Univalent Baker domains

Krzysztof Barański*

Institute of Mathematics
Warsaw University
ul. Banacha 2
02-097 Warsaw, Poland
baranski@mimuw.edu.pl

Núria Fagella†

Dep. de Mat. Aplicada i Anàlisi
Universitat de Barcelona
Gran Via 585
08007 Barcelona, Spain
fagella@maia.ub.es

Abstract

We classify Baker domains U for entire maps with $f|_U$ univalent into three different types, giving several criteria which characterize them. Some new examples of such domains are presented, including a domain with disconnected boundary in $\mathbb{C}$ and a domain which spirals towards infinity.

1 Introduction

Let $f : \mathbb{C} \to \mathbb{C}$ be an entire transcendental map. Then f induces a partition of the complex plane into two completely invariant sets: the *Fatou set* and the *Julia set*. The first one, $F(f)$, is defined as the set of points $z \in \mathbb{C}$ for which the sequence of iterates $\{f^n\}_{n\geq 0}$ forms a normal family in some neighbourhood of z. Its complement is the Julia set, $J(f)$. Clearly, the Fatou set is an open set of $\mathbb{C}$ while the Julia set is closed. It is a special property of entire transcendental maps that both sets are unbounded. Refer, for example, to [Ber2, BR] for the general description of the dynamics of these maps.

Since $F(f)$ is completely invariant, its connected components must map among themselves. We say that a connected component U of $F(f)$ is *periodic* of period $p \geq 1$, if $f^p(U) \subset U$. Note that unlike the case of rational maps, it is possible to have $f^p(U) \neq U$ (see e.g. [Ber2]).

If U is a periodic component of $F(f)$ of period $p \geq 1$, there are only four possible cases:

(a) U is a component of the *attracting basin* of an attracting periodic point $z_0 \in U$ and $f^{np}(z) \underset{n\to\infty}{\longrightarrow} z_0$ for all $z \in U$,

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(b) U is a component of the *parabolic basin* of a parabolic point $z_0 \in \partial U$ and $f^{np}(z) \underset{n\to\infty}{\longrightarrow} z_0$ for all $z \in U$,

(c) U is a *Siegel disc*, i.e. U is conformally equivalent to a disc and $f^p|_U$ is analytically conjugate to a rigid rotation,

(d) $f^{np}(z) \underset{n\to\infty}{\longrightarrow} \infty$ for all $z \in U$. In this case, U is called a *Baker domain.*

We refer to [Ber2] for a complete exposition about this classification. We observe that Baker domains do not exist for entire transcendental maps, for which $\mathrm{Sing}(f^{-1})$ is bounded, where $\mathrm{Sing}(f^{-1})$ denotes the closure in $\mathbb{C}$ of the set of all critical and asymptotic values of f (see [EL1]).

The first example of an entire function with a Baker domain was given by Fatou in [Fa], who considered the function $f(z) = z + 1 + e^{-z}$ and showed that the right half-plane is contained in an invariant Baker domain. Since then, plenty of other examples have been found, showing various properties that are possible for this type of Fatou components (see [BD2, Ber1, BW, EL2, H1, RS1, RS2]). It was proved in [Ba2] that all Baker domains for entire transcendental maps are simply connected.

In this paper we deal with univalent Baker domains, i.e. Baker domains on which f^p is univalent. Examples of such domains can be found among the references above (see also Section 5).

Let $f : \mathbb{C} \to \mathbb{C}$ be an entire transcendental map and let $U \subset \mathbb{C}$ be a univalent Baker domain. Replacing f by its iteration, we assume from now on that U is invariant, i.e. $f(U) \subset U$. It is known (see [Ber2]) that $U \setminus f(U)$ consists of at most one point. But the case $f(U) = U \setminus \{z_0\}$ is impossible, because U is simply connected and $f|_U$ is a homeomorphism. Hence, in this case we have $f(U) = U$.

We will classify invariant univalent Baker domains and study the relation between the dynamics of f and the geometry of U.

Definition 1.1. A point $\zeta \in \widehat{\mathbb{C}}$ in the boundary of a simply connected domain $U \subset \mathbb{C}$ is called *accessible* from U if there exists a curve $\gamma : [0, +\infty) \to U$ which *lands* at ζ, i.e. $\gamma(t)$ tends to ζ as $t \to +\infty$. If the boundary of U is locally connected, then all its points are accessible.

Following [Go, Pe] we say that two such curves γ_1 and γ_2 are in the same *access* to ζ, if for every neighbourhood $V \subset \widehat{\mathbb{C}}$ of ζ there exists a curve $\alpha : [0,1] \to U \cap V$, such that $\alpha(0) \in \gamma_1$ and $\alpha(1) \in \gamma_2$. Equivalently, an access is a homotopy class within the family of curves $\tilde{\gamma} : [0,1] \to \widehat{\mathbb{C}}$, such that $\tilde{\gamma}((0,1)) \subset U$ and $\tilde{\gamma}(1) = \zeta$. It is obvious that accesses define an equivalence relation.

It is known (see [Ba1]) that for every Baker domain U, the point at infinity is accessible from U. In fact, for any point $z \in U$, if we choose $\gamma_z : [0,1] \to U$ to be any curve connecting z and $f(z)$, then the standard estimates of the hyperbolic metric on U (see Lemma 2.2) show that the invariant curve $\Gamma_z = \bigcup_{n\geq 0} f^n(\gamma_z)$ lands at ∞, so it defines an access to infinity. We say that z tends to ∞ through this access. It is easy to show that this access does not depend on the choice of the curve γ_z. In fact, it is also independent of the choice of the point $z \in U$, as stated in the following lemma.

Lemma A. *The forward iterates of all points in U tend to infinity through the same access.*

We will call this access the *forward dynamical access* to infinity. The proof of this lemma is contained in Section 3.

Let R be a Riemann mapping from the open unit disc $\mathbb{D} \subset \mathbb{C}$ onto U. In the case of Baker domains, it is convenient to work with the conformal mapping Ψ from the upper half-plane $\mathbb{H}^+ = \{w \in \mathbb{C} \mid \operatorname{Im}(w) > 0\}$ onto U, setting $\Psi(w) = R(\frac{w-i}{w+i})$. Let $g = \Psi^{-1} \circ f \circ \Psi$ and $h = R^{-1} \circ f \circ R$. Then g (resp. h) must be a hyperbolic or parabolic automorphism of $\mathbb{H}^+$ (resp. $\mathbb{D}$). Hence, we can assume that g is of one of the two following forms:

$$g(w) = \begin{cases} aw \ (a > 1) & \text{(hyperbolic type)}, \\ w + 1 & \text{(parabolic type)}. \end{cases}$$

Then $h(u) = \frac{(a+1)u+a-1}{(a-1)u+a+1}$ or $h(u) = \frac{(1-2i)u-1}{u-1-2i}$ respectively for $u \in \mathbb{D}$. See Figures 1 and 2.

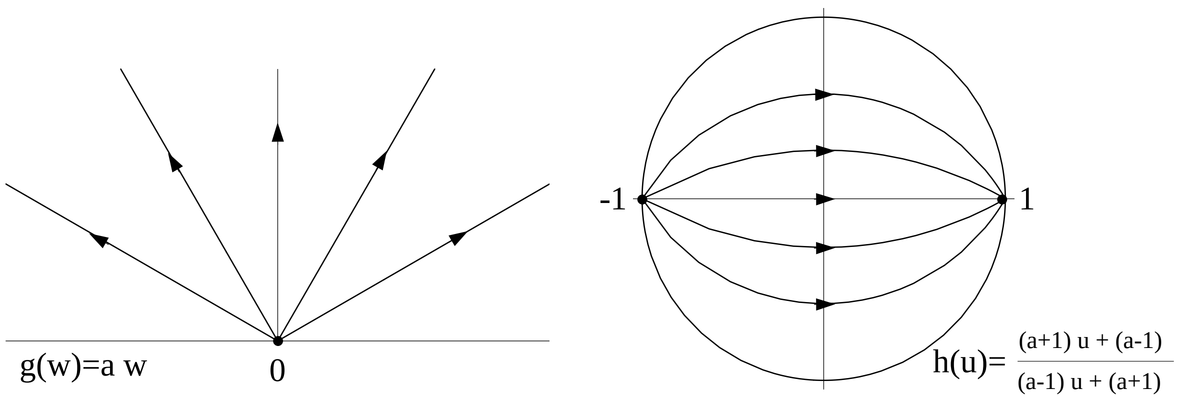

Figure 1: Invariant curves under a hyperbolic automorphism of $\mathbb{H}^+$ and $\mathbb{D}$.

