Invariant Manifold Dynamics
Advanced Aspects of Spacecraft Control and Mission Design
AstroNet-II Summer Training School and Annual Meeting

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- Computation of a family of periodic orbits
- Computation of a homoclinic or heteroclinic orbit
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Introduction and motivation
The Soho mission
The Genesis mission
Properties of Libration Point Orbits

1. They are easy and “inexpensive” to reach from Earth
2. They provide good observation sites of the Sun
3. The libration orbits around the $L_1$ and $L_2$ points of the Sun–Earth system always remain close to the Earth, with a near-constant communications geometry
4. Orbits around the $L_2$ point of the Sun–Earth system provide a constant geometry for observation with half of the entire celestial sphere available at all times, since the Sun, Earth and Moon are always behind the spacecraft
5. For missions with heat sensitive instruments, the $L_2$ environment of the Sun–Earth system is highly favourable for non-cryogenic missions requiring great thermal stability, suitable for highly precise visible light telescopes
Properties of Libration Point Orbits

1. The libration orbits around the $L_2$ point of the Earth–Moon system, can be used to establish a permanent communications link between the Earth and the hidden part of the Moon.

2. Suitable combinations of libration orbits of the ES and EM systems can provide ballistic planetary captures, such as for the one used by the Hiten mission.

3. The libration point orbits provide Earth transfer and return trajectories, such as the one used for the Genesis mission.

4. The libration point orbits provide interplanetary transport which can be exploited in the Jovian and Saturn systems to design a low energy cost mission to tour several of their moons (Petit Grand Tour Mission / JIMO / Laplace / EJSM).

5. Formation flight with a rigid shape is possible using libration point orbits.
Main Topics related to LPO dynamics

1. The phase space around the libration points in the CRTBP
2. Analytical and numerical computation of libration point orbits in the CRTBP
3. Computation of hyperbolic invariant manifolds
4. Determination of dynamical substitutes of libration points and LPO in $N$-body models seen as a perturbation of the CRTBP
5. Computation of nominal libration point orbits in accurate solar system models
6. Station-keeping strategies
7. Transfers from the Earth to libration point orbits
8. Transfers between libration point orbits
9. Low energy transfers and weak stability boundaries
10. Formation flight using libration point orbits
Discrete and continuous dynamical systems
Discrete and continuous dynamical systems

- Dynamical systems are systems that evolve with time

- Time may be thought as:
  - Continuous (clock), mathematically $t \in \mathbb{R}$
    $\rightarrow$ Continuous dynamical systems
  - Discrete (days, months, . . . ), mathematically $t \in \mathbb{Z}$
    $\rightarrow$ Discrete dynamical systems

- Continuous dynamical systems are given by a system of Ordinary Differential Equations (ODE).

- Discrete dynamical systems are given by an smooth, 1–1 map.
Continuous dynamical systems

Defined by a system of autonomous (i.e. time-independent) ODE.

\[
\begin{align*}
\dot{x}_1 &= f_1(x_1, x_2, \ldots, x_n), \\
\dot{x}_2 &= f_2(x_1, x_2, \ldots, x_n), \\
&\quad \vdots \\
\dot{x}_n &= f_n(x_1, x_2, \ldots, x_n).
\end{align*}
\]

In short,

\[
\dot{x} = f(x), \quad \text{for} \quad x \in \mathbb{R}^n, \quad f : \mathbb{R}^n \to \mathbb{R}^n.
\]

with

\[
x = \begin{pmatrix} x_1 \\
\vdots \\
x_n \end{pmatrix} \in \mathbb{R}^n, \quad f(x) = \begin{pmatrix} f_1(x) \\
\vdots \\
f_n(x) \end{pmatrix} \in \mathbb{R}^n.
\]
Continuous dynamical systems

An autonomous system of ODE

\[ \dot{x} = f(x), \quad \text{for} \quad x \in \mathbb{R}^n, \quad f : \mathbb{R}^n \to \mathbb{R}^n, \]

allows to define the flow (general solution of the system)

\[ \phi_t(x), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^n, \]

that satisfies

\[ \begin{align*}
\frac{d}{dt} \phi_t(x) &= f(\phi_t(x)), \\
\phi_0(x) &= x,
\end{align*} \]

and

\[ \phi_{t+s}(x) = \phi_s(\phi_t(x)). \]
Variational equations

We often need the derivative of the flow with respect to initial conditions; it will be denoted by \( D\phi_t(x) \).

