# Dynamics on Hubbard Trees

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**Abstract**

It is well known that the Hubbard tree of a postcritically finite complex polynomial contains all the combinatorial information of the polynomial. In fact, an abstract Hubbard tree as defined in [24] uniquely determines the polynomial up to affine conjugation. In this paper we study how much of the "dynamical information" is captured by the Hubbard tree of a quadratic Misiurewicz polynomial. More precisely, we study dynamical features such as entropy, transitivity or periodic structure of the polynomial restricted to the Hubbard tree, and compare them with the properties of the polynomial on its Julia set. Our results show that there is a strong connection between the renormalization properties of the polynomial and the mentioned dynamical features of the polynomial on its Hubbard tree. As a consequence of this relation we obtain criteria to check if a quadratic Misiurewicz polynomial is renormalizable by means of its Hubbard tree.

## 1 Introduction

In this paper we study the dynamical information that is captured by the Hubbard tree of a postcritically finite polynomial in the complex plane. In what follows we introduce notation and summarize the basic facts which are necessary to motivate and state the main results of this paper. For more details and general background we refer to [7, 8, 12, 16, 20, 26].

Let $f$ be a complex polynomial and let $z \in \mathbb{C}$. We will denote by $\mathcal{O}_f^+(z)$ the forward orbit of $z$ under $f$, i.e. the set $\{f^n(z) \mid n \in \mathbb{N}\}$. If $f^k(z) = z$ for some $k \in \mathbb{N}$ then $z$ is a *periodic point*. The smallest such $k$ is called the *period* of $z$. If $z$ is not periodic but $\mathcal{O}_f^+(z)$ is finite we say that $z$ is *preperiodic*.

For a complex polynomial $f$ of degree $d \geq 2$ the point at infinity is a superattracting fixed point. This allows us to define the *filled Julia set of* $f$, denoted by $K(f)$, as the complement

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of the basin of attraction of infinity. That is,

$$K = K(f) = \{z \in \mathbb{C} \mid \mathcal{O}_f^+(z) \text{ is bounded}\}.$$

The filled Julia set is clearly compact and totally invariant (that is, $f(K) = f^{-1}(K) = K$). The common boundary of $K(f)$ and the basin of attraction of infinity, is called the *Julia set of $f$* and denoted by $J(f)$. The Julia set is also totally invariant and is the set of points where "chaotic dynamics" occur.

Next we are going to describe the basic properties of the Julia set with respect to periodic points, topological entropy and transitivity. The notion of topological entropy was introduced by Adler, Konheim and McAndrew in [1], which we refer to for a precise definition and basic properties (see also, for instance, [13] or [2]). In what follows, the topological entropy of a map $f$ will be denoted by $h(f)$. Next we recall the concept of transitivity. Let $f : X \longrightarrow X$ be a continuous map of a compact metric space. We say that $f$ is *(topologically) transitive* if for any two non-empty open sets $U$ and $V$ in $X$, there is a positive integer $k$ such that $f^k(U) \cap V \neq \emptyset$. It is well known (see [29]) that if either $X$ has no isolated points or $f$ is onto, then $f$ is transitive if and only if it has a dense orbit (i.e. if there exists $x \in X$ such that $\mathcal{O}_f^+(x)$ is dense in $X$). When $f^n$ is transitive for each $n \in \mathbb{N}$ then $f$ is called *totally transitive* (see [6]).

The following proposition states some basic properties of the Julia set, all of which are well known (see for example [7]).

**Proposition 1.1.** *For a complex polynomial $f$ of degree $d \geq 2$ the following statements hold.*

*(a) Periodic points are dense in $J(f)$.*

*(b) $f\big|_{J(f)}$ is totally transitive.*

*(c) The topological entropy of $f\big|_{J(f)}$ is $\log(d)$.*

*(d) $f\big|_{J(f)}$ has periodic points of each period except maybe period 2.*

We say that a point $\omega \in \mathbb{C}$ is a *critical point of $f$* if $f'(\omega) = 0$. Then, its orbit is called a *critical orbit.* The behavior of the critical points under iteration determine in many ways the topology of $K(f)$ and the dynamics of $f$. As an example, let $C(f)$ denote the set of critical points of $f$. Then, $K(f)$ is known to be connected if and only if $C(f) \subset K(f)$.

We are interested in the special case where the critical orbits are finite. We call these polynomials *postcritically finite* (or PCF for short) and they can be of three types. If all critical orbits are periodic then $f$ is called a *center.* If the critical orbits are all preperiodic then $f$ is called a *Misiurewicz polynomial.* In this case $K(f)$ has empty interior and hence $K(f) = J(f)$ (see [14]). Finally, a PCF polynomial could exhibit both types of critical orbits. In all cases $K(f)$ is connected and locally connected (see [14]). In this paper, we are mainly concerned with Misiurewicz polynomials of degree two. Generalizations to higher degrees will be the object of a later paper.

A *tree* is a topological space which is uniquely arcwise connected and homeomorphic to the union of finitely many copies of the unit interval. Douady and Hubbard in [14] introduced a combinatorial description of the dynamics of PCF polynomials by representing each filled

Julia set as a tree, called the Hubbard tree. For Misiurewicz polynomials this is defined as follows. Given a subset $A$ of $J(f)$ we denote by [A] the *convex hull* of $A$ in $J(f)$, i.e. the smallest closed connected subset of $J(f)$ that contains $A$.

**Definition.** Let $f$ be a Misiurewicz polynomial and let $\Omega(f) = \bigcup_{\omega \in C(f)} \mathcal{O}_f^+(\omega)$ be the post-critical set which in this case is finite and contained in $J(f)$. We define the *Hubbard tree* of $f$ as

$$H(f) = [\Omega(f)].$$

We note that each Hubbard tree is a tree. Indeed, for any two points $x, y \in J(f)$, there is a unique Jordan arc in $J(f)$ that joins $x$ and $y$. The existence of this arc follows from the fact that $J(f)$ is connected and locally connected in $\mathbb{S}^2$ and the uniqueness from the fact that $J(f)$ has empty interior and is simply connected. Therefore, since $H(f)$ is the union of the Jordan arcs in $J(f)$ joining $x, y \in \Omega(f)$, it follows that $H(f)$ is a tree.

If $T$ is a tree and $x \in T$, the *valence* of $x$ is defined to be the number of components of $T \setminus \{x\}$. A point of valence 1 is called an *endpoint* and a point of valence greater than 2 is called a *branching point*. We define the set of *vertices of* $H(f)$ as

$$V(f) = \Omega(f) \cup \{v \in H(f) \mid v \text{ is a branching point}\}.$$

The closure of the arc in $H(f)$ in between two consecutive vertices is called an *edge*. Note that any endpoint of $H(f)$ belongs to $\Omega(f)$ and, hence, $V(f)$ contains all points of $H(f)$ with valence different from 2.

It is easy to check that the Hubbard tree is a forward invariant subset of the Julia set (see Lema 1.10 in [24]). The set of vertices is also forward invariant since $\Omega(f)$ is forward invariant and non-critical branching points must be mapped to branching points (because the map is a local homeomorphism). For this and other basic properties of Hubbard trees we refer to [24] although, in Section 2, we study the features that we use in proving the main results of this paper.

**Remark 1.2.** Hubbard trees are in fact defined for any PCF polynomial (see [14, 24]). If $f$ is not Misiurewicz, it follows from the definition that the Hubbard tree intersects the basins of attraction of points of $\Omega(f)$ in $K(f) \setminus J(f)$.

The interest of Hubbard trees lies on the fact that they contain all the combinatorial information of the polynomial. Indeed, Douady and Hubbard showed that if we retain the dynamics and the local degree of $f$ on the set of vertices, the way the tree is embedded in the complex plane and a little bit of extra information (which we will not make precise here), then different PCF polynomials (not conjugated as dynamical systems) give rise to different Hubbard trees. A variation of the converse is also true and was proved in a general version by A. Poirier in [24].

In this context it is natural to ask the following question. Apart from the topological and combinatorial structure of the Julia set, which dynamical information is contained in the Hubbard tree? In this case, by dynamical information we mean those dynamical features as transitivity, topological entropy, density of periodic points and periodic structure. More

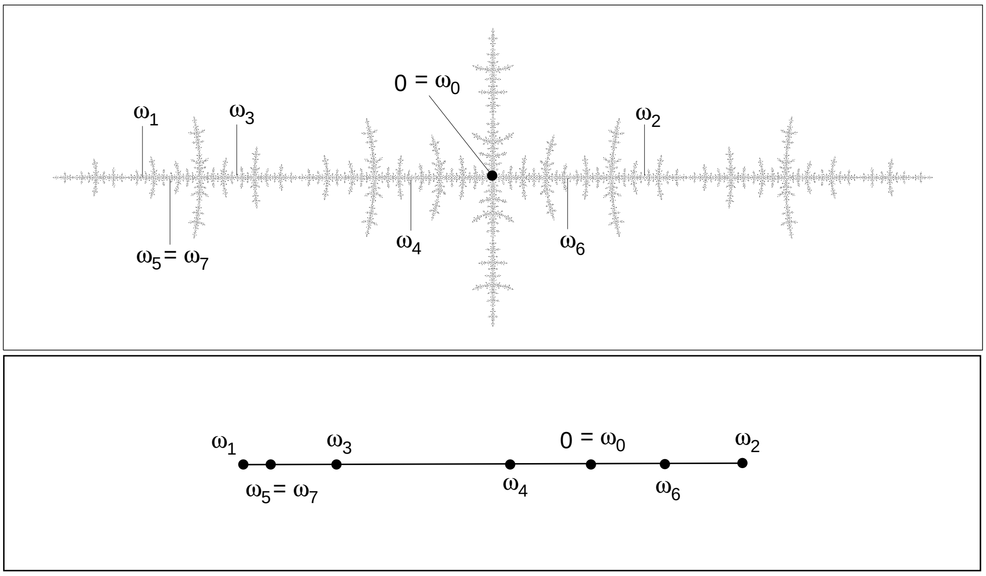


Figure 1: The Julia set and the Hubbard tree of $f_{c_1}$, where $c_1 = -1.430357\ldots$.

precisely, which of the properties in Proposition 1.1 (if any) are still true when we consider the map $f|_{H(f)}$? If the answer depends on the polynomial, what is a characterization of those polynomials for which those properties hold? We will see that these features are intimately related with the renormalization properties of the polynomial.

From now on the polynomial $f$ is of degree two and therefore it is affine conjugated to one of the form $f(z) := f_c(z) = z^2 + c$ for some $c \in \mathbb{C}$. A quadratic polynomial $f_c$ has only one critical point $\omega_0 = 0$. We will denote the critical orbit by $\omega_0, \omega_1, \ldots$ where $\omega_i = f^i(0)$. If $f$ is a center, as mentioned in Remark 1.2, the Hubbard tree has edges in the basin of attraction of the critical orbit. These edges cannot contain periodic points in their interiors. Thus, periodic points cannot be dense in this case and hence, $f|_{H(f)}$ is not transitive (see Proposition 2.8). With respect to the topological entropy and the set of periods, one can find examples of quadratic centers having zero entropy and finitely many periodic points. For this reason, from now on we consider only Misiurewicz polynomials. Hence, we assume that there exist $n > 2$ and $0 < k < n$ such that $\omega_n = \omega_k$.