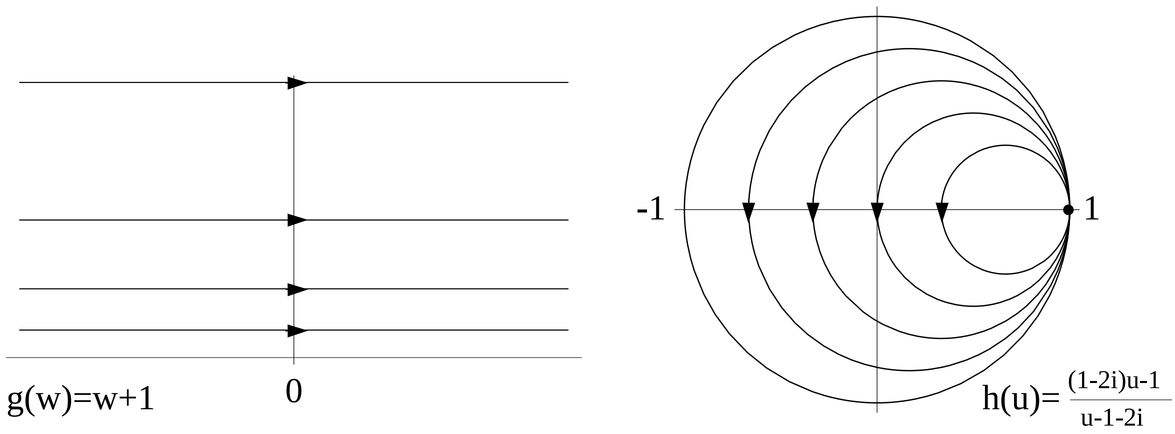

Figure 2: Invariant curves under a parabolic automorphism of $\mathbb{H}^+$ and $\mathbb{D}$.

The two cases give rise to dynamically different types of Baker domains. It is then natural to look for some geometric criteria which characterize hyperbolic and parabolic types. In

Section 3 we show that the two types can be distinguished by the behaviour of the orbits of points in U in relation with the distance to the boundary of U (Corollary 3.4). Moreover, we prove the following.

Theorem B. *Let $f : \mathbb{C} \to \mathbb{C}$ be an entire transcendental map and let $U \subset \mathbb{C}$ be an invariant univalent Baker domain. Then there exists a point $\zeta \in \widehat{\mathbb{C}}$, such that the backward iterates under $(f|_U)^{-1}$ of all points in U tend to ζ through the same access (which we call the* backward dynamical access*). Moreover, exactly one of the following occurs:*

(a) *$\zeta \neq \infty$ is a fixed point in the boundary of U, attracting or parabolic with multiplier* 1 *and U is of hyperbolic type.*

(b) *$\zeta = \infty$, the backward dynamical access is different from the forward one and U is of hyperbolic type.*

(c) *$\zeta = \infty$, the backward dynamical access is equal to the forward one and U is of parabolic type.*

If case (a) occurs we say that U is of *hyperbolic type I*, while U is of *hyperbolic type II* if (b) is satisfied. In Section 5 we give examples of univalent Baker domains of each of the three types.

Theorem B has interesting corollaries which can be found in Section 4. It appears that in the parabolic case, there exist points in U for which both forward and backward iterates tend to infinity relatively slowly and the entire trajectory lies arbitrarily close to infinity (Corollaries 4.2 and 4.6). Other consequences concerning the limit behaviour of the Riemann map are stated in Corollaries 4.4 and 4.5. In Corollary 4.7 we show that if the boundary of U is locally connected, then it contains at most one periodic point.

It is interesting to study the topology of the boundary of Baker domains. Non-univalent Baker domains have highly complicated boundaries. Indeed, Baker and Weinreich in [BW] showed that such domains can never have Jordan curves as their boundaries. A stronger result from [BD2] implies that in the non-univalent case there always exist infinitely many accesses to infinity. Although there are examples of univalent Baker domains whose boundaries are Jordan curves (see Section 5), this is not always the case. In Subsection 5.2 we present an example of a Baker domain whose boundary in $\mathbb{C}$ is disconnected. In fact, the following is a corollary of Theorem B.

Corollary C. *If U is a univalent Baker domain of hyperbolic type II, then the boundary of U in $\mathbb{C}$ is disconnected.*

We are not aware of any example of a univalent Baker domain with more than two accesses to infinity (and therefore with more than two components of its boundary in $\mathbb{C}$ – see Lemma 3.1). The problem of constructing such a domain is related to finding a holomorphic self-map of $\mathbb{C}^*$ with a Siegel disc or Herman ring around 0 having an access to infinity (see Section 5).

A different set of natural questions concerns the geometry of Baker domains. All previously known examples of Baker domains either contain a half-plane or are contained in a straight band. This suggests the question whether it is possible to find a Baker domain whose "asymptotic direction" to infinity is not a straight line. In Section 6 we show the existence of an entire map with a univalent Baker domain that spirals towards infinity. This construction is done using an approximation theory method used by Eremenko and Lyubich in [EL2].

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Notation. The terms "the closure" and "the boundary" of a set $A \subset \mathbb{C}$ refer to the closure and boundary in $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, but the symbols $\overline{A}$ and ∂A denote the closure and boundary in $\mathbb{C}$. For a point $z \in \mathbb{C}$ and sets $A, B \subset \mathbb{C}$ we use the notation $\operatorname{diam} A = \sup\{|z - w| \mid z, w \in A\}$, $\operatorname{dist}(z, A) = \inf\{|z - w| \mid w \in A\}$, $\operatorname{dist}(A, B) = \inf\{|z - w| \mid z \in A, w \in B\}$. If U is a univalent Baker domain, $z \in U$ and $n > 0$, then we denote by $f^{-n}(z)$ the preimage of z under inverse branches of f^{-1} leading into U.

2 Preliminaries

Hyperbolic metric and distortion estimates

We will use the following Koebe distortion theorem (see e.g. [CG]).

Koebe's Theorem. *Let $\phi : \mathbb{D} \to \mathbb{C}$ be a univalent holomorphic map. Then:*

$$|\phi'(0)| \frac{|z|}{(1 + |z|)^2} \le |\phi(z) - \phi(0)| \le |\phi'(0)| \frac{|z|}{(1 - |z|)^2},$$
$$|\phi'(0)| \frac{1 - |z|}{(1 + |z|)^3} \le |\phi'(z)| \le |\phi'(0)| \frac{1 + |z|}{(1 - |z|)^3}.$$

In particular, this implies that $\phi(\mathbb{D})$ contains the open disc centred at $\phi(0)$ of radius $|\phi'(0)|/4$.

Combining this with the Riemann mapping theorem, we get

Corollary 2.1. *For a given $\varrho > 0$ there exists $c(\varrho) > 0$, such that for every simply connected domain $U \subset \mathbb{C}$, every bounded set $V \subset U$ with $\operatorname{dist}(\overline{V}, \partial U) > \varrho \operatorname{diam} V$, every univalent holomorphic map $\phi : U \to \mathbb{C}$ and every $z_1, z_2 \in V$,*

$$\frac{|\phi'(z_1)|}{|\phi'(z_2)|} \le c(\varrho).$$

Moreover, $c(\varrho) \to 1$ as $\varrho \to +\infty$.

We will also use the following standard estimate of the hyperbolic metric (see e.g. [CG]).

Lemma 2.2. *Let $U \subset \mathbb{C}$ be a simply connected domain, such that $\mathbb{C} \setminus U$ contains at least two points and let ρ_U be the hyperbolic metric on U. Then*

$$\frac{1}{2 \operatorname{dist}(z, \partial U)} \le \rho_U(z) \le \frac{2}{\operatorname{dist}(z, \partial U)}$$

for every $z \in U$.

Boundary behaviour of the Riemann map

Let $U \subset \mathbb{C}$ be a simply connected domain such that its complement is infinite and let $R : \mathbb{D} \to U$ be a Riemann mapping. In this subsection we recall some basic facts concerning the behaviour of R at the boundary of $\mathbb{D}$. For details we refer the reader to [CL, M] and [P]. The following is the classical Fatou theorem.

Fatou's Theorem. *For almost every $\theta \in \mathbb{R}/(2\pi\mathbb{Z})$ there exists the radial limit*

$$\lim_{r \to 1} R(re^{i\theta}).$$

Moreover, if we fix θ such that this limit exists, then for almost every θ' the radial limit at θ' is different from the radial limit at θ.

The following Lindelöf-type theorem is very useful.

Theorem 2.3. *Let $\gamma : [0, +\infty) \to U$ be a curve which lands at a point ζ in the boundary of U. Then the curve $R^{-1} \circ \gamma$ in $\mathbb{D}$ lands at some point v of $\partial\mathbb{D}$. Moreover, R has the non-tangential limit at v equal to ζ. In particular, curves which land at different points of the boundary of U necessarily correspond to curves which land at different points of $\partial\mathbb{D}$.*

Finally, we recall the Carathéodory theorem.

