Notes on Renormalization of Complex Polynomials

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1 Introduction

Renormalization is a topic of interest in many areas of mathematics. The following are some informal notes on renormalization of complex polynomials and on the motivation that makes this subject important in the field of complex dynamics. They are intended for mathematicians whose research is in some other areas where renormalization is also a topic of interest. They are in no way a complete or deep exposition of the subject though we try to give a general idea of what complex renormalization is about and how it relates to the main problems that are currently open in holomorphic dynamics, together with several examples.

This paper is mainly based on parts of the book *Complex Dynamics and Renormalization* by Curtis T. McMullen [Mc] which we refer to for a complete exposition and many of the proofs.

2 Preliminaries on the dynamics of complex polynomials

For a complex polynomial $P(z)$ of degree $d \geq 2$, infinity is always a superattracting fixed point, i.e., a fixed point where the derivative vanishes. Let $A(\infty) = A_P(\infty)$ be its *basin of attraction*, that is,

$$A(\infty) = \{z \in \mathbb{C} \mid P^n(z) \to \infty\}.$$

This is an open set and it has only one connected component, since the point at infinity has no other preimage than itself. The *filled Julia set* of P is defined as

$$K(P) = \mathbb{C} \setminus A(\infty).$$

This set is totally invariant, compact and full (does not disconnect the plane) by the observation above. The boundary of the filled Julia set is called the *Julia set*, $J(P)$, and intuitively it is the set of points where "chaotic" dynamics occur.

A *critical point* of P is a point $\omega \in \mathbb{C}$ such that $P'(\omega) = 0$. In a neighborhood of a critical point the map is not locally injective. Many global dynamical properties can be derived from the behaviour of the critical points. For example,

$$K(P) \text{ is connected} \iff \text{the orbits of all critical points are bounded.}$$

$$\text{The orbits of all critical points tend to } \infty \implies K(P) \text{ is totally disconnected.}$$

Renormalization works best when the filled Julia set of the polynomial is connected. Hence, in these notes, we will be concerned mainly with polynomials with connected filled Julia set.

In those cases, the dynamics in $A(\infty)$ are very simple. One can find a holomorphic change of variables $\psi_P : \mathbb{C} \setminus \mathbb{D} \to A(\infty)$ (called the *Böttcher coordinates* at infinity) that conjugates $P|_{A(\infty)}$ to the map $z \mapsto z^d$ on the complement of the closed unit disk. This change is unique if we require the derivative at infinity to be one.

The image under ψ_P of a circle of radius $\exp(\eta) > 1$ in $\mathbb{C} \setminus \mathbb{D}$ is a simple closed curve in $A(\infty)$ called an *equipotential* of potential η. Hence an equipotential of potential η is mapped d to 1 under P to an equipotential of potential $d\eta$ (see Fig. 1). Parametrizing the arguments of the unit circle between 0 and 1, the image under ψ_P of a ray of argument t is called an *external ray of argument* t and denoted by $R_P(t)$. Again, since ψ_P is a conjugacy, an external ray of argument t is mapped to an external ray of argument $dt \pmod 1$. Equipotentials and external rays give us orthogonal coordinates in the attracting basin.

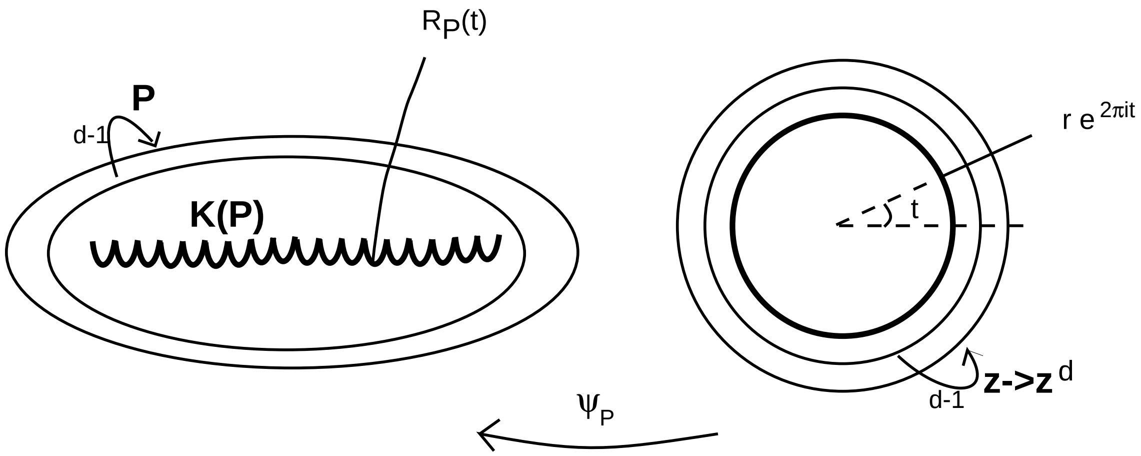

Figure 1: Böttcher coordinates, equipotentials and esternal rays.