In order to get some intuition on the different possible behaviors one can expect, we first look at some examples.

**Example 1.** Let $c_1 = -1.430357\ldots$. This parameter value is the last point of the period two copy of the Mandelbrot set on the real axis. The Julia set and Hubbard tree of the polynomial $f_{c_1}$ are shown in Figure 1. Note that $E = [\omega_1, \omega_3] \cup [\omega_4, \omega_2]$ is a proper forward invariant subset of $H = H(f_{c_1})$ and each point in the complement of $E$ (in $H$), except a repelling fixed point of $f_{c_1}$ in $(\omega_3, \omega_4)$, eventually falls in $E$. Consequently, periodic points cannot be dense in $H$ and $f_{c_1}|_H$ is not transitive (see Proposition 2.8). Moreover, any periodic point in $H$ must have period multiple of two. On the other hand, by using Theorem 2.9 (see also [9]), it follows that $h(f_{c_1}|_H) = \frac{\log 2}{4}$.

**Example 2.** Let $c_2 = -1.790327\ldots$. The polynomial $f_{c_2}$, whose Julia set and Hubbard tree are shown in Figure 2, can be found at the end of the small period three copy of the Mandelbrot set on the real axis. Note that $E = [\omega_1, \omega_4] \cup [\omega_6, \omega_3] \cup [\omega_5, \omega_2]$ is a proper forward invariant subset of $H = H(f_{c_2})$. However, in this case, by using standard arguments (see [9]) one can check that there is an invariant Cantor set $C$ in the complement of $E$ (in $H$) which contains periodic points of all periods. Still there is an open dense set of preimages of $E$ in $H \setminus E$. Consequently, periodic points cannot be dense in $H$ and $f_{c_2}\big|_H$ is not transitive. By using the arguments above it follows that $h(f_{c_2}\big|_H) = \log \frac{\sqrt{5}+1}{2}$.

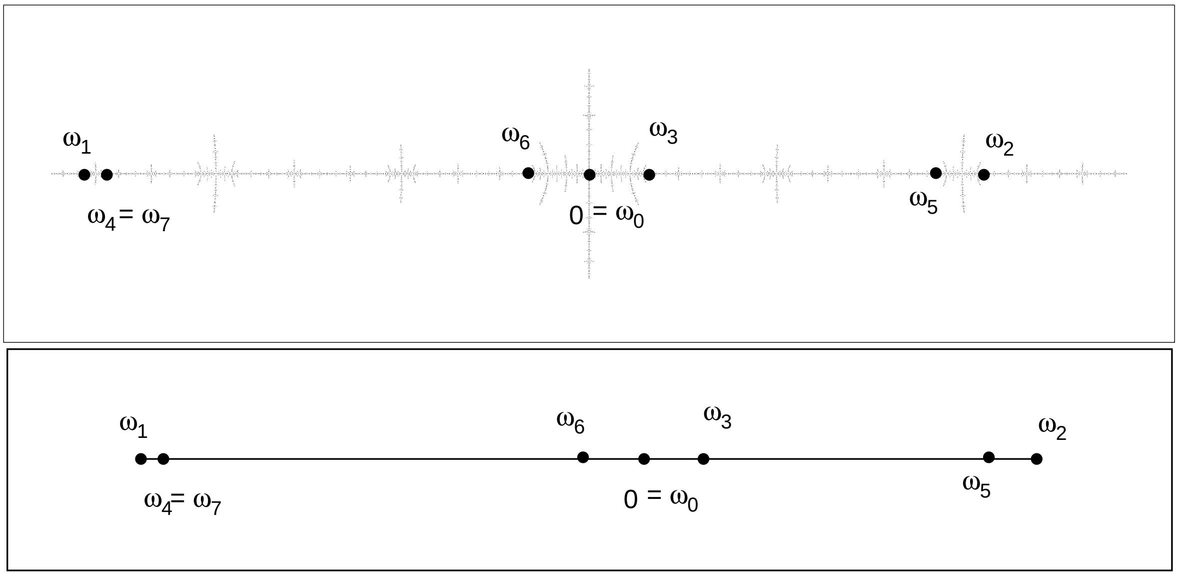


Figure 2: The Julia set and the Hubbard tree of $f_{c_2}$, where $c_2 = -1.790327\ldots$.

**Example 3.** Let $c_3 = -1.222863\ldots + i\,0.316882\ldots$. The Julia set and the Hubbard tree of $f_{c_3}$ are shown in Figure 3. It can be seen that in this case, the subtrees $T_0 = [\omega_5, \omega_2, \omega_4]$ and $T_1 = [\omega_5, \omega_1, \omega_3]$ are mapped to each other cyclically. Therefore, each of these trees is forward invariant by $f_{c_3}^2$ which implies that $f_{c_3}^2$ is not transitive. Hence, $f_{c_3}$ is not totally transitive. However, by using straightforward arguments it can be seen that $f_{c_3}^2\big|_{T_1}$ (and, hence, $f_{c_3}^2\big|_{T_0}$) is totally transitive, which implies the transitivity of $f_{c_3}$ (in fact, the transitivity of $f_{c_3}$ can be checked easily by using Theorem A). Consequently, periodic points are dense in $H(f_{c_3})$ (see Proposition 2.8) but their periods must be multiples of two. Concerning the entropy, by using the methods of the previous example, one can see that $h(f_{c_3}\big|_{H(f_{c_3})}) = \log 1.302160040\ldots$.

**Example 4.** Let $c_4 = 0.419643\ldots + i\,0.606291\ldots$. In Figure 4 the Julia set and Hubbard tree of $f_{c_4}$ are shown. As in the previous example one can see that $f_{c_4}$ is transitive on $H(f_{c_4})$; $f_{c_4}^2$ is not transitive because $[\omega_4, \omega_2]$ is mapped by $f_{c_4}$ to $[\omega_3, \omega_1]$ and viceversa; periodic points are dense but their periods can only be multiples of two, and $h(f_{c_4}\big|_{H(f_{c_4})}) = \frac{\log 2}{2}$.

**Example 5.** Let $c_5 = -0.228155\ldots + i1.115142\ldots$ (see Figure 5). We note that the map $f_{c_5}$ is conjugate to $f_{c_3}^2\big|_{T_0}$. So, $f_{c_5}\big|_{H(f_{c_5})}$ is totally transitive (this also can be checked by

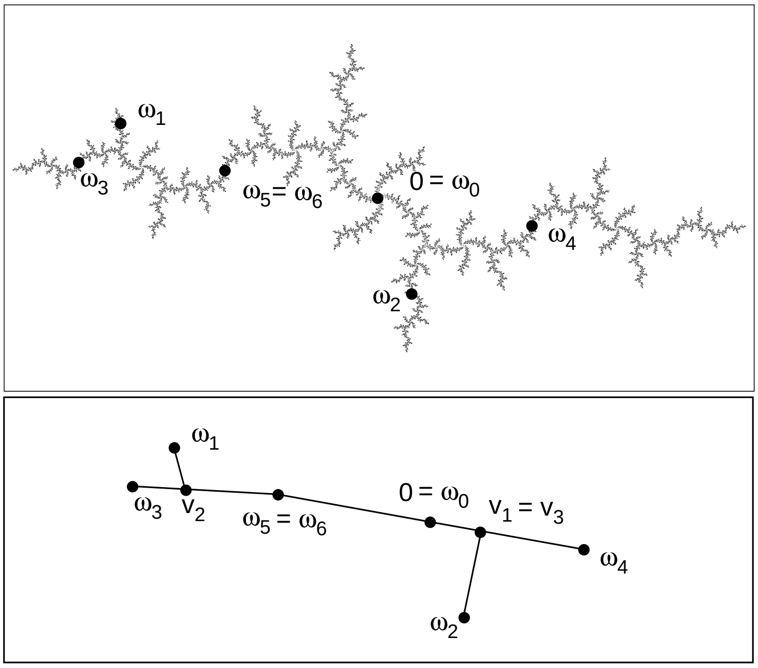


Figure 3: The Julia set and the Hubbard tree of $f_{c_3}$, where $c_3 = -1.222863\ldots + i\,0.316882\ldots$.

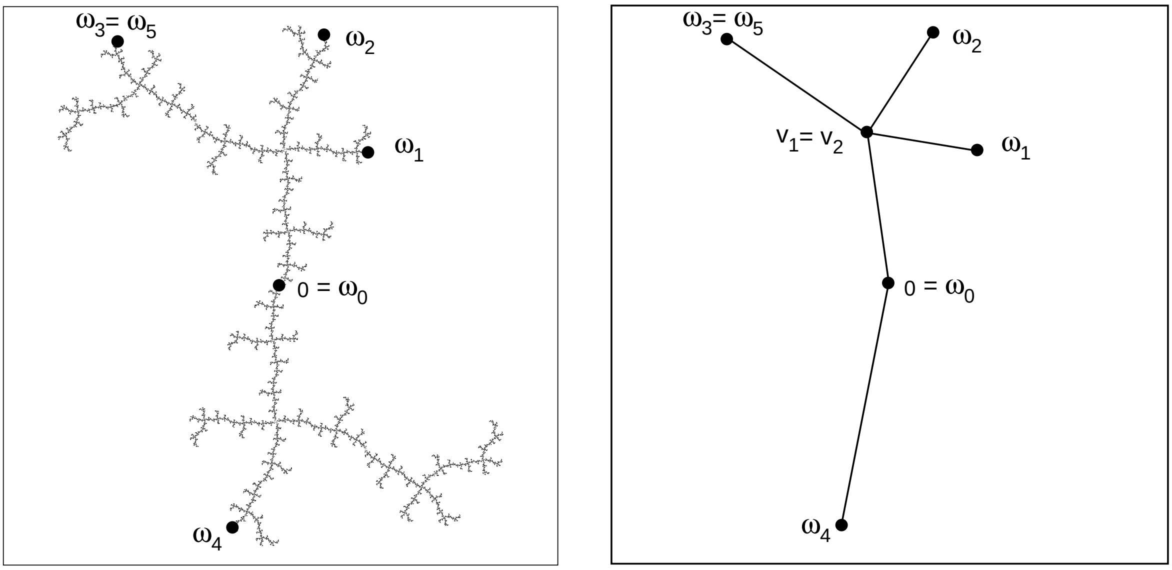


Figure 4: The The Julia set and the Hubbard tree of $f_{c_4}$, where $c_4 = 0.419643\ldots + i\,0.606291\ldots$.

using Theorem B). Hence, periodic points are dense in $H(f_{c_5})$ and there is not, in principle, any restriction on the periods that may appear. Concerning the entropy though, we have $h(f_{c_5}|_{H(f_{c_5})}) = 2h(f_{c_3}|_{T_0}) = 2\log 1.302160040\ldots$.