Carathéodory's Theorem. *The Riemann mapping R can be extended continuously to a map from $\overline{\mathbb{D}}$ onto the closure of U if and only if the boundary of U is locally connected. This extension is a homeomorphism if and only if the boundary of U is a Jordan curve.*

3 Classification of univalent Baker domains: proof of Theorem B

First we prove some useful lemmas which give necessary and sufficient conditions for two curves to be in one access. The proof of Lemma 3.2 can be found also in [Go].

Lemma 3.1. *Let $\Gamma : (0, 1) \to U$ be a simple arc with both ends landing at the same point ζ in the boundary of U (i.e. $\Gamma \cup \{\zeta\}$ is a Jordan curve). Let $\gamma_1 = \Gamma((0, 1/2])$, $\gamma_2 = \Gamma([1/2, 1))$. Then the following statements are equivalent:*

(a) *γ_1 and γ_2 are in the same access to ζ,*

(b) *$R^{-1}(\gamma_1)$ and $R^{-1}(\gamma_2)$ land at the same point of $\partial\mathbb{D}$,*

(c) *$\Gamma \cup \{\zeta\}$ does not dissect the boundary of U.*

Proof.
(a) $\Rightarrow$ (b)

For $n > 0$ let V_n be a sequence of neighbourhoods of ζ in $\widehat{\mathbb{C}}$, such that $\bigcap_{n>0} V_n = \{\zeta\}$. By assumption, there exist curves $\alpha_n : [0, 1] \to U \cap V_n$, such that $\alpha_n(0) \in \gamma_1$, $\alpha_n(1) \in \gamma_2$. By

Theorem 2.3, $R^{-1}(\gamma_k)$ for $k = 1, 2$ lands at some point $v_k = e^{i\theta_k} \in \partial\mathbb{D}$. Suppose $v_1 \neq v_2$. For $\theta \in [0, 2\pi)$ let l_θ denote the radial segment $\{re^{i\theta} \mid r \in [0, 1]\}$. Using the Fatou theorem, we can find two points $\tilde{v}_k = e^{i\tilde{\theta}_k}, k = 1, 2$, different from v_1, v_2, such that R has the radial limit at $\tilde{v}_k$ equal to $\zeta_k \neq \zeta$ and the points v_1, v_2 are in different components of the set $\overline{\mathbb{D}} \setminus L$, where $L = l_{\tilde{\theta}_1} \cup l_{\tilde{\theta}_2}$. Then $R^{-1}(\alpha_n)$ for large n must intersect L at some point $w_n \in \mathbb{D}$. Taking a subsequence, we can assume $w_n \to \tilde{v}_k$ as $n \to \infty$ for $k = 1$ or 2, so $R(w_n) \to \zeta_k$. This is a contradiction, since $\alpha_n \subset V_n$ implies $R(w_n) \to \zeta$.

(b) $\Rightarrow$ (c)

Let $v \in \partial\mathbb{D}$ be the common landing point of $R^{-1}(\gamma_1)$ and let $R^{-1}(\gamma_2)$ and S be the component of $\mathbb{D} \setminus R^{-1}(\Gamma)$, such that $\partial S \cap \partial\mathbb{D} = \{v\}$. Suppose the Jordan curve $\Gamma \cup \{\zeta\}$ dissects the boundary of U. Then the boundary of $R(S)$ contains points from the boundary of U different from ζ. Hence, we can take a curve $\beta : [0, +\infty) \to R(S)$ landing at a point $\zeta' \in \partial U$, $\zeta' \neq \zeta$ (e.g. we can take $\beta(0)$ close to the boundary of U and connect it by a straight line segment to a suitable point of the boundary of U). By Theorem 2.3, the curve $R^{-1}(\beta)$ must land at some point $v' \in \partial S \cap \partial\mathbb{D}$, $v' \neq v$. This leads to a contradiction.

(c) $\Rightarrow$ (a)

Let $\tilde{S}$ be the component of $\widehat{\mathbb{C}} \setminus (\Gamma \cup \{\zeta\})$, which does not contain points from the boundary of U. Then $\tilde{S} \cup \Gamma \subset U$, $\tilde{S} \cup \Gamma \cup \{\zeta\}$ is homeomorphic to $\overline{\mathbb{D}}$ and we can easily construct the suitable curves α from the definition of the access. $\square$

Lemma 3.2. *Let $\gamma_1, \gamma_2 : [0, +\infty) \to U$ be curves landing at a common point ζ in the boundary of U. Then γ_1, γ_2 are in the same access to ζ if and only if $R^{-1}(\gamma_1)$ and $R^{-1}(\gamma_2)$ land at the same point of $\partial\mathbb{D}$.*

Proof. By the definition of an access, it is easy to check that if a curve $\gamma : [0, +\infty) \to U$ lands at ζ, then there exists an open set $W \subset U$, such that $\gamma \subset W$ and every curve $\tilde{\gamma} : [0, +\infty) \to W$ landing at ζ is in the same access to ζ as γ. Using this, we can assume that γ_1, γ_2 are homeomorphic to $[0, 1)$. Note also that if γ_1 and γ_2 intersect at a sequence of points converging to ζ, then they are in the same access to ζ and $R^{-1}(\gamma_1)$, $R^{-1}(\gamma_2)$ land at the same point of $\partial\mathbb{D}$. Therefore, we can assume additionally that γ_1 and γ_2 are disjoint. Then connect $\gamma_1(0)$ to $\gamma_2(0)$ by a simple arc in U disjoint from $\gamma_1 \cup \gamma_2$ and use Lemma 3.1. $\square$

Remark. Note that all the results listed in Subsection 2 together with the two above lemmas apply also for the map Ψ from $\mathbb{H}^+$ defined in Section 1 instead of the Riemann mapping R from $\mathbb{D}$.

The proof of Lemma A follows now easily from Lemma 3.2.

Proof of Lemma A. Recall that Lemma 2.2 easily implies that each curve Γ_z defined in Section 1 lands at infinity. By Theorem 2.3, the curve $R^{-1}(\Gamma_z)$ lands at some point v of $\partial\mathbb{D}$. The dynamics of $h = R^{-1} \circ f \circ R$ implies that $v = 1$. By Lemma 3.2, we conclude that all curves Γ_z must be in the same access to ∞. This also shows that the forward dynamical access is well defined. $\square$

To prove Theorem B, we need some preliminary estimates reflecting the relation between the distance of a point to the boundary of U and the distance between successive iterates. Given $z \in U$, let

$$\delta(z) = \frac{\operatorname{dist}(z, \partial U)}{|f(z) - z|}.$$

The following lemma is a slightly different version of Theorem 1 in [RS2], suited to the univalent case.

Lemma 3.3. *Let U be a univalent Baker domain and let $w = \Psi^{-1}(z)$ for $z \in U$. If U is of hyperbolic type (and hence $f|_U$ is conjugate by Ψ to $g(z) = az$, $a > 1$), then there exists $\eta = \eta(a) > 0$ and for every $z \in U$ there exists $\varrho_1 = \varrho_1(a, \operatorname{Arg}(w)) > 0$, such that*

$$\varrho_1 \leq \delta(z) \leq \eta.$$

If U is of parabolic type (and hence $f|_U$ is conjugate by Ψ to $g(z) = z+1$), then for every $z \in U$ there exist $\varrho_2 = \varrho_2(\operatorname{Im}(w)) > 0$ and $\varrho_3 = \varrho_3(\operatorname{Im}(w)) > 0$, such that $\lim_{\operatorname{Im}(w)\to+\infty} \varrho_2 = +\infty$ and

$$\varrho_2 \leq \delta(z) \leq \varrho_3.$$

In particular, in both cases for every $z \in U$ there exist $c_1, c_2 > 0$, such that for every $n \in \mathbb{Z}$,

$$c_1 \leq \delta(f^n(z)) \leq c_2.$$

Proof. Let $z \in U$. Since $\mathbb{H}^+$ contains the disc centred at w of radius $\operatorname{Im}(w)$, the Koebe Theorem implies that $\operatorname{dist}(z, \partial U) \geq \frac{1}{4}|\Psi'(w)|\operatorname{Im}(w) = \frac{1}{4}|\Psi'(w)||w|\sin(\operatorname{Arg}(w))$. Connect w to $g(w)$ by a straight line segment $S \subset \mathbb{H}^+$. Note that $\operatorname{dist}(S, \partial\mathbb{H}^+) = \operatorname{Im}(w)$.

