Dynamics of the Complex Standard Family

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Abstract

The complexification of the standard family of circle maps $\theta \mapsto \theta + \alpha + \beta \sin(\theta) \bmod (2\pi)$, whose parameter space contains the well-known Arnold tongues, is given by $F_{\alpha\beta}(\omega) = \omega e^{i\alpha} e^{(\beta/2)(\omega - 1/\omega)}$, a holomorphic map of $\mathbb{C}^*$ with essential singularities at 0 and ∞. For real values of the parameters, we study the dynamical plane of the family $F_{\alpha\beta}$. Near the essential singularities we prove the existence of *hairs* in the Julia set, an invariant set of curves organized by some symbolic dynamics, and whose points (that are not endpoints) tend exponentially fast to 0 or ∞ under iteration. For $\beta < 1$, we give a complex interpretation of the bifurcations of the family of circle maps. More precisely, we give a new characterization of the rational Arnold tongues in terms of some of the hairs attaching to the unit circle. For certain irrational rotation numbers, we show that the Fatou set consists exclusively of a Herman ring and its preimages. For $\beta > 1$ we prove that, under certain conditions, all hairs end up attached to the unit circle as we increase the parameter.

1 Introduction

The standard family of maps of the circle is the two parameter family given by

$$F_{\alpha\beta}(\theta) = \theta + \alpha + \beta \sin(\theta) \bmod (2\pi), \quad \theta \in \mathbb{R}, \tag{1}$$

where α and β are real parameters. We restrict to the case $0 \leq \alpha \leq 2\pi$ to obtain a degree one map and to $\beta \geq 0$ because other real values of β give rise to dynamical systems equivalent to the ones we consider. These maps are interesting because they are simple perturbations of rigid rotations and it is possible to study how the dynamical properties such as orbit structure vary with the parameters. For example, for a given map in the family, it is possible to assign a *rotation number* to each point on the circle, which measures the asymptotic rate of rotation. For a map with $\beta \leq 1$ this rate is the same for all points on the circle.

Real techniques have been used fairly successfully to describe the decomposition of the parameter plane into subsets on which the rotation number is constant; these are called

*This work was done while the author was at Boston University and later at Univ. Autònoma de Barcelona. Partially supported by DGICYT grant number PB96-1153

Arnold Tongues (see Figure 1). They are curves when the rotation number is irrational but they have interior when the rotation number is rational. When the parameter β reaches the value 1, the maps are no longer homeomorphisms of the circle and the parameter plane structure becomes more complicated (see [Bo]).

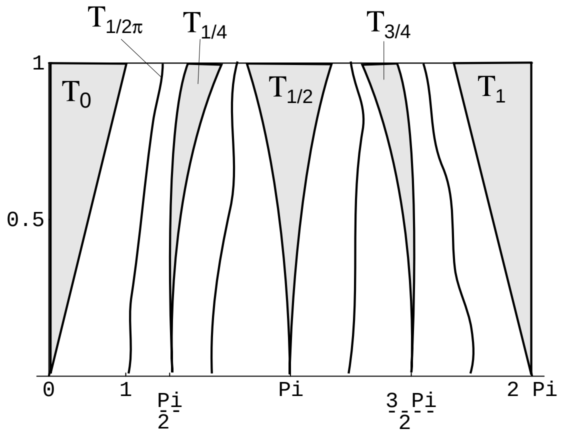

Figure 1: Sketch of the Arnold's Tongues for $\beta < 1$.

In this paper, we use complex techniques to find new invariants to characterize the Arnold Tongues. The family

$$F_{\alpha\beta} : \mathbb{C}^* \to \mathbb{C}^*; \quad F_{\alpha\beta}(w) = w\, e^{i\alpha}\, e^{\frac{\beta}{2}(w - \frac{1}{w})},$$

defined on the punctured plane $\mathbb{C}^*$ restricts to the standard family on the unit circle. For each pair of parameter values the iterates of $F_{\alpha\beta}$ define a holomorphic dynamical system on $\mathbb{C}^*$.

As we will describe in Section 3, the dynamics of $F_{\alpha\beta}$ near the essential singularities have a structure similar to that of entire functions of finite type (like the exponential map) as it was studied in [DT]. Some modifications of their arguments (unfortunately, their theorems cannot be directly applied to this type of maps) will show that the Julia set (or chaotic set) contains an invariant set of curves, the *tails*, with a well defined combinatorial structure. In fact, this combinatorial structure characterizes the Arnold Tongues. Points on these tails tend exponentially fast to the essential singularities under iteration (see Theorem 3.3).

These tails often terminate in an endpoint, whose orbit is bounded away from the essential singularity. This fact was proved in [DT] for entire functions of finite type (satisfying certain conditions). In Section 3.3 we show that, under some conditions, periodic tails always land at an endpoint, which of course must be a repelling (or parabolic) periodic point (see Proposition 3.6 and Corollary 3.7). In fact, the combinatorial information is coded by these periodic points. This proof is general, i.e., it does not depend on the map we consider. It is an open question if, in general, every repelling periodic point is the landing point of at least one tail. Each of these tails, together with its endpoint, is called a *hair* of the Julia set.

The hairs of $F_{\alpha\beta}$ can intersect the invariant circle only at their endpoints, for all points of the tail must escape to 0 or ∞ under iteration. In Section 4 we show that the bifurcations that occur when we vary the parameters α and β, can be seen from the point of view of the complex plane as certain hairs attaching to the unit circle, then pulling away. More precisely (see Theorem 4.2), we show that for $\beta < 1$, the parameters (α, β) belong to a p/q-Arnold tongue if and only if a periodic cycle of q hairs lands on the circle. In this case the Fatou set (or stable set) consists exclusively of the basin of attraction of the attracting q-cycle. Figure 2 (right) shows a sketch for (α, β) in the 1/2-tongue. Figures 2 (left) and 3 (left) are numerical observations of the dynamical plane for (α, β) in the 1/2-tongue and in the 3/4-tongue respectively, where the orbits of points in black have not come close to the essential singularities after 100 iterates, and hence are assumed to belong to the stable set. Truncated black regions are a consequence of numerical truncation. Hairs live in the grey areas and the unit circle has been emphasized by a dashed white curve.

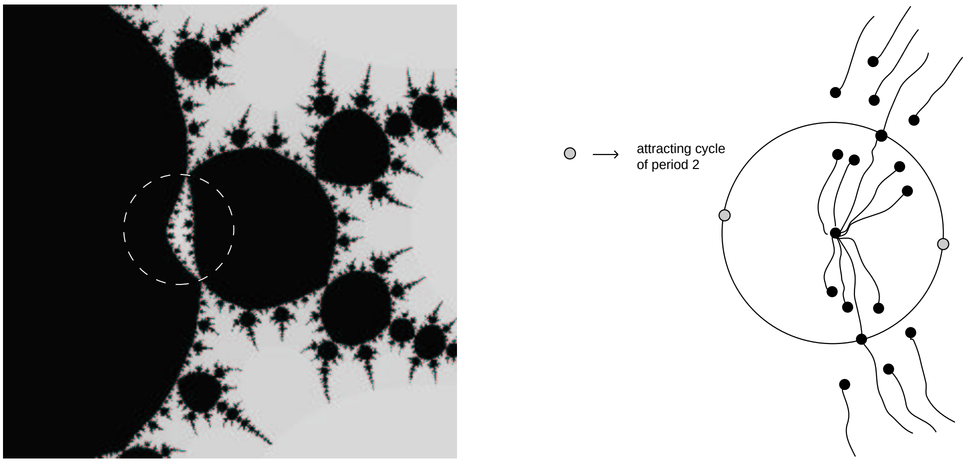

Figure 2: Left: ω-plane for (α, β) in the 1/2 tongue. $\alpha = 3.1, \beta = 0.8$. Range: $[-2, 2] \times [-2, 2]$. Right: Sketch of the hairs of $J(F_{\alpha\beta})$, for (α, β) in the 1/2 tongue.

If (α, β) lie on an irrational curve of rotation number r satisfying some arithmetic conditions, it follows from a theorem of Yoccoz that the Fatou set contains a *Herman ring*, i.e., an annulus where the map is holomorphically conjugate to a rigid rotation. We show that in fact, the ring and its preimages are the only components of the stable set. For all other values of α and β, we conjecture that the Julia set is the whole plane.

In Section 5 we describe the dynamical plane of $F_{\alpha\beta}$ for values of β larger than one. As β increases, numerical experiments indicate that more and more hairs attach to the circle (see Figure 3 (right)). We give necessary and sufficient conditions for a hair to attach to the circle and never pull away again (see Theorem 5.1).

Acknowledgments

I wish to thank Bob Devaney for introducing me to the subject and for many helpful conversations. I am grateful to Adrien Douady for providing the idea for a better proof of Proposition 4.1, and to Carsten Petersen and Tan Lei for their help in the proof of Proposition 3.6, which is in fact due to Tan Lei and Y. Yongcheng. I am grateful to Boston University, the Universitat Autònoma de Barcelona and the Centre de Recerca Matemàtica for their hospitality and to Paul Blanchard and Christian Mannes for their help in handling the various computer systems involved in this work. Finally, I wish to thank the referees for carefully reading the paper and making many suggestions that made the paper better.

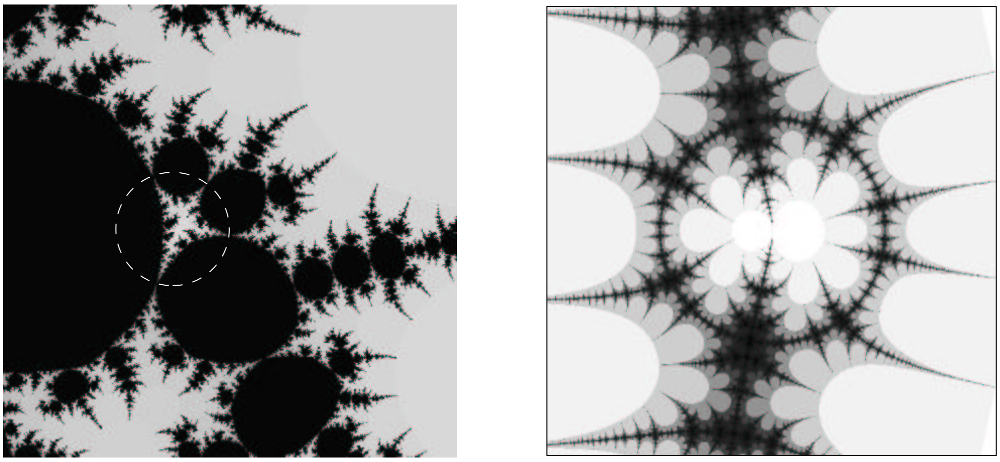

Figure 3: Left: ω-plane for (α, β) in the 3/4 tongue. $\alpha = 4.57, \beta = 0.8$. Range: $[-1.7, 1.7] \times [-1.7, 1.7]$. Right: ω-plane for $\alpha = 3.1$ and $\beta = 5$. Range: $[-2, 2] \times [-2, 2]$.

2 Preliminaries

In this section we give some preliminary definitions and results, together with the notation that will be used throughout the paper.

2.1 Circle maps

We refer to [A, Bo, He] and [Y] for the contents of this section.