\( D\phi_t(x) \) can be computed by solving the variational equations:

\[
\begin{aligned}
\frac{dx}{dt} &= f(t, x), \\
\frac{dA}{dt} &= \frac{\partial f}{\partial x}(t, x)A,
\end{aligned}
\]

where \( x \) is an \( n \)-dimensional vector and \( A \) is a \( n \times n \) matrix.

If \( x(t) \) and \( A(t) \) are solutions of the previous system with

\[ x(t_0) = x_0, \quad A(t_0) = I_n, \]

then

\[ D\phi_{t_0}^t(x_0) = A(t). \]
The variational equations can be written as a system of $n + n^2$ ODE as

\[
\begin{align*}
\dot{x}_1 &= f_1(t, x_1, \ldots, x_n), \\
& \quad \vdots \\
\dot{x}_n &= f_n(t, x_1, \ldots, x_n), \\
\dot{a}_{1,1} &= \sum_{k=1}^{n} \frac{\partial f_1}{\partial x_k}(t, x) a_{k,1}, \quad \ldots \quad \dot{a}_{1,n} &= \sum_{k=1}^{n} \frac{\partial f_1}{\partial x_k}(t, x) a_{k,n}, \\
& \quad \vdots \\
\dot{a}_{n,1} &= \sum_{k=1}^{n} \frac{\partial f_n}{\partial x_k}(t, x) a_{k,1}, \quad \ldots \quad \dot{a}_{n,n} &= \sum_{k=1}^{n} \frac{\partial f_n}{\partial x_k}(t, x) a_{k,n}.
\end{align*}
\]
Some terminology

- Given an initial condition $x_0$, its **orbit** is
  \[ \{ \phi_t(x_0) \}_{t \in \mathbb{R}}. \]

- A **fixed point** of a continuous dynamical system is a point $x_0$ whose orbit is
  \[ \phi_t(x_0) = x_0, \quad \forall t \in \mathbb{R}, \]
  this can only happen if $f(x_0) = 0$.

- An orbit $\{ \phi_t(x_0) \}_{t \in \mathbb{R}}$ is said to be **periodic** if there is $T > 0$ such that
  \[ \phi_T(x_0) = x_0, \]
  \[ \phi_t(x_0) \neq x_0, \quad \text{for} \quad 0 < t < T, \]
  then $T$ is said to be its period.

- An **invariant set** $A$ is a set of initial conditions invariant by the flow
  \[ \phi_t(x_0) \in A, \quad \forall x_0 \in A. \]
Example: the restricted three body problem

The restricted three body problem is an autonomous Hamiltonian system

\[
\begin{align*}
\dot{x} &= \frac{\partial H}{\partial p_x}, & \dot{p}_x &= -\frac{\partial H}{\partial x}, \\
\dot{y} &= \frac{\partial H}{\partial p_y}, & \dot{p}_y &= -\frac{\partial H}{\partial y}, \\
\dot{z} &= \frac{\partial H}{\partial p_z}, & \dot{p}_z &= -\frac{\partial H}{\partial z},
\end{align*}
\]

with

\[
H(x, y, z, p_z, p_y, p_z) = \frac{1}{2} (p_x^2 + p_y^2 + p_z^2) - xp_y + yp_x - \frac{1 - \mu}{r_1} - \frac{\mu}{r_2},
\]

and

\[
r_1 = \sqrt{(x - \mu)^2 + y^2 + z^2}, \quad r_2 = \sqrt{(x - \mu + 1)^2 + y^2 + z^2}.
\]
Non-autonomous system of ODE

A system of non-autonomous ODE is a system of ODE that depends on time

\[ \dot{x} = f(t, x), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^n, \quad f : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n. \]