Suppose U is of hyperbolic type. Then $\operatorname{diam} S = (a-1)|w|$, so by Corollary 2.1, the distortion of Ψ on S is bounded by a constant $\tilde{\varrho}_1$ depending only on a and $\operatorname{Arg}(w)$. Hence, $|f(z) - z| \leq (a-1)\tilde{\varrho}_1|\Psi'(w)||w|$. This gives the first inequality.

To obtain the second one, suppose that $\delta(z) > M$ for a large $M > 0$. Then $\operatorname{dist}(z, \partial U) > M|f(z) - z|$, so by the Koebe theorem,

$$\operatorname{Im}(w) = \operatorname{dist}(w, \partial\mathbb{H}^+) \geq \frac{M}{4}|(\Psi^{-1})'(z)||f(z) - z|$$

and

$$(a-1)|w| = \operatorname{diam} S \leq 2|(\Psi^{-1})'(z)||f(z) - z|$$

(if M is sufficiently large). This easily implies $M \leq 8/(a-1)$.

In the parabolic case, the proof is similar and we leave it to the reader. $\square$

Directly from this lemma we can deduce the following characterization of hyperbolic and parabolic type.

Corollary 3.4. *U is of hyperbolic type if and only if* $\sup_{z\in U}\delta(z)<\infty$. $\square$

We are now ready to prove Theorem B.

Proof of Theorem B. Let $z\in U$, $w=\Psi^{-1}(z)\in\mathbb{H}^+$ and define $\widetilde{\Gamma}$ to be the g-invariant curve

$$\widetilde{\Gamma}=\begin{cases}\{tw\mid t\in(0,+\infty)\} & \text{if } U \text{ is of hyperbolic type,}\\ \{t+i\mathrm{Im}(w)\mid t\in\mathbb{R}\} & \text{if } U \text{ is of parabolic type.}\end{cases}$$

Let $\widetilde{\gamma}_1(t)=\widetilde{\Gamma}(t)$ for $t\in(0,1]$ (in the hyperbolic case) or $t\in(-\infty,1]$ (in the parabolic case) and $\widetilde{\gamma}_2(t)=\widetilde{\Gamma}(t)$ for $t\in[1,+\infty)$. Finally, let $\Gamma=\Psi\circ\widetilde{\Gamma}$, $\gamma_1=\Psi\circ\widetilde{\gamma}_1$ and $\gamma_2=\Psi\circ\widetilde{\gamma}_2$. Suppose that $\gamma_1(t)$ does not tend to infinity as $t\to 0$ (resp. $t\to-\infty$). Then there exists $\zeta\in\mathbb{C}$ which is an accumulation point of the curve γ_1 for $t\to 0$ (resp. $t\to-\infty$). It is obvious that ζ is in the boundary of U. Take a sequence $z_n\in\gamma_1$ such that $z_n\to\zeta$. By Lemma 3.3, we have $|f(z_n)-z_n|\to 0$, so passing to the limit we get $f(\zeta)=\zeta$. Suppose γ_1 has another limit point for $t\to 0$ (resp. $t\to-\infty$), different from ζ. Then there is a continuum of such points and repeating the above arguments, we would have a continuum of fixed points of f. This shows that γ_1 has a limit ζ for $t\to 0$ (resp. $t\to-\infty$), in particular $f^{-n}(z)\to\zeta$ as $n\to+\infty$. By the Snail Lemma (see e.g. [M]), ζ is either repelling or parabolic with multiplier 1.

The same argument as in the proof of Lemma A shows that $f^{-n}(z)\to\zeta$ through the same access for every $z\in U$ (the backward dynamical access). Recall that γ_2 is in the forward dynamical access to ∞.

Suppose U is of parabolic type. Then, by definition, both $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ land at ∞. Hence, Theorem 2.3 implies $\zeta=\infty$. Moreover, by Lemma 3.1, γ_1 and γ_2 are in the same access to ∞, so the backward dynamical access is equal to the forward one. On the other hand, if U is of hyperbolic type, then $\widetilde{\gamma}_1$ lands at 0 and $\widetilde{\gamma}_2$ land at ∞, so by Lemma 3.1, γ_1 and γ_2 cannot be in the same access to ∞. Hence, in the hyperbolic case if $\zeta=\infty$, then the forward and backward dynamical accesses are different. $\square$

4 Consequences

Theorem B has several consequences concerning both the boundary behaviour of the Riemann map and the geometry of the boundary of the Baker domain.

The first consequence is as stated in the introduction:

Corollary C. *If U is a univalent Baker domain of hyperbolic type II, then ∂U is disconnected.*

Proof. This follows directly from Theorem B and Lemma 3.1. $\square$

Consider now the speed of convergence to ∞ of forward or backward iterates of points from U. By Lemma 3.3, it is related to the shape of the domain U. Moreover, by Bergweiler's result about the distance of points from the post-singular set $P(f)=\overline{\bigcup_{n\geq 0}f^n(\mathrm{Sing}(f^{-1}))}$

to ∂U (Lemma 3 in [Ber1]) and Lemma 3.3, for every compact set $K \subset U$ there exists a constant $q > 0$ and n_0, such that for every $z \in K$ and every $n > n_0$,

$$\operatorname{dist}(f^n(z), P(f)) \leq q|f^{n+1}(z) - f^n(z)|.$$

The estimates of the hyperbolic metric from Lemma 2.2 imply (see e.g. [Ber2]) that for any invariant Baker domain U (not necessarily univalent), we have

$$|f^n(z)|/C \leq |f^{n+1}(z)| \leq C|f^n(z)|$$

for every $z \in U$, where $C > 0$ is a constant depending on z. This implies

$$\log |f^n(z)| = O(n).$$

In the univalent case, we can slightly improve these estimates. Note first that by Corollary 2.1, the distortion of f^n on a compact set $K \subset U$ is bounded by a constant depending only on K. Therefore, the Koebe theorem implies that for every compact set $K \subset U$ there exists a constant $c > 0$, such that for every $z \in K$ and every $n \in \mathbb{Z}$,

$$|(f^n)'(z)|/c \leq |f^{n+1}(z) - f^n(z)| \leq c|(f^n)'(z)|.$$

Hence, if the forward or backward iterates tend to infinity, then the speed of convergence is closely related to the speed of the growth of the derivative. In particular, by Theorem B,

Corollary 4.1. *For every $z \in U$, the series $\sum_{n=0}^{\infty} |(f^n)'(z)|$ is diverging. Moreover, if U is of parabolic type or hyperbolic type II, then the series $\sum_{n=-\infty}^{0} |(f^n)'(z)|$ is diverging for every $z \in U$ and if U is of hyperbolic type I, then the series $\sum_{n=-\infty}^{0} |(f^n)'(z)|$ is converging for every $z \in U$.* □

Using the Koebe theorem for the Riemann map $R : \mathbb{D} \to U$, it is easy to check that for every compact set $K \subset U$ there exist constants $c_1, \dots, c_4 > 0$, such that for every $z \in K$ and every $n \in \mathbb{Z}$,

$$\left.\begin{aligned} \frac{1}{c_1 a^{2|n|}} \leq |(f^n)'(z)| &\leq c_1 a^{2|n|} \\ |f^n(z)| &\leq c_2 a^{2|n|} \end{aligned}\right\} \quad \text{if } U \text{ is of hyperbolic type,}$$

$$\left.\begin{aligned} \frac{1}{c_3 n^4} \leq |(f^n)'(z)| &\leq c_3 n^4 \\ |f^n(z)| &\leq c_4 n^4 \end{aligned}\right\} \quad \text{if } U \text{ is of parabolic type.}$$

(In the hyperbolic case the constant a is such that $f|_U$ is conjugate by Ψ to $z \mapsto az$.)

Theorem B gives another corollary concerning the speed of convergence to ∞ of forward and backward iterates in the parabolic case.

Corollary 4.2. *If U is of parabolic type, then for every $\varepsilon > 0$ there exists $z \in U$ and n_0, such that $|f^{n+1}(z) - f^n(z)| < \varepsilon|f^n(z)|$ for every $n \in \mathbb{Z}$ with $|n| > n_0$.*

Proof. Fix a point $z_0 \in \partial U$ and take $\varepsilon > 0$. By Lemma 3.3, for $z \in U$ with sufficiently large $\mathrm{Im}(w)$, we have $\delta(f^n(z)) > 2/\varepsilon$ for every $n \in \mathbb{Z}$. This implies

$$|f^{n+1}(z) - f^n(z)| < \frac{\varepsilon}{2}|f^n(z) - z_0|.$$

Moreover, by Theorem B, $f^n(z) \to \infty$ as $n \to \pm\infty$. Hence, we have $|f^n(z) - z_0| < 2|f^n(z)|$ for large $|n|$, which ends the proof. □

Note that the reciprocal of this corollary does not hold (see the example in Subsection 5.2).