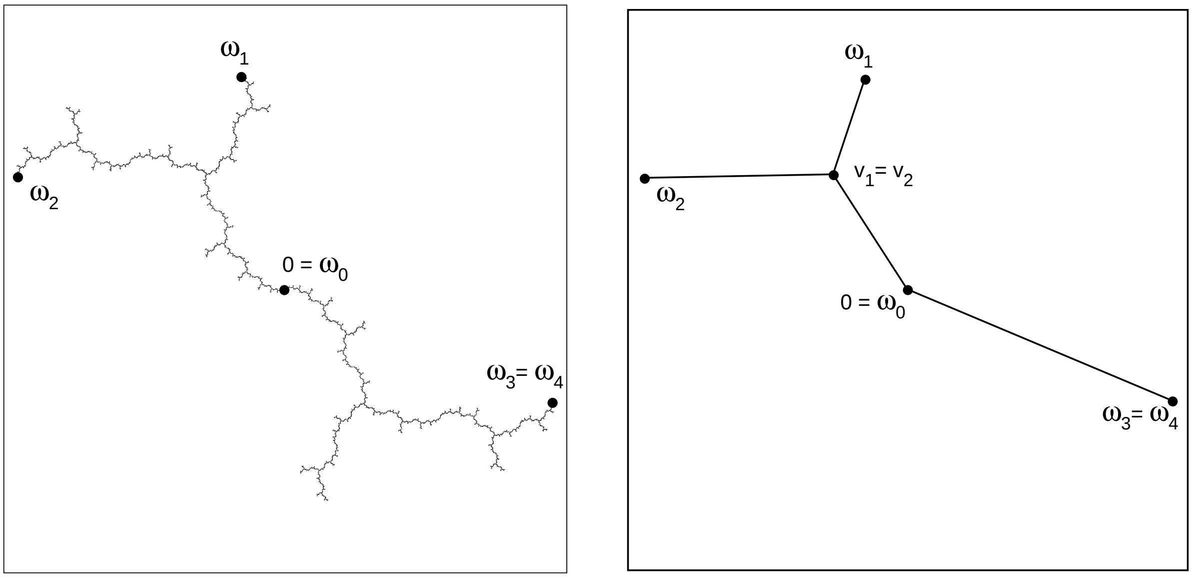


Figure 5: The Hubbard tree of $f_{c_5}$, where $c_5 = -0.228155\ldots + i\,1.115142\ldots$

From the above examples we deduce that, in general, periodic points are not dense, $f|_{H(f)}$ is not totally transitive (sometimes not even transitive) and the entropy can be strictly smaller than $\log 2$. However, they also show that this is not the case for *all* polynomials.

Before stating the main results of this paper we still need to recall some definitions (mainly following [19]).

**Definition.** Let $f(z) = z^2 + c$ be such that $J(f)$ is connected. For $n > 1$ we say that $f^n$ is *renormalizable* (or that $f$ is *renormalizable for* $n > 1$) if there exist open bounded sets $U$ and $V$ isomorphic to disks such that

(1) $\overline{U} \subset V$,
(2) $f^n(U) = V$ and $f^n : U \longrightarrow V$ is proper of degree two, i.e. every point in $V$ has two preimages in $U$ counted with multiplicity.
(3) $f^{kn}(0) \in U$ for all $k \geq 0$.

We define the *small filled Julia set* of the renormalization, $K_n$, as the points that never leave $U$ under iteration of $f^n$.

It follows from the Straightening Theorem (see [15]) that there exists a unique (up to affine conjugation) quadratic polynomial $Q$ such that $f^n$ and $Q$ are hybrid equivalent in neighborhoods of $K_n$ and $K(Q)$ respectively. We refer to [15] for a definition of a hybrid equivalence but we remark that, in particular, it means that $f^n$ and $Q$ are quasiconformally conjugate on the mentioned domain.

Hence the small filled Julia set $K_n$ is homeomorphic to the actual filled Julia set of a quadratic polynomial (see Figure 6). We define the cycle

$$K_n(0) = K_n, K_n(1) = f(K_n(0)), \dots, K_n(n-1) = f(K_n(n-2)),$$

where $f(K_n(n-1)) = K_n(0)$. We call each set $K_n(i)$ a *small filled Julia set* of the renormalization. McMullen in [19] showed that these sets can intersect each other in at most one single point which of course must be a fixed point of $f^n$.

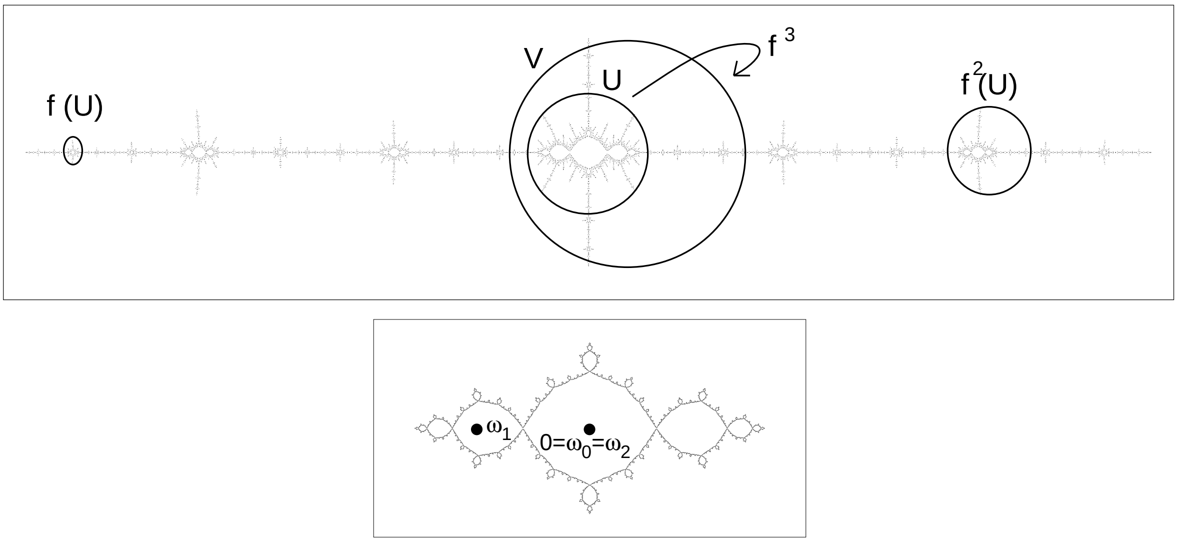


Figure 6: The Julia set of $f_c$ with $c = -1.772892\dots$. The map $f_c^3$ is renormalizable and the small filled Julia set is quasiconformally homeomorphic to the filled Julia set of $f_{-1}$ (lower figure). This is an example of renormalization of disjoint type.

**Definition.** Let $f(z) = z^2+c$ be such that $K(f)$ is connected. We say that $f$ is *renormalizable of disjoint type* if $f$ is renormalizable for some $n > 1$ and all the small filled Julia sets $K_n(i)$, $0 \le i < n$ are disjoint.

Let $\mathrm{Per}(f) = \{n \in \mathbb{N} \mid f \text{ has a periodic point of period } n\}$. Recall that a set $A \subset \mathbb{N}$ is called *cofinite in* $\mathbb{N}$, if $\mathbb{N} \setminus A$ is finite. We are now ready to state the main results of the paper.

**Theorem A.** *Let $f$ be a Misiurewicz polynomial of degree 2 and $H(f)$ be its Hubbard tree. Then, the following statements are equivalent:*

*(a) $f$ is not renormalizable of disjoint type.*

*(b) Periodic points are dense in $H(f)$.*

*(c) The map $f|_{H(f)}$ is transitive.*

*(d) For all edge $l$ of $H(f)$, $\bigcup_{n\ge 0} f^n(l) = H(f)$.*

**Theorem B.** *Let $f$ be a Misiurewicz polynomial of degree 2 and let $H(f)$ be its Hubbard tree. Then, the following statements are equivalent:*

*(a) $f$ is non renormalizable.*

*(b)* $f|_{H(f)}$ *is totally transitive.*

*(c) For all edge* $l$ *of* $H(f)$*, there exists* $n > 0$ *such that* $f^n(l) = H(f)$*.*

*Moreover, if* $f|_{H(f)}$ *is totally transitive then* $\mathrm{Per}(f|_{H(f)})$ *is cofinite in* $\mathbb{N}$*.*

We remark that the converse of the last statement of Theorem B is not true. Indeed, the map $f_{c_2}|_{H(f_{c_2})}$ from Example 2 has periodic points of all periods while it is not even transitive. However, one can show that if $f|_{H(f)}$ is transitive but not totally transitive then its set of periods is not cofinite in $\mathbb{N}$.

We observe that Theorems A and B give an easy criterium to check if the polynomial $f$ is renormalizable and, in that case, of which type. One only needs to construct the Hubbard tree and check the images of its edges. We also note that, in general, density of periodic points does not imply transitivity. It does though in the above cases.

**Theorem C.** *Let* $f$ *be a Misiurewicz polynomial of degree* $d \geq 2$ *and let* $H(f)$ *be its Hubbard tree. Then,* $h(f|_{H(f)}) \leq \log d$ *and the equality holds if and only if* $H(f) = J(f)$*.*

From [4, Theorem B] it follows immediately the following corollary of Theorem A.

**Corollary 1.3.** *Let* $f$ *be a Misiurewicz polynomial of degree 2 such that it is not renormalizable of disjoint type and let* $H(f)$ *be its Hubbard tree. Then,*

$$h(f|_{H(f)}) \geq \frac{\log 2}{\mathrm{End}(H(f))}$$

*where* $\mathrm{End}(H(f))$ *denotes the number of endpoints of* $H(f)$*.*

Hence if a Misiurewicz polynomial $f$ of degree 2 is not renormalizable of disjoint type, the entropy of the map $f|_{H(f)}$ is always bounded below by a positive quantity which depends on each particular polynomial. One might then ask whether there exists a universal lower bound for the topological entropy of such polynomials. The answer to this question is negative as shown in the following proposition.

**Proposition D.** *There exists a family of non renormalizable quadratic Misiurewicz polynomials* $\{g_n\}_{n\geq 1}$ *such that* $h(g_n|_{H(g_n)})$ *tends to zero as* $n$ *tends to infinity.*

In fact one can also find sequences of quadratic Misiurewicz polynomials verifying the same property which are either renormalizable of disjoint type or renormalizable of $\beta$–type (see Subsection 2.1 for a definition of these terms).

The rest of the paper is organized as follows. Section 3 contains the proofs of Theorems A, B and C, and Proposition D, while Section 2 is meant to be a summary of the definitions and tools needed for those proofs. These preliminaries are distributed into independent subsections, according to the subject they belong to.

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## 2 Definitions and Preliminaries

### 2.1 Quadratic polynomials, renormalization and the Yoccoz puzzle

The Yoccoz puzzle (see for example [18, 21]) is a useful tool to deduce renormalization properties of quadratic polynomials. To define the Yoccoz puzzle we use an orthogonal set of coordinates in the basin of attraction of infinity: equipotentials and external rays. These coordinates are defined in the same way for polynomials of any degree $d \geq 2$ with a connected Julia set.

Let $f(z) = z^2 + c$ have a connected Julia set. Since the point at $\infty$ is a superattracting fixed point, the dynamics in its basin of attraction, $A(\infty)$, are very simple. One can find a holomorphic change of variables $\psi_f : \mathbb{C} \setminus \mathbb{D} \to A(\infty)$ (called the *Böttcher coordinates* at infinity) that conjugates $f\big|_{A(\infty)}$ to the map $z \mapsto z^2$ on the complement of the closed unit disk. This change is unique if we require the derivative at infinity to be one.

