On the Dynamics of the Trojan Asteroids

Menelaus, Paris, Diomedes, Odysseus, Nestor, Achilles and Agamemnon.

Frederic Gabern Guilera

29 d’abril de 2003
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  – What are the Trojan asteroids?
  – Review of the existing results.
  – Motivation for the present work.

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  – The Bicircular Coherent Problem.
  – The Elliptic Restricted Three Body Problem.

• Quasi-periodic models.
  – The Bianular Problem.
  – The Tricircular Coherent Problem.

• OSS model.

• Conclusions and Future Work.
What are the Trojan asteroids? (I)

$L_4$:
- Achilles
- Hector
- Nestor
- Agamemnon

$L_5$:
- Patroclus
- Aeneas
- Memnon
- Paris

L4: Greek asteroids
L5: Trojan asteroids
What are the Trojan asteroids? (II)
What are the Trojan asteroids? (III)
Previous Work

The Restricted Three Body Problem

- Analytic Tools.
- Semi-analytic tools.

The Outer Solar System

- Frequency analysis.
- Lyapunov exponents.

Theoretical Results

Numerical Results
The Restricted Three Body Problem

The dynamics around the triangular points of the Sun-Jupiter system has been studied using analytical and semi-analytical tools (transformation of the Hamiltonian to a normal form, computation of some approximate first integrals) by several authors: Giorgilli, Delshams, Fontich, Galgani & Simó, 1989; Simó, 1989; Celletti & Giorgilli, 1991; Jorba & Simó, 1994 (in the elliptic case); Giorgilli & Skokos, 1997; Skokos & Dokoumetzidis, 2000.
The dynamics of some jovian Trojan asteroids has been studied in several papers using the Outer Solar System: Bien & Schubart, 1987; Milani, 1993; Levison, Shoemaker & Shoemaker, 1997; Pilat-Lohinger, Dvorak & Burger, 1999; Tsiganis, Dvorak & Pilat-Lohinger, 2000.
Some New Models

We want to build and study models more sophisticated than the RTBP. These models will try to simulate in a better way the relative Sun-Jupiter motion (that it turns out to be one of the key points in a Trojan simulation) and they will be written as a RTBP perturbation in order the semi-analytical tools can be applied.

In this work, we focus on 4 models:

- The Bicircular Coherent Problem (BCCP).
- The Elliptic Restricted Three Body Problem (ERTBP).
- The Bianular Problem (BAP).
- The Tricircular Coherent Problem (TCCP).
The BCCP Model (I)

- We look for a **periodic** solution of the planar Sun-Jupiter-Saturn Three Body Problem.
- The **period** is chosen to be the relative period of **Saturn** in the Sun-Jupiter RTBP system: $T_{sat}$.
- The osculating eccentricity of **Jupiter** is small (about 50 times smaller than the real one) but the **semi-major axis** and the **period** are quite well adjusted to the actual ones.
The BCCP Model (II)

It is possible to write the equations of a massless particle that moves under the attraction of the three primaries. The corresponding Hamiltonian is:

\[
H_{BCCP} = \frac{1}{2} \alpha_1(\theta)(p_x^2 + p_y^2 + p_z^2) + \alpha_2(\theta)(xp_x + yp_y + zp_z) \\
+ \alpha_3(\theta)(yp_x - xp_y) + \alpha_4(\theta)x + \alpha_5(\theta)y - \alpha_6(\theta) \left[ \frac{1 - \mu}{q_S} + \frac{\mu}{q_J} + \frac{m_{sat}}{q_{sat}} \right]
\]

where

\[
q_S^2 = (x - \mu)^2 + y^2 + z^2, \\
q_J^2 = (x - \mu + 1)^2 + y^2 + z^2, \\
q_{sat}^2 = (x - \alpha_7(\theta))^2 + (y - \alpha_8(\theta))^2 + z^2,
\]

and \(\theta = \omega_{sat}t + \theta^0\). The auxiliary \(\alpha_i(\theta)\) functions are \(2\pi\)-periodic and are obtained by means of Fourier analysis.