Remark 4.3. We do not know any good lower estimates for the series from Corollary 4.1. For instance, it would be interesting to check whether it is possible to have $|(f^n)'(z)| \to 0$ as $n \to +\infty$ (which is equivalent to $|f^{n+1}(z) - f^n(z)| \to 0$). This can happen for non-univalent Baker domains (e.g. for the map $f(z) = z + e^{-z}$ studied in [BD2]), but in all known univalent examples we have $|f^{n+1}(z) - f^n(z)| > \mathrm{const}$. Geometrically, this is related to the problem of finding a Baker domain forming a cusp at infinity (see Lemma 3.3).

Concerning the boundary behaviour of the Riemann map we have the following corollaries.

Corollary 4.4. *If U is of hyperbolic type, then Ψ has non-tangential limits at 0 (equal to ζ from Theorem B) and ∞ (equal to ∞).* □

Corollary 4.5. *If U is of parabolic type, then Ψ has a limit equal to ∞ along every curve $\gamma : [0, +\infty) \to \mathbb{H}^+$, such that $\lim_{t\to+\infty} \gamma(t) = \infty$ and $\mathrm{Im}(\gamma(t)) > y_0$ for some constant $y_0 > 0$.*

Proof. Let $\Gamma = \Psi(\{\mathrm{Im}(w) = y_0\})$. By Theorem B, $\Gamma \cup \{\infty\}$ is a Jordan curve. Let S be the component of $\widehat{\mathbb{C}} \setminus (\Gamma \cup \{\infty\})$ containing $\Psi(\{\mathrm{Im}(w) > y_0\})$. Since $\Psi(\gamma(t))$ must approach the boundary of U as $t \to +\infty$, it is sufficient to show that S does not contain points from ∂U. This follows from Lemma 3.1. □

The following corollary states that in the parabolic case there exist trajectories lying arbitrarily close to infinity.

Corollary 4.6. *If U is of parabolic type, then for every $M > 0$ there exists $z \in U$, such that the entire orbit of z in U is outside the disc of radius M, i.e. $|f^n(z)| > M$ for every $n \in \mathbb{Z}$.*

Proof. Fix $M > 0$. Let $\Gamma_n = \Psi(\{\mathrm{Im}(w) = n\})$ for $n > 0$. Then Γ_n contains the entire trajectory of the point $\Psi(n\,i)$ in U. We prove that if n is sufficiently large, then $|z| > M$ for every $z \in \Gamma_n$. Suppose this is not true. Then there exists a sequence $z_{n_k} \in \Gamma_{n_k}$, such that $n_k \to +\infty$ and $|z_{n_k}| \le M$. But $\mathrm{Im}(\Psi^{-1}(z_{n_k})) = n_k$, so by Corollary 4.5, $z_{n_k} \to \infty$, which is a contradiction. □

The following is a curious observation.

Corollary 4.7. *If U is of parabolic type and Ψ has a limit at ∞, then there are no periodic points in ∂U. If U is of hyperbolic type and Ψ has limits at 0 and ∞, then either there are no periodic points in ∂U (provided U is of type II) or there is exactly one periodic point in ∂U (provided U is of type I), i.e. the fixed point ζ from Theorem B.*

In particular, by the Carathéodory theorem, this holds when the boundary of U is locally connected.

Proof. Let $\zeta_0 \in \mathbb{C}$ be a periodic point of period p in ∂U. Take a sequence of points $z_n \in U$, such that $z_n \to \zeta_0$ as $n \to +\infty$. Passing to a subsequence, we can assume additionally that $f^{pn}(z_n) \to \zeta_0$. Let $w_n = \Psi^{-1}(z_n)$ and $\tilde{w}_n = \Psi^{-1}(f^{pn}(z_n))$. Then $|w_n| \to 1$ and passing to a subsequence, we can assume $w_n \to v$ for some v in the boundary of $\mathbb{H}^+$. In the parabolic case, we have $\tilde{w}_n = w_n + pn$, so one of the sequences w_n, $\tilde{w}_n$ tends to ∞. By assumption, Ψ has a limit at ∞, which must be equal to ∞ by the dynamics of f. Hence, one of the sequences z_n, $f^{pn}(z_n)$ tends to ∞, which is a contradiction. In the hyperbolic case, $\tilde{w}_n = a^{pn}w_n$ and similar argument implies that one of the sequences w_n, $\tilde{w}_n$ tends either to ∞ or to 0. The first possibility, as previously, leads to a contradiction. If the second one holds, note that by assumption, Ψ has a limit at 0 which is equal to ζ by Theorem B. Therefore, $\zeta_0 = \zeta$. □

Remark 4.8. It was proved in [PZ] that if U is the immediate basin of attraction of an attracting or parabolic point for a rational map f, then periodic sources are dense in the boundary of U. To our knowledge it is not known, whether this holds for entire maps. For rotation domains of rational or entire maps, it is not known whether they must contain periodic sources.

We remark that, by Corollary 4.7 and the density of periodic sources in the Julia set, it can never occur that the Julia set coincides with the boundary of a univalent Baker domain. This is possible for the immediate basin of an attracting fixed point, e.g. for the map $f(z) = \lambda e^z$, $\lambda \in (0, 1/e)$.

5 Examples of three types of Baker domains

5.1 Hyperbolic type I

Let $f(z) = 2 - \log 2 + 2z - e^z$. It was shown in [Ber1] that f has an invariant univalent Baker domain U containing the half-plane $\{\mathrm{Re}(z) < -2\}$ and the boundary of U is a Jordan curve.

One can observe these facts by considering the map $F(w) = \frac{1}{2}w^2 e^{2-w}$, which is a projection of f by $w = e^z$, i.e. $F(w) = e^{f(z)}$. A simple computation shows that F has only two critical points (at $z = 0$ and $z = 2$) which are fixed, and an asymptotic value at $z = 0$. Let V be the immediate basin of (super)attraction of $z = 0$ and let U be the preimage of V under exp. By [Ber3], the Fatou set of F lifts to the Fatou set of f, so U is a Fatou component. Note that $|F(w)| < |w|$ for $|w| < e^{-2}$ and the circles $\{|w| = r\}$ are lifted to the lines $\{\mathrm{Re}(z) = \log r\}$. Hence, U is invariant, contains the half-plane $\{\mathrm{Re}(z) < -2\}$ and the orbits of all points from U escape to infinity. It follows that U is an invariant Baker domain. It is shown in [Ber1] (using polynomial-like maps) that the boundary of V is a Jordan curve in $\mathbb{C}$, so the boundary of U is a Jordan curve passing through infinity. By [BW], this implies that $f|_U$ is univalent.

Proposition 5.1. *The Baker domain U is of hyperbolic type I.*

Proof. Consider the map f as a map of the real line into itself. It is easy to check that f has two fixed points: $q = \log 2$, which is superattracting and $p \sim -1$, which is repelling (see Figure 3). The map is strictly increasing on the infinite segment $S = (-\infty, p)$ and $f(x) < x$

for every $x \in S$. Hence, for any $x \in S$ we have $f^n(x) \underset{n\to+\infty}{\longrightarrow} \infty$ and $(f|_S)^{-n}(x) \underset{n\to+\infty}{\longrightarrow} p$. This shows that $S \subset U$ and p is a repelling fixed point in ∂U. By Theorem B, U is of hyperbolic type I and the iterates of all points from U behave in the same way as for points in S. $\square$

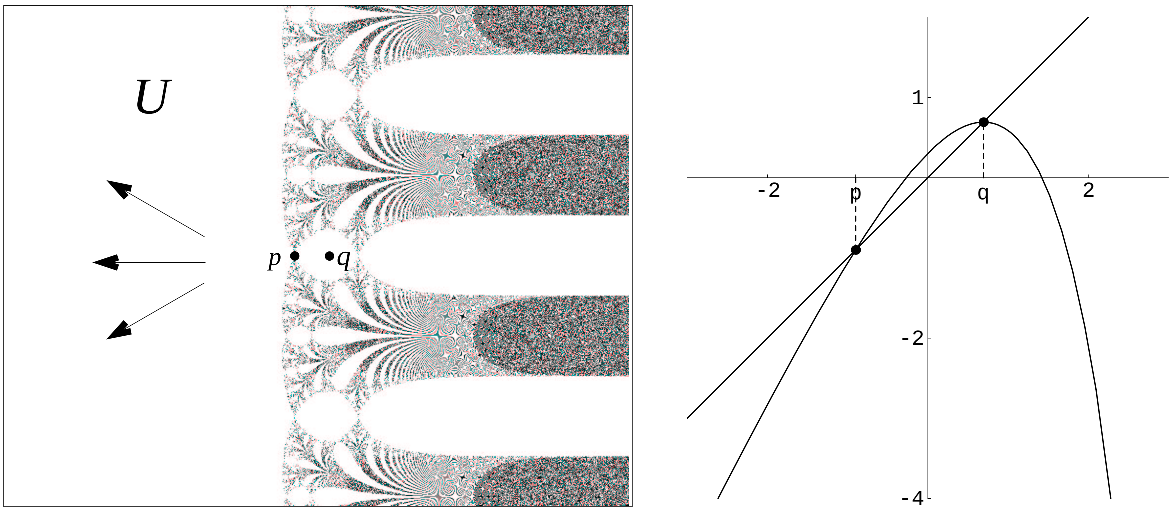

Figure 3: Left: The dynamical plane of $f(z) = 2 - \log 2 + 2z - e^z$ with the Baker domain U of hyperbolic type I. Right: The map $f(x)$ on the real line.