For a map $F_{\alpha\beta}$ in the Standard family (1) we denote by $\rho_{\alpha\beta}(\theta)$ the rotation number of a point $\theta \in S^1$. Given $r \in \mathbb{R}$, the r–Arnold Tongue is defined as the subset of parameter space for which there is a point on the circle with rotation number r. That is,

$$T_r = \{(\alpha, \beta) \mid \exists \quad \theta \in S^1 \text{ s.t. } \rho_{\alpha\beta}(\theta) = 2\pi r\}.$$

If $\beta \leq 1$, all points on the circle have the same rotation number. Hence, Arnold Tongues of different rotation number do not overlap (see Figure 1). For a rational number p/q, the following are well known facts:

- The p/q-Arnold tongue is a closed region with nonempty interior;
- If $(\alpha, \beta) \in T_{p/q}$, the map $F_{\alpha\beta}$ has an attracting periodic orbit of period q. The converse is also true.

Let r be an irrational number. Then,

- The r-Arnold tongue is a curve;
- If $(\alpha, \beta) \in T_r$, the map $F_{\alpha\beta}$ is topologically conjugate to the rigid rotation $R_r(\theta) = \theta + r$ mod (2π). Depending on the arithmetic properties of r, this conjugacy can be shown to be analytic.

When $\beta > 1$, the rotation number is no longer the same for all points of S^1. Several tongues overlap but they do so in an orderly fashion.

2.2 Holomorphic maps of $\mathbb{C}$ or $\mathbb{C}^*$

For a holomorphic map g of $\mathbb{C}$ or $\mathbb{C}^*$, the *Fatou set* (or stable set) of g is defined as

$$N(g) = \{z \mid \{g^n\}_{n=0}^{\infty} \text{ is a normal family in some neighborhood } U \text{ of } z\}$$

The complement of the stable set is called the *Julia set*, $J(g)$. Alternatively, one can define $J(g)$ as the closure of the repelling periodic points of g.

It is well known that singular values play an essential role in the dynamics of holomorphic maps. We recall that v is a *singular value* for a holomorphic map g if g^{-1} is not well defined in any neighborhood of v. We say that an entire function, g, of $\mathbb{C}$ or $\mathbb{C}^*$ is of *finite type* if g has a finite number of singular values. For entire maps of finite type, a singular value can be of two kinds: v is a *critical value* if $v = g(c)$ where $g'(c) = 0$ (in this case c is called a *critical point*); or else v is an *asymptotic value* if there exists a path $\gamma(t) \to \infty$ such that $g(\gamma(t)) \to v$ as $t \to \infty$. (For example, $v = 0$ is an asymptotic value for the exponential map.) The importance of singular values lies on the fact that each periodic component of the stable set must have the orbit of a singular value associated to it.

Functions of finite type have been studied extensively in [Ba, Be, DT, EL2, GK, Ke, Ko1] and [Ko2]. In particular, it is known that all connected components of the stable set must be periodic or preperiodic. Moreover, if U is a periodic component of period p of $N(g)$, then we have one of the following possibilities:

- U contains an attracting periodic point z_0 of period p. Then $g^{np} \to z_0$ for $z \in U$ as $n \to \infty$ and U is called the *immediate basin* of attraction of z_0. Some iterate of U must contain a singular value of g.
- The boundary of U contains a periodic point z_0 of period p and $g^{np} \to z_0$ for $z \in U$ as $n \to \infty$. Then, $(f^p)'(z_0) = 1$ and U is called a *parabolic* domain. Some iterate of U must contain a singular value of g.
- The map g^p on U is analytically conjugate to an irrational rigid rotation R_r on the unit disc (then U is called a *Siegel disc*) or on an annulus (then U is called a *Herman ring*). The latter can only occur if g is a rational map or a holomorphic map of $\mathbb{C}^*$. The orbit of a singular value of g must accumulate on the boundary of U.

For an entire transcendental map g of $\mathbb{C}^*$, we have an alternative definition of the Julia set as the closure of the set of points that tend to 0 or ∞ under iteration.

2.3 Complexification of the Standard family

To find an expression for the complexification of the standard map we first define its corresponding lift in the covering space by

$$f_{\alpha\beta} : \mathbb{C} \to \mathbb{C}; \quad f_{\alpha\beta}(z) = z + \alpha + \beta \sin(z). \tag{2}$$

Note that $f_{\alpha\beta}(z + 2\pi) = f_{\alpha\beta}(z) + 2\pi$, for all $z \in \mathbb{C}$. Semiconjugating these functions by the projection e^{iz}, we send the invariant real line to the unit circle and obtain

$$F_{\alpha\beta} : \mathbb{C}^* \to \mathbb{C}^*; \quad F_{\alpha\beta}(w) = w\, e^{i\alpha}\, e^{\frac{\beta}{2}(w - \frac{1}{w})}.$$

These functions, when restricted to the unit circle, are exactly those described in equation (1). Note that this is a family of entire functions of $\mathbb{C}^*$, with essential singularities at 0 and ∞.

Clearly, the stable set of $F_{\alpha\beta}$ lifts to the stable set of $f_{\alpha\beta}$ and the same holds for the Julia set. Hence it is equivalent to work with either family of maps. Throughout the paper, we will usually state results for $F_{\alpha\beta}$ but prove them on the covering space.

The maps $f_{\alpha\beta}$ are entire functions of $\mathbb{C}$ although not of finite type. Indeed, $f_{\alpha\beta}$ has an infinite number of critical values at

$$f_{\alpha\beta}(\pm \arccos(\frac{-1}{\beta}) \pm 2k\pi), \quad k \in \mathbb{Z}.$$

However, since $f_{\alpha\beta}$ is the lift of $F_{\alpha\beta}$ which only has two critical values, this infinite number is only due to the branches of the logarithm. Thus, we essentially have two critical values and their translations by $2k\pi$, where $k \in \mathbb{Z}$.

Functions of the family $f_{\alpha\beta}$ may have *wandering domains*. These are components of the stable set which are neither periodic nor eventually periodic. However, these wandering domains for $f_{\alpha\beta}$ exist because of the branches of the logarithm, and project to basins of attraction of attracting (or parabolic) cycles for the functions $F_{\alpha\beta}$ under e^{iz}. As a consequence, the Julia set of $f_{\alpha\beta}$ is the closure of the points whose orbits have imaginary part tending to ∞ or $-\infty$ (the two asymptotic directions).

3 Dynamics near the essential singularities

Our goal in this section is to show that, near the essential singularities, the Julia sets of the maps of the standard family contain a collection of curves, the *tails*, where the function acts essentially as the well-understood shift automorphism. That is, $f_{\alpha\beta}$ simply permutes these tails according to some symbolic dynamics. At the same time, points on the tails always have orbits tending to the essential singularities.

This structure of the Julia set is usual for entire transcendental maps. For periodic functions like $\exp(z)$, $\sin(z)$ or $\cos(z)$ this phenomenon was described mainly in [DK]. The existence of these Cantor sets of curves together with their endpoints was then proved in [DT] for a geometrically defined class of entire functions of finite type. Following that paper, we call the curves with their endpoints Cantor bouquets.

The maps of the standard family do not fall into this class. We believe that no modification of the arguments in [DT] can be made in order to make their theorems hold for entire maps of $\mathbb{C}^*$ or entire maps of $\mathbb{C}$ with an unbounded set of singular values. However, the existence of the tails needs to be established since they play an essential role in the bifurcations of the standard family.

The main theorem in this section is Theorem 3.3. First we must set up the symbolic dynamics (Subsection 3.1), after which we can give a precise statement and its proof (Subsection 3.2). Finally, in Subsection 3.3 we study some cases for which these tails (for a general map) have a well defined endpoint.

3.1 Fundamental domains

In what follows we construct some fundamental domains in the complex plane, that is, regions of $\mathbb{C}$ which map under $f_{\alpha\beta}$ to the whole plane in a one to one fashion. Thus on these domains, a branch of $f_{\alpha\beta}^{-1}$ is well defined.

Consider the curves on the plane defined by the equation

$$y + \beta\cos(x)\sinh(y) = 0 \tag{3}$$

Note that these curves map to the real line under one iteration of $f_{\alpha\beta}$. The strip $\{-\pi < \mathrm{Re}(z) < \pi\}$ contains two of them which we denote by η_0 and μ_0 respectively, as shown in Figure 4. We denote the remainder of the solution curves by η_k and μ_k where $k \in \mathbb{Z}$ and

$$\mu_k = \mu_{k-1} + 2\pi \tag{4}$$

$$\eta_k = \eta_{k-1} + 2\pi. \tag{5}$$

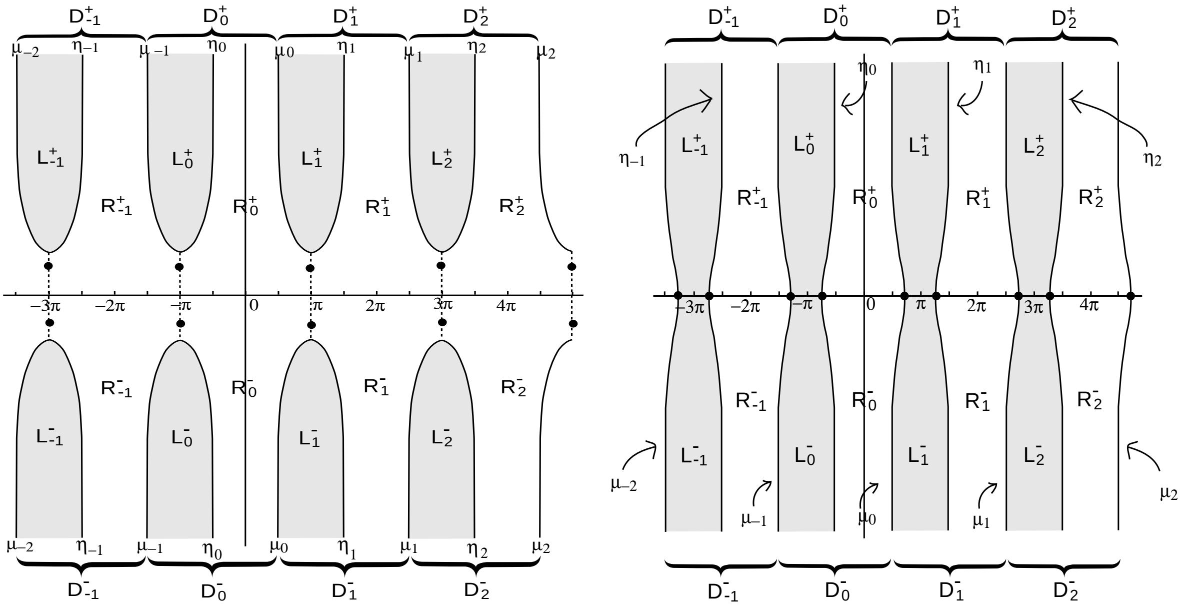

Figure 4: Fundamental domains for $f_{\alpha\beta}$ with $\beta < 1$ (left) and with $\beta > 1$ (right). Black dots denote the critical points.

It is easy to check that for all $k \in \mathbb{Z}$, the curve η_k (resp. μ_k) is asymptotic to the vertical line $\{\mathrm{Re}(z) = 3\pi/2 + 2(k-1)\pi\}$ (resp. $\{\mathrm{Re}(z) = \pi/2 + 2k\pi\}$). Their shape is different for the cases $\beta > 1$ and $\beta < 1$. In the first case, each curve is continuous and crosses the real axis at a critical point. In the second case, it has a jump discontinuity from a point of the form $z = (2n+1)\pi + i\tilde{y}$ to its conjugate, where $\tilde{y}$ is the positive solution to the equation $y = \beta\sinh(y)$. When $\beta = 1$ the intermediate case occurs; each μ_k meets η_{k+1} on the real line, exactly at the critical point.