Note: A Bicircular Coherent problem was first developed by Andreu (PhD Thesis, 1998) for the Earth-Moon-Sun case to study the Eulerian points.
The ERTBP

The elliptic RTBP is a classical model for studying the dynamics of a small particle in the Sun-Jupiter system. We rewrite it in such a way that the physical time is the independent variable.

\[ H_{ERTBP} = \frac{1}{2} \alpha_1(\bar{\theta})(p_x^2 + p_y^2 + p_z^2) + \alpha_2(\bar{\theta})(xp_x + yp_y + zp_z) \]

\[ + \alpha_3(\bar{\theta})(yp_x - xp_y) - \alpha_4(\bar{\theta}) \left[ \frac{1 - \mu}{q_S} + \frac{\mu}{q_J} \right] \]

where

\[ q_S^2 = (x - \mu)^2 + y^2 + z^2, \]
\[ q_J^2 = (x - \mu + 1)^2 + y^2 + z^2, \]

\[ \bar{\theta} = t + \bar{\theta}^0, \text{ and} \]

\[ \alpha_j(\bar{\theta}) = \sum_{k \geq 0} \alpha_{jk} \cos(k\bar{\theta}), \quad j = 1, 3, 4; \quad \alpha_2(\bar{\theta}) = \sum_{k \geq 1} \alpha_{2k} \sin(k\bar{\theta}), \]

are \(2\pi\)-periodic functions that are obtained from the elliptic orbit.
Local Study around $L_5$

In the BCCP and ERTBP systems, the $L_5$ point is replaced by a periodic orbit.

(BCCP case: x-y projection)

The linear dynamics around these orbits is totally elliptic.
Expansion of the Hamiltonian

We compose three linear changes of variables:

1. A **Periodic** Translation.
2. A **Symplectic Floquet** Transformation.
3. A **Complexification**.

to write the second degree of the Hamiltonians as:

\[
H_2(q, p) = i\omega_1 q_1 p_1 + i\omega_2 q_2 p_2 + i\omega_3 q_3 p_3.
\]

The frequencies are

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Finally, we expand the Hamiltonians in **Fourier-Taylor** series.
Truncated Normal Forms

Using the Lie series method implemented as in Jorba 99, we transform the expanded Hamiltonians to truncated normal forms up to high degree:

\[
H_{ERTBP} = I\bar{\theta} + \omega_1 I_1 + \omega_2 I_2 + I_3
\]
\[
+ \sum_{n=2}^{[N/2]} H^{(n)}(I_1, I_2, I_3, \varphi_3 - \bar{\theta}) + \sum_{k \geq N} H_k(q, p, \bar{\theta})
\]

\[
H_{BCCP} = \omega_{sat} I_\theta + \omega_1 I_1 + \omega_2 I_2 + \omega_3 I_3
\]
\[
+ \sum_{n=2}^{[N/2]} H^{(n)}(I_1, I_2, I_3) + \sum_{k \geq N} H_k(q, p, \theta)
\]

where \( I_1, I_2 \) and \( I_3 \) are the actions \( I_j = q_j \cdot p_j \).

\( I\bar{\theta} \) and \( I_\theta \) are the “fictitious” momenta corresponding to the variables \( \bar{\theta} = t + \bar{\theta}^0 \) and \( \theta = \omega_{sat} t + \theta^0 \).
Local Non-Linear Dynamics (I)

- **ERTBP**: (Jorba & Simó, 94) The phase space

\[(J_3, \psi_3) = (I_{\bar{\theta}} + I_3, \varphi_3 - \bar{\theta})\]

corresponds essentially to a perturbed pendulum depending on the parameters \(I_1\) and \(I_2\). In particular, near the periodic orbit replacing \(L_5\) we have normally hyperbolic invariant tori of dimensions 2 and 3.