In order to consider the associated autonomous dynamical system define

\[ y = (y_0, y_1, \ldots, y_n)^T := (t, x)^T, \]

and consider

\[
\begin{align*}
\dot{y}_0 &= 1, \\
\dot{y}_1 &= g_1(y_0, y_1, \ldots, y_n), \\
&\quad \ldots \\
\dot{y}_n &= g_n(y_0, y_1, \ldots, y_n).
\end{align*}
\]

in short

\[ \dot{y} = g(y). \]

An non-autonomous system of ODE has never fixed points (\(g_0 = 1\), so \(g \neq 0\)).
Example: the bicircular four body problem

\[
\dot{x} = \frac{\partial H}{\partial p_x}, \quad \dot{p}_x = -\frac{\partial H}{\partial x}, \quad ...
\]

with

\[
H = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) - xp_y + yp_x - \frac{1 - \mu}{((x - \mu)^2 + y^2 + z^2)^{1/2}} - \frac{\mu}{((x - \mu + 1)^2 + y^2 + z^2)^{1/2}} - \frac{m_P}{((x - a_P \cos \theta)^2 + (y + a_P \sin \theta)^2 + z^2)^{1/2}} - \frac{m_P}{a_P^2} (y \sin \theta - x \cos \theta),
\]

with \( \theta = \omega_P t + \theta_0 \), where \( \omega_P \) is the mean angular velocity of the third primary, \( m_P \) its mass and \( a_P \) the distance from barycentre of the first two primaries to the third one. We can autonomize it by introducing the new variable \( \theta \).
Discrete dynamical systems

Discrete dynamical systems are defined by diffeomorphisms (smooth 1–1 maps)

\[ F : \mathbb{R}^n \rightarrow \mathbb{R}^n \]

\[ x \rightarrow F(x) \]

We denote by \( F^{-1} \) the inverse map of \( F \), and use superscript notation for the composition of map

\[
\begin{align*}
F^0(x) &= x, \\
F^1(x) &= F(x), \\
F^2(x) &= F(F(x)), \\
F^3(x) &= F(F(F(x))), \\
&\vdots
\end{align*}
\]
Some concepts

- Given an initial condition, its related orbit is the set \( \{ F^i(x) \}_{i \in \mathbb{Z}} \), that is \( \{ ..., F^{-3}(x), F^{-2}(x), F^{-1}(x), F^0(x), F^1(x), F^2(x), ... \} \).

- A fixed point is an initial condition such that its orbit is itself

\[ F(x_0) = x_0. \]

- An \( n \)-periodic point is an initial condition \( x_0 \) such that

\[
\begin{align*}
F^n(x_0) &= x_0, \\
F^i(x_0) &\neq x_0, \quad i = 1, ..., n - 1.
\end{align*}
\]

- A set of initial conditions \( A \in \mathbb{R}^n \) is said to be an invariant set if

\[ F^n(x) \in A, \quad \forall n \in \mathbb{Z}, \quad \forall x \in A. \]
Example: the standard map

\[ F : \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} x + a \sin(x + y) \\ x + y \end{pmatrix} \]

for \( x, y \in T = \mathbb{R} \setminus [0, 2\pi] \)

Iterates of the standard map for \( a = -0.7 \)
Example: the time-period flow of the bicircular problem

Denote by

\[ \dot{y} = F(y), \]

the autonomized differential equations of the bicircular four body problem, where \( y = (\theta, x, y, z, p_x, p_y, p_z)^T = (\theta, x)^T \), and by \( \Phi_t(y) \) its flow

\[ \begin{cases} 
\frac{d}{dt} \Phi_t(y) &= F(\Phi_t(y)), \\
\Phi_0(y) &= y, \quad \forall y \in \mathbb{R}^7.
\end{cases} \]

Then

\[ \Phi_t(y) = \Phi_t \left( \begin{array}{c} \theta \\ x \end{array} \right) = \left( \begin{array}{c} \theta + \omega_p t \\ \Phi^\theta_t(x) \end{array} \right), \]

here \( \Phi^\theta_t(x) \) is the flow from time 0 to time \( t \) of the BCP with phase \( \theta \).
Example: the time-period flow of the bicircular problem