We denote by D_k the regions bounded by the curves μ_{k-1} and μ_k. Inside these strips, we label different subregions as follows: let L_k and R_k be the left and right components of D_k, that is, the regions bounded respectively by μ_{k-1} and η_k and by η_k and μ_k. Let D_k^+ and D_k^-

be the components of D_k in the upper and lower half plane respectively. Also, following the same notation we may talk about the components L_k^+, L_k^-, R_k^+ or R_k^-.

Proposition 3.1 *For all k, $f_{\alpha\beta}$ maps each L_k^+ and R_k^- to the lower half plane in a one to one fashion, while it maps each L_k^- and R_k^+ to the upper half plane. Thus, each region $D_k^\pm$ forms a fundamental domain for $f_{\alpha\beta}$.*

Before we prove Proposition 3.1, we need some intuition (and estimates) on how these regions are mapped to their images.

Definition For any $k \in \mathbf{Z}$ and $y_0 \in \mathbb{R} - \{0\}$, a (k, y_0)-*skewed ellipse* is the image by $f_{\alpha\beta}$ of the segment of the horizontal line $\mathrm{Im}(z) = y_0$ whose end points lie on the curves μ_{k-1} and μ_k respectively.

In order to describe the shape of a (k, y_0)-skewed ellipse we first consider the image of the horizontal segment by the function $z \mapsto \beta \sin(z)$. We parametrize the segment by $h(t) = t + i\, y_0$ for $x_{k-1} \le t \le x_{k-1} + 2\pi = x_k$. Then,

$$\beta \sin(h(t)) = \beta \sin(t) \cosh(y_0) + i\, \beta \cos(t) \sinh(y_0)$$

is an ellipse centered at zero with major and minor axis equal to $\beta \cosh(y_0)$ and $\beta \sinh(y_0)$ respectively. When we now apply $f_{\alpha\beta}$ to the segment $h(t)$ we are adding the vector $h(t) + \alpha$ to each point on the ellipse $\beta \sin(h(t))$. That is

$$e(t) := f_{\alpha\beta}(h(t)) = t + \alpha + \beta \sin(t) \cosh(y_0) + i\,(y_0 + \beta \cos(t) \sinh(y_0)). \qquad (6)$$

For y_0 large enough, the radii of the ellipses are much larger than 2π. Therefore, the (k, y_0)-skewed ellipse is a slight perturbation of the translation of the actual ellipse $\beta \sin(h(t))$ (see Figure 5). Note that the three intersections with the real line correspond to the images of the three points on $h(t)$ that are also on η_k, μ_{k-1} and μ_k in this order.

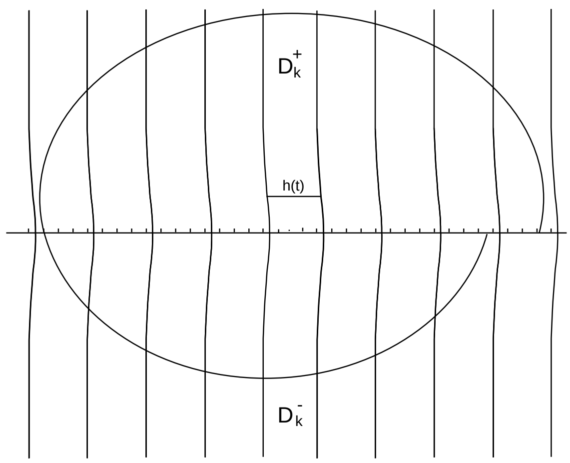

Figure 5: A (k, y_0)-skewed ellipse, that is the image of a segment $h(t) = t + iy_0$. The same figure holds for $\beta < 1$ if $h(t)$ is high enough.

We now give some estimates about the size of a (k, y_0)-skewed ellipse.

Proposition 3.2 *Let $e(t)$ be a (k, y_0)-skewed ellipse. Let* $M(\beta) = \frac{2\pi+\sqrt{4\pi^2-\beta^2}}{\beta}$ *if* $\beta \leq 2\pi$ *and* $M(\beta) = 1$ *if* $\beta \geq 2\pi$. *Then,*

(a) $y_0 - \beta \sinh(y_0) \leq \mathrm{Im}(e(t)) \leq y_0 + \beta \sinh(y_0)$.

(b) *If* $\cosh(y_0) > \max\{M(\beta), \frac{\beta}{\pi}\}$, *then the circle of radius* $\frac{\beta \sinh(y_0)}{\sqrt{2}}$ *and center* $c = x_{k-1} + \pi + \alpha + i\, y_0$ *is entirely contained inside the (k, y_0)-skewed ellipse.*

Proof : Recall the parametrization of the (k, y_0)-skewed ellipse, $e(t)$ as shown in equation (6). Statement (a) is obvious from the fact that $-1 \leq \cos(t) \leq 1$. To show statement (b) we must prove:

$$|e(t) - c|^2 > \frac{\beta^2 \sinh^2(y_0)}{2} \quad \text{for all} \quad x_{k-1} \leq t \leq x_{k-1} + 2\pi.$$

We know

$$\begin{aligned} |e(t) - c|^2 &= (t + \beta \sin(t) \cosh(y_0) - x_{k-1} - \pi)^2 + \beta^2 \cos^2(t) \sinh^2(y_0) \\ &= (t - x_{k-1} - \pi)^2 + 2\beta \sin(t) \cosh(y_0)(t - x_{k-1} - \pi) + \beta \sin^2(t) + \beta^2 \sinh^2(y_0). \end{aligned}$$

Since $-\pi \leq t - x_{k-1} - \pi \leq \pi$, we get

$$\begin{aligned} |e(t) - c|^2 &\geq -2\pi\beta |\sin(t)| \cosh(y_0) + \beta^2 \sin^2(t) + \beta^2 \sinh^2(y_0) \\ &\geq -2\pi\beta \cosh(y_0) + \beta^2 + \beta^2 \sinh^2(y_0), \end{aligned}$$

where the last inequality is due to the fact that the expression

$$-2\pi\beta |\sin(t)| \cosh(y_0) + \beta^2 \sin^2(t)$$

has its minimum when $|\sin(t)| = 1$ (since by hypothesis we have $\cosh(y_0) > \beta/\pi$). Hence,

$$|e(t) - c|^2 - \frac{\beta^2 \sinh^2(y_0)}{2} \geq -2\pi\beta \cosh(y_0) + \beta^2 + \frac{\beta^2 \sinh^2(y_0)}{2}.$$

Finally, it is easy to check that, if $\beta \geq 2\pi$, the right hand side of this inequality is greater than zero for all y_0. In the case when $\beta < 2\pi$, the same holds for all y_0 such that $\cosh(y_0) > \frac{2\pi+\sqrt{4\pi^2-\beta^2}}{\beta} = M(\beta)$. $\square$

Proof of Proposition 3.1: We prove the proposition for the case of R_k^+ and $\beta > 1$ since all the other cases can be proved using the same arguments. Consider the segment of the horizontal line $\mathrm{Im}(z) = y_0$ that is contained in R_k^+. Note that the end points of this segment are on the curves η_k and μ_k and thus they are mapped to points on the real line. As we described above, the image of the entire segment consists of the top half of a (k, y_0)-skewed ellipse (see figure 6). As $y_0 \longrightarrow \infty$, the images of these end points grow farther apart, tending to $-\infty$ and $+\infty$ respectively. The half (k, y_0)-skewed ellipse grows with them, covering the whole upper half plane as $y_0 \longrightarrow \infty$. $\square$

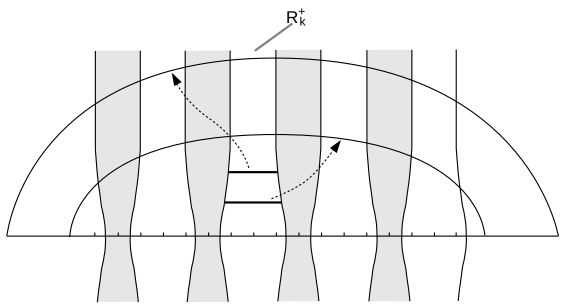

Figure 6: Image of a horizontal segment in R_k^+ by $f_{\alpha\beta}$.

3.2 Existence of the tails

Definition The *(plain) itinerary* of a point $z \in \mathbb{C}$ for $f_{\alpha\beta}$ is the infinite sequence

$$S(z) = (s_0, s_1, s_2 \dots)$$

where $s_j = k$ if $f_{\alpha\beta}^j(z) \in D_k$. The *signed itinerary* of a point $z \in \mathbb{C}$ for $f_{\alpha\beta}$ is the infinite signed sequence

$$S^\epsilon(z) = (s_0^{\epsilon_0}, s_1^{\epsilon_1}, s_2^{\epsilon_2} \dots)$$

where $s_j^{\epsilon_j} = k^+$ if $f_{\alpha\beta}^j(z) \in D_k^+$ and $s_j^{\epsilon_j} = k^-$ if $f_{\alpha\beta}^j(z) \in D_k^-$.

Note that every signed itinerary has a corresponding plain itinerary while every plain itinerary corresponds to infinitely many signed ones.

Definition A sequence $s = (s_0, s_1, \dots)$ is an *allowable* sequence for $f_{\alpha\beta}$ if there exists a real number $y > 0$ such that

$$2\pi\,|s_j| \le g^j(y) \quad \text{for all } \; j \in \mathbb{N}$$

where $g(y) = y + \beta\sinh(y) = \mathrm{Im}(f_{\alpha\beta}(iy))$. A signed sequence is allowable if its corresponding plain sequence is allowable.

We will show later on that allowable sequences are those which correspond to actual orbits of $f_{\alpha\beta}$. The main theorem below, states that the points that share the same itinerary and whose orbits tend to the essential singularities under iteration, form a continuous curve called the *tail* of the given itinerary, parametrized by the absolute value of its imaginary part.

Theorem 3.3 *For each allowable signed sequence s^ϵ, there exists $y_* \in \mathbb{R}$ and a continuous one to one curve $Z = Z_{s^\epsilon} : [y_*, \infty] \longrightarrow D_{s_0^{\epsilon_0}}$, such that:*

(a) $\mathrm{Im}(Z(y)) = \epsilon_0 y$, *for* $y \ge y_*$.

(b) $S^\epsilon(Z(y)) = s^\epsilon$, *for* $y \ge y_*$.

(c) $\lim_{n\to\infty} |\mathrm{Im}(f_{\alpha\beta}^n(Z(y)))| = \infty$, *for* $y \ge y_*$

(d) $f_{\alpha\beta}(Z_{s^\epsilon}) = Z_{\sigma(s^\epsilon)}$ *where σ is the shift automorphism* $\sigma(s_0, s_1, s_2, \dots) = (s_1, s_2, \dots)$.

(e) $Z_{s^\epsilon} \subset J(f_{\alpha\beta})$.

We have divided the proof into three separate parts.