- **BCCP**: The phase space around the periodic orbit is completely foliated by 1, 2 and 3-parametric families of invariant tori. In particular, in the initial BCCP coordinates, they give rise to 2, 3 and 4-dimensional tori (the perturbation adds an extra frequency).
Local Non-Linear Dynamics (II): ERTBP
Local Non-Linear Dynamics (III): BCCP
**Approximate First Integrals**

Given the expanded Hamiltonians

\[ H^M(q, p, \theta, p_\theta) = \omega^M p_\theta + H_2^M(q, p) + \sum_{n \geq 3} H_n^M(q, p, \theta), \]

where \( \omega^{BCCP} = \omega_{sat} \) and \( \omega^{ERTBP} = 1 \), we look for functions

\[ F^M(q, p, \theta) = \sum_{n \geq 2} F_n^M(q, p, \theta) \]

such that

\[ \{H^M, F^M\} = 0. \]

This equation gives a recursive way of computing the approximate first integrals \( F^M \). We solve it, for both models, up to order 16.

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<th>BCCP</th>
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<td>( F_2^M )</td>
<td>( iq_1 p_1 + iq_2 p_2 )</td>
<td>( iq_1 p_1 + iq_2 p_2 + iq_3 p_3 )</td>
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BCCP: Zones of Effective Stability

Normal Form

First Integral

(x-y section for \( t = 0 \))

(x-y section for \( t = 0 \))

(x-y projection)
Quasi-Periodic Models

- The Bianular Problem (BAP).
- The Tricircular Coherent Problem (TCCP).
- A Preliminary Study of the TCCP.
The Bianular Problem (I)

- We compute a quasi-periodic solution, with two basic frequencies, of the planar Sun-Jupiter-Saturn Three Body Problem.

- This quasi-periodic solution lies on a torus. As the problem is Hamiltonian, this torus belongs to a family of tori.

- We look for a torus on this family for which the osculating eccentricity of Jupiter’s orbit is quite well adjusted to the actual one.
The Bianular Problem (II)

We split the construction of the model in four parts:

1. The reduced Hamiltonian of the planar Three Body Problem.
3. Finding the desired torus.
   Continuation of the family of invariant curves.
4. The Hamiltonian of the Bianular Problem.
The reduced Hamiltonian of the Three Body Problem

We take the Hamiltonian of the planar Three Body Problem written in the Jacobi coordinates in a uniformly rotating reference frame and we make a canonical change of variables (using the angular momentum first integral) in order to reduce this Hamiltonian from 4 to 3 degrees of freedom.

\[
H = \frac{1}{2\alpha} \left( P_1^2 + \frac{A^2}{Q_1^2} \right) + \frac{1}{2\beta} \left( P_2^2 + P_3^2 \right) - K - \frac{\alpha}{r} \left( \frac{(1 - \mu)m_{sat}}{r_{13}} \right) - \frac{\mu m_{sat}}{r_{23}}
\]

where

\[
\alpha = \mu(1 - \mu) \quad \beta = \frac{m_{sat}}{(1 + m_{sat})} \quad A = Q_2P_3 - Q_3P_2 + K
\]
Computing 2-D invariant tori (I)

- The method can be found in *Castellà & Jorba, CelMech, 2000*. Basically, it is a method for computing invariant curves of maps.

- The map is chosen as the Poincaré map of the time-$T$ flow corresponding to the reduced Three Body Problem (let us call it $\Phi_T(\cdot)$), where the fixed time $T$ is the relative period of Saturn in the Sun-Jupiter system ($T_{sat} = \frac{2\pi}{\omega_{sat}}$).
Computing 2-D invariant tori (II)