The image under $\psi_f$ of a circle of radius $\exp(\eta) > 1$ in $\mathbb{C} \setminus \mathbb{D}$ is a simple closed curve in $A(\infty)$ called an *equipotential* of potential $\eta$. We denote the potential function defined this way by $G(z) := G_f(z)$. Thus an equipotential of potential $\eta$ is mapped 2 to 1 under $f$ to an equipotential of potential $2\eta$ (see Figure 7). Parameterizing the arguments of the unit circle between 0 and 1, the image under $\psi_f$ of a ray of argument $t$ is called an *external ray of argument* $t$ and denoted by $R_f(t)$. Again, since $\psi_f$ is a conjugacy, an external ray of argument $t$ is mapped to an external ray of argument $2t$ (mod 1). Equipotentials and external rays give us orthogonal coordinates in the superattracting basin.

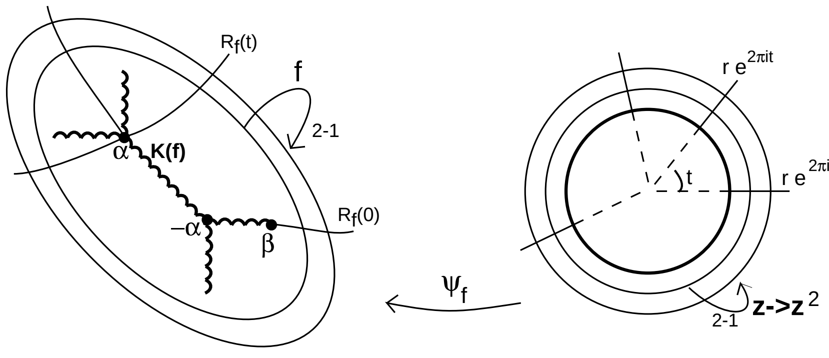


Figure 7: Böttcher coordinates, equipotentials and external rays.

From now on, we assume that both fixed points of $f$ are repelling. Then one fixed point (to be called $\beta$) is the landing point of the external ray of argument zero and the other (called $\alpha$) is the landing point of a cycle of $q$ external rays where $q \geq 2$ (see for example [20, 22]).

**Remark 2.1.** By a simple combinatorial argument on external rays, using that $\theta \mapsto 2\theta$ (mod 1) is order preserving, one can show that $0, f(0), \ldots, f^{q-1}(0)$ lie in different components of $J(f) \setminus \{\alpha\}$.

Suppose $f$ is renormalizable for $n > 1$ as defined in Section 1 and let $K_n(i)$ for $0 \leq i < n$ be the small filled Julia sets. For $0 \leq i < n$ we denote the boundary of $K_n(i)$ by $J_n(i)$. The sets $J_n(i)$ are called the *small Julia sets*. Each of these small Julia sets is homeomorphic to the actual Julia set of the renormalized polynomial. Thus, each $J_n(i)$ is $f^n$–forward invariant, and hence $f(J_n(i)) = J_n(i+1 \pmod n)$. We denote the $\alpha$ and $\beta$ fixed points of $J_n(i)$ (under $f^n$) by $\alpha_n(i)$ and $\beta_n(i)$ for $0 \le i < n$. From [19], the renormalization can only be of one of the following types.

(i) $f^n$ is renormalizable of *disjoint type* if the small Julia sets are all disjoint.

(ii) $f^n$ is renormalizable of *$\beta$–type* if all intersections among small Julia sets occur at their $\beta$ fixed points.

(iii) $f^n$ is renormalizable of *$\alpha$–type* or *crossed type* if all intersections among small Julia sets occur at their $\alpha$ fixed points.

We now give some examples (without proofs) of renormalizable polynomials of each type above.

**Examples.**

(i) The polynomial in Example 1 (see Figure 1) is renormalizable of disjoint type for $n = 2$ and of $\beta$–type for $n = 4$. Indeed, the subset $E$ defined in the example has two disjoint components which are the Hubbard trees of the small Julia sets for $f^2_{c_1}$. This map restricted to the right most component of $E$ (which contains the critical point) is conjugate to $z^2 - 1.543689\ldots$. On the other hand, $f^4_{c_1}$ restricted to $[\omega_4, \omega_6]$ is conjugate to $z^2 - 2$.

(ii) The polynomial in Example 2 (see Figure 2) is renormalizable of disjoint type for $n = 3$. Indeed, the subset $E$ defined in the example has three disjoint components which are the Hubbard trees of the small Julia sets. The third iterate of $f_{c_2}$ restricted to the middle component (which contains the critical point) is conjugate to $z^2 - 2$.

(iii) The polynomial in Example 3 (see Figure 3) is renormalizable of $\beta$–type for $n = 2$. The second iterate of $f_{c_3}$, restricted to the small Julia set, is conjugate to the polynomial in Example 5. The tree $T_0$ defined in the example is, under $f^2_{c_3}$, the Hubbard tree of the renormalized map. Note that the small Julia set $K_2(0)$ and its image, meet at their corresponding $\beta$–fixed points ($\beta_2(0) = \beta_2(1)$), which is the $\alpha$–fixed point of $f_{c_3}$.

(iv) The polynomial in Example 4 (see Figure 4) is renormalizable of crossed type for $n = 2$. The arc $[\omega_2, \omega_4]$ is the small Julia set $J_2(0)$, homeomorphic to the Julia set of $z^2 - 2$. The two sets $J_2(0)$ and $J_2(1) = f(J_2)$ "cross" at their $\alpha$ fixed points, which is also the $\alpha$–fixed point of $f_{c_4}$.

(v) The polynomial in Example 5 (see Figure 5) is not renormalizable for any $n > 1$. As it was shown in [19], a small Julia set $K_n(i)$, cannot contain the $\beta$–fixed point of the original polynomial, since that would make $K_n(i)$ for $i > 0$ intersect $K_n(0)$ in more than one point. In particular, when some iterate of the critical point hits the $\beta$–fixed point (as in this example), the polynomial is not renormalizable.

In what follows we define the Yoccoz puzzle construction and summarize its basic properties and applications, mainly following [21]. As always, let $\omega_0, \omega_1, \ldots$ be the critical orbit, where $\omega_i = f^i(0)$. Let $G(z)$ be the potential function and set $D = \{z \in \mathbb{C} \mid G(z) \le 1\}$. This is a compact set isomorphic to a disk that contains the filled Julia set. The *Yoccoz puzzle of depth zero* consists of the pieces $P_0(\omega_0), P_0(\omega_1), \ldots, P_0(\omega_{q-1})$ obtained by cutting the region

$D$ along the $q$ rays landing at $\alpha$ and labeling them so that the piece $P_0(\omega_i)$ contains the point $\omega_i$. Each piece is a compact set whose boundary contains the $\alpha$ fixed point, two segments of external rays and a piece of $\partial D$ (see Figure 8).

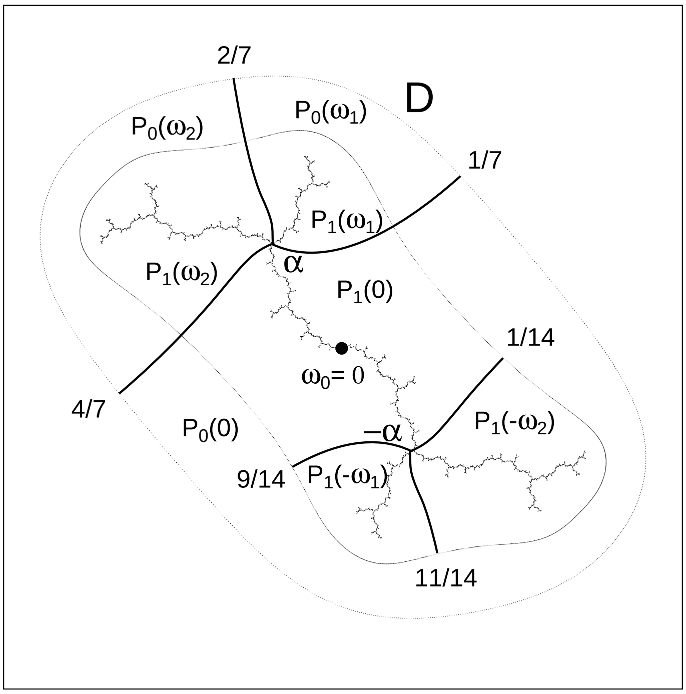


Figure 8: Some pieces of the Yoccoz puzzle of $f_{c_5}$ of Example 5.

The puzzle pieces of depth $d > 0$ are defined by induction as the connected components of $f^{-1}(P)$ where $P$ ranges over all puzzle pieces of depth $d-1$. The puzzle pieces of depth $d$ have disjoint interiors and each of them is contained in a unique piece of depth $d-1$. Any point $z \in K(f)$ which is not a preimage of $\alpha$ is contained in a unique puzzle piece at each depth, which we denote by $P_d(z)$.

In the next section we will use the following three lemmas to deduce renormalization of disjoint type, $\beta$–type and crossed type respectively. We always assume that $f$ is a quadratic polynomial with a connected Julia set, with both fixed points repelling and $q$ external rays landing at $\alpha$.

**Lemma 2.2 (Lemma 2 of [21]).** *Suppose the orbit of the critical point avoids the $\alpha$ fixed point. If $P_d(0) = P_d(\omega_p)$ for all depths $d$ and some $p > 1$ then $f^p$ is renormalizable.*

**Lemma 2.3 (Lemma 3 of [21]).** *If the critical orbit is entirely contained in the closure of $P_1(\omega_0) \cup P_1(\omega_1) \cup \cdots \cup P_1(\omega_{q-1})$ (that is, of the union of the puzzle pieces of depth one that touch the fixed point $\alpha$), then $f^q$ is renormalizable of disjoint type or $\beta$–type.*

**Lemma 2.4.** *If there exists $n \in \mathbb{N}$ such that $n$ divides $q$, $2n \leq q$ and $\{\omega_{nk}\}_{k\in\mathbb{N}}$ lies entirely in the closure of*

$$P_1(0) \cup P_1(\omega_n) \cup \cdots \cup P_1(\omega_{q-n}) \cup P_1(-\omega_n) \cup P_1(-\omega_{2n}) \cup \cdots \cup P_1(-\omega_{q-n}),$$

*then $f^n$ is renormalizable.*

The proofs of the lemmas above make use of the so called *thickened puzzle pieces.* Intuitively, a thickened puzzle piece $\widehat{P}_0(\omega_i)$ (for $0 \leq i \leq q$) is a slight enlargement of the puzzle piece $P_0(\omega_i)$ (see Figure 9[1] and [21] for details). By the usual inductive procedure, the thickened puzzle pieces of depth $d > 0$ are the connected components of $f^{-1}(\widehat{P})$, where $\widehat{P}$ ranges over all the thickened puzzle pieces of depth $d-1$. The main virtue of these thickened pieces is the following: *If a puzzle piece $P_d(z)$ contains $P_{d+1}(z)$ then the corresponding thickened puzzle piece $\widehat{P}_d(z)$ contains $\widehat{P}_{d+1}(z)$ in its interior.* Indeed, in all the the three lemmas above one can find puzzle pieces ($p_d(0)$ for $d$ large enough in the case of Lemma 2.2 and $P_1(0)$ in the case of Lemma 2.3) that satisfy all the requirements in the definition of a renormalizable map, except for the fact that these pieces are contained, but not compactly contained, in their images under the appropriate iterate of $f$. The use of thickened pieces takes care of this problem.