Lemma 3.4 *A signed sequence s^ϵ is the signed itinerary for some point $z \in \mathbb{C}$ under the function $f_{\alpha\beta}$ if and only if it is allowable.*

Proof : First we prove that all itineraries corresponding to actual orbits of $f_{\alpha\beta}$ are allowable sequences. In order to do so pick a point $z = x_0 + i\, y_0$ with itinerary $S(z) = (s_0, s_1, s_2 \ldots)$. We must produce y such that $2\pi\,|s_j| \leq g^j(y)$ for all $j \in \mathbb{N}$, where $g(y) = y + \beta \sinh(y)$. By construction of the strips note that $2\pi|s_j| < |\mathrm{Re}(f^j_{\alpha\beta}(z)| + 2\pi$. Therefore, it suffices to find y such that

$$g^j(y) \geq |\mathrm{Re}(f^j_{\alpha\beta}(z)| + 2\pi. \tag{7}$$

We prove this statement by induction. Let $c \in \mathbb{R}^+$ be the solution to $\beta(\sinh(c) - 1) = \alpha$ and let

$$y \;=\; \max\{|x_0| + 2\pi, |y_0| + c\} \;=\; \max\{|\mathrm{Re}(z)| + 2\pi, |\mathrm{Im}(z)| + c\}.$$

Choosing y this way, inequality (7) is proven for $j = 0$. Assume it is proven for j and also that

$$g^j(y) \geq |\mathrm{Im}(f^j_{\alpha\beta}(z))| + c.$$

Then,

$$\begin{aligned}
g^{j+1}(y) &= g^j(y) + \beta \sinh(g^j(y)) \\
&\geq |\mathrm{Re}(f^j_{\alpha\beta}(z)| + 2\pi + \beta \sinh(|\mathrm{Im}(f^j_{\alpha\beta}(z))| + c) \\
&\geq |\mathrm{Re}(f^j_{\alpha\beta}(z)| + 2\pi + \beta(\cosh|\mathrm{Im}(f^j_{\alpha\beta}(z))| + \sinh(c) - 1) \\
&= |\mathrm{Re}(f^j_{\alpha\beta}(z)| + \beta(\sinh(c) - 1) + \beta \cosh(\mathrm{Im}(f^j_{\alpha\beta}(z))) + 2\pi \\
&\geq |\mathrm{Re}(f^j_{\alpha\beta}(z)| + \alpha + \beta|\sin(\mathrm{Re}(f^j_{\alpha\beta}(z)))| \cosh(\mathrm{Im}(f^j_{\alpha\beta}(z))) + 2\pi \\
&\geq |\mathrm{Re}(f^j_{\alpha\beta}(z)) + \alpha + \beta \sin(\mathrm{Re}(f^j_{\alpha\beta}(z))) \cosh(\mathrm{Im}(f^j_{\alpha\beta}(z)))| + 2\pi \\
&= |\mathrm{Re}(f^{j+1}_{\alpha\beta}(z))| + 2\pi,
\end{aligned}$$

where we have used that $\sinh(y + c) \geq \cosh(y) + \sinh(c) - 1$ for all $y > 0$ and $c > 0$. This proves inequality (7) for $j + 1$. Also,

$$\begin{aligned}
g^{j+1}(y) &= g^j(y) + \beta \sinh(g^j(y)) \\
&\geq |\mathrm{Im}(f^j_{\alpha\beta}(z))| + c + \beta \sinh(|\mathrm{Im}(f^j_{\alpha\beta}(z))| + c) \\
&\geq |\mathrm{Im}(f^j_{\alpha\beta}(z))| + \beta|\cos(\mathrm{Re}(f^j_{\alpha\beta}(z)))| \sinh(|\mathrm{Im}(f^j_{\alpha\beta}(z))|) + c \\
&\geq |\mathrm{Im}(f^j_{\alpha\beta}(z)) + \beta \cos(\mathrm{Re}(f^j_{\alpha\beta}(z))) \sinh(\mathrm{Im}(f^j_{\alpha\beta}(z)))| + c \\
&= |\mathrm{Im}(f^{j+1}_{\alpha\beta}(z))| + c.
\end{aligned}$$

Therefore inequality (7) is proved for all j.

We proceed now to show that any allowable sequence is the actual itinerary of some point z on the complex plane. This construction is fundamental for the proof of Theorem 3.3.

We start with a signed sequence s^ϵ for which there is a positive real number y such that $2\pi\,|s_j| \leq g^j(y)$ for all $j \in \mathbb{N}$. We restrict ourselves to the case where $\beta > 1$ and all the signs of the sequence are positive. Since we are near the essential singularities, all other cases are equivalent. Indeed, if β were small, one should reproduce the argument below by starting with $g^k(y)$ for some k instead of y. We construct a sequence of "squares" in the following way: for all j, let B_j be the "square" of height 2π inside $D^+_{s_j}$ with its lateral sides on the curves μ_{j-1} and μ_j respectively and its lower side at height $g^j(y)$ (see Figure 7).

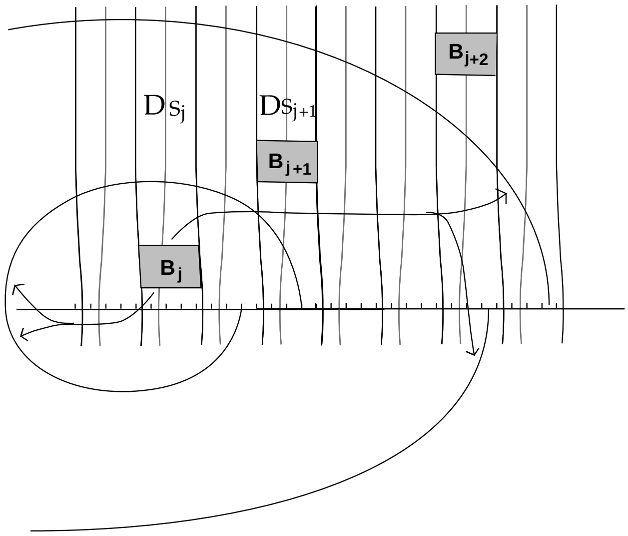

Figure 7: Construction of the B_j's and its images.

We claim that, for all j, the image of each B_j under $f_{\alpha\beta}$ covers completely B_{j+1}. To see that, recall that the image of horizontal segments like the ones we are considering are skewed ellipses (see Figure 7). In particular, the upper side of B_j maps to a $(j, g^j(y) + 2\pi)$-skewed ellipse, the lower side maps to a $(j, g^j(y))$-skewed ellipse and the lateral sides map to the real line. Therefore the image of the square B_j is the region between the two skewed ellipses. By statement (a) of proposition 3.2, the interior one has a maximum height of $y + \beta\sinh(g^j(y))$. Hence B_{j+1} is entirely outside of it.

It remains to be shown that B_{j+1} is entirely inside the $(j, g^j(y) + 2\pi)$-skewed ellipse. By statement (b) of proposition 3.2, it contains the circle with center at the middle point of the upper side of B_j, which we denote by C, and radius $\beta\sinh(g^j(y) + 2\pi)/\sqrt{2}$. We will prove that B_{j+1} is entirely contained in this circle. Without loss of generality we assume that B_{j+1} is located right of B_j, that is $s_{j+1} \geq s_j$. It suffices to show that the upper right corner of B_{j+1}, which we denote by R, is inside the circle. Thus we must show:

$$|R - C|^2 < \frac{\beta^2\sinh^2(g^j(y) + 2\pi)}{2}. \tag{8}$$

If we write down the coordinates for R and C we obtain:

$$\begin{aligned}|R-C|^2 &= (\frac{\pi}{2}+2\pi s_{j+1}-(-\frac{\pi}{2}+2\pi s_j+\alpha))^2+(g^{j+1}(y)+2\pi-(g^j(y)+2\pi))^2\\ &= (\pi-\alpha+2\pi(s_{j+1}-s_j))^2+\beta^2\sinh^2(g^j(y))\\ &\leq (\pi-\alpha+2g^j(y)+\beta\sinh(g^j(y)))^2+\beta^2\sinh^2(g^j(y))\\ &\ll (10\beta\sinh(g^j(y)))^2+\beta^2\sinh^2(g^j(y))\\ &= 101\beta^2\sinh^2(g^j(y))\\ &\ll e^{4\pi}\beta^2\sinh^2(g^j(y))/2\\ &< \beta^2\sinh^2(g^j(y)+2\pi)/2,\end{aligned}$$

where we have used that

$$2\pi|s_{j+1}-s_j| \leq 2g^j(y)+\beta\sinh(g^j(y))$$

and also that $\sinh(y+2\pi) > e^{2\pi}\sinh(y)$ for all $y \in \mathbb{R}$. Hence inequality (8) is proven.

Thus we have shown that for all j, the image of each B_j under $f_{\alpha\beta}$ covers completely B_{j+1}. Also, since we chose y larger than 2π we have that $|f'_{\alpha\beta}(z)| > 1$ on all the $B_j{}'s$. Therefore, the preimage of each B_{j+1}, is a compact connected region inside B_j. Hence, the sequence of preimages of the $B_j{}'s$, is a nested sequence of compact sets inside $D_{s_0^{\epsilon_0}}$, that converges to a unique point z with itinerary s^ϵ. □

The argument above, produces a point $Z = Z(y)$ with the prescribed itinerary for each y bigger than some y_*. We now prove that $y \mapsto Z(y)$ is a continuous curve:

Lemma 3.5 *For each allowable sequence s^ϵ, there exists $y_* \in \mathbb{R}$ such that*

$$Z_{s^\epsilon} : [y_*, \infty) \longrightarrow D_{s_0^{\epsilon_0}}$$

is continuous and one to one.

Proof : Since s^ϵ is allowable, there is some y_* for which Z_{s^ϵ} is defined. Then Z_{s^ϵ} is defined for all $y > y_*$. To prove its continuity at any point $y_0 > y_*$ consider a ball of radius $\varepsilon > 0$ around $Z_{s^\epsilon}(y_0)$, and denote it by $\mathcal{B}_\varepsilon(Z_{s^\epsilon}(y_0))$. We must find $\delta > 0$ so that if $|y_0 - y| < \delta$ then $Z_{s^\epsilon}(y) \in \mathcal{B}_\varepsilon(Z_{s^\epsilon}(y_0))$.

In the box construction above there exists $n \in \mathbb{N}$ such that the preimage by $f^n_{\alpha\beta}$ of the box B_n inside $D_{s_0^{\epsilon_0}}$ is contained in $\mathcal{B}_{\varepsilon/2}(Z_{s^\epsilon}(y_0))$. We also may assume there exists a δ' so that the preimage by $f^n_{\alpha\beta}$ of the box $B_n+\delta'$ inside $D_{s_0^{\epsilon_0}}$ is contained in $\mathcal{B}_\varepsilon(Z_{s^\epsilon}(y_0))$, where $B_n+\delta'$ is the box B_n elongated by δ' both on the top and on the bottom.

By continuity of $g(y) = y+\beta\sinh(y)$, there exists $\delta > 0$ so that if $|y_0 - y| < \delta$ then $B_n(y) \subset B_n(y_0)+\delta'$. Hence $Z_{s^\epsilon}(y) \in \mathcal{B}_\varepsilon(z_{s^\epsilon}(y_0))$.

The fact that Z_{s^ϵ} is 1-1 follows from the fact that if $y_1 < y_2$ then $g^n(y_1) \ll g^n(y_2)$ for large n. Hence the boxes containing $f^n_{\alpha\beta}(Z_{s^\epsilon}(y_1))$ and $f^n_{\alpha\beta}(Z_{s^\epsilon}(y_2))$ are disjoint. □

We now conclude the proof of Theorem 3.3.