Let

$$\varphi(\theta) = A_0 + \sum_{k>0} (A_k \cos(k\theta) + B_k \sin(k\theta))$$

be a parameterization of the invariant curve and $\omega$ its rotation number. Let, also, $C(T^1, \mathbb{R}^n)$ be the space of continuous functions from $T^1$ in $\mathbb{R}^n$, and let us define the map $F : C(T^1, \mathbb{R}^n) \rightarrow C(T^1, \mathbb{R}^n)$ as

$$F(\varphi)(\theta) = \Phi_T(\varphi(\theta)) - (T_\omega \varphi)(\theta) \quad \forall \varphi \in C(T^1, \mathbb{R}^n),$$

where $T_\omega : C(T^1, \mathbb{R}^n) \rightarrow C(T^1, \mathbb{R}^n)$ is the translation by $\omega$:

$$(T_\omega \varphi)(\theta) = \varphi(\theta + \omega).$$

It is clear that the zeros of $F$ in $C(T^1, \mathbb{R}^n)$ correspond to invariant curves of rotation number $\omega$. 
Computing 2-D invariant tori (III)

As we do the computations numerically, we fix a truncation value for the series \((N_f)\) and we construct a discretized version of the function \(F\) on the following mesh of \(2N_f + 1\) points on \(\mathbb{T}^1\):

\[
\theta_j = \frac{2\pi j}{2N_f + 1}, \quad 0 \leq j \leq 2N_f.
\]

Let \(F_{N_f}\) be this discretization of \(F\):

\[
\Phi_T(\varphi(\theta_j)) - \varphi(\theta_j + \omega), \quad \forall 0 \leq j \leq 2N_f.
\]

So, given a set of Fourier coefficients \(A_0, A_k\) and \(B_k\) \((1 \leq k \leq N_f)\), we can compute the points \(\varphi(\theta_j)\), then \(\Phi_T(\varphi(\theta_j))\) and next the points \(\Phi_T(\varphi(\theta_j)) - \varphi(\theta_j + \omega), \quad 0 \leq j \leq N_f\).
Finding the desired torus (I)

For solving the equations, we use the well known Newton method.

The initial approximation to the unknowns in the Newton method is given by the linearization of the Poincaré map around a fixed point (a periodic orbit, for the flow) $X_0$. We use the periodic orbit computed in the BCCP model.

$$X_0 = \Phi_{T_{sat}}(X_0)$$

As there are two different non-neutral normal directions, there will be two families of tori starting at the periodic orbit.
Finding the desired torus (II)

1. We compute a first torus for each family.

2. We add to the invariant curve equations the following one:

$$\text{eccen}(x_1, x_2, x_3, x_4, x_5, x_6, K) = e$$

where $\text{eccen}(\cdot)$ is a function that gives us Jupiter’s osculating eccentricity (we evaluate it when Sun, Jupiter and Saturn are in a particular collinear configuration); and $e$ is a fixed constant that will be used as a control parameter.

3. We make continuation of each family increasing the parameter $e$ to its actual value. This is equivalent to use the angular momentum $K$ as parameter.
Finding the desired torus (III)

The results differ depending on the family.

- **Family 1**: As the parameter $e$ is increased, the number of harmonics ($N_f$) increases very much. We stop the continuation when $N_f = 90$ (181 harmonics). The solution’s orbital elements are not as desired.
Finding the desired torus (IV)

- **Family 2**: Here, it is possible to continue the family of tori until Jupiter’s eccentricity is \( e = 0.0484 \). The two frequencies of the final torus are:

  \[
  \omega_1 = 0.597039074021947 \text{ (Saturn’s frequency)}. \]

  \[
  \omega_2 = 0.194113943490717 \left( \frac{\omega_1 \cdot \omega}{2\pi} \right). \]

The **Bianular** Solution:

(Rotating Ref. Frame) (Inertial Ref. Frame)
The Hamiltonian of the BAP Model

It is possible to write the equations of a massless particle that moves under the attraction of the three primaries. The corresponding Hamiltonian is:

\[
H_{BAP} = \frac{1}{2} \alpha_1(\theta_1, \theta_2)(p_x^2 + p_y^2 + p_z^2) + \alpha_2(\theta_1, \theta_2)(x p_x + y p_y + z p_z) \\
+ \alpha_3(\theta_1, \theta_2)(y p_x - x p_y) + \alpha_4(\theta_1, \theta_2)x + \alpha_5(\theta_1, \theta_2)y \\
- \alpha_6(\theta_1, \theta_2) \left[ \frac{1 - \mu}{q_S} + \frac{\mu}{q_J} + \frac{m_{sat}}{q_{sat}} \right]
\]

where

\[
q_S^2 = (x - \mu)^2 + y^2 + z^2, \quad q_J^2 = (x - \mu + 1)^2 + y^2 + z^2, \\
q_{sat}^2 = (x - \alpha_7(\theta_1, \theta_2))^2 + (y - \alpha_8(\theta_1, \theta_2))^2 + z^2, \\
\theta_1 = \omega_1 t + \theta_1^{(0)}, \quad \theta_2 = \omega_2 t + \theta_2^{(0)}.
\]
The functions $\alpha_1(\theta_1, \theta_2)$ and $\alpha_2(\theta_1, \theta_2)$

\[
\alpha_1(\theta_1, \theta_2) = \sum_{k \geq (0,0)} \alpha_{1k}^+ \cos(k_1 \theta_1 + k_2 \theta_2) + \sum_{k \geq (1,1)} \alpha_{1k}^- \cos(k_1 \theta_1 - k_2 \theta_2)
\]

\[
\alpha_2(\theta_1, \theta_2) = \sum_{k > (0,0)} \alpha_{2k}^+ \sin(k_1 \theta_1 + k_2 \theta_2) + \sum_{k \geq (1,1)} \alpha_{2k}^- \sin(k_1 \theta_1 - k_2 \theta_2)
\]

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The Tricircular Coherent Problem

- We compute a quasi-periodic solution, with two basic frequencies, of the planar Sun-Jupiter-Saturn-Uranus Four Body Problem. We use the generalized Jacobi coordinates.

- We take the frequencies of Saturn and Uranus in the rotating Sun-Jupiter system.

- Finally, we write the Hamiltonian of a fifth massless particle that moves under the action of the four primaries, supposing that they are following the solution found before.
The generalized Jacobi coordinates
The Hamiltonian of the SJSU problem

We take the Hamiltonian of the planar 4BP written in the generalized Jacobi coordinates in a uniformly rotating reference frame and we make a canonical change of variables (using the angular momentum first integral) in order to reduce this Hamiltonian from 6 to 5 degrees of freedom.

\[ H = \frac{1}{2\alpha} \left( P_1^2 + \frac{A^2}{Q_1^2} \right) + \frac{1}{2\beta} (P_2^2 + P_3^2) + \frac{1}{2\gamma} (P_4^2 + P_5^2) - K \]

\[ -\frac{\alpha}{r} - \frac{(1-\mu) m_{sat}}{r_{13}} - \frac{\mu m_{sat}}{r_{23}} \]

\[ -\frac{(1-\mu) m_{ura}}{r_{14}} - \frac{\mu m_{ura}}{r_{24}} - \frac{m_{sat} m_{ura}}{r_{34}} \]

where

\[ \alpha = \mu (1 - \mu) \]

\[ \gamma = \frac{(1 + m_{sat}) m_{ura}}{1 + m_{sat} + m_{ura}} \]

\[ \beta = m_{sat} / (1 + m_{sat}) \]

\[ A = Q_2 P_3 - Q_3 P_2 + Q_4 P_5 - Q_5 P_4 + K \]
Computation of a first torus (I)

First, we look for a quasi-periodic solution of SJSU equations for $m_{ur} = 0$. We use the same method as before. That is, to compute an invariant curve of the map:

$$\Phi_T(\varphi(\theta)) = \varphi(\theta + \omega), \ \forall \theta \in \mathbb{T}.$$ 

As a first approximation for the Newton method, we use the previously computed periodic orbit for SJS and the Keplerian orbit for Uranus.
Computation of a first torus (II)