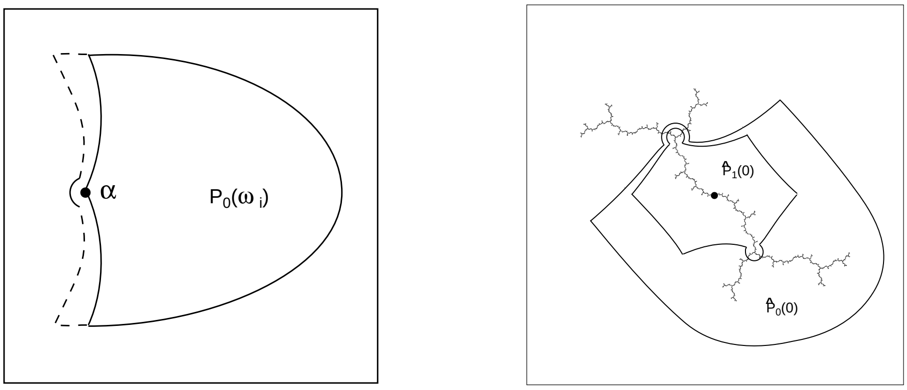


Figure 9: Sketch of a puzzle piece $P_0(\omega_i)$ and its corresponding thickened puzzle piece $\widehat{P}_0(\omega_i)$.

**Remark 2.5.** Lemma 3 of [21] (Lemma 2.3 here) is included in a chapter where it is generally assumed that the critical orbit does not hit the fixed point $\alpha$. To prove it though, one only needs to work with thickened pieces up to level one and hence this assumption is not necessary.

**Proof of Lemma 2.4.** Set

$$U' = P_n(0) \cup P_n(\omega_n) \cup P_n(\omega_{2n}) \cup \cdots \cup P_n(\omega_{q-n}) \cup P_n(-\omega_n) \cup P_n(-\omega_{2n}) \cup \cdots \cup P_n(-\omega_{q-n})$$

and

$$V' = P_0(\omega_n) \cup P_0(\omega_{2n}) \cup P_0(\omega_{3n}) \cup \cdots \cup P_0(0).$$

Notice that the pieces $P_0(\omega_n), P_0(\omega_{2n}), \ldots, P_0(\omega_{q-n})$ do not contain any $n^{\text{th}}$ (or smaller) preimage of $\alpha$, so the rays bounding $P_n(\omega_i)$ are the same as the ones bounding $P_0(\omega_i)$, for $i = n, 2n, \ldots, q-n$ (see Figure 10).

[1] Although this figure has been made by the authors, the design is obtained from [21].

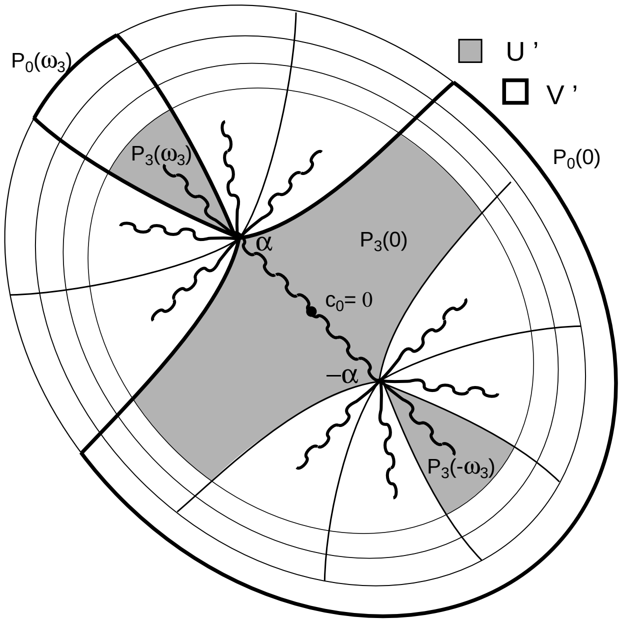


Figure 10: Sketch of the construction in the proof of Lemma 2.4 for $n = 3$ and $q = 6$.

It is easy to check that $U' \subset V'$ and that $f^n$ maps $U'$ in $V'$ with degree two. Moreover, the orbit of the critical point (under $f^n$) is contained in $U'$. To conclude that $f^n$ is renormalizable we would need to see that $U'$ is contained *in the interior* of $V'$ and that the critical orbit (under $f^n$) is entirely contained in the interior of $U'$ (which is not the case if the orbit hits the fixed point $\alpha$). This is obviously not true but if we replace all puzzle pieces by thickened puzzle pieces then all requirements are satisfied. $\square$

## 2.2 Hubbard Trees

Let $f$ be a Misiurewicz polynomial of degree $d \geq 2$ and let $H(f)$ be its Hubbard tree. To ease notation, in the rest of this subsection, we set $H = H(f)$. The following proposition states some properties of Hubbard trees that we will need in the proofs of the main theorems. For a complete description we refer to [24]. In what follows, when speaking about the *interior* of $[x, y]$ we will rather mean the set $[x, y] \setminus \{x, y\}$ instead of the usual topological interior.

**Proposition 2.6.** *Let $f$ be a Misiurewicz polynomial of degree $d \geq 2$ and let $H$ be its Hubbard tree.*

*(a) If the interior of $[x, y]$ does not contain a critical point then $f$ is one to one on $[x, y]$.*

*(b) Preperiodic points are dense in $H$.*

*(c) Let $x, y \in H$ be two preperiodic points. Then, there exists $n > 0$ such that the interior of $[f^n(x), f^n(y)]$ contains a critical point.*

*(d) Given $x, y \in H$ there exists $n > 0$ such that $[f^n(x), f^n(y)]$ contains a whole edge of $H$.*

*Moreover, when* $d = 2$,

(e) $H$ *has no invariant subtree.*

(f) *The fixed point* $\alpha$ *of* $f$ *is a point of* $H$ *of valence greater than one.*

*Proof.* Statement (a) is trivial. To show (b), let $x, y \in H$ and assume that $[x, y]$ contains no preperiodic point in its interior. Choose $t$ in the interior of $[x, y]$ and choose $q \in J(f)$, periodic and sufficiently close to $t$ so that $q$ can be joined with $H$ by an arc in $J(f)$ through a point in the interior of $[x, y]$ (this is possible because $J(f)$ is connected and locally connected, $H \subset J(f)$ and periodic points are dense in $J(f)$). Let $p$ be the joining point in the interior of $[x, y]$. It follows that $J(f) \setminus \{p\}$ has more than two connected components. For a PCF polynomial, every such point is preperiodic (see Prop. 3.2 in [24]). Hence $p$ is preperiodic contradicting the assumption that $[x, y]$ contained no preperiodic point in its interior.

Statement (c) is Proposition 1.18 in [24] but we include its proof for completeness. Assume the conclusion is false. Then, the interior of $[x, y]$ contains no preimage of a critical point and hence $f^m$ is injective on $[x, y]$ for all $m > 0$. By taking high enough iterates we may assume that $x$ and $y$ are periodic. Let $m$ be the least common multiple of the periods of $x$ and $y$. Since there are a finite number of fixed points of $f^m$, we may assume that $[f^m(x), f^m(y)] = [x, y]$ does not contain any other one. But both endpoints are repelling (since $f$ is Misiurewicz) and $f^m$ is a homeomorphism of $[x, y]$ onto itself. It follows that there must be another fixed point of $f^m$ in its interior in contradiction with what was assumed.

To show (d), take two different preperiodic points in the interior of $[x, y]$. This is possible by part (b). By Statement (c), there exists $k > 0$ such that $[f^k(x), f^k(y)]$ contains a vertex in its interior (recall that each critical point is a vertex by definition). If it contains two vertices we are done. Otherwise, let $v$ be the unique vertex in the interior of $[f^k(x), f^k(y)]$ and apply the above procedure again to $[v, f^k(y)]$, to obtain $n > 0$ such that $[f^n(v), f^{n+k}(y)]$ contains a vertex $v'$ in its interior. Then, since the set of vertices is forward invariant, we have that $[f^n(v), v']$ contains the desired edge.

To see (e) let $T \subseteq H$ be an invariant subtree of $H$ (i.e. a nonempty, compact, connected, forward invariant subset of $H$). Applying part (c) we obtain that $T$ must contain the critical point and hence the critical orbit. Thus, since $T$ is connected, the convex Hull of the critical orbit, i.e. the Hubbard tree $H$ must be contained in $T$ and we are done.

As we saw in the preceding section (see Remark 2.1), $\omega_0$ and $\omega_1$ belong to different components of $J(f) \setminus \{\alpha\}$. Hence, by definition, $\alpha \in [\omega_0, \omega_1] \subset H$. This proves (f). □

## 2.3 Transitive maps on trees

This subsection is summarizes some results and techniques about continuous maps on trees. The first proposition shows that a transitive non-totally transitive map gives a useful decomposition of the space. Its proof follows from a more general theorem of Blokh (for non-connected graphs) stated in [11] and proved in [10] (see also [6] for a version of this result for locally connected compact metric spaces).

**Proposition 2.7.** *Let* $T$ *be a tree and let* $f : T \longrightarrow T$ *be transitive. Then, either* $f$ *is totally transitive or there exist* $X_0, X_1, \ldots, X_{k-1}$ *closed, connected subsets of* $T$ *with nonempty* *interior and a fixed point $y$ of $f$ of valence larger than or equal to $k$ such that*

(a) $T = \bigcup_{i=0}^{k-1} X_i$,

(b) $X_i \cap X_j = \{y\}$ *for all* $i \neq j$,

(c) $f(X_i) = X_{i+1 \pmod{k}}$ *for* $i = 1, \ldots, k-1$.

*In particular,* $f^k\big|_{X_i}$ *is transitive for all* $i \in \{0, 1, \ldots, k-1\}$.

The next result is proved by Blokh in [11] when the space is a graph, by using a spectral decomposition (and in [5] for other types of metric spaces).

**Proposition 2.8.** *Let $T$ be a tree and let $f : T \longrightarrow T$ be transitive. Then the set of periodic points of $f$ is dense.*

The rest of this subsection outlines a common technique to compute the topological entropy of tree maps which are "monotone" restricted to each of its edges.

