda/e$ belongs to γ_{-1}^{λ}.

Proposition 4.4 *P^t maps Λ^+ into its interior. Hence, for each $t > 0$ there is a unique fixed point p_t (which may be infinity).*

Proof : Recall we defined the boundary of Λ^+ as the curve of parameters for which the critical value belongs to the boundary of L, together with the real line. Since the preimage γ_{-1}^{λ} is always below the preimage of $\mathbf{R}^+$ contained in D_{-1}^{λ}, we have:

$$Im(\gamma_{-1}^{\lambda}(t)) \leq -2\pi - \arg(\gamma_{-k}^{\lambda}(t)) - \arg(\lambda)$$

for all $t > 0$. Therefore,

$$\begin{aligned} Im(P^t(\lambda)) &\geq 2e\pi + e\arg(\gamma_{-1}^{\lambda}(t)) + e\arg(\lambda) \\ &> 2e\pi + e(-\pi) + e\arg(\lambda) \\ &= e\pi + e\arg(\lambda), \end{aligned}$$

which by definition implies that $P^t(\lambda)$ belongs to the interior of Λ^+ (see Fig. 25).

Now consider P^t as a map from $\Lambda^+ \cup \{\infty\}$ to itself. Topologically, this is a map from the closed disk to itself which, by the Brouwer fixed point theorem, must have a fixed point p_t. If p_t is finite, consider P^t as a map from the interior of Λ^+ to itself. By a Möbius transformation that sends the interior of Λ^+ to the open disk and the p_t to 0, it can be considered as a map from the open disk to itself that fixes 0. Applying Schwarz lemma, p_t is a global attractor for P^t and therefore it must be unique.

$\square$

Proposition 4.5 *There exists $T^* > 0$ such that for $t > T^*$, the fixed point p_t is finite.*

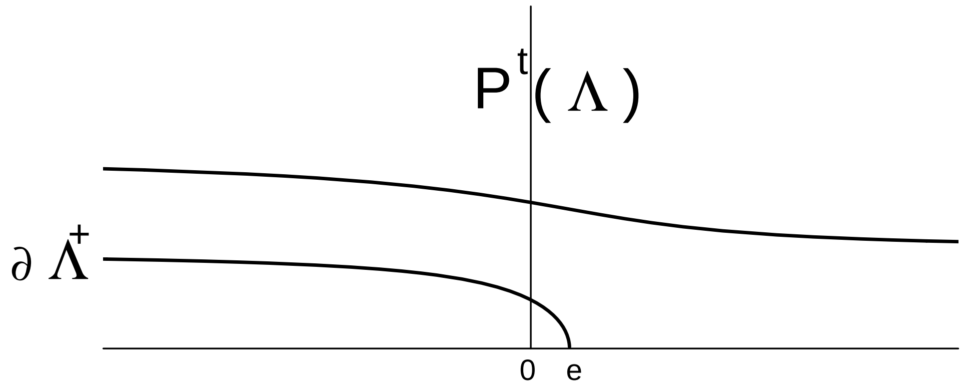

Figure 25: Parameter plane: image of Λ^+ by P^t.

Proof : We will show that for values of t larger than a certain T^*, the function P maps a bounded region in Λ totally inside itself. That will give as a finite fixed point for P which must coincide with the unique p_t. The following lemmas will make this construction precise.

Lemma 4.8 *Given $t > \pi$, there exists $M_1 > 0$ s.t. if $|\lambda| < M_1$ then $Re(H^t(\lambda)) > 0$. Hence $H^t(\lambda) \notin \mathbf{R}^-$ for any $|\lambda| < M_1$.*

Proof : By proposition 4.3 we know :

$$Re(H^t(\lambda)) \geq t - \log(|\lambda|) - \pi.$$

Hence, as long as $|\lambda| < e^{t-\pi} \equiv M_1$ we have that the real part of $H^t(\lambda)$ must be positive.

□

Let Λ_1 be the set of λ's inside $Lambda^+$ with modulus less than M_1. Then, for those $\lambda \in \Lambda_1$, the fixed hair at t has not touched the negative real line yet. This means that its preimage $\gamma^\lambda_{-1}(t)$ must still be inside the fundamental domain D^λ_{-1}. Hence the imaginary part of $P^t(\lambda)$ is bounded above and below (see Fig. 26). Note that this constant M_1 increases with t while these bounds do not. We want to take t big enough so that the circle with radius M_1 covers the strip as shown in the figure .