Proof of Theorem 3.3: By construction of these curves, the first three statements are clear. In order to prove (d), fix $\hat{y} \in [y_*, \infty)$. Note that the itinerary of $f_{\alpha\beta}(Z_{s^\epsilon}(\hat{y}))$ is $(s_1^{\epsilon_1}, s_2^{\epsilon_2}, \dots) = \sigma(s^\epsilon)$. Also by construction, $Z_{s^\epsilon}(\hat{y})$ is the only point in the box $B_0(\hat{y})$ that has itinerary s^ϵ and such that $f_{\alpha\beta}^n(Z_{s^\epsilon}(\hat{y})) \in B_n(\hat{y})$. Now construct the curve $Z_{\sigma(s^\epsilon)}$. Clearly, $g(\hat{y})$ makes $\sigma(s^\epsilon)$ allowable. Hence $Z_{\sigma(s^\epsilon)}(g(\hat{y}))$ is a point on the curve $Z_{\sigma(s^\epsilon)}$ with itinerary $\sigma(s^\epsilon)$. Moreover, by construction, it is the only point on $Z_{\sigma(s^\epsilon)}$ such that $f_{\alpha\beta}^n(Z_{\sigma(s^\epsilon)}(g(\hat{y}))) \in B_n(g(\hat{y}))$. But now note that $B_n(\hat{y}) = B_{n-1}(g(\hat{y}))$ for all n and therefore $f_{\alpha\beta}(Z_{s^\epsilon}(\hat{y})) = Z_{\sigma(s^\epsilon)}(g(\hat{y}))$.

To show that the curves are contained in the Julia set we observe that these are points whose imaginary part tends exponentially fast to infinity under iteration, and hence they cannot belong to any component of the Fatou set. □

3.3 Landing of the tails

Up to this point we have proven that the Julia set contains a special type of curves, the tails, whose points escape very fast in the imaginary direction under iteration. Note that if we (semi)conjugate the function by e^{iz} these curves map to the corresponding tails around the circle that we described in the Introduction. In this case, these points escape to infinity or zero under iteration.

For entire functions of finite type (and satisfying certain conditions) it has been shown in [DK, DT] that tails with a bounded itinerary terminate in endpoints which also belong to the Julia set, but whose orbit is bounded away from the essential singularities. Obviously, an endpoint and its corresponding tail must share the same itinerary. The set of endpoints is called the *crown* and contains many repelling periodic points. It is not clear if, in general, all repelling periodic points must be the landing point of some tail.

Definition A sequence s is *essentially periodic* if there exist integers K and N such that $s_i = s_{i-N} + K$ for all $i \geq N$.

It is easy to check that the essentially periodic sequences correspond to those tails which are periodic after the projection by e^{iz}.

In what follows we will show that tails whose itinerary is essentially periodic land at an endpoint, assuming that the critical orbits stay bounded away from the tail. This proof is completely general, that is, it does not depend on the map we consider. A similar result for the case of rational maps can be found in [TY]. The precise statement is as follows (see Figure 8).

Proposition 3.6 *Let f be an entire map of $\mathbb{C}$ (or $\mathbb{C}^*$). Suppose $h : (-\infty, \infty) \to \mathbb{C}$ is a continuous one to one curve such that $f(h(t)) = h(t+1)$ and $h(t) \in J(f)$ for all $t \in \mathbb{R}$. Assume there exists $z_0 = h(t_0) \in \mathbb{C}$ and U, a neighborhood of z_0, such that $h(t) \in U$ for all $t_0 \leq t \leq t_0 + 2$ and such that the orbit of the singular values of f never enters U. Then, there exists a unique fixed point $z_\infty \in \mathbb{C} \cup \{\infty\}$ such that $h(t) \to z_\infty$ as $t \to -\infty$.*

Note that we must allow the case $z_\infty = \infty$ as it can be seen from the example $f(z) = z + i + e^{-iz}$. Indeed, the vertical lines $\{(2k+1)\pi + iy, y \in \mathbb{R}\}$ are invariant by f and f restricted to those is $y \mapsto y + 1 + e^y$. Hence those are the fixed hairs coming from imaginary ∞, but their pullback converges clearly to imaginary $-\infty$, which in this case acts as a repelling fixed point.

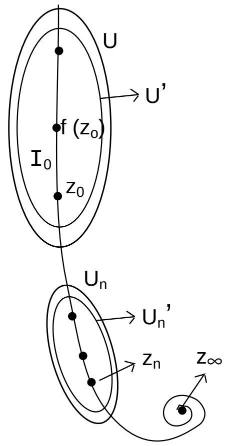

Figure 8: Sketch of an invariant curve in the hypothesis of Proposition 3.6.

Corollary 3.7 *Let s^ϵ be an essentially periodic sequence and $Z = Z_{s^\epsilon}$ be its corresponding tail obtained by Theorem 3.3. Assume that Z is bounded away from the critical orbits. Then Z can be extended and converges to a unique point $z_* = z_*(s^\epsilon)$. Moreover, either $z_* = \infty$ or e^{iz_*} is a repelling periodic point by $F_{\alpha\beta}$.*

Proof : Let s^ϵ be an essentially periodic sequence and $Z = Z_{s^\epsilon} : [y_*, \infty) \to \mathbb{C}$ be its corresponding tail obtained by Theorem 3.3. Without loss of generality we may assume that Z is an invariant curve (i.e. s^ϵ is periodic of period one). (Otherwise, we project by e^{iz} and consider an iterate of $F_{\alpha\beta}$.) We can reparametrize this curve by setting $t = 0$ at $Z(y_*)$ and $t = 1$ at $f_{\alpha\beta}(Z(y_*))$, and extending to $t = \infty$ by the conjugacy $f(Z(t)) = Z(t+1)$. Choosing an appropriate branch of the inverse, we can proceed in the same fashion to define $Z(t)$ for $t < 0$. Since the tail is bounded away from the critical orbits, this pull-back never hits any critical value of $f_{\alpha\beta}$, and hence it is well defined. It is clear then that we are under the hypothesis of Proposition 3.6 and hence there exists a unique point z_* such that $Z(t) \to z_*$ as $t \to -\infty$. Note that e^{iz_*} must be a repelling fixed point unless $z_* = \infty$. □

Proof of Proposition 3.6: See Figure 8. Let $I_0 = \{h(t) \in \mathbb{C} \mid t_0 \le t \le t_0 + 1\}$. Take a sequence $\{t_n\} \to -\infty$ such that $z_n := h(t_n)$ has an accumulation point in the compact set $\mathbb{C} \cup \infty$ and let z_∞ be this point. Define the function $k(n)$ to be the integer such that $f^{k(n)}(z_n) \in I_0$ and let U_n be the connected component of $f^{-k(n)}(U)$ containing z_n.

Let U' be an open set inside U with the same properties, and define U_n' in the obvious way. Then, $f^{k(n)}$ is an isomorphism from U_n to U and such that $f^{k(n)}(U_n') = U'$. As a consequence of Koebe's theorem the distortion must be bounded, i.e. there exists $c > 0$ such that $D(z_n, cr_n) \subseteq U_n$, where r_n is the diameter of U_n' and $D(z_n, cr_n)$ is the disc centered at z_n of radius cr_n .

To conclude the proof we must show that the diameter of U_n tends to zero. Indeed, in this case all points in U_n tend to z_∞ and since z_n and $f(z_n)$ belong to U_n for all n we have that z_∞ is a fixed point and the unique limit for $h(t)$.

So assume this is not the case. Then, there exists $\varepsilon > 0$ such that the disc $D(z_n, \varepsilon) \subseteq U_n$ for all $n \ge 1$. Since $z_n \to z_\infty$, we can find $N > 0$ such that for all $n \ge N$, $|z_n - z_\infty| < \varepsilon/2$

and hence the disc $D(z_\infty, \varepsilon/2) \subseteq U_n$.

By Montel's theorem, the family of iterates $\{f^{k(n)}\}_{n \geq N}$ is a normal family on $D(z_\infty, \varepsilon/2)$, since $f^{k(n)}(D(z_\infty, \varepsilon/2)) \subseteq U$ and hence it omits more than three points. Thus z_∞ does not belong to the Julia set, which contradicts the fact that it is an accumulation point of points of $J(f)$. $\square$

The tails and the crown together form what we call the *hairs* of the Julia set. This structure turns out to be "typical" for Julia sets of entire functions. These Julia sets contain what is called an infinite union of Cantor Bouquets, which are essentially Cantor sets of hairs like the ones we have encountered here. For details see [DK].

4 Dynamics for $\beta \leq 1$

Our goal in this section is to describe the dynamical plane of $F_{\alpha\beta}$ for different types of parameter values. The main result in this Section is Theorem 4.2, which gives a characterization of the Arnold tongues in terms of the hairs we constructed in the last section.

We start with those parameters $(\alpha, \beta) \in T_r$, where r is an irrational number. That is, the rotation number $\rho_{\alpha\beta}$ of $F_{\alpha\beta}|_{S^1}$ is irrational.

It is well known by a theorem of Denjoy that any $\mathcal{C}^2$-diffeomorphism of the circle with an irrational rotation number r is topologically conjugate to an irrational rigid rotation of rotation number r, i.e., $R_r(\theta) = \theta + r \bmod (2\pi)$. The conditions to impose on the map and on the rotation number to increase the smoothness of the conjugacy have been studied by Herman, Arnold, Russmann and Yoccoz among others. In particular, J. C. Yoccoz in [Y] shows that, if r belongs to a certain set $\mathcal{H} \in \mathbb{R} \setminus \mathbb{Q}$, then the conjugacy is $\mathbb{R}$-analytic. This set contains all the Diophantine numbers and it is optimal in the following sense: one can find an $\mathbb{R}$-analytic circle map g with rotation number $r \notin \mathcal{H}$ such that g is not analytically conjugate to $\mathcal{H}$.

The following proposition, deals with those maps of the Standard family with irrational rotation number for which this conjugacy is analytic. Its first part can be found in [Ba].

We recall that a Herman ring is a connected component of the stable set isomorphic to an annulus on which the map is analytically conjugate to an irrational rotation (see Section 2).

Proposition 4.1 *Let $F_{\alpha\beta}$ be such that $r = \rho_{\alpha\beta}$ is irrational and $F_{\alpha\beta}|_{S^1}$ is analytically conjugate to the rigid rotation of rotation number r. (In particular, this is true if $r \in \mathcal{H}$). Then, there exists a Herman ring around S^1 of rotation number r. Moreover, the stable set $N(f)$ consists exclusively of this ring and its preimages.*

Proof : Let h be the map which conjugates $F_{\alpha\beta}|_{S^1}$ to R_r. Since h is $\mathbb{R}$-analytic, it extends holomorphically to a neighborhood of S^1. Let A' be this neighborhood and note that the holomorphic map $h \circ F_{\alpha\beta} \circ h^{-1}$ is equal to R_r on S^1. Since the two maps agree on S^1, they must agree on the whole domain of definition, in particular on A'. Let A be the maximum domain of extension of h. By Theorem 3.1 in [Ke], any component of the stable set is either simply connected or an annulus. Clearly, there exists a neighborhood of the essential singularity at 0 that is not in A, but $S^1 \subset A$. Hence A is an annulus, and the required Herman ring.

To show the last statement, we recall the Sullivan classification of components of the stable set. Since $F_{\alpha\beta}$ is of finite type (only two critical points) all components are periodic or preperiodic [Ke, Ko2]. The periodic ones can only be attracting (or superattracting), parabolic, or rotation domains (Siegel discs or Herman rings). Attracting, superattracting and parabolic domains must contain a critical orbit in its interior, while the boundary of rotation domains must be contained in the ω-limit of a critical orbit. Since $F_{\alpha\beta}$ has only two singular values (with symmetric orbits) and no asymptotic values, it follows that only other rotation domains can coexist with the Herman ring A. It is shown in [Ba] that there can be at most one multiply connected component in the Fatou set, hence other Herman rings cannot exist for $F_{\alpha\beta}$. This leaves us only with the possibility of having one (or several) Siegel discs. The idea to show that this is also impossible is as follows.