Let $f : T \longrightarrow T$ be a tree map and let $P \subset T$ be a finite forward invariant set of $f$ which contains all endpoints and branching points of $f$. The closure of a connected component of $T \setminus P$ will be called a *$P$–basic interval.* Notice that each $P$–basic interval is homeomorphic to a closed interval of the real line. The *$f$–graph of $P$* is the oriented generalized graph having the $P$–basic intervals as vertices and arrows working as follows. If $K$ and $L$ are $P$–basic intervals and $K$ has $m$ subintervals with pairwise disjoint interiors such that the $f$–image of each of these intervals contains $L$, then there are $m$ arrows from $K$ to $L$. The *transition matrix* of the $f$–graph of $P$ is the matrix of size equal to the number of $P$–basic intervals such the the $i, j$–entry is the number of arrows from the vertex $i$ to the vertex $j$. If $M$ is such a matrix, let its largest eigenvalue be denoted by $\rho(M)$. We note that, since $M$ is a non-negative integral matrix, in view of the Perron–Frobenius Theorem (see [17]), $\rho(M)$ is in fact the spectral radius of $M$. The map $f$ is called *$P$–monotone* if the image of each $P$–basic interval is homeomorphic to a closed interval of the real line and is monotone considered as an interval map.

The next result gives the desired formula for the topological entropy of a $P$–monotone map. It can be proved in a similar way to [2, Theorems 4.4.3 and 4.4.5].

**Theorem 2.9.** *Let $f : T \longrightarrow T$ be a tree map and let $P \subset T$ be a finite forward invariant set of $f$ which contains all endpoints and branching points of $f$. Let $M$ denote the transition matrix of the $f$–graph of $P$. Then $h(f) \geq \log(\rho(M))$. Moreover, if $f$ is $P$–monotone then $h(f) = \max\{0, \log \rho(M)\}$.*

## 3 Proofs of the Main Results

Let $f$ be a quadratic Misiurewicz polynomial and set $H = H(f)$. For the proof of Theorem A we need the following three lemmas.

A closed subset $I \subsetneq H$ is called *proper* if $\mathrm{Int}(I) \neq \emptyset$. Note that the interior of the complement of any proper set is also non-empty.

**Lemma 3.1.** *Let $I$ be a proper, forward invariant subset of $H$. Then, there exists a forward invariant set $E \subseteq I$ which is a finite union of edges of $H$ (in particular $E$ is also proper).*

*Proof.* Since the interior of $I$ is nonempty, we can choose $x, y \in I$ such that $[x, y] \subset I$. By part (d) of Proposition 2.6, there exists $n > 0$ such that $[f^n(x), f^n(y)] \subset f^n([x, y])$ contains a whole edge of $H$. Since $I$ is invariant it follows that $I$ contains an edge, which we denote by $l$. Then, since the set of vertices is invariant, the set $E = \bigcup_{n \geq 0} f^n(l)$ is obviously the union of a finite number of edges and is a proper, forward invariant subset of $I$. $\square$

**Lemma 3.2.** *Any proper, forward invariant set $E \subset H$ must contain the critical point of $f$. Moreover, $E$ is not connected.*

*Proof.* The first statement follows from parts (b) and (c) of Proposition 2.6. Moreover, if $E$ were connected, it would be an invariant subtree of $H$ (since it is proper and closed); in contradiction with part (e) of Proposition 2.6. $\square$

**Lemma 3.3.** *Suppose $f^n$ is renormalizable for some $n > 1$ and let $J_n(i)$ for $0 \leq i \leq n-1$ be the small Julia sets. Then, the set $J_n(i) \cap H$ has non-empty interior for all $0 \leq i \leq n-1$.*

*Proof.* It suffices to show that $J_n(0) \cap H$ contains at least two points. Indeed, since $J_n(i) \cap H$ is connected and simply connected it follows that if $x, y \in J_n(0) \cap H$ with $x \neq y$, then $[x, y] \subset J_n(0) \cap H$. Moreover, $f^k([x, y]) \subset J_n(k) \cap H$ has non-empty interior for all $0 \leq k \leq n-1$ because $f$ is non constant.

So, let us assume that $J_n(0) \cap H$ contains only one point. Since $0 \in J_n(0) \cap H$ it follows that this point must be $\omega = 0$. But $J_n(0)$ is invariant by $f^n$ and so is $H$. Hence, $f^n(0) = 0$ which contradicts the fact that $f$ is Misiurewicz. $\square$

Now we are ready to prove Theorem A.

**Proof of Theorem A.** We denote by (a'), (b'), (c') and (d') the opposite statements to (a), (b), (c) and (d) respectively. We will prove Theorem A by showing

$$\text{(a')} \Longrightarrow \text{(b')} \Longrightarrow \text{(c')} \Longrightarrow \text{(d')} \Longrightarrow \text{(a')}.$$

To show (a') $\Rightarrow$ (b'), assume that $f^n$ is renormalizable of disjoint type for some $n > 1$. For $i = 0, 1 \ldots, n-1$ let $J_n(i)$ be the small Julia sets and set $E_i = J_n(i) \cap H$ which have non-empty interior because of Lemma 3.3. Set $E = \bigcup_{0 \leq i \leq n-1} E_i$. Then, $E$ is a closed, forward invariant subset of $H$. Moreover, $E \neq H$ because the $E_i$'s are disjoint (since the renormalization is of disjoint type) and $H$ is connected.

Since $f(E_i) = E_{i+1 \pmod n}$ and the $E_i$'s are disjoint it follows that $\alpha \notin E$. Let $C$ be the connected component of $H \setminus E$ that contains the fixed point $\alpha$. Since $E \neq H$ and $E$ is closed it follows that $C$ is open (in $H$). If $C$ is invariant then the closure of $C$ is a proper invariant subtree, in contradiction with Proposition 2.6(e). Hence, $f(C) \neq C$. Note that $f(C)$ is open, connected and intersects $C$. Therefore $C$ contains open sets whose image is contained in $E$. These sets cannot contain any periodic point of $f$, so periodic points are not dense in $H$.

The fact that (b') implies (c') follows from Proposition 2.8.

To show (c') $\Rightarrow$ (d'), assume $f$ is not transitive on $H$. Then, there exist two open sets $U$ and $V$ in $H$ such that $f^k(U) \cap V = \emptyset$ for all $k \in \mathbb{N}$. Then, the set $I = \overline{\bigcup_{k \in \mathbb{N}} f^k(U)}$ is proper (since $V \cap I = \emptyset$) and forward invariant. By Lemma 3.1 we may assume that $I$ contains an edge $l$. Then, $\bigcup_{k\in\mathbb{N}} f^k(l) \subseteq I \subsetneq H$.

Finally we prove (d') $\Rightarrow$ (a'). Let $l \subset H$ be the edge such that $E := \bigcup_{n\in\mathbb{N}} f^n(l) \subsetneq H$. Since the set of vertices is forward invariant we have that $E$ is a finite union of edges and hence closed. Also note that $E$ is proper and forward invariant. Thus, by Lemma 3.2, $E$ is disconnected and contains the critical point. Let $E_0, E_1, \ldots, E_{p-1}$ be a cycle of (pairwise different) connected components of $E$ such that $\omega_0 \in E_0$ and $f(E_i) = E_{i+1}$ for $i = 0, 1 \ldots, p-2$. To see that such a cycle exists note that since $E$ has finitely many connected components there must be a cycle among them. Also, by Proposition 2.6(c) this cycle has to contain the critical point. Without loss of generality we may assume that $p$ is the smallest possible number satisfying these properties and that $\omega_0 \in E_0$. Moreover, $p$ must be larger than one for $E_0$ cannot be an invariant subtree by Proposition 2.6(e). Therefore $\omega_0$ and $\omega_p = f^p(\omega_0)$ lie inside $E_0$.

Now $E$, and in particular $E_0$, cannot contain any preimage of $\alpha$. Indeed, that would imply that $\alpha$ belongs to all of the $E_i's$, contradicting the fact that they are disjoint. Hence the arc $[\omega_0, \omega_p]$ does not contain any preimage of $\alpha$. This implies that $P_d(\omega_p) = P_d(0)$ for all depths $d$, where these are the Yoccoz puzzle pieces defined in Section 2. Indeed, pieces with disjoint interiors only have preimages of $\alpha$ as common boundary on the Julia set. By Lemma 2.2, $f^p$ is renormalizable. It is easy to check that this renormalization is of disjoint type since the puzzle piece $\widehat{P}_d(0)$, for $d$ large enough, does not contain $E_i$ for any $i \neq 0$. $\square$

**Proof of Theorem B.** Let (a'), (b'), (c') denote the opposite statements to (a), (b) and (c). We will prove Theorem B by showing

$$\text{(a')} \Longrightarrow \text{(c')} \Longrightarrow \text{(b')} \Longrightarrow \text{(a')},$$

and (b) implies that the set of periods of $f\big|_H$ is cofinite in $\mathbb{N}$.

To see that (a') $\Rightarrow$ (c'), suppose that $f^n$ is renormalizable for some $n > 1$ and let $J_n(i)$, $0 \le i < n$ be the small Julia sets. Define $E_i = J_n(i) \cap H$ and let $E = \bigcup_i E_i$. Recall that the sets $E_i$ have non-empty interior by Lemma 3.3. Since the union of the small Julia sets is invariant and so is $H$, it follows that $E$ is invariant. Moreover, $E_i \cap E_j$ consists of at most one point, for all $i \neq j$ (see Subsection 2.1) and hence, $E_i \neq H$ for all $i$. By part (d) of Proposition 2.6, $E$ must contain an edge which we denote by $l$. If, say, $l \in E_0$, it follows that $f^k(l) \in E_{k \pmod n}$ for all $k \ge 0$. Hence we have $f^k(l) \neq H$ for all $k \in \mathbb{N}$.

To show (c') $\Rightarrow$ (b'), suppose that there exists $l$ an edge of $H$ such that $f^n(l) \neq H$ for all $n \in \mathbb{N}$. If $l$ is not contained in $\bigcup_{n\in\mathbb{N}} f^n(f(l))$ (which is a union of edges because the set of vertices is invariant), then $f$ is not transitive (by definition) and we are done. So, there exists $t \ge 1$ such that $l \subset f^t(l)$. We will show that $f^t$ is not transitive. Clearly we have an increasing sequence of sets $l \subset f^t(l) \subset f^{2t}(l) \subset f^{3t}(l) \subset \cdots$, such that, by hypothesis, $f^{kt}(l) \neq H$ for all $k \in \mathbb{N}$. Therefore, we have

$$\bigcup_{i=0}^{k} (f^t)^i(l) = f^{kt}(l) \neq H$$

for each $k \in \mathbb{N}$ and each of the sets $\bigcup_{i=0}^{k} (f^t)^i(l)$ is a union of edges. Therefore, $\bigcup_{i\in\mathbb{N}} (f^t)^i(l) \neq$

$H$ because $H$ has a finite number of edges. Consequently, $f^t$ is not transitive by definition and so $f$ is not totally transitive.

Finally we show (b') $\Rightarrow$ (a'). We note that if $f$ is not transitive then it is renormalizable of disjoint type by Theorem A. So, we may assume that $f$ is transitive. We divide the proof in two cases: the case where $-\alpha$ is not in the interior of $H$ and the case where it is.

**Case 1.** If $-\alpha \notin \mathrm{Int}(H)$ it follows that the orbit of the critical point is entirely contained in the closure of the puzzle pieces of depth one that touch the fixed point $\alpha$ (see Section 2). Indeed, if $\omega_n \in \bigcup_{i=1}^{q-1} P_1(-\omega_i)$ for some $n$, then $[\omega_0, \omega_n]$ must contain $-\alpha$ in its interior. Hence we may apply Lemma 2.3 to conclude that $f^q$ is renormalizable. We remark that this renormalization is of $\beta$–type for it cannot be of disjoint type since $f$ is transitive (see Theorem A).