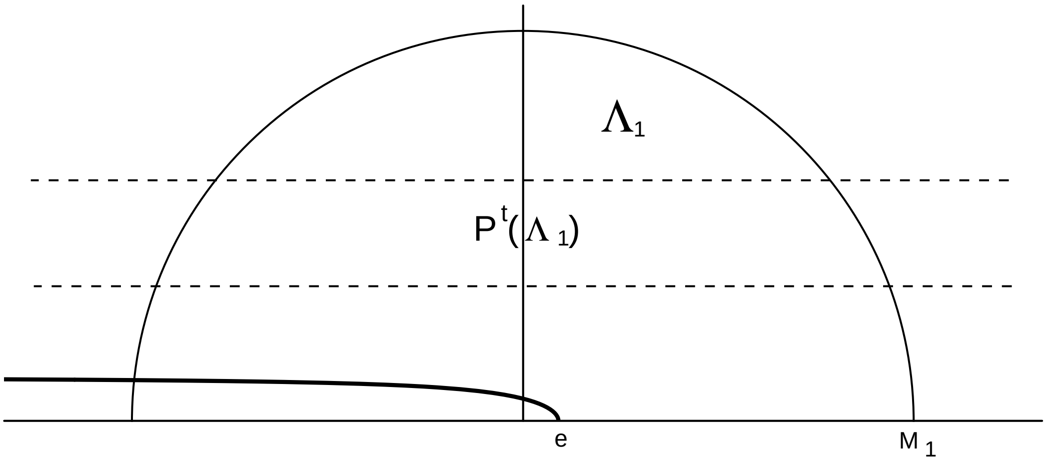

Figure 26: Parameter plane: image of Λ_1 by P.

Lemma 4.9 *There exists $M_2 < M_1$ such that if $Re(P^t(\lambda)) \leq 0$ then $|P^t(\lambda)| < M_2$.*

Proof : We defined the function P^t as $P^t(\lambda) = -e\gamma_{-1}(t)$. This implies that, if we apply G_λ to both sides:

$$G_\lambda(-P^t(\lambda)/e) = H^t(\lambda).$$

Therefore, by proposition 4.3, the inequality

$$|G_\lambda(-P^t(\lambda)/e)| \leq t + N + \pi$$

must be satisfied, where N is defined as in the mentioned proposition. Since $\lambda \in \Lambda$ implies $|\lambda| > e$, and by assumption , the real part of $P(\lambda)$ is negative,

$$|G_\lambda(-P^t(\lambda)/e)) = (1/e)|\lambda||P^t(\lambda)|e^{Re(\frac{-P^t(\lambda)}{e})} \geq |P^t(\lambda)|.$$

Hence, a necessary condition is

$$|P^t(\lambda)| \leq t + 1 + \pi \equiv M_2$$

for which we have used that N must always be smaller than 1. If t is big enough ($t > 5.5$), then $M_2 < M_1$.

□

The lemma above works for all λ. However, in particular it tells us that the image of Λ_1 by P must be bounded also in its left side. So, in order to show that $P^t(\Lambda_1)$ is totally contained inside Λ_1, it remains to find a bound for its right side.

Lemma 4.10 *There exists $M_3 < M_1$ such that if $|\lambda| < M_1$ then $Re(P^t(\lambda)) < M_3$.*

Proof : As in the proof above, by proposition 4.3 we must have:

$$|G_\lambda(-P^t(\lambda)/e)| \geq t - \log(|\lambda|) \leq \pi$$

where, for the last inequality we have imposed $|\lambda| < M_1 = e^{\pi - t}$. By lemma 4.8, this condition also implies that the imaginary part of $P^t(\lambda)$ is bounded by some constant c. Hence

$$|P^t(\lambda)| \leq Re(P^t(\lambda)) + c$$

and therefore

$$|G_\lambda(-P^t(\lambda)/e)| \leq (1/e)e^{t-\pi}(Re(P^t(\lambda)) + c) \;\; e^{Re(\frac{-P^t(\lambda)}{e})}.$$

The condition then becomes

$$\frac{Re(P^t(\lambda)) + c}{e} \, e^{Re(\frac{-P^t(\lambda)}{e})} \geq \pi e^{\pi - t}.$$

The left hand side is a function that decreases with $Re(P^t(\lambda)$, tending to zero as $Re(P^t(\lambda)$ tends to infinity. Hence, there exists M_3, such that the condition is satisfied only when $Re(P^t(\lambda) \leq M_3$. One can check that for t large enough , M_3 is always smaller than M_1.