We can consider the time–$2\pi/\omega_P$ flow for the BCP equations

$$G(y) = \Phi_{2\pi/\omega_P}(y),$$

this is

$$G\left( \begin{array}{c} \theta \\ x \end{array} \right) = \left( \begin{array}{c} \theta + \frac{2\pi}{\omega_P} \\ \Phi_{\theta}^{\omega_P}(x) \end{array} \right) = \left( \begin{array}{c} \theta \\ \Phi_{2\pi/\omega_P}^{\theta}(x) \end{array} \right).$$

Since the coordinate $\theta$ is invariant, we can skip it and define

$$F_\theta(x) := \Phi_{2\pi/\omega_P}^{\theta}(x).$$

Note that a fixed point of $F_\theta$ is a periodic orbit of the BCP with initial phase $\theta$. 
Assume we are given:

- A dynamical system, either continuous or discrete.
- An initial condition, \( x_0 \in \mathbb{R}^n \).

We want to numerically generate the orbit of \( x_0 \).

- In the discrete case, we just have to write a routine that implements the map and iterate it.
- In the continuous case: we need to use numerical methods for systems for ODE.
- Some families of methods for nonstiff ODEs are Runge-Kutta, Taylor or extrapolation ones.
- One popular method is the RKF78 method by Fehlberg (1968), developed under contract by NASA for the Apollo missions. We will just comment the use of a RKF routine as a black box.
Use of a black-box RKF routine

Consider a system of \( n \) possibly non-autonomous ODE

\[
\dot{x} = f(t, x),
\]

and denote by \( \phi_{t_0}^t(x) \) its flow from time \( t_0 \) to time \( t \)

\[
\begin{align*}
\frac{d}{dt} \phi_{t_0}^t(x) &= f(t, \phi_{t_0}^t(x)), \\
\phi_{t_0}^t(x) &= x, \quad \forall x \in \mathbb{R}^n.
\end{align*}
\]

Given \( t_0 \in \mathbb{R}, x_0 \in \mathbb{R}^n, h_0 \in \mathbb{R} \) (small), and a tolerance \( tol \), a RKF routine returns \( t_1 \in \mathbb{R}, x_1 \in \mathbb{R}^n, h_1 \in \mathbb{R} \) verifying:

(a) \( \| x_1 - \phi_{t_0}^{t_1}(x_0) \| < tol \),

(b) \( t_1 \) is as close to \( t_0 + h_0 \) as possible,

(c) \( h_1 \) is the recommended stepsize for the next call.

We will denote a call to such an RKF routine as

\[
(t_1, x_1, h_1) = \text{RKFstep}(t_0, x_0, h_0, f, tol).
\]
Invariant objects: fixed points, periodic orbits, tori,...
Fixed points

Consider a differential system

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n$$

Let the general solution of this system be $\phi_t(x)$.

- An equilibrium solution $\phi_t(y)$ is such that $\phi_t(y) = y$ for all $t$. Obviously $\phi_t(y)$ is an equilibrium solution if and only if $f(y) = 0$.
- The eigenvalues of

$$\frac{\partial f(y)}{\partial x},$$

are called the **characteristic exponents** of the equilibrium.

- If $\partial f(y)/\partial x$ is nonsingular, or equivalently the exponents are all nonzero, the equilibrium point is called **elementary**.
- By the implicit function theorem it follows that **elementary equilibrium points are isolated**.
Stability of fixed points

The analysis of stability, bifurcations, etc of equilibrium points starts with an analysis of the linearized equations.