Remarks 2.1

- If $K(P)$ is not connected, the Böttcher coordinates still exist. The difference is that they are not well defined in the whole attracting basin but in some neighborhood of infinity.

- Böttcher coordinates can be defined always in a neighborhood of any superattracting periodic point. Indeed, if z_0 is such that $P(z) = z_0 + (z - z_0)^n + \text{h.o.t.}$ for $n \geq 2$, then z_0 is a superattracting fixed point of multiplicity n. The dynamics in some neighborhood

U of z_0 are holomorphically conjugate to those of the map $z \mapsto z^n$ in the unit disk. If in addition, z_0 is the only critical point in the immediate superattracting basin, then, the neighborhood U is the whole immediate basin. In this case, the images of the rays in the unit disk are called *internal rays.*

We will use internal and external rays when we speak about tuning in Sect. 3.1.

2.1 Quadratic polynomials and the Mandelbrot set

Any quadratic polynomial is conjugate by an affine map to a polynomial of the form $Q_c(z) = z^2 + c$ for some $c \in \mathbb{C}$. Hence the space of quadratic polynomials is represented by the complex plane, which we call the *parameter plane.* We set $K_c = K(Q_c)$, $J_c = J(Q_c)$, etc. A quadratic polynomial Q_c has only one critical point, $\omega = 0$, in the plane. Hence, we have the following dichotomy.

$$\text{If } \{Q_c^n(0)\} \text{ is bounded then } K_c \text{ is connected.}$$

$$\text{If } Q_c^n(0) \to \infty \text{ then } K_c \text{ is totally disconnected.}$$

The *Mandelbrot set* is defined as the connectedness locus of the sets K_c. That is,

$$M = \{c \in \mathbb{C} \mid K_c \text{ is connected}\}.$$

A picture of M is shown in Fig. 2. The Mandelbrot set is a very complicated object which is almost, but not yet completely understood. It is known that M is compact, connected and full. Presently, one of the main conjectures in complex dynamics is the following.

Conjecture 2.2 *M is locally connected.*

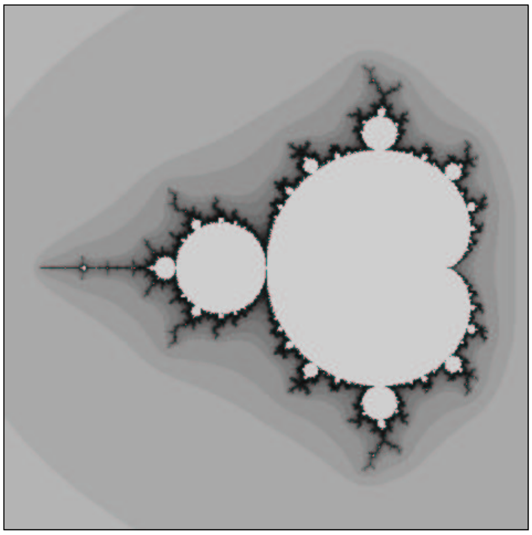

Figure 2: The Mandelbrot set

The Mandelbrot set exhibits some kind of self-similarity. When we blow up around random points of the boundary of M we are sure to find plenty of smaller copies of the whole set. Moreover, one could keep doing this up to infinite scale. This phenomenom can be explained by the theory of polynomial-like maps and, in particular, renormalization.

In general, a rational map is *hyperbolic* if the orbits of the critical points do not accumulate on the Julia set or, equivalently, if they all tend to attracting cycles. In particular

$$Q_c \text{ for } c \in M \text{ is hyperbolic} \Longleftrightarrow Q_c \text{ has an attracting cycle in } \mathbb{C}.$$

This is clear since $\omega = 0$ is the only finite critical point while the other one, infinity, is always superattracting.