The angle between the initial and final position of \textit{Uranus} in the last trajectory is very close to:

\[ \tilde{\omega} = \frac{2\pi \omega_{ura}}{\omega_{sat}} \pmod{2\pi} = 2.750807556. \]

Thus, it is possible to choose the \textit{frequencies} of the first torus as

- \( \omega_{sat} \) (\textit{Saturn}’s relative frequency in the Sun-Jupiter system)
- \( \omega_{ura} \) (\textit{Uranus}’ relative frequency in the Sun-Jupiter system).
The Tricircular Coherent Solution of the SJSU Problem

Once a solution for $m_{ura} = 0$ is computed, by means of a continuation method we proceed to increase the parameter $m_{ura}$ up to its actual value, maintaining the two internal frequencies of the torus.
The Hamiltonian of the TCCP Model

Finally, it is possible to write the equations of a massless particle that moves under the attraction of the four primaries. The corresponding Hamiltonian is:

$$
H_{TCCP} = \frac{1}{2} \alpha_1(\theta_1, \theta_2)(p_x^2 + p_y^2 + p_z^2) + \alpha_2(\theta_1, \theta_2)(xp_x + yp_y + zp_z)
+ \alpha_3(\theta_1, \theta_2)(yp_x - xp_y) + \alpha_4(\theta_1, \theta_2)x + \alpha_5(\theta_1, \theta_2)y
- \alpha_6(\theta_1, \theta_2) \left[ \frac{1 - \mu}{q_S} + \frac{\mu}{q_J} + \frac{m_{sat}}{q_{sat}} + \frac{m_{ura}}{q_{ura}} \right]
$$

where

$$
q_S^2 = (x - \mu)^2 + y^2 + z^2, \quad q_J^2 = (x - \mu + 1)^2 + y^2 + z^2,
q_{sat}^2 = (x - \alpha_7(\theta_1, \theta_2))^2 + (y - \alpha_8(\theta_1, \theta_2))^2 + z^2,
q_{ura}^2 = (x - \alpha_9(\theta_1, \theta_2))^2 + (y - \alpha_{10}(\theta_1, \theta_2))^2 + z^2,
\theta_1 = \omega_{sat} t + \theta_1^{(0)}, \quad \theta_2 = \omega_{ura} t + \theta_2^{(0)}.
$$
Preliminary Study of the TCCP

- Local linear analysis near the triangular points.
  - Symplectic quasi-periodic Floquet Transformation.

- High-order normal form.
  - Non-linear dynamics.

- Approximate first integral.
  - Effective Stability.
Local Study around $L_5$

Jorba & Simó, 96:
In the TCCP system, the $L_5$ point is replaced by a 2-D invariant torus: $T_5$.

We will see that the linear normal modes of $T_5$ have modulus exactly 1
$\implies T_5$ is Linearly Stable
Symplectic Quasi-Periodic Floquet Transformation (I)

The linear flow around the 2-D invariant torus $T_5$:

\[
\begin{align*}
\dot{z} &= Q(\theta_1, \theta_2) z \\
\dot{\theta}_1 &= \omega_{sat} \\
\dot{\theta}_2 &= \omega_{ura}
\end{align*}
\]

$(\theta_1 = 2\pi)$-Poincaré section $\Longrightarrow$ Linear quasi-periodic skew product:

\[
\begin{align*}
\bar{z} &= A(\theta) z \\
\bar{\theta} &= \theta + \omega
\end{align*}
\]

where $\omega = 2\pi \left( \frac{\omega_{ura}}{\omega_{sat}} - 1 \right) = 2.75080755611202$. 
Symplectic Quasi-Periodic Floquet Transformation (II)

Using the method by Jorba 2001, we reduce this quasi-periodic skew product to

\[ \bar{y} = \Lambda y, \]

by implementing a linear change of variables \( z = C'(\theta)y. \)

- Diagonal complex \( 6 \times 6 \) matrix.
- Constant coefficients.
- \( A(\theta)C(\theta) = C(\theta + \omega)\Lambda. \)