- For this reason ones shifts the equilibrium point to the origin, and the differential equation is rewritten

\[
\dot{x} = f(x) = Ax + g(x),
\]

where

\[
A = \frac{\partial f(0)}{\partial x}, \quad g(x) = f(x) - Ax
\]

so

\[
g(0) = 0 \quad \text{and} \quad \frac{\partial g(0)}{\partial x} = 0.
\]

- As has been said, the eigenvalues of \(A\) are called the characteristic exponents.
- The reason of this is that the linearized equations \((g(x) = 0)\) have solutions which contain terms like

\[
\exp(\lambda t),
\]

where \(\lambda\) is an eigenvalue of \(A\).
Periodic orbits

As we have already said, a solution $\phi_t(y)$ is said to be a periodic solution with period $T > 0$ (equilibrium solutions are not considered periodic) if

$$
\phi_T(y) = y, \\
\phi_t(y) \neq y, \text{ if } 0 < t < T.
$$

Let $\phi_t(y)$ be a periodic solution. The matrix

$$
\frac{\partial \phi_T(y)}{\partial x},
$$

solution of the variational equations at $t = T$, is called the monodromy matrix, and its eigenvalues are called the characteristic multipliers of the periodic solution.

The reason the eigenvalues $\lambda_i$ of the monodromy matrix are called multipliers is that the linearized Poincaré map takes an eigenvector to $\lambda_i$ times itself.
Periodic orbits

It is tempting to use the implicit function theorem to find a condition for local uniqueness of a periodic solution. Since for a periodic orbit $\phi_T(y) = y$, to be able to do that the matrix

$$\frac{\partial \phi_T(y)}{\partial x} - I,$$

would have to be nonsingular, or $+1$ would not be a multiplier.

In fact: $f(y)$ is always an eigenvector of the monodromy matrix corresponding to the eigenvalue $+1$.

**Proof:** Differentiating $\phi_\tau(\phi_t(y)) = \phi_{t+\tau}(y)$ with respect to $x$, and setting $t = 0$ and $\tau = T$ gives (using the notation: $\phi_t(y) := \phi(t, y)$)

$$\frac{\partial \phi}{\partial x} (T, y) \dot{\phi}(0, y) = \dot{\phi}(T, y),$$

$$\frac{\partial \phi}{\partial x} (T, y) f(y) = f(y).$$
Cross sections and Poincaré maps

Because of the above property one introduces a cross section.

- Let $\phi_t(y)$ be a periodic solution. A cross section to the periodic solution, or simply a section, is a hyperplane $\Sigma$ of codimension 1 through $y$ and transverse to $f(y)$.

- For example, $\Sigma$ would be the hyperplane

$$\{ x : a^T(x - y) = 0 \},$$

where $a$ is a constant vector with $a^T f(y) \neq 0$.

- The periodic solutions starts on the section and, after a time $T$, returns to it. By the continuity of solutions with respect to initial conditions, nearby solutions do the same.

- If $y'$ is close to $y$ on $\Sigma$, there is a time $\tau(y')$ close to $T$ such that $\phi_{\tau(y')}(y')$ is on $\Sigma$. 

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Cross sections and Poincaré maps

The \textit{Poincaré map} is defined as the map

\[
P : N \rightarrow \Sigma
\]

\[
y' \rightarrow \phi_{\tau(y')}(y')
\]

which is a map of a neighborhood $N$ of $y$ in $\Sigma$ into $\Sigma$.

- If this neighborhood is small enough the Poincaré map is smooth.
- The periodic solution now appears as a fixed point of $P$.
- Indeed, any fixed point, $z$, of $P$ is the initial condition for a periodic solution of period $\tau(z)$.
- A point $z \in N$ such that $P^k(z) = z$ for some integer $k > 0$ is called a periodic point of $P$ of period $k$. 
Poincaré maps: numerical computation

We are given:

- A Poincaré map: \( P(x) = \phi_{\tau(x)}(x) \).
- A surface of section: \( \Sigma = \{ x \mid g(x) = 0 \} \), to be traversed from \( \{ g(x) < 0 \} \) to \( \{ g(x) > 0 \} \) or vice versa.