A connected component Ω of the interior of M is called a *hyperbolic component* if Q_c is hyperbolic for some $c \in \Omega$ and it is known that in that case, Q_c is hyperbolic for all $c \in \Omega$. By the Implicit Function Theorem, the period of the attracting cycle is constant and it is called the *period of the hyperbolic component*. The multiplier of the attracting cycle of Q_c determines a biholomorphic map from Ω to the unit disk. The preimage of 0 is called the *center* of Ω and corresponds to a polynomial, for which the critical point itself is periodic (and hence superattracting). It is a quite surprising fact that any point in the boundary of M can be approximated by a sequence of centers of hyperbolic components. As we will see in Sect. 4, it is not known if all connected components of the interior of M are hyperbolic.

3 Polynomial-like maps and renormalization

As their name indicates, polynomial-like maps are those maps that behave locally like a polynomial.

Definition A *polynomial-like map* of degree $d \geq 2$ is a triple (f, U', U) where U and U' are open sets of $\mathbb{C}$ isomorphic to disks with $\overline{U'} \subset U$ and $f : U' \longrightarrow U$ is a holomorphic map such that every point in U has exactly d preimages in U' when counted with multiplicity.

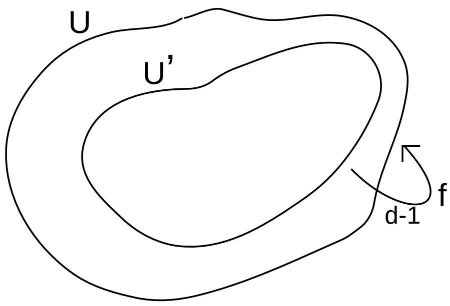

Figure 3: The three elements (f, U', U) that form a polynomial-like map.

Notice that any polynomial of degree d is a polynomial-like map of degree d. Indeed, one can take U' as the open set bounded by an equipotential of some potential η, in which case U would be the open set bounded by the equipotential of potential $d\eta$.

The most common examples of polynomial-like maps are given by polynomial, rational or entire transcendental maps restricted to an appropiate open set (see [F] for particular examples). On the other hand, restrictions of some iterate of a polynomial can also give a polynomial-like map of the same or lower degree in which case we will speak of renormalization.

The *filled Julia set* and the *Julia* set are defined for polynomial-like maps in the same fashion as for polynomials, keeping in mind that a polynomial-like map is defined only in an open subset of $\mathbb{C}$.

Definition Let $f : U' \longrightarrow U$ be a polynomial-like map. The *filled Julia set* of f is defined as the set of points in U' that never leave U' under iteration, i.e.,

$$K(f) := \{z \in U' \mid f^n(z) \in U' \text{ for all } n \geq 0\}.$$

An equivalent definition is

$$K(f) = \bigcap_{n\geq 0} f^{-n}(\overline{U'}),$$

and from this expression it is clear that $K(f)$ is a compact set. As for polynomials, we define the *Julia set of* f as

$$J(f) := \partial K(f).$$

Notice that any polynomial-like map (f, U', U) must have a critical point in U'.

The importance of polynomial-like maps is their correspondence to actual polynomials. This is explained by the Straightening Theorem which we state after the following definition.

Definition T wo polynomial-like maps f and g are *hybrid equivalent* if there is a quasiconformal conjugacy h defined in a neighborhood of their respective filled Julia sets and such that $\partial h = 0$ on $K(f)$.

Theorem 3.1 ([DH3]) *Let (f, U', U) be a polynomial-like map of degree $d \geq 2$. Then, f is hybrid equivalent to a polynomial P of degree d. Moreover, if $K(f)$ is connected, then P is unique up to affine conjugation.*

In particular, a hybrid equivalence implies that the corresponding Julia sets are homeomorphic. Hence, the theorem explains why we find copies of filled Julia sets of polynomials when we look at the dynamical planes of other types of holomorphic maps.

For convenience, when we speak about renormalization we restrict to quadratic polynomials with connected filled Julia sets.

Definition Let Q_c be a quadratic polynomial with connected filled Julia set. We say that Q_c^n is renormalizable if one can find open bounded sets U' and U such that $0 \in U'$ and $(P^n|_{U'}, U', U)$ is a polynomial-like map of degree 2 with connected Julia set.