We will construct an entire transcendental map $G : \mathbb{C} \to \mathbb{C}$ such that G has a fixed Siegel disc containing $\overline{\mathbb{D}}$, but G is conjugated to $F_{\alpha\beta}$ outside $\mathbb{D}$. Note that, in particular, G has only one critical point. Hence, suppose $F_{\alpha\beta}$ had at least one cycle of Siegel discs $S_1, \ldots, S_n$, apart from the ring A. If the whole cycle is outside the unit disc, then G has two distinct Siegel cycles and only one critical point. It is shown in [EL2] that for an entire transcendental map, the number of critical points is an upper bound for the number of irrational cycles, and hence this case is impossible. Finally, if the Siegel cycle of $F_{\alpha\beta}$ had a component inside $\mathbb{D}$, then some iterate of the critical point of G would have to enter $\mathbb{D}$ and hence never exit again. This is also impossible since this orbit must accumulate on the boundaries of the Siegel disks. The remainder of this proof is the construction of such a map G.

Since the conjugacy h is $\mathbb{R}$-analytic, it is in particular quasi-symmetric and hence it can be extended to a quasiconformal map H of the unit disc $\overline{\mathbb{D}}$. Define a new map $\widetilde{G} : \mathbb{C} \to \mathbb{C}$ as follows.

$$\widetilde{G} = \begin{cases} F_{\alpha\beta} & \text{on } \mathbb{C} \setminus \overline{\mathbb{D}} \\ H^{-1} \circ R_r \circ H & \text{on } \overline{\mathbb{D}}. \end{cases}$$

This map is conjugate to $F_{\alpha\beta}$ outside $\mathbb{D}$ and it has the desired properties, but it is not holomorphic. We can make it holomorphic as follows. Let σ_0 denote the standard complex structure of $\mathbb{C}$. We define a new almost complex structure σ on $\mathbb{C}$ as

$$\sigma = \begin{cases} H_*(\sigma_0) & \text{on } \overline{\mathbb{D}} \\ (G^n)_*(\sigma) & \text{on } F_{\alpha\beta}^{-n}(\overline{\mathbb{D}}) \text{ for all } n \geq 1, \\ \sigma_0 & \text{on } \mathbb{C}^* \setminus \bigcup F_{\alpha\beta}^{-n}(\overline{\mathbb{D}}). \end{cases}$$

By construction, σ has bounded distortion and is invariant under $\widetilde{G}$. We may then apply the Ahlfors–Bers theorem (see [A]) to obtain a quasiconformal homeomorphism $\varphi : \mathbb{C} \to \mathbb{C}$ such that φ integrates the complex structure σ, i.e., $\varphi_*(\sigma) = \sigma_0$. Finally, we define $G = \varphi \circ G \circ \varphi^{-1}$, which is an entire transcendental map of $\mathbb{C}$ with the desired properties. □

For all other cases of irrational rotation number we have the following conjecture.

Conjecture *Suppose $F_{\alpha\beta}|_{S^1}$ has rotation number $r \in \mathbb{R} \setminus \mathbb{Q}$, but $F_{\alpha\beta}|_{S^1}$ is not analytically conjugated to the irrational rotation R_r. Then, $J(F_{\alpha\beta}) = \mathbb{C}$.*

We proceed now with the rational cases. Let $P(z) = e^{iz}$. If s^ϵ is an allowable signed sequence and Z_{s^ϵ} its associated hair, we define the *projected hair* associated to s^ϵ to be the image of Z_{s^ϵ} under e^{iz}, i.e., $P(Z_{s^\epsilon})$.

Given $r \in \mathbb{R}$, denote by $[r]$ its integer part.

Theorem 4.2 *Let $p, q \in \mathbb{Q} \setminus [0,1)$ such that $(p,q) = 1$, and let $T_{p/q}$ be the p/q-Arnold tongue. Let $s^\epsilon := s^\epsilon(p/q)$ denote the sequence such that $s_n^{\epsilon_n} = [np/q]^+$ for all $n \geq 0$. Then, a pair of parameter values (α, β) belongs to $T_{p/q}$ if and only if $P(Z_{s^\epsilon})$ lands on the unit circle.*

Observe that, as a consequence, all images of $P(Z_{s^\epsilon})$ under $F_{\alpha,\beta}$ must also land on the unit circle. It is easy to check that $s^\epsilon(p/q)$ is essentially periodic of period q and rotation number p/q. Hence, $P(Z_{s^\epsilon})$ and its images form a periodic cycle of period q which then must land at a repelling (or parabolic) periodic orbit of period q (by Corollary 3.7). Note that this observation already proves the "only if" part since for $\beta \leq 1$ there can only be one such orbit on the unit circle and only in the case when the parameters belong to the p/q-tongue.

It follows from the symmetry of the map, that the q symmetric hairs coming from the essential singularity at $w = 0$ also land on the repelling orbit. Numerical observations are shown in Figures 2 and 3.

For illustration purposes, assume $\alpha = 0$. Then, (α, β) belong to the fixed tongue T_0 and, indeed, there are two fixed points on the unit circle: an attracting one at $w = -1$ and a repelling one at $w = 1$. Since $p = 0$, the associated sequence is $s^\epsilon = (0^+, 0^+, \ldots)$ and hence corresponds to a fixed hair (here and on the covering space). The tail is simply the interval $(1, \infty)$, which is invariant and whose points tend exponentially fast to infinity. The symmetric tail $(0,1)$ also lands at $w = 1$ and its points tend exponentially fast to 0. All points on the unit circle, except for $w = 1$, are contained in the immediate basin of attraction of $w = -1$.

The remainder of this section will be dedicated to the proof of Theorem 4.2.

We prove Theorem 4.2 on the covering space, that is, we will show that under the hypothesis above, the hair Z_{s^ϵ} lands on the real line. Figure 9 shows part of the dynamical plane in the covering space for p/q equal 1/2 and 3/4, to be compared with Figures 2 (left) and 3 (left).

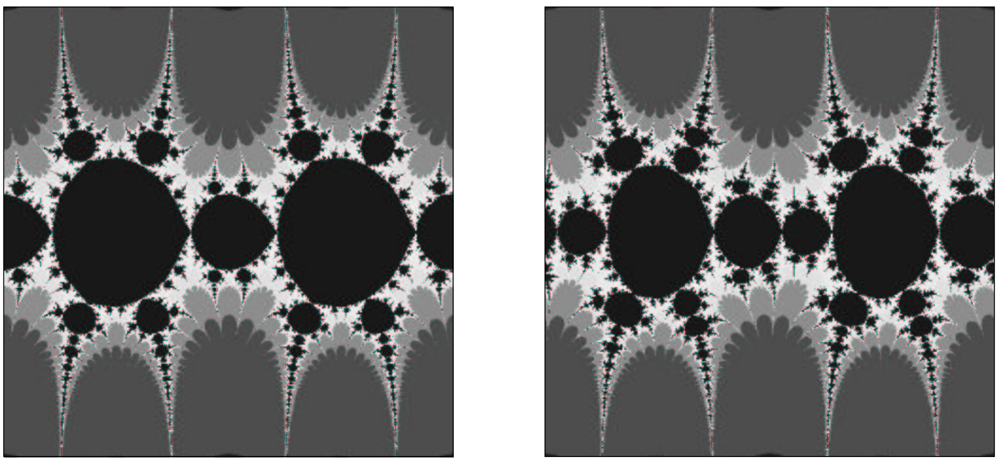

Figure 9: (to be compared with Figures 2, 3, 11 and 12). Left: z-plane for (α, β) in the 1/2 tongue. $\alpha = 3.1, \beta = 0.8$. Range: $[-2\pi, 2\pi] \times [-2\pi, 2\pi]$. Right: z-plane for (α, β) in the 3/4 tongue. $\alpha = 4.57, \beta = 0.8$. Range: $[-2\pi, 2\pi] \times [-2\pi, 2\pi]$. The escaping colors have been inverted to better see the hairs.

Let $(\alpha, \beta) \in T_{p/q}$ and denote by W the repelling periodic orbit on the unit circle. For technical reasons, we will assume that $-1 \notin W$ for any $0 \leq i \leq q-1$. Although we believe that, in fact, this case never occurs, if it did, a slight perturbation of the parameters inside $T_{p/q}$ would make it disappear. Let $a, b \in P^{-1}(W) \subset \mathbb{R}$ be such that $a < -\pi < b$ and there is no other point of $P^{-1}(W)$ in the interval (a, b). For the remainder of the section, set $f = f_{\alpha\beta}$. Finally, for any $k \in \mathbb{Z}$ we define the map $T_k(z) = z + 2k\pi$.

Our goal is to show that Z_{s^ϵ} lands at the point b and the strategy of the proof will be as follows. We will construct an open, bounded set B_0 isomorphic to a disc, completely contained in R_0^+, which contains all points that share the same itinerary with Z_{s^ϵ}, up to q iterates. The boundary of B_0 will contain the point b, and no other point of the real line. We will construct another open, bounded set B_1 isomorphic to a disc, such that $B_0 \subset B_1$ and $f^q(B_0) = T_p(B_1)$. Considering this map on the cylinder (that is, $z \sim T_k(z)$ for every k) we have that B_0 is mapped outside itself under f^q. Taking an appropriate branch of f^{-q} we may apply Schwartz lemma to conclude that any fixed point of f^q in B_0 must be a global attractor under f^{-q}. But the point b in the boundary of B_0 is an attracting fixed point of f^{-q} and hence attracts some of the points of B_0. Therefore, B_0 cannot contain any other fixed point of f^q. Finally, we will see that Z_{s^ϵ} must land at some fixed point of f^q in the closure of B_0. This endpoint cannot be inside B_0 by the argument above and hence it can only be the point b. We now proceed to make this construction.

Let c and $\overline{c}$ be the two critical points of f whose real part equals $-\pi$. We define A to be the lift of the immediate basin of attraction of the attracting cycle and denote by A_0 the connected component of A such that c and $\overline{c}$ belong to A_0. It follows easily that $(a, b) \subset A_0$ (see Figure 10).

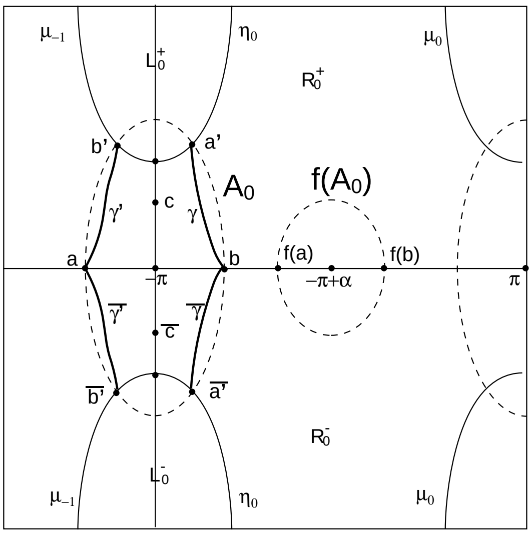

Figure 10: Sketch of the component A_0 and the curves γ and γ' of Lemma 4.3.

Let a' (resp. b') be the only preimage by f of $f(a)$ (resp. $f(b)$) that lies on the curve η_0 (resp. μ_{-1}) of the upper half plane. Indeed, the curve μ_{-1} is mapped one to one to the semiline $[-\pi + \alpha, \infty)$ while η_0 is mapped one to one to $(-\infty, -\pi + \alpha]$. Since $f(a) < -\pi + \alpha < f(b)$ it follows that a' and b' must belong to η_0 and μ_{-1} respectively. With this notation, we can state the following lemma.