**Case 2.** To deal with this case we need to introduce some notation. Let $L_0$ be the closure of the connected component of $J(f)\setminus\{\alpha, -\alpha\}$ that contains the critical point, let $L_1, L_2, \dots, L_{q-1}$ be the closures of the connected components of $J(f) \setminus \{\alpha\}$ that do not contain $L_0$ labeled in such a way that $f(L_i) = L_{i+1}$ for $0 \le i < q-1$. Let also $L'_i = -L_i$. Then, $f(L_{q-1}) = L_0 \cup \left(\bigcup_{i=1}^{q-1} L'_i\right)$. Clearly, $H \cap L_i$ is nonempty for all $0 \le i \le q-1$ and since we are assuming that $-\alpha$ belongs to the interior of $H$, we have that $H \cap L'_i$ is nonempty for some $0 \le i \le q-1$. For $i$ in this range, let $H_i = H \cap L_i$ and $H'_i = H \cap L'_i$ (which might be empty).

Since $f\big|_H$ is transitive but not totally transitive, in view of Proposition 2.7 there exist $X_0, X_1, \dots, X_{n-1}$ closed, connected subsets of $H$ with nonempty interior such that $H = \cup_{i=0}^{n-1} X_i$, $X_i \cap X_j = \{\alpha\}$ for all $i \neq j$ and $f(X_i) = X_{i+1 \pmod n}$ for $i = 1, \dots, n-1$. We also may assume that $X_0$ is such that $0 = \omega_0 \in X_0$. Clearly, each connected component of $H \setminus \{\alpha\}$ is contained in some $X_i$. Since $\omega_i = f^i(0) \in H_i$ for all $0 \le i \le q-1$, it follows by a simple combinatorial argument that the partition must be as follows.

$$\begin{aligned}
X_0 &= H_0 \cup H_n \cup H_{2n} \cup \dots \cup H_{q-n} \cup H'_n \cup H'_{2n} \cup \dots \cup H'_{q-n} \\
X_1 &= H_1 \cup H_{n+1} \cup H_{2n+1} \cup \dots \cup H_{q-n+1} \\
&\vdots \\
X_{n-1} &= H_{n-1} \cup H_{2n-1} \cup H_{3n-1} \cup \dots \cup H_{q-1},
\end{aligned}$$

where all the $H'_i$ except one could be empty. It is clear that for the partition to exist we need that $n$ divides $q$. If $q = n$ then any $H'_i$ contained in $X_0$ must be mapped to $X_1 = H_1$. Since there does not exist such $H'_i$ it follows that $q \ge 2n$. It follows easily that we are under the hypothesis of Lemma 2.4, and hence $f^n$ is renormalizable. We remark that this renormalization is of $\alpha$–type. Indeed, $\omega_n = f^n(0)$ belongs to $H_n$ and hence $[0, \omega_n] \subset J_n(0)$. Since $H_i \cap H_j = \{\alpha\}$ for all $i \neq j$, we have that $\alpha$ belongs to the interior of this arc. Hence, it follows that $\alpha$ must be the $\alpha$–fixed point of $J_n(0)$. On the other hand $\alpha$ belongs to $f([0, \omega_n])$ which is included in $J_n(1)$, so $J_n(0) \cap J_n(1) = \{\alpha\}$.

The last step of the proof of Theorem B is to show that (b) implies that the set of periods of $f\big|_H$ is cofinite in $\mathbb{N}$. In order to show this we need some preliminary definitions. Given a periodic orbit $P$ of $f\big|_H$ we define the map $f_{[P]} : [P] \longrightarrow [P]$ as $f_{[P]} = r \circ f\big|_H$ where $r : H \longrightarrow [P]$ is the natural retraction from $H$ onto $[P]$. Choose a fixed point $y$ of $f_{[P]}$ where we let $P$ be a periodic orbit of $f\big|_H$ with period larger than one. We say that $P$ has a *division with respect to $y$* if there exist $M_0, M_1, \dots, M_{l-1}$ such that

(i) $l \geq 2$,

(ii) each $M_i$ is a union of connected components of $[P] \setminus H_{P,y}$, where $H_{P,y}$ denotes the connected component of $[P] \setminus P$ which contains $y$,

(iii) $f(M_i \cap P) \subset M_{i+1 \pmod{l}}$.

From [3, Theorem A], if a continuous map from a tree to itself has a periodic orbit $P$ having *no* division with respect to any fixed point of $f_{[P]}$, then the set of periods of such a map is cofinite in $\mathbb{N}$. So, in view of [3] we only have to show that such a periodic orbit exists for $f\big|_H$. The strategy will be to construct a periodic orbit $P$ that shadows the orbits of all vertices of $H$. For such an orbit, $\alpha$ will be the only fixed point of $f_{[P]}$. If $P$ had division with respect to $\alpha$ it would imply that a partition as in Proposition 2.7 can be obtained, contradicting the assumption of total transitivity. To construct this orbit we proceed as follows.

Set $\widetilde{V} = V(f) \cup \{\alpha\}$, where $\alpha$ denotes the $\alpha$–fixed point of $J(f)$ and $V(f)$ is the set of vertices of $H$. In view of the fact that $V(f)$ is invariant and finite there exist $x_1, x_2, \dots, x_{k^*} \in \widetilde{V}$ such that $x_i \notin \mathcal{O}^+(x_j)$ for $i \neq j$ and $\widetilde{V} = \cup_{i=1}^{k^*} \mathcal{O}^+(x_i)$. Moreover, for each $i = 1, 2, \dots, k^*$ denote by $n_i$ the cardinal of $\mathcal{O}^+(x_i)$. Now let us choose a neighborhood $U$ of $\alpha$ in $H$ such that $U \cap \widetilde{V} = \{\alpha\}$. If $v$ denotes the valence of $\alpha$, then $U$ is homeomorphic to a star of $v$ branches having $\alpha$ as a branching point. Let us choose $v$ points, one in each connected component of $U \setminus \{\alpha\}$, and denote them by $x_{k^*+1}, x_{k^*+2}, \dots, x_{k^*+v}$. We also set $k = k^* + v$, $n_{k^*+1} = n_{k^*+2} = \dots = n_{k^*+v} = 0$ and $V^+ = \widetilde{V} \cup \{x_{k^*+1}, x_{k^*+2}, \dots, x_k\}$. Note that, for each $i \in \{1, 2, \dots, k\}$ and $j \in \{0, 1, \dots, n_i\}$ we have $f^j(x_i) \in V^+$. Moreover, $f^j(x_i) = f_m(x_i)$ for some $m < j$ if and only if $j = n_i$ and $i \in \{1, 2, \dots, k^*\}$.

By continuity of $f$ and the fact that $f$ is holomorphic it follows that for each $i \in \{1, 2, \dots, k\}$ we can chose an open neighborhood $U_i$ of $x_i$ (in $H$) such that $f^j(U_i)$ is open and $f^j(U_i) \cap V^+ = \{f^j(x_i)\}$ for each $j \in \{0, 1, \dots, n_i\}$. Moreover we also require $f^l(U_i) \cap f^m(U_j) = \emptyset$ for all $i, j \in \{1, 2, \dots, k\}$, $l \in \{0, 1, \dots, n_i\}$ and $m \in \{0, 1, \dots, n_j\}$ such that $f^l(U_i) \cap V^+ \neq f^m(U_j) \cap V^+$.

Now set $W_k = U_k$. Then, since $f\big|_H$ is transitive, there exists $l_k > 1$ such that $f^{-l_k}(W_k) \cap f^{n_{k-1}}(U_{k-1}) \neq \emptyset$. So, the set $W_{k-1} = f^{-l_k - n_{k-1}}(W_k) \cap U_{k-1}$ (when chosen the appropriate branches of the inverse) is open and non-empty. We iterate this procedure until we get an open set $W_1 \subset U_1$. By Proposition 2.8 there exists a periodic point $z \in W_1$ of $f\big|_H$ with period larger than 1. Set $P = \mathcal{O}^+(z)$. By construction of the sets $W_i$ we see that for each $i \in \{1, 2, \dots, k\}$ and $j \in \{0, 1, \dots, n_i\}$ there exists a point $t_{i,j} \in P$ such that $t_{i,j} \in f^j(U_i)$ and $t_{i,j} = f(t_{i,j-1})$ when $j > 0$. Therefore, by the choice of $x_{k^*+1}, x_{k^*+2}, \dots, x_k$ and $U_{k^*+1}, U_{k^*+2}, \dots, U_k$ it follows that $\alpha \in [t_{k^*+1,0}, t_{k^*+2,0}, \dots, t_{k,0}] \subset [P]$. Moreover, it is not difficult to see that $\alpha$ is the only fixed point of $f_{[P]}$.

To end the proof of the theorem we only have to show that $P$ has no division with respect to $\alpha$. Otherwise, there exist $M'_0, M'_1, \dots, M'_{l-1}$ verifying (i–iii) above. Note that, by construction, $H_{P,\alpha} \subset [t_{k^*+1,0}, t_{k^*+2,0}, \dots, t_{k,0}]$ and it is homeomorphic to a star of $v$ branches having $\alpha$ as the branching point. For each $i \in \{0, 1, \dots, l-1\}$ let $M_i$ be the unique union of connected components of $H \setminus H_{P,y}$ such that $M_i \cap [P] = M'i$.

We claim that $f(M_i \cap V(f)) \subset M_{i+1 \pmod{l}} \cup \{\alpha\}$ for $i = 0, 1, \ldots, l-1$. To prove this fact note that, by choice of the sets $U_i$, $H_{P,\alpha} \cap f^j(U_i) = \emptyset$ for each $i \in \{1, 2, \ldots, k^*\}$ and $j \in \{0, 1, \ldots, n_i\}$ such that $\alpha \notin f^j(U_i)$. Consequently, since each of the sets $M_r$ is in the complement of $H_{P,\alpha}$, it follows that

(*) if for some $r \in \{0, 1, \ldots, l-1\}$, $i \in \{1, 2, \ldots, k^*\}$ and $j \in \{0, 1, \ldots, n_i\}$ we have $\alpha \notin f^j(U_i)$ and $M_r \cap f^j(U_i) \neq \emptyset$ then $f^j(U_i) \subset M_r$.

Now fix $r \in \{0, 1, \ldots, l-1\}$ and a point in $M_r \cap V(f)$. Since $\alpha \in H_{P,\alpha}$ and $M_r$ is contained in the complement of $H_{P,\alpha}$, this point is of the form $x_{i,j} = f^j(x_i) \in f^j(U_i)$ with $i \in \{1, 2, \ldots, k^*\}$, $j \in \{0, 1, \ldots, n_i - 1\}$ and $\alpha \neq x_{i,j}$. In particular we have $\alpha \notin f^j(U_i)$. We have to show that $f(x_{i,j}) \in M_{r+1 \pmod{l}} \cup \{\alpha\}$. If $f(x_{i,j}) = \alpha$ we are done. So, we assume that $f(x_{i,j}) \neq \alpha$. By (*), $f^j(U_i) \subset M_r$. Hence, $t_{i,j} \in M_r$ and, thus, $f(t_{i,j}) \in M_{r+1 \pmod{l}} \cap f^{j+1}(U_i)$ by (iii). Also, since $f^{j+1}(U_i) \cap V^+ = \{f(x_{i,j})\}$ and $f(x_{i,j}) \neq \alpha$ we get $\alpha \notin f^{j+1}(U_i)$. Hence, again by (*), $f(x_{i,j}) \in f^{j+1}(U_i) \subset M_{r+1 \pmod{l}}$. This ends the proof of the claim.