5.2 Hyperbolic type II: the standard family

Let $f(z) = z + \alpha + \beta \sin(z)$ for $0 < \alpha < 2\pi$ and $0 < \beta < 1$. Projecting f by $w = e^{iz}$, we obtain the map

$$F(w) = e^{i\alpha} w e^{\frac{\beta}{2}(w-1/w)}$$

which is a holomorphic self-map of $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. It is easy to check that F restricted to the unit circle S^1 is the well-known standard family of circle maps, studied initially by Arnold in [A].

It has been shown that for appropriately chosen values of the parameters α and β, the map F has a Herman ring V symmetric with respect to S^1 (see e.g. [Ba3, F, H1]). This means that V is conformally equivalent to an annulus and $F|_V$ is conformally conjugate to an irrational rigid rotation.

Like in the previous example, it is easy to check that lifting V by e^{iz} we obtain a Fatou component U of f, which is an invariant Baker domain, symmetric with respect to the real axis. Since V is a rotation domain, the map F is univalent in V. Using the fact $f(z + 2k\pi) = f(z) + 2k\pi$ one can easily show that $f|_U$ must also be univalent.

On the real axis, the forward iterates under f tend to $+\infty$ and the backward ones to $-\infty$. Since ∂U is symmetric with respect to $\mathbb{R}$, Lemma 3.1 implies that the backward dynamical access must be different from the forward one. Therefore, U is a Baker domain of hyperbolic type II and ∂U is disconnected.

We remark that the boundary of U has at least two connected components but it is not known whether these are the only two. As noted in [BD1], it is not known whether the Herman ring V contains 0 and ∞ in its boundary. If that were the case and these points were accessible from V, then U would have infinitely many accesses to infinity and ∂U would have infinitely many components. Numerical estimates seem to indicate this does not hold (see Figure 4).

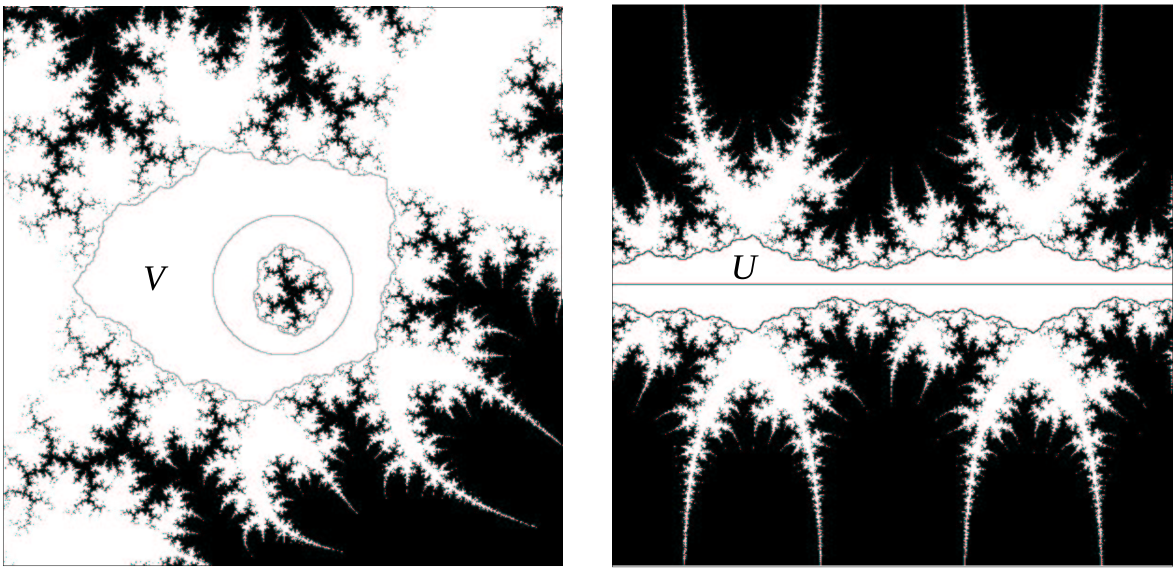

Figure 4: Left: The dynamical plane of $F(w) = we^{i\alpha}e^{(\beta/2)(w-1/w)}$ with $\alpha = 1.8$ and $\beta = 0.6$. The boundaries of the Herman ring V have been obtained by plotting the orbit of the critical points. The unit circle is indicated. Right: The dynamical plane of $f(z) = z + \alpha + \beta\sin(z)$ with the Baker domain U of hyperbolic type II.

However, there exists an example of a univalent Baker domain U, such that ∂U has exactly two components. In Theorem 7 of [BD1] the authors modify a construction of M. Herman [H2] and E. Ghys [Gh] to obtain an example of an analytic transcendental self-map f of $\mathbb{C}^*$, such that f has a Herman ring V symmetric with respect to S^1 and ∂V consists of two quasicircles in $\mathbb{C}^*$. The lift by exp of such a Herman ring gives an invariant univalent Baker domain U, such that ∂U has exactly two components.

5.3 Parabolic type: the semistandard map

Consider $F(w) = e^{\alpha}we^{w}$, known as the semistandard map. Then 0 is the fixed point of F and $F'(0) = e^{\alpha}$. Under the exponential map, F lifts to the map $f(z) = z + \alpha + e^{z}$. Note that if we choose the value of α appropriately, so that the fixed point $w = 0$ is say, attracting, then its immediate basin of attraction lifts to a domain containing a left half-plane $\{\mathrm{Re}(z) < x_0\}$ for some x_0, on which all iterates under f must tend to ∞ (like in Bergweiler's example). However, the Baker domain obtained in this way is non-univalent.

The works of Herman [H2] and Ghys [Gh] imply (as explained in [BW]), that for appropriate values of the parameter α the map F has a Siegel disc V around $w = 0$, bounded by a quasicircle. In the same way as in Subsection 5.2 one can show that lifting V by the exponential, we obtain a univalent Baker domain U containing a left half-plane, such that U is bounded by a Jordan curve (the lift of ∂V). See Figure 5.

Observe that if $z \in U$ and $\mathrm{Re}(z) \to -\infty$, then $\mathrm{dist}(z, \partial U)/|f(z) - z| \to \infty$. Hence, by Corollary 3.4, U is of parabolic type.

Figure 5: Left: The dynamical plane of $F(w) = e^{\alpha} w e^{w}$ with $e^{\alpha} = (\sqrt{5} - 1)/2$. The orbit of the critical point and some orbits in the Siegel disc V are indicated. Right: The dynamical plane of $f(z) = z + \alpha + e^{z}$ with the Baker domain U of parabolic type.

6 Example of a univalent spiraling Baker domain

For any $a > 0$ we construct an entire transcendental map $F : \mathbb{C} \to \mathbb{C}$ with an invariant Baker domain U, such that F is univalent on an open neighbourhood of $\overline{U}$ and

$$U \supset \{re^{i\theta} \mid a \log r + \alpha_1 < \theta < a \log r + \beta_1, r > r_1\},$$
$$U \subset \{re^{i\theta} \mid a \log r + \alpha_2 < \theta < a \log r + \beta_2, r > r_2\}$$

for some constants $\alpha_1, \alpha_2, \beta_1, \beta_2, r_1, r_2$, such that $0 < \beta_i - \alpha_i < 2\pi$, $r_i > 0$, $i = 1, 2$. First we prove a general lemma about constructing invariant Baker domains using the approximation theory, based on the method from [EL2].