We can numerically evaluate \( P(x) \) by the following algorithm:

**input:** \( x, g, f, \text{tol} \)

**do:**

\[
\begin{align*}
& t := 0, \quad y := x, \quad h := \text{tol} \\
& \text{while} \ (g(y) \geq 0) \\
& \quad (t, y, h) := \text{RKFstep}(t, y, h, f, \text{tol}) \\
& \text{while} \ (g(y) < 0) \\
& \quad (t, y, h) := \text{RKFstep}(t, y, h, f, \text{tol}) \\
& \text{while} \ (|g(y)| < \text{tol}) \\
& \quad \delta := -g(y)/(Dg(y)f(y)) \\
& \quad (t, y, h) := \text{RKFflow}(t, t + \delta, y, h, f, \text{tol})
\end{align*}
\]

**output:** \( t, y \).

At the end of the algorithm, \( y = P(x) \) and \( t = \tau(x) \).
Poincaré maps
Poincaré maps can display local properties around a reference orbit
Poincaré maps

Poincaré maps can display the global structure of the phase space.
Poincaré maps: differential

The differential of $P(x)$ is

$$DP(x) = \frac{d}{dx} \phi_\tau(x)(x) = \frac{d}{d\tau} \phi_\tau(x)(x) + D\phi_\tau(x)(x) = f(P(x))D\tau(x) + D\phi_\tau(x)(x),$$

so we need the differential of the time-return map $D\tau(x)$. From

$$g(P(x)) = g(\phi_\tau(x)(x)) = 0,$$

we get

$$D\tau(x) = -\frac{Dg(P(x))D\phi_\tau(x)(x)}{Dg(P(x))f(P(x))}.$$

$$DP(x) = -f(P(x))\frac{Dg(P(x))D\phi_\tau(x)(x)}{Dg(P(x))f(P(x))} + D\phi_\tau(x)(x).$$

Note that $D\phi_\tau(x)(x)$ is given by the variational equations
Fixed points of Poincaré maps

- If the multipliers of the periodic solution are
  \[1, \lambda_2, \ldots, \lambda_n,\]
  then the multipliers of the corresponding fixed point of the Poincaré map are
  \[\lambda_2, \ldots, \lambda_n.\]

- A periodic orbit of period \(T\) is *isolated* if there is a neighborhood \(L\) of it with no other periodic orbits in \(L\) with period near to \(T\).

- A periodic orbit is isolated if and only if the corresponding fixed point of the Poincaré map is an isolated fixed point.

- There may be periodic solutions of much higher period near an isolated periodic orbit.

- A periodic orbit is called *elementary* if none of its nontrivial multipliers are \(+1\).

- By the implicit function theorem it follows that *elementary fixed points and elementary periodic orbits are isolated.*
The Hamiltonian case

We have already seen that the monodromy matrix of a periodic solution has $+1$ as a multiplier.

If the differential equations of motion are Hamiltonian

$$\dot{z} = J \nabla H(z),$$

where

$$z = \begin{pmatrix} q \\ p \end{pmatrix}, \quad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \quad \nabla H = \begin{pmatrix} \partial H / \partial z_1 \\ \vdots \\ \partial H / \partial z_{2n} \end{pmatrix},$$

then the monodromy matrix

$$\frac{\partial \phi_T(z')}{\partial z},$$

would be symplectic

$$\left( \frac{\partial \phi_T(z')}{\partial z} \right)^T J \left( \frac{\partial \phi_T(z')}{\partial z} \right) = J.$$

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The Hamiltonian case

- The simplectic character of the monodromy matrix implies that the algebraic multiplicity of the eigenvalue $+1$ would be even and so at least equal to 2.

- Actually, this is simply due to the fact that an autonomous Hamiltonian system has a first integral.

- In fact if $\phi_t(z')$ is a periodic solution of the Hamiltonian system, the row vector $\partial H(z')/\partial z$ is a left eigenvector of the monodromy matrix corresponding to the eigenvalue $+1$.

- There is a good geometric reason for this property: the periodic solution lies in a $(2n - 1)$–dimensional level set of the Hamiltonian, and typically in nearby level sets of the Hamiltonian there is a periodic orbit. So, *in Hamiltonian systems, periodic orbits are not isolated.*
The Poincaré map in the Hamiltonian case

- Consider the Poincaré map

\[ P : N \rightarrow \Sigma, \]

where \( N \) is a neighborhood of \( z' \) in \( \Sigma \) (initial conditions of a periodic orbit of the system).