If we define

- \( T_\omega : \Psi(\theta) \in \mathcal{C}(\mathbb{T}^1, \mathbb{C}^n) \to \Psi(\theta + \omega) \in \mathcal{C}(\mathbb{T}^1, \mathbb{C}^n). \)
- \( \Psi_j(\theta) \): \( j \)-th column of the matrix \( C(\theta). \)

we obtain a generalized eigenvalue problem:

\[ A(\theta)\Psi_j(\theta) = \lambda_j T_\omega \Psi_j(\theta). \]
Symplectic Quasi-Periodic Floquet Transformation (III)

\[ \dot{z} = Q(\theta_1, \theta_2)z \quad \overset{z=P^c(\theta_1, \theta_2)y}{\rightarrow} \quad \dot{y} = D_B y \]  

Complex change:

\[ \bar{z} = A(\theta)z \quad \overset{z=C(\theta)y}{\rightarrow} \quad \bar{y} = \Lambda y \]  

(1) \[ \dot{P}^c(\theta_1, \theta_2) = Q(\theta_1, \theta_2)P^c(\theta_1, \theta_2) - P^c(\theta_1, \theta_2)D_B ; \quad P^c(t = 0) = C(\theta_2^{(0)}) \]

(2) \[ A(\theta)C(\theta) = C(\theta + \omega)\Lambda \]

Real change:

\[ \dot{z} = Q(\theta_1, \theta_2)z \quad \overset{z=P^r(\theta_1, \theta_2)y}{\rightarrow} \quad \dot{x} = Bx \]

(3) \[ \dot{P}^r(\theta_1, \theta_2) = Q(\theta_1, \theta_2)P^r(\theta_1, \theta_2) - P^r(\theta_1, \theta_2)B ; \quad P^r(t = 0) = R(\theta_2^{(0)}) \]
Symplectic Quasi-Periodic Floquet Transformation (IV)

Matrices:

\[
R(\theta) = \frac{1}{2} C(\theta) \begin{pmatrix}
I_3 & -iI_3 \\
I_3 & iI_3
\end{pmatrix}
\]

\[
B = \begin{pmatrix}
0 & 0 & 0 & \omega_1 & 0 & 0 \\
0 & 0 & 0 & 0 & \omega_2 & 0 \\
0 & 0 & 0 & 0 & 0 & \omega_3 \\
-\omega_1 & 0 & 0 & 0 & 0 & 0 \\
0 & -\omega_2 & 0 & 0 & 0 & 0 \\
0 & 0 & -\omega_3 & 0 & 0 & 0
\end{pmatrix}
\]

\(P^r(\theta_1, \theta_2)\) is a symplectic matrix.

The second degree terms of the Hamiltonian:

\[
H^r_2(x, y) = \frac{1}{2} \omega_1 (x_1^2 + y_1^2) + \frac{1}{2} \omega_2 (x_2^2 + y_2^2) + \frac{1}{2} \omega_3 (x_3^2 + y_3^2)
\]

where the frequencies are \(\omega_1 = -0.080473064872369\),
\(\omega_2 = 0.996680625156409\) and \(\omega_3 = 1.00006269133083\).
High-Order Normal Form

- **Expansion** of the Hamiltonian by means of the Legendre polynomials recurrence.

- **Normalizing** procedure with the Lie series method implemented as in Jorba ’99.

\[
H = \langle \varpi, p_\theta \rangle + N(q_1 p_1, q_2 p_2, q_3 p_3) + R(q_1, q_2, q_3, p_1, p_2, p_3, \theta_1, \theta_2)
\]