Lemma 4.3 *There exist simple curves γ and γ' completely contained in A_0, such that γ joins a' and b and γ' joins b' and a,*

Proof : We need to show that a, a', b and b' are accessible from the interior of A_0 (and hence belong to ∂A_0). Since A_0 contains two critical points, it is mapped onto its image with degree three. Now recall that $(a, b) \subset A_0$, hence a and b are clearly accessible. Since (a, b) is mapped one to one onto its image, we have that A_0 must contain two other preimages of $(f(a), f(b))$. Clearly, one of those can only be the arc on $\mu_{-1} \cup \eta_0$ joining b' and a', being its conjugate the remainder one. Hence b' and a' (and also their conjugates) are endpoints of arcs contained in A_0. It follows that they are accessible and we may take γ to be any arc in A_0 that joins a' and b, and γ' any arc in A_0 that joins b' and a. $\square$

We remark that as a consequence of this lemma, if a tail belongs to R_k^+, for some $k \in \mathbb{Z}$, its pullback can never cross to another strip different from R_k^+, for else it would have to cross A_0 or one of its translates. Hence the whole hair, including its landing point, must be contained entirely in R_k^+. In other words, we can rule out the case of Corollary 3.7 where the landing point is infinity.

For clarity's sake we make the following construction in the case $p/q = 1/2$, for which $s^\epsilon = (0^+, 0^+, 1^+, 1^+, 2^+, 2^+, \dots)$. A sketch is shown in Figure 11, which we recommend to have in mind while reading the remainder of the proof. It will be clear that all other cases can be handled with the same argument. Figure 12 shows the case $p/q = 3/4$.

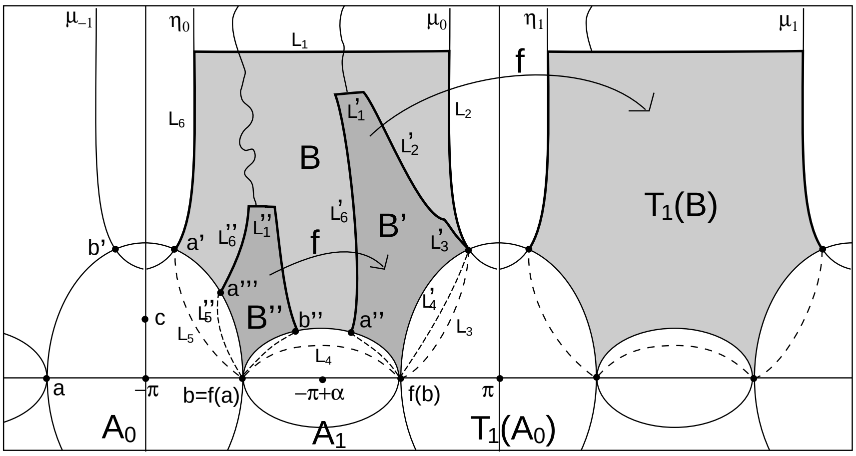

Figure 11: Sketch of the construction in the proof of Theorem 4.2, for the case $p/q = 1/2$.

Let y_* be as in Theorem 3.3 for the sequence $\sigma^2(s^\epsilon) = (1^+, 1^+, 2^+, 2^+, \dots)$. Then, $Z_{\sigma^2(s^\epsilon)} : [y_*, \infty) \to R_1^+$ is parametrized by its imaginary part.

Set $A_1 = f(A_0)$ and hence $f(A_1) = T_1(A_0)$ (since $p/q = 1/2$).

We define the boundary of the set B (in the upper half plane) to be made of the following pieces: a horizontal straight line, L_1, in R_0^+ of imaginary part y_*, joining the curves η_0 and μ_0; a piece of μ_0, to be called L_2, joining the right most point of L_1 with the point $T_1(b')$;

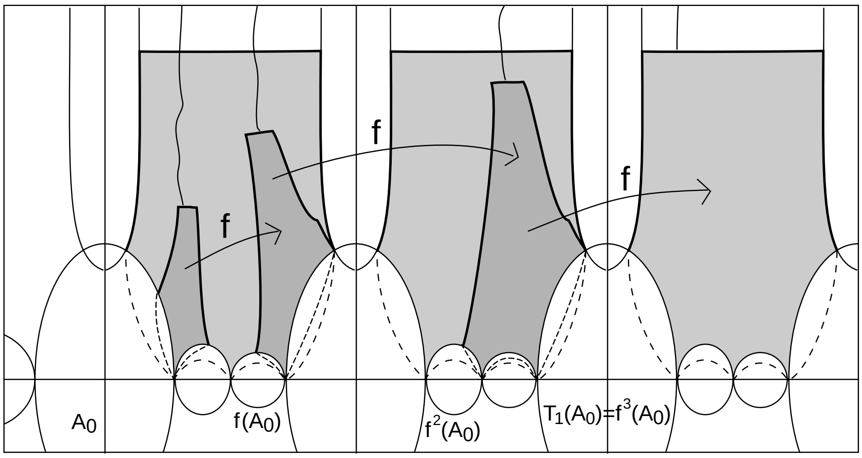

Figure 12: Sketch of the construction in the proof of Theorem 4.2, for the case $p/q = 3/4$.

the curve $L_3 = T_1(\gamma')$ that joins $T_1(b')$ and $T_1(a) = f(b)$ and is contained in $T_1(A_0)$; a curve L_4 in A_1 joining b and $f(b) = T_1(a)$; the curve $L_5 = \gamma$ in A_0 joining a' and $b = f(a)$; and finally the piece of η_0, to be called L_6, that joins a' with the left most point of L_1. The curve $L := \bigcup_{i=1}^6 L_6$ is clearly homeomorphic to a circle and we let B be the bounded connected component of the complement of L. Since we have freedom in choosing L_3, L_4 and L_5, we choose them so that no critical point or critical value is contained in the closure of B. Hence f is injective in the closure of B.

We define B' to be the set of points in $B \cup A_1 \cup T_1(A_0)$ that are mapped to $T_1(B)$ under one iteration of f. Let a'' be the only preimage of $T_1(a')$ on the boundary of A_1. We see that $\partial B'$ must be mapped to $T_1(\partial B)$ and hence is made of the following pieces: a curve L_5' in A_1 joining a'' and $T_1(a) = f(b)$, which maps to $T_1(L_5)$; a curve in $T_1(A_0)$, to be called L_4', that joins $T_1(a)$ with $T_1(b')$ and is mapped to $T_1(L_4)$; and four curves inside B to be called L_3', L_2', L_1' and L_6' that are mapped respectively to $T_1(L_3), T_1(L_2), T_1(L_1)$ and $T_1(L_6)$. We observe that these last three pieces must be compactly contained in B, except for the point $T_1(b')$. By construction, all points in B' have itineraries that start with $(0^+, 1^+, \dots)$.

Finally, we define B'' to be the set of points in $B \cup A_0 \cup A_1$ that are mapped to B' under one iteration of f. Let a''' and b'' be the only points in $\partial A_0 \cap B$ and ∂A_1 respectively that are mapped to a'' and $T_1(b')$ in this order. Then, $\partial B''$ is made of the following pieces: a curve L_5'' in A_0 joining a''' and b that maps to L_5'; a curve L_4'' in A_1 joining b and b'' that maps to L_4'; and four curves L_3'', L_2'', L_1'' and L_6'' compactly contained in B that map to L_3', L_2', L_1' and L_6'. By construction, all points in B' have itineraries that start with $(0^+, 0^+, 1^+, \dots)$.

Let $\widetilde{A}$ be the complement of $P^{-1}(A)$ and set $B_1 = B \cap \widetilde{A}$ and $B_0 = B'' \cap \widetilde{A}$. Note that $B_0 \subset B_1$ and since A is simply connected, so are B_0 and B_1. Moreover, $f^2(B_0) = T_1(B_1)$ and points in B_0 are the only ones in $R_0^+ \cap \widetilde{A}$ that are mapped to $T_1(B_1)$ under f^2 with itinerary starting with $(0^+, 0^+, 1^+, \dots)$. Finally, $\partial B_0 \cup \mathbb{R} = \{b\}$ and it is easy to check that no other point than b in ∂B_0 can satisfy $f^2(z) = T_1(z)$. Hence B_0 and B_1 satisfy the requirements we stated at the beginning of the proof. Thus, after identifying any point z with $T_k(z)$ for any k, we can conclude that B_0 does not contain any fixed point of f^2.

It only remains to show that Z_{s^ϵ} must land at some point of the closure of B_0. Recall that $Z_{\sigma^2(s^\epsilon)}$ must intersect the line $T_1(L_1)$ at a single point by the choice of y_*. Moreover, $Z_{\sigma^2(s^\epsilon)}[y_*, \infty)$ is contained in $R_1^+ \setminus T_1(B_1)$ and hence, the remainder of the tail is contained in $T_1(B_1)$. By taking one preimage, we have that $Z_{\sigma(s^\epsilon)}$ intersects L_1' in only one point, and the remainder of this tail must be contained in B'. Taking one last preimage, we conclude that Z_{s^ϵ} intersects L_1'' at a single point, and the remainder of the tail must be contained in B_0. Thus, we conclude that Z_{s^ϵ} lands at some point of the closure of B_0, which by the arguments above, must be b. This concludes the proof of Theorem 4.2

5 Dynamics for $\beta > 1$

The dynamics of the standard family when $\beta > 1$ are considerably more complicated than in the case $\beta \leq 1$. The maps on the circle are no longer diffeomorphisms since the two critical points are now on S^1. Consequently, their orbits are no longer symmetric and hence different types of stable components may now coexist, each associated to a different critical orbit. The rotation number of $F_{\alpha\beta}|_{S^1}$ is no longer the same for all points on S^1. It is true though that for each pair (α, β), the possible rotation numbers form a closed interval [I].

In this region of parameter space, the Arnold tongues are still well defined, as the set of parameter values for which there exists a point in S^1 with the given rotation number. Hence, the irrational curves now open up into tongues and tongues of several rotation numbers overlap (see [Bo]). As in the case $\beta < 1$, a pair of parameters (α, β) belong to a rational tongue $T_{p/q}$ if and only if there exists a periodic orbit on S^1 of period q and rotation number p/q. However, this orbit does no longer need to be attracting.

The dynamics on the complex plane must also be very different. Since the critical orbits are completely contained on the unit circle, we may rule out the existence of some types of stable components. Indeed, Herman rings or Siegel discs are no longer possible, since the critical orbits could in no way accumulate in their boundaries. Hence only attracting, superattracting or parabolic domains can exist and, as before, such cycles must be on the unit circle.

With respect to symbolic dynamics on the covering space, we observe that in this case the fundamental domains constructed in Section 3.1 are now vertical strips (see Figure 4, right). Hence, the pull back of any tail is now bound to stay within one of these strips. Moreover we observe that this pullback always exists. Indeed, since the critical orbits stay on the circle, all hairs are bounded away from them. Hence all branches of $f_{\alpha\beta}^{-1}$ are well defined on D_k^+ and D_k^- for all $k \in \mathbb{Z}$.

Since many different itineraries may now coexist on the unit circle, and this number increases with β, it is natural to ask if many different hairs simultaneously land on the unit circle and if the number of such hairs increases with β as well. Numerical observations (see Figure 3 (right)) indicate that this is indeed the case. The answer to this question is the main theorem in the current section.

Given an allowable sequence s^ϵ let $Z_{s^\epsilon} : [y_*(s^\epsilon), \infty) \to D_{s_0^{\epsilon_0}}$ be the tail given by Theorem 3.3, parametrized by its imaginary part. Since this tail is bounded away from the critical orbit we may pull back Z_{s^ϵ} as follows (see Corollary 3.7 for the essentially periodic case). For clarity, we drop the ϵ's assuming they are all $+$. Consider $Z_{\sigma(s)} : [y_*(\sigma(s)), \infty] \to D_{s_1}$. By construction, $y_*(s)$ and $y_*(\sigma(s))$ are comparable and hence $\text{Im}(f_{\alpha\beta}(Z_s(y_*(s)))) >> y_*(\sigma(s))$.