□

Consequently, for any t larger than a certain T^* (the maximum from the three lemmas above), the function P^t maps the set Λ_1 (which gets larger with t) into itself. By the Brouwer fixed point theorem, Λ_1 must contain a fixed point. By the Schwarz lemma, if Λ has a finite fixed point, this must be unique. Hence this is the fixed point p_t.

□

Theorem 4.5 *For all $t > T^*$, the function $\Gamma_1 : t \longmapsto p_t$ defines a continuous curve of parameter values. Moreover, $\Gamma_1(t)$ has bounded imaginary part and its real part tends to $-\infty$ when t tends to ∞.*

Proof : The maps P^t form a continuous family of analytic functions from $\mathbf{C}$ to $\mathbf{C}$. The fixed points p_t correspond to the intersection of the graph of P^t with the diagonal of $\mathbf{C^2}$. Since for all $t \geq T^*$, the p_t's are global attractors , these intersections are at least topologically transverse and hence, the fixed points must vary continuously with t.

By lemma 4.8, we know that the imaginary part of $\Gamma_1(t)$ is bounded. By proposition 4.3 , it is necessary that

$$|\lambda \frac{-\Gamma_1(t)}{e} e^{\frac{-\Gamma_1(t)}{e}}| \geq t - \log(|\Gamma_1(t)|).$$

In order for $\Gamma_1(t)$ to satisfy this condition, its real part must tend to $-\infty$ as t increases.

$\square$

It is clear that we can substitute T^* by any value T_1 for which p_t is finite for $t > T_1$. This extends the curve of fixed points $\Gamma_1(t)$ to some interval (T_1, ∞) where $T_1 \geq 0$ and the limit as t tends to T_1 is infinity (because we know it cannot tend to the finite boundary of Λ).

The same construction also applies for any $k \neq 0$. Thus we have a family $\{\Gamma_k : (T_k, \infty) \longrightarrow \Lambda\}$ of continuous curves in parameter plane (see Fig. 27) which satisfies that if $\lambda = \Gamma_k(t)$ for some k and some t, then the critical value for G_λ escapes to infinity. Hence $J(G_\lambda) = \mathbf{C}$.

Although it is not proven at this point, we expect T_k to be equal to zero for all k as well as the imaginary part of $\Gamma_k(t)$ to be bounded for all t.

Remarks :

1. The hairs $\Gamma_k(t)$ correspond to λ-values for which the critical value lands on the fixed hair $H^\lambda(t)$ after one iteration. We conjecture that there exist other hairs in Λ that correspond to critical values landing on the fixed hair after more than one iterations.

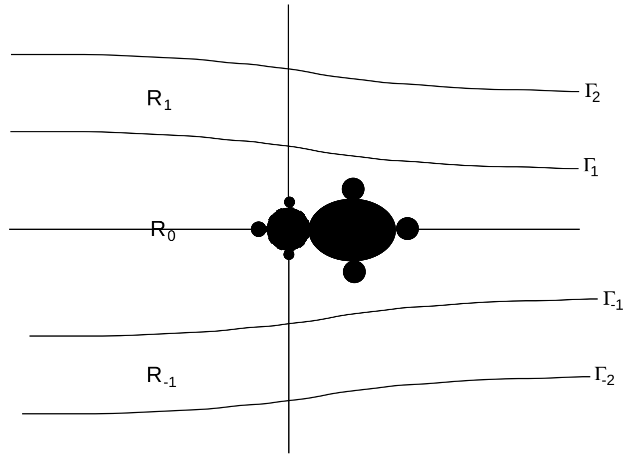

Figure 27: Parameter plane: sketch of the hairs $\Gamma_k(t)$.

The itinerary of the orbit of the critical value before it lands on the fixed hair, determines the position of the hair in parameter space.