In real action-angle coordinates, \( N \) does not depend on the angles \( \varphi_j \) but only on the actions \( I_j \):

\[
N = \sum_{|k| = 1}^{[N/2]} h_k I_1^{k_1} I_2^{k_2} I_3^{k_3}, \quad k \in \mathbb{Z}^3, \quad h_k \in \mathbb{R}
\]
Non-Linear Dynamics (I)
Approximate First Integral

\[ \{H, F\} = 0 \]

\[ \Downarrow \]

\[ F(q, p, \theta_1, \theta_2) = i(q_1 p_1 + q_2 p_2 + q_3 p_3) + \sum_{n=3}^{N} \sum_{|k|=n} \sum_{j=(j_1, j_2)} \left( f_{n,j} e^{i(j_1 \theta_1 + j_2 \theta_2)} \right) q^k p^k \]

Zone of Effective Stability:
The Trojan Asteroids in the Outer Solar System

- The **Outer Solar System** and the **Symplectic Integrator**.
- **Frequency** Analysis of the **Trojan** Asteroids in the Outer Solar System.
- Comparison between the **Outer Solar System** and the semi-analytical models.
The Outer Solar System and the Symplectic Integrator (I)

- **OSS**: Sun + Jupiter + Saturn + Uranus + Neptune.

- **Symplectic Integrator**: Laskar & Robutel, 2001.

The Hamiltonian of the OSS + particles in Jacobi coordinates:

\[ H = H_0 + \epsilon H_1 \]

- **\( H_0 \)**: Sum of Kepler problems (integrable).
- **\( H_1 \)**: Perturbative part (depending only in positions \( \longrightarrow \) integrable).

To Integrate exactly a Hamiltonian close to \( H \):

\[
SABA_1 : e^{\tau LH} = e^{\frac{\tau}{2} LH_0} e^{\tau L\epsilon H_1} e^{\frac{\tau}{2} LH_0} + O(\tau^2 \epsilon + \tau^2 \epsilon^2)
\]

\[
SABA_4 : e^{\tau LH} = e^{c_1 \tau LH_0} e^{d_1 \tau L\epsilon H_1} e^{c_2 \tau LH_0} e^{d_2 \tau L\epsilon H_1} e^{c_3 \tau LH_0} e^{d_2 \tau L\epsilon H_1} e^{c_2 \tau LH_0} e^{d_1 \tau L\epsilon H_1} e^{c_1 \tau LH_0} + O(\tau^8 \epsilon + \tau^2 \epsilon^2)
\]

where \( \tau \) is the integration step and \( \epsilon \sim 10^{-3} \).
The Outer Solar System and the Symplectic Integrator (II)

\[ \log_{10} \left| \frac{E(t) - E(0)}{E(0)} \right| \]
Frequency analysis of the Trojans in the OSS (I)

- Laskar’s Frequency Map Analysis (Laskar 1988, 89, 90,...)

- Basic Frequencies:

\[
\begin{align*}
\alpha_k(t) &= a_k(t) \exp(i(\lambda_k(t) - \lambda_5(t))) \quad \longrightarrow \quad \nu \\
\beta_k(t) &= e_k(t) \exp(i\varpi_k(t)) \quad \longrightarrow \quad g \\
\gamma_k(t) &= \sin\left(\frac{i_k(t)}{2}\right) \exp(i\Omega_k(t)) \quad \longrightarrow \quad s
\end{align*}
\]

\(\nu\) : Frequency of Libration

\(g\) : Perihelion’s Frequency

\(s\) : Node’s Frequency
Frequency analysis of the Trojans in the OSS (II)

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<th>Name</th>
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<th>$g$ (sec)</th>
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## Trojan asteroids: Semi-analytical models versus OSS

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<th>Name</th>
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<th>BCCP</th>
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Conclusions of the Present Work

• First Step of a Long-term project which Ultimate Goal is:
  
  To compute an approximate quasi-periodic solution for a Trojan asteroid.
  To conclude its practical stability for the estimated life time of the Solar system.

• Models:
  
  RTBP → BCCP → TCCP
  ↓    ↓
  ERTBP → BAP

• Numerical methods and software:
  
  – Quasi-periodic solutions.
  – Symplectic Quasi-Periodic Floquet Theory.
  – High-Order Normal Forms.
  – Approximate First Integrals.
  – Regions of Effective Stability.