We then take the branch of $f_{\alpha\beta}^{-1}$ that takes values on D_{s_0} and consider

$$f_{\alpha\beta}^{-1}(Z_{\sigma(s)}[y_*(\sigma(s)), \mathrm{Im}(f_{\alpha\beta}(Z_s(y_*(s))))]).$$

This is a curve that extends Z_s, whose points share itinerary with those on Z_s, and tend to infinity exponentially fast under iteration. Hence we consider this curve as the (first) pullback of Z_s. We could repeat this procedure to obtain the hair $Z_s : [-\infty, \infty] \to D_{s_0}$. Note that $\mathrm{Im}(Z_s(t)) = t$ only for $t \geq y_*(s)$.

We say that a hair Z_{s^ϵ} *lands* if $\lim_{t\to-\infty} Z_{s^\epsilon}(t)$ exists, and the landing point will be denoted by $z_* = z_*(s^\epsilon)$. By Corollary 3.7, all essentially periodic hairs always land. For $\beta \geq 1$ we will see that many others do too.

Definition A sequence $s = (s_0, s_1, s_2 \dots)$ is a sequence of *bounded jump* if $|s_{j+1} - s_j| \leq M$ for some integer $M \geq 0$ and for all $j \geq 0$.

We also say that signed sequences are of bounded jump when the corresponding plain sequences are of bounded jump. Note that sequences of bounded jump are those growing at an approximately linear rate and therefore they are always allowable sequences.

Theorem 5.1 *Given a hair $Z_{s^\epsilon} : (-\infty, \infty) \longrightarrow D_{s_o^\epsilon}$ where $s^\epsilon = (s_0^{\epsilon_0}, s_1^{\epsilon_1}, \dots)$, the following statements are equivalent:*

(a) *s^ϵ is of bounded jump.*

(b) *for each $\alpha \geq 0$, there exists $\beta_0 \geq 0$ such that for all $\beta \geq \beta_0$ the hair Z_{s^ϵ} lands on the real line.*

Hence the bounded jump condition of s^ϵ is necessary and sufficient in order for the corresponding hair (and all its images) to land at the real line, and remain there as we increase the parameter β.

Proof of Theorem 5.1: First, we show that the bounded jump condition is necessary. This is clear since, by construction of the regions D_j, we have

$$2\pi(s_j - 1) < \mathrm{Re}(f_{\alpha\beta}^j(z)) \leq 2\pi(s_j + 1) \tag{9}$$

$$\mathrm{Re}(f_{\alpha\beta}^j(z)) - 2\pi < 2\pi s_j < \mathrm{Re}(f_{\alpha\beta}^j(z)) + 2\pi \tag{10}$$

Also, since $\sin(x)$ is bounded on $\mathbb{R}$, we have

$$|f^{j+1}(x) - f^j(x)| \leq \alpha + \beta \text{ for all } x \in \mathbb{R} \tag{11}$$

and hence, combining estimates 9 and 10 with equation 11, we get

$$|s_{j+1} - s_j| \leq \frac{\alpha + \beta}{2\pi} + 2$$

Therefore, in order for $z_*(s^\epsilon)$ to be in $\mathbb{R}$, its itinerary should be of bounded jump.

To prove the sufficiency of the condition we will use heavily the existence of certain itineraries on the real line. Note that on the real line, signed itineraries do not make sense. The equivalent to signed itineraries in this case is as follows.

Definition The *rl itinerary* of a point $z \in \mathbb{C}$ for $f_{\alpha\beta}$ is the infinite *rl* sequence

$$S_{rl}(z) = (\delta_0, \delta_1, \delta_2 \dots)$$

where $\delta_j = r_k$ if $f^j_{\alpha\beta} \in R_k$ and $\delta_j = l_k$ if $f^j_{\alpha\beta} \in L_k$

Note that, because of proposition 3.1, every signed itinerary corresponds to a unique *rl* itinerary while each *rl* itinerary corresponds to two different signed itineraries. As with the signed sequences we will say that an *rl* sequence is of bounded jump if its corresponding plain sequence is of bounded jump. Also, we will use the notation δ^ϵ in the obvious way.

Lemma 5.2 *Given any rl sequence s_{rl} of bounded jump, and $\alpha > 0$, there exists $\beta_1 > 0$ such that for all $\beta \geq \beta_1$ there is a point $x_\beta \in \mathbb{R}$ such that $S_{rl}(x_\beta) = s_{rl}$. Moreover, there is $\beta_0 > \beta_1$ such that for all $\beta > \beta_0$, x_β is unique.*

Proof of lemma 5.2: Throughout this proof let I_k, I_{L_k} and I_{R_k} denote the intervals of the real line corresponding to the regions D_k, L_k and R_k respectively.

Let M be the bound such that

$$|s_{j+1} - s_j| \leq M \text{ for all } n$$

given by the assumption that s_{rl} is of bounded jump.

Define $\beta_1 = \sqrt{((M+1)2\pi + \alpha)^2 + 1}$. This is the β-value for which critical points with positive second derivative get mapped to themselves minus $(M+1)2\pi$. Indeed, the critical points c_i satisfy $\cos(c_i) = \frac{-1}{\beta_1}$. Since $\sin(c_i) = \frac{-1}{\beta}\sqrt{\beta_1^2 - 1}$, we get

$$f_{\alpha\beta_0}(c_i) = c_i + \alpha + \beta_1 \sin(c_i) = c_i + \alpha - \sqrt{\beta_1^2 - 1} = c_i - 2\pi(M+1).$$

A similar calculation shows that the maxima of the function are mapped to themselves plus $2\alpha + 2\pi(M+1)$ (see Figure 13).

Consequently, for any $\beta \geq \beta_1$, the image of each interval I_k covers twice the intervals

$$I_{k-M}, \dots, I_k, I_{k+1}, \dots, I_{k+M}.$$

Looking at the *rl* subintervals, the image of each I_{R_k} or I_{L_k} covers once the subintervals

$$I_{L_{k-M}}, I_{R_{k-M}}, \dots, I_{L_k}, I_{R_k}, I_{L_{k+1}}, I_{R_{k+1}}, \dots, I_{L_{k+M}}, I_{R_{k+M}}$$

Therefore, the preimage of any closed interval inside or equal to one of the *rl* subintervals, is a closed interval in each of the *rl* companions. Given that, standard arguments show that there exists a point $x_{\beta_1} \in I_{\delta_0}$ such that $S_{rl}(x_{\beta_1}) = s_{rl}$. Clearly, the same argument applies for any $\beta > \beta_1$, since the larger β is, the more intervals are covered by the image of I_k.

Now choose $\beta_0 > \beta_1$ as the first value of β for which $|f'_{\alpha,\beta}(x)| \geq 1$ on the set $\{x \in I_{\delta_0} \mid f_{\alpha,\beta}(x) \in \bigcup_{i=-M}^{M} I_{k+i}\}$. Then, if $\beta > \beta_0$, the point x_β is unique. $\square$

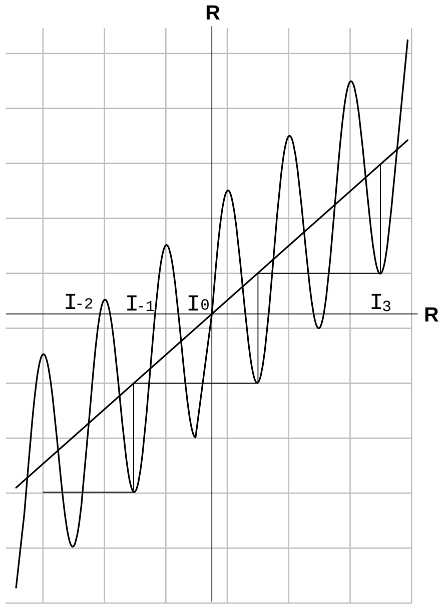

Figure 13: Graph of $f_{\alpha\beta_0}$ on the real line. The plot is for the values $\alpha = 0$ and $M = 1$.

Now we must prove the sufficiency of the bounded jump condition. We will show that $Z_{s^\epsilon}(y^*) \in \mathbb{R}$ for all $\beta \geq \beta_0$. In order to do that, fix $\beta \geq \beta_0$ and consider the rl sequence s_{rl} corresponding to s^ϵ and the point $x_\beta \in \mathbb{R}$, given by the lemma, which has itinerary s_{rl}.

Consider the connected compact set K_0 in $D_{\delta_0^{\epsilon_0}}$ bounded by the real line and a horizontal segment of the line $\mathrm{Im}(z) = y$, where y is large enough so that the image of the segment (half of a (s_0, y)-skewed ellipse) crosses M strips to the left and M to the right (see Figure 14). On the sides the bounds are the same as those of the strip. The upper bound of the strip cuts the tail associated to s^ϵ at a single point $Z_{s^\epsilon}(y')$, for some $y' > y_*$ (if that were not the case we may increase y until it happens).

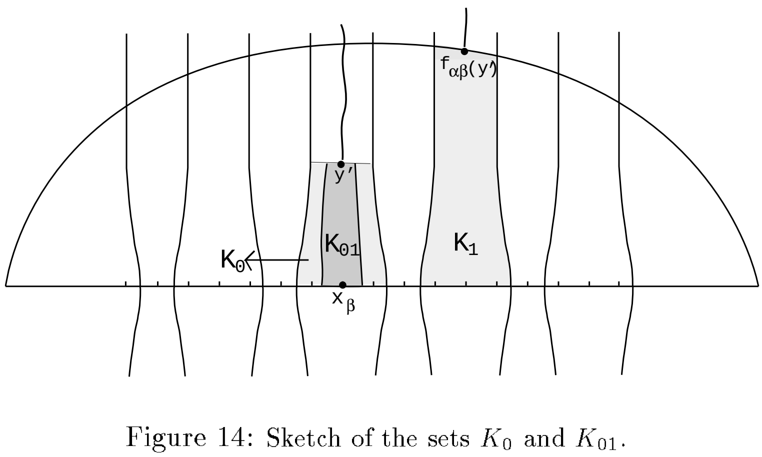

Figure 14: Sketch of the sets K_0 and K_{01}.

The image by $f_{\alpha\beta}$ of K_0, is the inside of half of a (s_0, y)-skewed ellipse. The subset of this ellipse contained in $D_{\delta_1^{\epsilon_1}}$ is a compact connected set $K_1 \subset D_{\delta_1^{\epsilon_1}}$, similar to K_0.

The preimage of K_1, inside K_0, consists of a compact connected subset K_{01} of K_0 (see Figure 14), containing x_β and $Z_{s^\epsilon}(y')$ in its bounds. All points in K_{01} have itineraries starting with $\delta_0^{\epsilon_0}$ and $\delta_1^{\epsilon_1}$.

We can repeat this process and obtain a sequence of nested compact connected sets $\{K_{012,\ldots,n}\}$ all of them containing x_β on the bottom and $Z_{s^\epsilon}(y')$ on the top. The intersection of all these sets

$$K = \bigcap_{n=0}^{\infty} K_{01\ldots n}$$

has to be a nonempty compact connected set, containing $Z_{s^\epsilon}(y')$ on the top and x_β on the bottom, which shows that x_β has to be equal to $z_*(s^\epsilon)$. This concludes the proof of Theorem 5.1. □

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