Remark 3.2 *The restriction to quadratic polynomials is just convenient though not necessary. It could be extended to polynomials P of any degree asking that U' contain a unique critical point of P.*

The choice of the pair (U', U) as above is called a *renormalization* of Q_c^n. So, as we see, a renormalizable polynomial Q_c^n and its renormalization is just a special case of a polynomial like map.

To guarantee that the iterate process of renormalization makes sense it is important to make sure that the choice of the open sets (U', U) does not affect the resulting polynomial (given by Theorem 3.1).

Theorem 3.3 *Any two renormalizations of Q_c^n have the same filled Julia set.*

Sketch of the proof If (U_1', U_1) and (U_2', U_2) are two renormalizations with filled Julia sets K_1 and K_2, one can show that $K = K_1 \cap K_2$ would be the filled Julia set of another renormalization (U', U) with $U' \subset U_1' \cap U_2'$. Since the degree of all the renormalized maps is the same and the preimages of any point of a Julia set are dense in that Julia set it follows that $K = K_1 = K_2$.

q.e.d.

The natural numbers n for which Q_c^n is renormalizable are called the *levels of renormalizations* of Q_c. We denote the set of levels of renormalization of Q_c by $\mathcal{R} = \mathcal{R}_c$. If $\mathcal{R}$ is an infinite set, we say that Q_c is *infinitely renormalizable.*

Example A: Let $c = -1.772892\ldots$. This is the center of a hyperbolic component of period 6 and hence $Q_c^6(0) = 0$. In this case Q_c^3 and Q_c^6 are renormalizable hence $\mathcal{R}_c = \{1, 3, 6\}$. For Q_c^3, the critical point $w = 0$ is periodic of period 2. Hence, the renormalized filled Julia set must be that of $z^2 - 1$. For Q_c^6, the critical point is fixed so the renormalized polynomial is z^2. Figure 4 shows the filled Julia set of Q_c and, below, a magnification of the filled Julia sets after renormalizing Q_c^3 and Q_c^6 respectively.

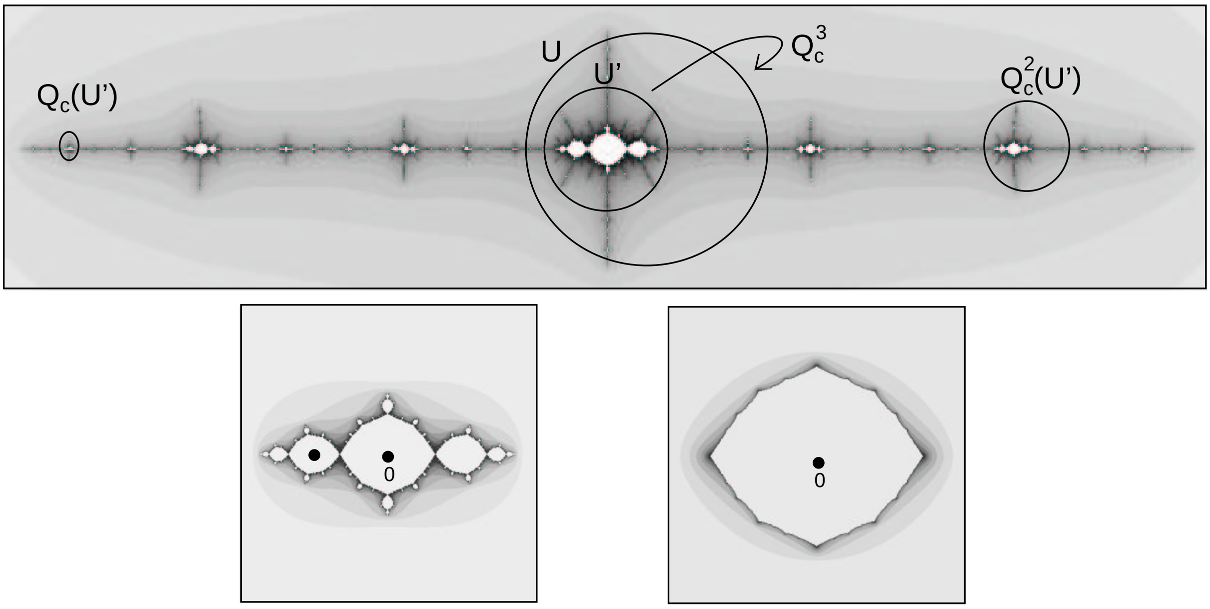

Figure 4: The filled Julia set of Q_c for $c = -1.772892$ and the filled Julia sets of the renormalizations of Q_c^3 and Q_c^6.