Lemma 6.1. *Let V_1, V_2, W_1, W_2 be unbounded domains in $\mathbb{C}$, such that $V_1 \subset V_2 \subset W_1$, $\overline{W_1} \cap \overline{W_2} = \emptyset$ and $\widehat{\mathbb{C}} \setminus (\overline{W_1} \cup \overline{W_2})$ is connected and locally connected at ∞. Let $G : \overline{W_1} \cup \overline{W_2} \to \mathbb{C}$ be a continuous map, holomorphic on $W_1 \cup W_2$, such that:*

(a) $G(\overline{V_1}) \subset V_1$ *and* $\mathrm{dist}(G(\overline{V_1}), \partial V_1) > c_1$,

(b) $G(\partial V_2) \subset W_2$ *and* $\mathrm{dist}(G(\partial V_2), \partial W_2) > c_2$,

(c) *$G(\overline{W_2})$ is a bounded subset of W_2 and* $\operatorname{dist}(G(\overline{W_2}), \partial W_2) > c_3$,

(d) $|G(z) - z| > c_4$ *for every* $z \in V_1$

for some constants $c_1, c_2, c_3, c_4 > 0$. Then for every sufficiently small $\varepsilon > 0$ there exists an entire transcendental map $F : \mathbb{C} \to \mathbb{C}$, such that $|F(z) - G(z)| < \varepsilon$ for every $z \in \overline{W_1} \cup \overline{W_2}$ and F has an invariant Baker domain U, such that $\overline{V_1} \subset U \subset \overline{U} \subset V_2$.

Moreover, if in addition:

(e) $\overline{V_2} \subset W_1$ *and* $\operatorname{dist}(\overline{V_2}, \partial W_1) > c_5$,

(f) *Every points $z_1, z_2 \in V_2$ can be joined by a curve in V_2 of length smaller than $c_6|z_1 - z_2|$,*

(g) $|G(z_1) - G(z_2)| \geq c_7|z_1 - z_2|$ *for every $z_1, z_2 \in V_2$ such that* $|z_1 - z_2| < 1$,

(h) $|G(z_1) - G(z_2)| > c_8$ *for every $z_1, z_2 \in V_2$ such that* $|z_1 - z_2| \geq 1$

for some constants $c_5, c_6, c_7, c_8 > 0$, then F is univalent on V_2, in particular on U.

Proof. To construct the map F we use the following theorem from approximation theory (see e.g. [Ga]).

Arakeljan's Theorem. *Let $E \subset \mathbb{C}$ be a closed set. The following properties are equivalent:*

1. *Every function ϕ continuous on E and analytic in the interior of E can be uniformly approximated by entire functions,*
2. *$\widehat{\mathbb{C}} \setminus E$ is connected and locally connected at ∞.*

Set $E = \overline{W_1} \cup \overline{W_2}$, $\phi = G$ and choose $\varepsilon > 0$ smaller than c_i for $i = 1, \ldots, 4$. By the Arakeljan theorem, there exists an entire map $F : \mathbb{C} \to \mathbb{C}$, such that $|F(z) - G(z)| < \varepsilon$ for every $z \in \overline{W_1} \cup \overline{W_2}$. This together with the assumption (a) implies that $F(\overline{V_1}) \subset V_1$ and $\overline{V_1}$ is contained in some F-invariant Fatou component U. By the assumption (d), there are no F-fixed points in $\overline{V_1}$, so by the classification of Fatou components, $F^n(z) \to \infty$ as $n \to +\infty$ for every $z \in U$. On the other hand, by the assumption (c), we have $F(\overline{W_2}) \subset W_2$ and $\overline{W_2}$ is contained in a basin of an attracting fixed point from W_2. This implies that $U \cap \overline{W_2} = \emptyset$ and also that F is not a polynomial (because W_2 is unbounded). Moreover, $F(\partial V_2) \subset W_2$ by the assumption (b), so $\overline{U} \subset V_2$. Thus, U is an invariant Baker domain, such that $\overline{V_1} \subset U$ and $\overline{U} \subset V_2$.

Suppose now that the additional assumptions of the lemma are satisfied. We show that for sufficiently small ε, F is univalent on V_2. Suppose that $F(z_1) = F(z_2)$ for some $z_1, z_2 \in V_2$. By the assumption (h), we can assume $|z_1 - z_2| < 1$. Let $\psi(z) = F(z) - G(z)$ for $z \in \overline{W_1} \cup \overline{W_2}$. By the assumption (e), ψ is defined on every disc centred at $z \in V_2$ of radius c_5. Since $|\psi| < \varepsilon$, by the Cauchy formula for ψ' we get

$$|\psi'(z)| < \varepsilon / c_5 \text{ for every } z \in V_2.$$

This together with the assumption (f) gives

$$|\psi(z_1) - \psi(z_2)| \leq (c_6/c_5)\varepsilon|z_1 - z_2|,$$

so using the assumption (g) we get

$$c_7|z_1 - z_2| \le |G(z_1) - G(z_2)| = |\psi(z_1) - \psi(z_2)| \le (c_6/c_5)\varepsilon|z_1 - z_2|.$$

This implies $z_1 = z_2$ provided $\varepsilon < c_5c_7/c_6$. $\square$

To construct our spiraling example, we define the suitable domains V_1, V_2, W_1, W_2 together with the map G, such that all the assumptions of Lemma 6.1 are satisfied.

Let $H(w) = 4(w-5)+5$ for $w \in \mathbb{C}$. Set $\theta_0 = \arctan a$, $s = \sin^2\theta_0$ and define

$$\begin{aligned} P_1 &= \{w \mid x > |y|^s + 6\}, & \tilde{P}_1 &= H(P_1) = \{w \mid x > 4^{1-s}|y|^s + 9\}, \\ P_2 &= \{w \mid x > -|y|^s + 4\}, & \tilde{P}_2 &= H(P_2) = \{w \mid x > -4^{1-s}|y|^s + 1\}, \\ S &= \{w \mid x > -2^{1-s}|y|^s + 3\}, & l &= \{w \mid x = -2^{1-s}|y|^s + 2\}, \end{aligned}$$

where $w = x+iy \in \mathbb{C}$. Then $\overline{\tilde{P}_1} \subset P_1 \subset \overline{P_1} \subset P_2 \subset \overline{P_2} \subset S \subset \overline{S} \subset \tilde{P}_2$ and $l \subset \tilde{P}_2 \setminus \overline{S}$. See Figure 6.

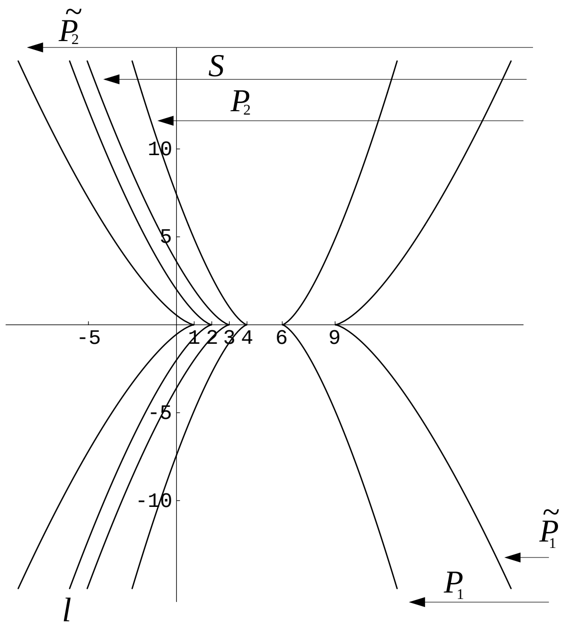

Figure 6: The regions P_1, $\tilde{P}_1$, P_2, $\tilde{P}_2$, S and the curve l.

Denote by Log the branch of the logarithm defined on $\mathbb{C} \setminus (-\infty, 0]$ leading to the strip $\{z \mid |\mathrm{Im}(z)| < \pi\}$. Note that $\mathrm{Log}(P_1) \supset \{x+iy \mid x > x_0, -\pi/3 < y < \pi/3\}$ for some $x_0 > 0$.

Let $Q(z) = \cos\theta_0 e^{i\theta_0} z$ and let

$$\Phi = \exp \circ Q \circ \mathrm{Log}\,.$$

Define

$$\begin{aligned} V_i &= \Phi(P_i), & \tilde{V}_i &= \Phi(\tilde{P}_i),\ i = 1, 2, \\ W_1 &= \Phi(S), & \Gamma &= \Phi(l). \end{aligned}$$

An easy calculation shows that the image of a horizontal line $\{x+iy_0 \mid x \in \mathbb{R}\}$ under $\exp \circ Q$ is a spiraling curve $\{re^{i\theta} \mid \theta = a \log r + y_0\}$. Hence, V_1 contains a suitable spiraling domain and V_2 is contained in another such domain. Moreover, the intersection of $Q(\{z \mid |\mathrm{Im}(z)| < \pi\})$ with any vertical line is an open interval of length 2π, so exp is univalent on $Q(\{z \mid |\mathrm{Im}(z)| < \pi\})$. Hence, Φ is univalent on $\mathbb{C} \setminus (-\infty, 0]$. Define

$$G(z) = \Phi(H(\Phi^{-1}(z)))$$

for $z \in \overline{W_1}$. Define also W_2 to be the component of $\mathbb{C} \setminus \Gamma$ disjoint from W_1. (See Figure 7). Then $\Gamma = \partial W_2$, $\overline{W_1} \cap \overline{W_2} = \emptyset$ and $\partial \tilde{V}_2 \subset W_2$. Choose a geometric disc D such that $\overline{D} \subset W_2$ and define G on $\overline{W_2}$ to be a homeomorphism onto $\overline{D}$, conformal on W_2.