Now set $\widetilde{M_i} = [M_i \cup \{\alpha\}]$. From the claim above, the fact that $V(f)$ contains the critical point of $f$ and Proposition 2.6(a) it follows that if $L$ is the closure of a connected component of $H \setminus \widetilde{V}$ then $L \subset \widetilde{M}_s$ for some $s \in \{0, 1, \ldots, l-1\}$ and $f(L) \subset \widetilde{M}_{s+1 \pmod{l}}$. In other words, $f(\widetilde{M_s}) \subset \widetilde{M}_{s+1 \pmod{l}}$. Consequently, $\widetilde{M}_0, \widetilde{M}_1, \ldots, \widetilde{M}_{l-1}$ give a partition as in Proposition 2.7 and hence $f\big|_H$ cannot be totally transitive. $\square$

**Proof of Theorem C.** By Proposition 1.1(c) and [2, Lemma 4.1.3] we get

$$h(f\big|_H) \leq h(f\big|_{J(f)}) = \log d,$$

because $H \subset J(f)$ and $H$ is forward invariant. So, we only have to show that $h(f\big|_H) < \log d$ whenever $H \neq J(f)$. To this end we will use the techniques from Section 2.3 (see also [9], [2, Section 4.4]). Let $M$ be the matrix of the $f$–graph of $V(f)$. From the fact that $V(f)$ contains the critical point of $f$ and Proposition 2.6(a), it follows that $f$ is $V(f)$–monotone. Hence, by Theorem 2.9, $h(f\big|_H) = \max\{0, \log(\rho(M))\}$. Thus, it is enough to prove that if $H \neq J(f)$ then $\rho(M) < d$. So, in what follows we assume that $H \subsetneq J(f)$ and, to simplify notation, we will denote $f\big|_H$ by $\varphi$.

We claim that for each $x \in H$ there exists $n(x) \in \mathbb{N}$ such that $\operatorname{Card}(\varphi^{-n(x)}(x)) < d^{n(x)}$, where $\operatorname{Card}(\cdot)$ denotes the cardinality of a set. To prove the claim we assume the contrary. Then, there exists $z \in H$ such that $\operatorname{Card}(\varphi^{-m}(z)) \geq d^m$ for each $m \in \mathbb{N}$. Note that $\operatorname{Card}(f^{-m}(z)) = d^m$ for each $m \in \mathbb{N}$. Hence, $\varphi^{-m}(z) = f^{-m}(z)$ for each $m \in \mathbb{N}$. On the other hand, since $H \subset J(f)$, it is closed, and the preimages of any point are dense in $J(f)$ (see for example [7]), then

$$J(f) = \overline{\cup_{m=1}^{\infty} f^{-m}(z)} = \overline{\cup_{m=1}^{\infty} \varphi^{-m}(z)} \subset H;$$

a contradiction. This ends the proof of the claim.

From Proposition 2.6(a) and the fact that $V(f)$ is invariant it follows that if $L$ is an edge of $H$ and $U$ is an open set (in $H$) contained in the interior of an edge then either $f(L) \cap U = \emptyset$ or $f(L) \supset U$. The inductive use of this fact shows that each point in the interior of an edge has the same number of preimages by $\varphi^m$ for each $m \in \mathbb{N}$. So, by the above claim, for each edge $L$ there exists a positive integer $n(L)$ such that for each $x$ in the interior of $L$ we have that $\operatorname{Card}(\varphi^{-n(L)}(x)) < d^{n(L)}$. Note that, for each $m \geq n(L)$ we have

$$\begin{aligned}\operatorname{Card}(\varphi^{-m}(x)) = \operatorname{Card}(\varphi^{-m+n(L)}(\varphi^{-n(L)}(x))) = \operatorname{Card}\left(\bigcup_{y\in\varphi^{-n(L)}(x)} \varphi^{-m+n(L)}(y)\right)\\ \leq d^{m-n(L)}\operatorname{Card}(\varphi^{-n(L)}(x)) < d^{m-n(L)}d^{n(L)} = d^m.\end{aligned}$$

Therefore, for all $\nu > \max n(L)$ where $L$ ranges over all edges of $H$, we have $\operatorname{Card}(\varphi^{-\nu}(x)) < d^\nu$ for each $x \in H \setminus V(f)$. Consequently, by [2, Lemma 4.4.1], the sum of each column of $M^\nu$ is smaller than $d^\nu$. So, there exists $\gamma < d$ such that the sum of each column of $M^\nu$ is smaller than $\gamma^\nu$. Let $v$ denote the vector of size $r$ (where $r$ denotes the order of the matrix $M$) which has all entries equal to 1. Clearly, $vM^\nu \leq \gamma^\nu v$. By induction we have $vM^{l\nu} \leq \gamma^{l\nu} v$ for each $l \in \mathbb{N}$. Hence, the sum of all entries of $M^{l\nu}$ is $vM^{l\nu}v' \leq \gamma^{l\nu}vv' = r\gamma^{l\nu}$, where $v'$ denotes the transpose of $v$. Thus, (see [25]),

$$\rho(M) = \lim_{l\to\infty} \sqrt[l]{vM^lv'} = \lim_{l\to\infty} \sqrt[l\nu]{vM^{l\nu}v'} \leq \lim_{l\to\infty} \sqrt[l\nu]{r\gamma^{l\nu}} = \left(\lim_{l\to\infty} \sqrt[l\nu]{r}\right)\gamma = \gamma < d.$$

This ends the proof of the theorem. □

Before proving Proposition D we recall the notion of an $n$–star. An *$n$–star* $X_n$ is a tree which has a unique branching point denoted by $b$ which has valence $n$. The closure of a connected component of $X_n \setminus \{b\}$ will be called a *branch of* $X_n$.

We endow each branch of $X_n$ with a linear ordering such that $b$ is the smallest point on the branch while the endpoint is the largest one.

**Proof of Proposition D.** For $n \in \mathbb{N}$ consider an $(n+1)$–star $X_{n+1}$ and a set $W \subset X_{n+1}$ such that (see Figure 11):

(i) $W = \{\omega_0, \omega_1, \ldots, \omega_{n+1}\}$.
(ii) The points $\omega_1, \omega_2, \ldots, \omega_{n+1}$ are the endpoints of $X_{n+1}$.
(iii) $b < \omega_0 < \omega_{n+1}$.

Now consider a continuous map $\varphi_n : X_{n+1} \longrightarrow X_{n+1}$ such that $\varphi_n(b) = b$, $\varphi_n(\omega_i) = \omega_{i+1}$ for $i = 0, 1, \ldots, n$, $\varphi_n(\omega_{n+1}) = \omega_{n+1}$ and $\varphi_n$ is injective on the closure of each connected component of $X_{n+1} \setminus (W \cup \{b\})$. Note that then $\varphi_n$ is $(W \cup \{b\})$–monotone.

It can be seen that $X_{n+1}$ is homeomorphic to the Hubbard tree of a non renormalizable quadratic complex polynomial $g_n$ and that $\varphi_n$ is conjugate to $g_n\big|_{H(g_n)}$. Moreover, $\omega_0$ is the critical point of $g_n$ and, consequently, $g_n$ is Misiurewicz. In fact, Example 5 shows the Julia set of $g_2$.

To end the proof of the proposition we only have to show that $\lim_{n\to\infty} h(\varphi_n) = 0$. To prove this we use Theorem 2.9. We start by computing the $\varphi_n$–graph of $W \cup \{b\}$. To this end we label the closures of the connected components of $X_{n+1} \setminus (W \cup \{b\})$ according to the largest endpoint. That is, let $[x, y]$ be the closure of a connected component of $X_{n+1} \setminus (W \cup \{b\})$. Clearly, $[x, y]$ is contained in a branch of $X_{n+1}$. So we may suppose that $x < y$. Hence,

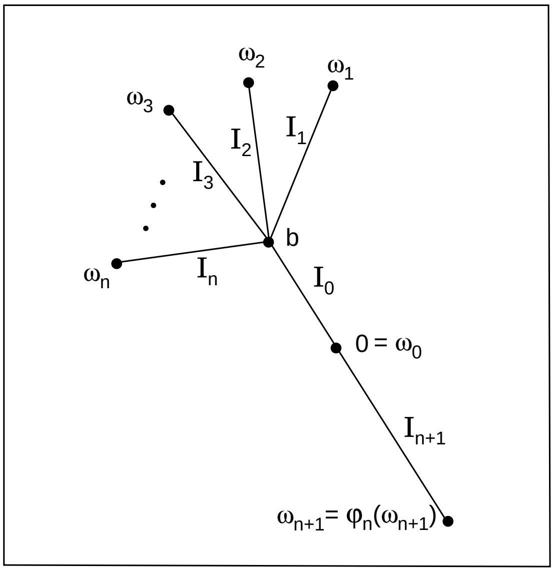


Figure 11: The $(n+1)$–star and the points $\omega_0, \omega_1, \ldots, \omega_{n+1}$.

$y = \omega_i$ for some $i \in \{0, 1, \ldots, n+1\}$. Then, $[x, y]$ will be denoted by $I_i$. With this notation, the $\varphi_n$–graph of $W \cup \{b\}$ is:

$$
\begin{array}{ccccccccccc}
 & & & & & & & I_{n+1} & \rightarrow & I_{n+1} & \\
 & & & & & & \nearrow & & \searrow & & \\
I_1 & \rightarrow & I_2 & \rightarrow & \ldots & \rightarrow & I_n & \downarrow & & I_1 & \\
 & & & & & & \searrow & & \nearrow & & \\
 & & & & & & & I_0 & & &
\end{array}
$$

Let $M_n$ denote the transition matrix of the $\varphi_n$–graph of $W \cup \{b\}$. To compute its characteristic polynomial we use the "rome" method from [9] (see also [2, Section 4.4]). Indeed, we take $I_1$ and $I_{n+1}$ as a "rome" and we get that the characteristic polynomial of $M_n$ is $(-1)^n x(x^{n+1} - x^n - 2)$. Note that $\rho(M_n)$ is the unique point larger than 1 where $x^n$ intersects the curve $\frac{2}{x-1}$. Since $x^n < x^m$ for all $m > n$, we see that $\rho(M_m) < \rho(M_n)$ and $\lim_{n\to\infty} \rho(M_n) = 1$. Therefore, by Theorem 2.9 it follows that

$$\lim_{n\to\infty} h(\varphi_n) = \lim_{n\to\infty} \max\{0, \log(\rho(M_n))\} = \lim_{n\to\infty} \log \rho(M_n) = 0.$$

This ends the proof of the proposition. □

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