2. We conjecture the existence of a different type of hair in parameter plane. This curve would be composed of λ-values for which the critical value itself lies on the fixed hair. This curve does not live in Λ nor in Λ' (see remarks in the section above). In fact it should separate the region of λ's for which the fixed hair attaches to $z = 0$, from the one for which it attaches to $z = -\log(\lambda)$. For real values of λ, it is easy to see that this boundary corresponds exactly to $\lambda = 1$. That is, for $0 < \lambda < 1$ the fixed hair coincides with $[-\log(\lambda), \infty)$, while for $\lambda > 1$ it coincides with $[0, \infty)$.

4.3 Baby Mandelbrot Sets

When we randomly magnify small regions in Fig. 8 we are very likely to see a picture like the one shown in Fig. 28. The appearance of small copies of the Mandelbrot set in the parameter plane is something that never occurs in the case of the exponential family.

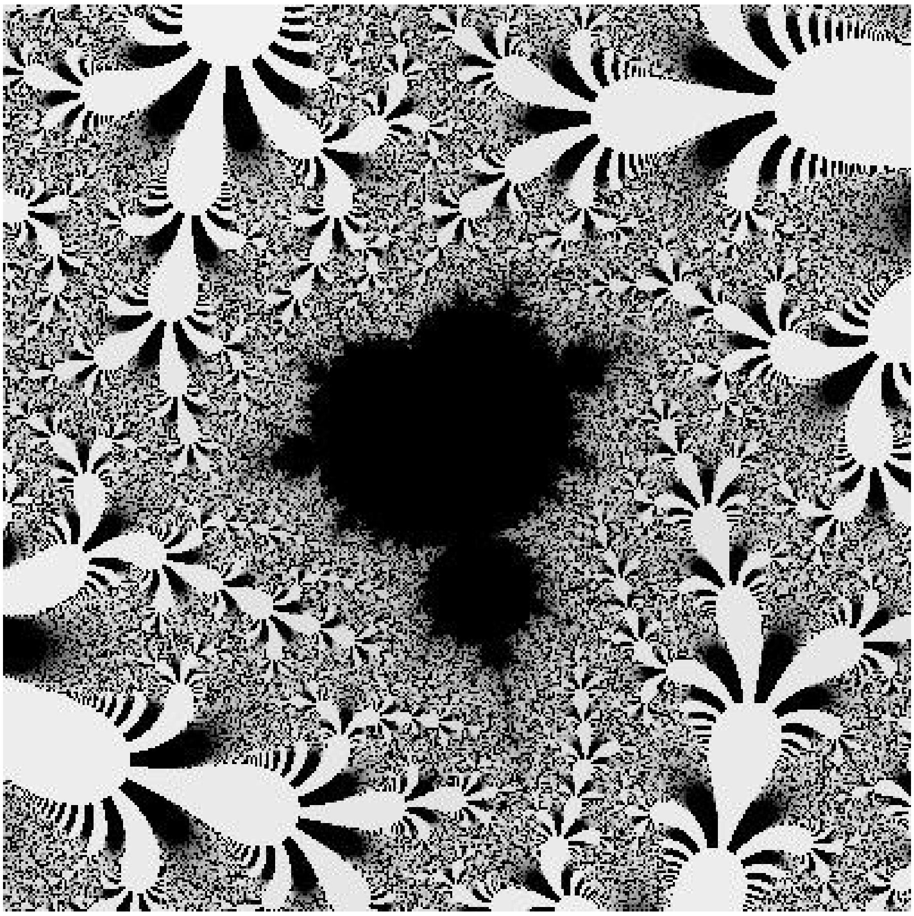

Figure 28: Parameter plane: magnification of small region of Fig. 7.

In this section, we will prove the existence of infinitely Mandelbrot-like sets, distributed according to some symbolic dynamics. To do so, we will use the theory of polynomial-like maps of Douady & Hubbard [1985], plus the construction of the hairs in the section above.

We start by defining the itinerary of a point under the function G_λ.