It is obvious that V_1, V_2, W_1, W_2 are simply connected unbounded domains in $\mathbb{C}$, such that $V_1 \subset V_2 \subset \overline{V_2} \subset W_1$, $\overline{W_1} \cap \overline{W_2} = \emptyset$ and $\widehat{\mathbb{C}} \setminus (\overline{W_1} \cup \overline{W_2})$ is connected and locally connected at ∞. See Figure 7.

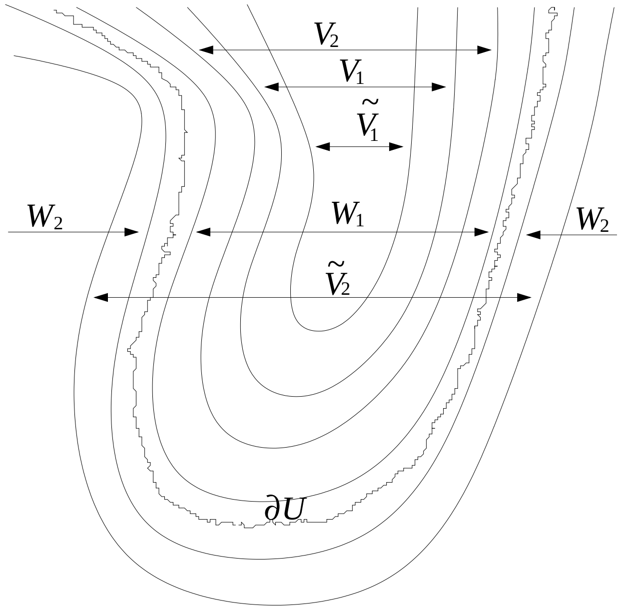

Figure 7: The image of Figure 6 under the map Φ.

By definition, $G(\overline{V_1}) = \overline{\tilde{V}_1} \subset V_1$ and $G(\partial V_2) = \partial \tilde{V}_2 \subset W_2$. Moreover, $G(\overline{W_2}) = \overline{D}$, so it is a bounded subset of W_2 and $\mathrm{dist}(G(\overline{W_2}), \partial W_2) > C_1$ for some constant $C_1 > 0$. Hence, the assumption (c) of Lemma 6.1 is satisfied.

A straightforward computation shows

$$|w|^{-s}/C_2 < |\Phi'(w)| < C_2|w|^{-s} \tag{1}$$

for some constant $C_2 > 0$ and every $w \in \mathbb{C} \setminus (-\infty, 0]$. Moreover, we have

Lemma 6.2. *There exists $c > 0$ such that*

$$\begin{aligned} \mathrm{dist}(w, \partial P_i) &> c|w|^s \textit{ for every } w \in \partial \tilde{P}_i,\ i = 1, 2, \\ \mathrm{dist}(w, l) &> c|w|^s \textit{ for every } w \in \partial \tilde{P}_2, \\ \mathrm{dist}(w, \partial P_2) &> c|w|^s \textit{ for every } w \in \partial S. \end{aligned}$$

Proof. Note that the sets $\partial P_i, \partial\tilde{P}_i, \partial S, l$ are disjoint curves described by equations of the form $x = A|y|^s + B$ for some $A, B \in \mathbb{R}$. Hence, clearly we can assume that $|w|$ is sufficiently large. Then the constants B are small compared to $|y|^s$, so to prove the lemma, it is sufficient to show that for given $A_1 > A_2 > 0$, if $w_1 = x_1 + iy_1$, $w_2 = x_2 + iy_2$ with $x_1 = A_1 y_1^s$, $x_2 = A_2 y_2^s$, then there exists $c > 0$ such that $|w_1 - w_2| > c|w_1|^s$ for sufficiently large $y_1, y_2 > 0$. Note that by definition, $s \in (0, 1)$.

If $|y_1 - y_2| > y_1^s$, then for large y_1,

$$|w_1|^s \le (x_1 + y_1)^s < (2y_1)^s < 2^s|y_1 - y_2| \le 2^s|w_1 - w_2|.$$

If $|y_1 - y_2| \le y_1^s$, then

$$|y_1^s - y_2^s| \le s(y_1^{s-1} + y_2^{s-1})|y_1 - y_2| \le s(y_1^{s-1} + y_2^{s-1})y_1^s,$$

so

$$\begin{aligned}|w_1 - w_2| \ge |x_1 - x_2| \ge A_1 y_1^s - A_2(y_1^s + s(y_1^{s-1} + y_2^{s-1})y_1^s)\\ = y_1^s(A_1 - A_2 - A_2 s(y_1^{s-1} + y_2^{s-1})).\end{aligned}$$

Since $A_1 - A_2 > 0$ and $y_1^{s-1} + y_2^{s-1}$ tends to 0 as $y_1, y_2 \to +\infty$, we get $|w_1 - w_2| > ((A_1 - A_2)/2)y_1^s > ((A_1 - A_2)/4)|w_1|^s$ for large y_1, y_2. □

Using (1), Lemma 6.2 and the Koebe theorem, we easily check that the assumptions (a), (b), (d), (e) and (f) of Lemma 6.1 are satisfied. Finally, we check the assumptions (g) and (h). Let $z_1, z_2 \in V_2$ and let $w_1 = \Phi^{-1}(z_1)$, $w_2 = \Phi^{-1}(z_2)$. Suppose $|z_1 - z_2| < 1$. Then by the assumptions (b), (f), Corollary 2.1 and the Koebe theorem for the map Φ^{-1}, we have

$$\operatorname{dist}(w_1, (-\infty, 0]) > C_3|w_1 - w_2|$$

for some constant $C_3 > 0$. Since H is affine, this together with Corollary 2.1 and the Koebe theorem for the map Φ implies $|G(z_1) - G(z_2)| \ge C_4|z_1 - z_2|$ for a constant $C_4 > 0$. Hence, the assumption (g) is satisfied. Assume now $|z_1 - z_2| \ge 1$ and suppose $|G(z_1) - G(z_2)| < \varepsilon_0$ for a small $\varepsilon_0 > 0$. Note that $G(z_1), G(z_2) \in \tilde{V}_2$ and by (1) and the Koebe theorem, we have

$$\operatorname{dist}(\tilde{V}_2, \Phi(\mathbb{C} \setminus (-\infty, 0])) > C_5$$

for a constant $C_5 > 0$. Hence, by Corollary 2.1 and the Koebe theorem for the map Φ^{-1},

$$\operatorname{dist}(H(w_1), (-\infty, 0]) > C_6|H(w_1) - H(w_2)| = 4C_6|w_1 - w_2|$$

for some constant $C_6 > 0$. Using this, Corollary 2.1 and the Koebe theorem for the map Φ, we get $|G(z_1) - G(z_2)| > C_7|z_1 - z_2| \ge C_7$ for a constant $C_7 > 0$. This is a contradiction for $\varepsilon_0 < C_7$. Therefore, the assumption (h) is satisfied. In this way we have shown that all the assumptions of Lemma 6.1 are fulfilled.

Remark 6.3. It is easy to check that in our example, U is of hyperbolic type I. Indeed, let $Y \subset P_2$ be a bounded domain containing the segment $[5, 13]$, such that $\overline{H^{-1}(Y)} \subset Y$. Then $\Phi(Y) \subset V_2$, so f is univalent on $\Phi(Y)$ and

$$\overline{(F|_{\Phi(Y)})^{-1}(\Phi(Y))} \subset \Phi(Y)$$

(if ε is small enough). Hence, $\Phi(Y)$ contains a repelling F-fixed point ζ. Since $\Phi([7, 13]) \subset \Phi(Y)$ is a curve joining $\Phi(7)$ with $F(\Phi(7))$ in U, it follows that $(F|_{\Phi(Y)})^{-n}(\Phi(7)) \in U$ for $n \geq 0$, so the backward iterates of $\Phi(7)$ under $(F|_U)^{-n}$ tend to ζ. By Theorem B, U is of hyperbolic type I.

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