Example B: Let $c = -1.401155\ldots$. Then, Q_c is the Feigenbaum polynomial, that is the limit of the cascade of period doublings in the real axis that starts at 0. For any $n \geq 1$, the polynomial $Q_c^{2^n}$ is renormalizable and all these renormalizations are hybrid equivalent to itself. Fig. 5 shows the filled Julia set of Q_c and the filled Julia set of the renormalization of Q_c^2.

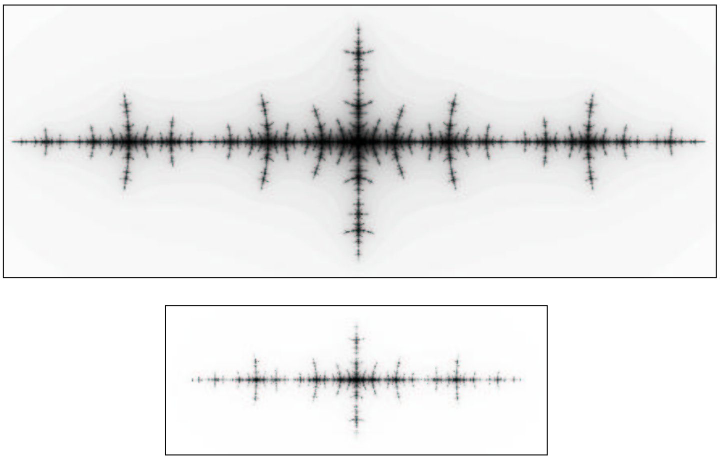

Figure 5: The filled Julia set of the Feigenbaum polynomial Q_c with $c = -1.401155\ldots$, and the filled Julia set of the renormalization of Q_c^2.

Example C: Let $c \sim 0.419643 + 0.60629i$, a Missurewicz point in the boundary of the Mandelbrot set. For this map $\omega = 0$ becomes periodic of period two after three iterations (see Fig. 6). Since Q_c^2 is renormalizable, $\omega = 0$ is fixed after two iterations of the renormalized map. Hence, the renormalized filled Julia set is hybrid equivalent to $z^2 - 2$, i.e., a quasi-conformal image of the interval $[-2, 2]$. No further iterates are therefore renormalizable.

3.1 Tuning

The process of tuning explains in a more geometric way how the Mandelbrot set contains infinitely many copies of itself and, in addition, it provides plenty of examples of renormalizable polynomials. It was developed by A. Douady and the details can be found in [D3]. This procedure could be viewed as the complex version of the *star product* in real dynamical systems.

To each center c_0 of a hyperbolic component there is associated a copy of the Mandelbrot set, M_{c_0} obtained by the *tuning homeomorphism*:

$$\begin{array}{ccc} M & \longrightarrow & M_{c_0} = c_0 \perp M \\ c & \longmapsto & c_0 \perp c \end{array}$$

which works intuitively as follows. Since c_0 is the center of a hyperbolic component, the critical point $\omega = 0$ is periodic under Q_{c_0}. Let p be the period of the cycle. Let A be the immediate basin of attraction of $\omega = 0$ and ψ be the unique holomorphic map that conjugates

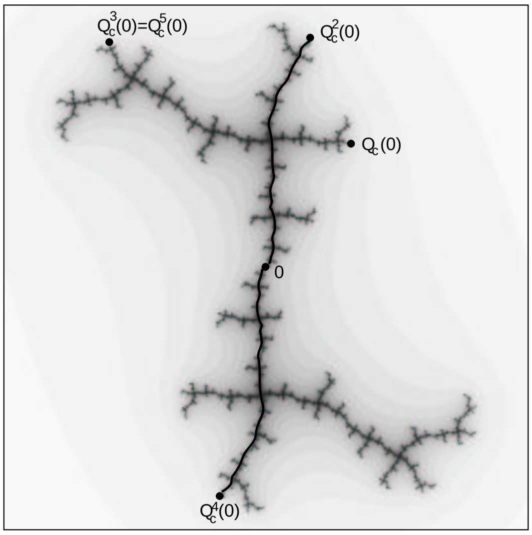

Figure 6: The Julia set of Q_c for $c \sim 0.419643 + 0.60629i$. Emphasized, the Julia set of the renormalization of Q_c^2.