Definition Fix $\lambda \in \mathbf{C}$. For each $z \in \mathbf{C}$ we define the itinerary of z as the infinite sequence $s_\lambda(z) = (s_0, s_1, \ldots)$, where

$$s_i = k \quad \longleftrightarrow \quad G_\lambda^i(z) \in S_k^\lambda.$$

For each $k > 0$, let R_k denote the region in parameter plane bounded by the hairs Γ_k and Γ_{k+1} (Γ_k and Γ_{k-1} if $k < 0$; Γ_1 and Γ_{-1} if $k = 0$) (see Fig. 27).

We have the following theorem:

Theorem 4.6 *For each $k \neq 0$, there exists a subset of R_k, $\mathcal{M}_k$ homeomorphic to the Mandelbrot set. Its main cardioid corresponds to λ-values for which there is an attracting cycle of period 2 with itinerary $\underline{s} = (\overline{0, -k}, \ldots)$.*

This theorem will follow from applying the polynomial-like mapping theory of Douady and Hubbard. Douady & Hubbard [1985] prove the following major result:

Theorem 4.7 *Suppose f_λ is an analytic family of analytic functions. Let D denote a set homeomorphic to the closed disk. Assume that for each $\lambda \in D$ the following conditions are satisfied:*

1. *There exist open disks U_λ, U'_λ depending continuously on λ, U'_λ relatively compact in U_λ, such that $f_\lambda : U'_\lambda \longrightarrow U_\lambda$ is proper of degree two. Any map with this property is said to be a polynomial-like map of degree 2.*

2. *There is a unique critical point z_λ in U_λ and as λ turns around ∂D once, $f_\lambda(z_\lambda) - z_\lambda$ turns around 0 once. A family of maps with this property is said to have parametric degree one.*

Then, there is a subset $\mathcal{M} \subset D$ which is homeomorphic to the standard Mandelbrot set via a map $c = c(\lambda)$. The set $\mathcal{M}$ is given by $\{\lambda \in D \mid f_\lambda^n(z_\lambda) \in U'_\lambda$ forall $n \geq 0\}$. Moreover, for each $\lambda \in \mathcal{M}$, the map $f_\lambda|_{U'_\lambda}$ is topologically conjugate to $z \mapsto z^2 + c(\lambda)$ on the filled in Julia sets of each, where the filled in Julia set of $f_\lambda|_{U'_\lambda}$ is defined as the set of points that do not leave U'_λ under iteration.

Proof of theorem 4.6: For simplicity of notation we will prove only the case $k = 1$, dropping all the subscripts that indicate dependence on k. It will be clear though, that the proof is the same for all the k-values.

Consider the set D of parameter values defined by

$$D = \{\lambda \in R_1 \mid Re(\lambda) > R\},$$

where R is a negative number to be defined. Clearly, D is homeomorphic to a disk. By construction of the hairs, for each $\lambda \in D$, the critical value lies in the fundamental domain S_{-1}^λ in dynamical plane. We denote by $\mathcal{O}_\lambda$ the set of points in S_0^λ that map to S_{-1}^λ under one iteration of G_λ (see Fig. 29). To determine the topological shape of this set consider the preimages of γ_{-1}^λ and γ_{-2}^λ inside S_0^λ. Since S_0^λ is a double fundamental domain, it contains two preimages of each of the mentioned curves. Given the position of the critical value, one computes easily that the preimages of γ_{-1}^λ consist of two "C-shaped" curves, each intersecting the negative real line in a single point. Also, the preimages of γ_{-2}^λ are two C-shaped curves, in the upper and lower half planes respectively. To check this, one may compute the images of vertical lines in S_0^λ and note their intersections with S_{-1}^λ.

Therefore, $\mathcal{O}_\lambda$ is a set homeomorphic to a disk that maps onto S_{-1}^λ with degree 2 under G_λ, and hence, to the whole plane with degree 2 under G_λ^2. Note that the fixed hair γ_0^λ never intersects the boundary of $\mathcal{O}_\lambda$.

We define the open sets

$$U'_\lambda = \{z \in \mathcal{O}_\lambda \mid Re(z) < R'\}$$

where R' is large enough so that the image of U'_λ by G_λ^2 covers itself. To see how that there is such an R' consider the image under G_λ of U'_λ. This image consists of the strip S_{-1}^λ cut

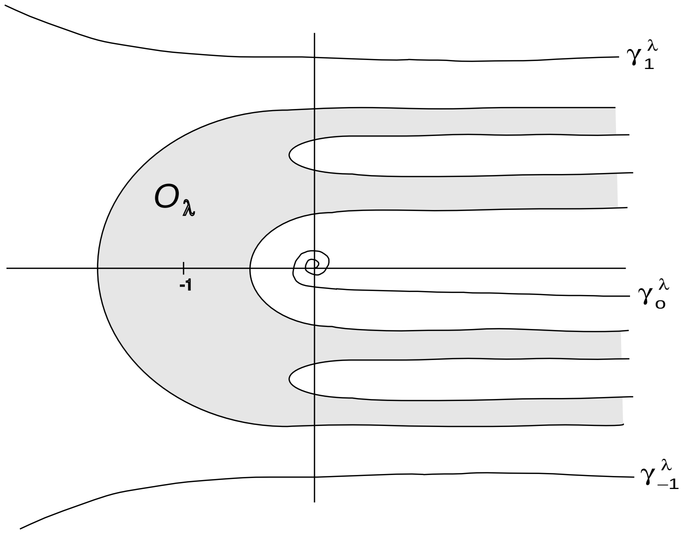

Figure 29: Dynamical plane: sketch of the set $\mathcal{O}_\lambda$.

right and left by the image of the vertical segment $\{Re(z) = R'\} \cup S_0^\lambda$ in S_{-1}^λ. By taking R' large enough, the image of the right bound is a huge disk that includes U'_λ, and the image of the left bound is a small neighborhood of $z = 0$ which does not intersect U'_λ. The image of the upper and lower bounds are, by construction, the fixed hair (see Fig. 30). Hence, if we define U_λ as the image of U'_λ by G_λ^2 , then U'_λ is relatively compact in U_λ and $G_\lambda^2 |_{U'_\lambda}$ is proper of degree 2. Therefore we have verified the first hypothesis of theorem 4.7.

The second hypothesis is also satisfied. As λ comes from infinity along the upper boundary of D, the critical value is on the fixed hair coming from zero. Taking R large enough, the critical value goes around a large circle when λ goes along the left boundary of D. The lower boundary corresponds again to $-\lambda/e$ on the fixed hair, this time returning to 0.

With this setup, the result follows from theorem 4.7.

□

We expect that similar techniques will allow us to extend this result to other itineraries.

Conjecture 2 *Given any repeating sequence of the form* $\underline{s} = (\overline{0, s_1, \ldots, s_k}, \ldots)$ *with* $s_i \neq 0$

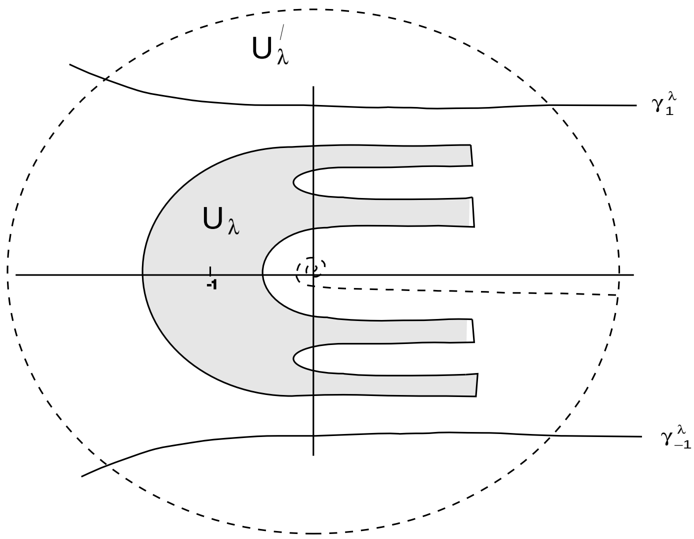

Figure 30: Dynamical plane: sketch of the sets U'_λ and U_λ.

for all i*, there exists a set* $\mathcal{M}_{s_1,\ldots,s_k}$ *homeomorphic to the Mandelbrot set. Its main cardioid corresponds to* λ*-values for which there is an attracting cycle of period* $k+1$*, with itinerary* $\underline{s}$.

This conjecture is based on the existence of the hairs in parameter space for which the critical value lands on the fixed hair after more than two iterations (see remarks in section above). Once the existence of those hairs is established, the result follows from theorem 4.7 with a very similar construction.

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