$Q_{c_0}|_A$ to $z \mapsto z^2$ in the unit disk (see Sect. 2). Then ψ provides internal rays in A and internal arguments in ∂A since ψ extends continuously to the boundary of the unit disk.

Now take Q_c any polynomial in the Mandelbrot set, with locally connected Julia set (for the non-locally connected case see [D3]). Its filled Julia set K_c has external rays in $A_c(\infty)$ that provide external arguments in $\partial K_c = J_c$. Topologically, the tuning process consists of replacing A by a copy of K_c, identifying points in ∂A with points in J_c whose internal and external angles are equal. Afterwards, replace the preimages of A by preimages of K_c accordingly.

One can show that the map can be modified to obtain a polynomial-like map whose filled Julia set is hybrid equivalent to a new polynomial $Q_{c'}$ with $c' = c_0 \perp c$, the tuning of c_0 by c. Clearly, $Q_{c'}^p$ is renormalizable and its filled Julia set is K_c.

The tuning map is a homeomorphism mapping centers of hyperbolic components to centers of hyperbolic components and Misiurewicz points to Misiurewicz points; it is analytic in the interior of M and $\partial M_{c_0} \subset \partial M$

Example : Suppose we choose $c_0 = -1.75488\ldots$, the center of the hyperbolic component of period three in the real axis whose filled Julia set is shown in Fig.7.

By tunning c_0 with all the points in M we obtain all the points in this copy. Some of them are shown in Fig. 8.

As an example of a tunned Julia set we refer to Example A above and in particular to Figure 4. That is the filled Julia set of Q_c where $c = c_0 \perp -1$. As we see, the "disks" in K_{c_0} (Fig. 7) have been substituted by filled Julia sets of $z^2 - 1$.

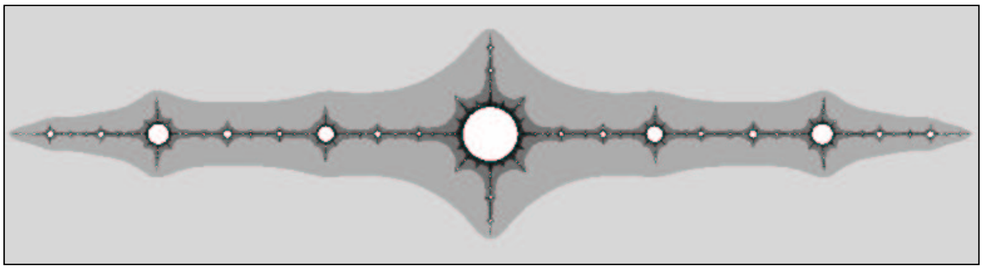

Figure 7: The filled Julia set of Q_c with $c_0 = -1.75488\ldots$, the center of the hyperbolic component of period three in the real axis.

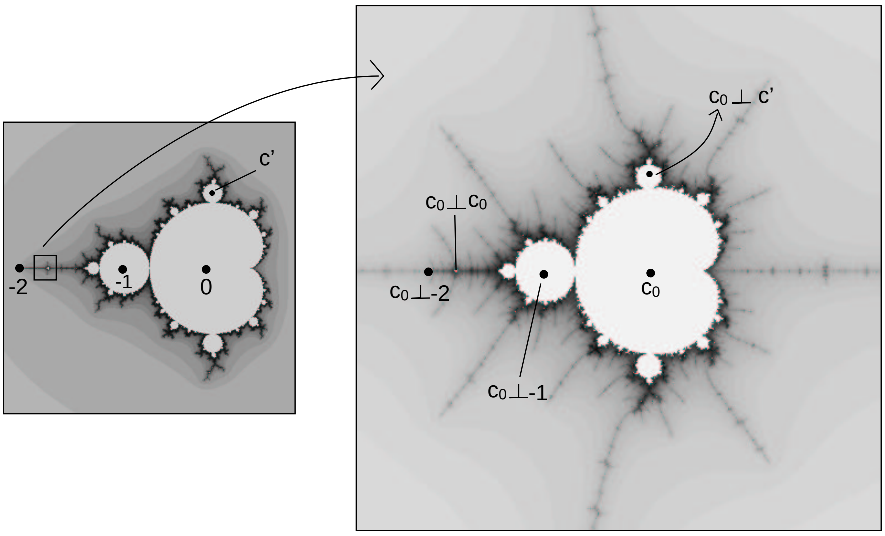

Figure 8: Left: The Mandelbrot set. Right: Magnification of the small period 3 copy of the Mandelbrot set in the real axis. Some of its points have ben marked as they would be obtained by tuning.

4 The importance of renormalization in complex dynamics

In this section we summarize the strong relation that exists between renormalization and the main present conjectures in the field of complex dynamics.

Let Rat_d denote the space of rational maps of degree d.

Conjecture 4.1 *The set of hyperbolic rational maps of degree d is dense in Rat_d.*

For quadratic polynomials this conjecture specializes to the following.

Conjecture 4.2 *The set of parameters $c \in \mathbb{C}$ such that Q_c is hyperbolic is dense in $\mathbb{C}$.*

The well known conjecture of local connectivity of M (Conj. 2.2) is stronger than the above, i.e.

Theorem 4.3 ([DH2]) *If M is locally connected then hyperbolic quadratic polynomials are dense in $\mathbb{C}$.*

It is known (see [MSS]) that the set of parameters in $\mathbb{C} \setminus \partial M$ form an open and dense subset of $\mathbb{C}$ (those are the maps that satisfy a special type of structural stability). Since the quadratic polynomials in $\mathbb{C} \setminus M$ are clearly hyperbolic (the orbit of the critical point tends to the superattracting fixed point at infinity), conjecture 4.2 is equivalent to

Conjecture 4.4 *Every connected component of the interior of M is hyperbolic.*

There are many routes one can take to relate this conjecture to renormalization. One of them is through the following theorem of Yoccoz.

Theorem 4.5 *Let $Q_c(z) = z^2 + c$ and suppose that Q_c*

- *has a connected Julia set,*
- *has no indifferent cycles and*
- *is not infinitely renormalizable.*

Then, J_c is locally connected. If moreover Q_c has no attracting cycle, then c lies in the boundary of the Mandelbrot set and M is locally connected at c.

The proof makes use of the *Yoccoz puzzle*, a Markov partition for the dynamics of quadratic polynomials using equipotentials and external rays (see [H, Mi2, Y]).

Corollary 4.6 *If M is not locally connected at $c \in \partial M$ or c belongs to a non-hyperbolic component of the Mandelbrot set then Q_c is infinitely renormalizable*

Sketch of the proof: The first statement is immediate. If c belongs to a non-hyperbolic component of the Mandelbrot set then c does not have any attracting cycle and does not lie in the boundary of M. Hence one of the three conditions in Yoccoz's theorem is not satisfied. Since $c \in M$, the Julia set of Q_c is connected. If Q_c had an indifferent cycle, c would lie in the boundary of M (since no such quadratic polynomial could be structurally stable). Hence Q_c must be infinitely renormalizable.

q.e.d.

Hence the complete understanding of the dynamics of infinitely renormalizable polynomials is of main importance in the field. In fact, we have

Theorem 4.7 *M is locally connected if and only if for any (strictly) decreasing sequence of tuning copies of M in M, their intersection is reduced to a point.*

For completeness we mention that Conjecture 4.4 is also related to the measure of the Julia set and to the existence of invariant line fields. Intuitively, a *line field* supported in a set $E \in J_c$ of positive measure is an assignment of a line through the origin in the tangent plane to each point $z \in E$ such that the slope is a measurable function of z. The line field is *invariant* if $f^{-1}(E) = E$ and the derivative of Q_c maps the line at z to the line at $f(z)$.

One can show (see [MSS]):

Theorem 4.8 *A parameter c belongs to a non-hyperbolic component of M if an only if J_c has positive measure and carries an invarient line field.*

Combining this result with Yoccoz's theorem we see that an equivalent conjecture to 4.4 is

Conjecture 4.9 *An infinitely renormalizable quadratic polynomial cannot support an invariant line field on its Julia set.*

A stronger statement would be that all Julia sets of quadratic polynomials have measure zero. Recently though, the existence of high degree polynomials with Julia sets of positive measure has been shown.

Finally, the main result in [Mc] is the following:

Theorem 4.10 *Every connected component of the interior of M that meets the real axis is hyperbolic.*

In fact, this theorem is a corollary of two stronger facts which we do not include here since they are beyond the scope of these notes.

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