- We can apply the Flow Box theorem, so there are coordinates \((u_1, u_2, ..., u_{2n})\) such that the system of differential equations, in these new coordinates becomes:

\[ \dot{u}_1 = 1, \quad \dot{u}_2 = 0, ..., \dot{u}_{2n} = 0, \]

and \( H(u) = u_2 \).

- In these coordinates, we may take \( \Sigma \) to be \( u_1 = 0 \). Since \( u_2 \) is the integral in these coordinates, \( P \) maps the level sets \( u_2 = \text{constant} \) into themselves; so, we can ignore the \( u_2 \) component of \( P \).
The Poincaré map in the Hamiltonian case

- Let $u_2 = h$, $\Sigma_h$ be the intersection of $\Sigma$ and the level set $H = h$ and let $y_1 = u_3, ..., y_{m-2} = u_m$ be coordinates in $\Sigma_h$. Here $h$ is considered as a parameter (the value of the integral).

- In these coordinates the Poincaré map $P$ is a function of $y$ and the parameter $h$. So

  $$P(h, y) = (h, Q(h, y)),$$

  where for fixed $h$, $Q(h, \cdot)$ is a mapping of a neighborhood $N_h$ of the origin in $\Sigma_h$ into $\Sigma_h$.

- $Q$ is called the Poincaré map in an integral surface (it is a symplectic map).
Cylinders of periodic orbits

- The eigenvalues of
  \[ \frac{\partial Q(0, 0)}{\partial y} \]
  are called the \textit{nontrivial multipliers} of the fixed point in the integral surface.
- If the multipliers of a periodic solution of a Hamiltonian system are
  \[ 1, 1, \lambda_3, \ldots, \lambda_{2n}, \]
  then the multipliers of the fixed point in the integral surface are
  \[ \lambda_3, \ldots, \lambda_{2n}. \]
- If none of the nontrivial multipliers are 1, then we will say that that the periodic solution is \textit{elementary}.
- If we apply the implicit function theorem to \( Q(h, y) - y = 0 \), we can proof that an \textit{elementary periodic orbit lies in a smooth cylinder of periodic solutions parametrized by the energy} \( H \).
Theorem (The Lyapunov Center Theorem) Assume that a Hamiltonian system has an equilibrium point with exponents

\[ \pm \sqrt{-1} \omega, \lambda_3, \ldots, \lambda_n, \bar{\lambda_3}, \ldots, \bar{\lambda_n}, \]

where \( \sqrt{-1} \omega \neq 0 \) is pure imaginary.

If

\[ \frac{\lambda_j}{\sqrt{-1} \omega} \notin \mathbb{Z}, \quad j = 3, 4, \ldots, n, \]

then there exist a one parameter family of periodic orbits emanating from the equilibrium point. Moreover, when approaching the equilibrium point along the family, the periods tend to \( 2\pi / \omega \) and the nontrivial multipliers tend to \( \exp(2\pi \lambda_j / \omega) \), \( j = 3, \ldots, n \).
Numerical computations
Numerical computation of fixed points

We have two cases to consider.

- In the case of a flow
  \[ \dot{x} = f(x), \]
  we look for \( p \) such that \( G(p) := f(p) = 0. \)

- In the case of a map,
  \[ x \to F(x), \]
  we look for \( p \) such that \( G(p) := F(p) - p = 0. \)

In both cases the problem is reduced to look for a zero of a function

\[ G : \mathbb{R} \to \mathbb{R}. \]

We can use Newton’s method in several variables
Numerical computation of fixed points

We can implement Newton’s method to look for a zero of

\[ G : \mathbb{R} \to \mathbb{R}, \]

as follows:

**input:** \( p_0, G, tol, maxit \)

**do:**

\[ p := p_0 \]

for \( it \) from 1 to \( maxit \) do

if \( |G(p)| < tol \) return \( p \)

solve \( DG(p) \Delta p = G(p) \) for \( \Delta p \)

\[ p := p - \Delta p \]

error (maxit exceeded)

**output:** \( p \) (if OK)
Consider a flow
\[ \dot{x} = f(x), \]
with a fixed point \( p \)
\[ f(p) = 0. \]
The Taylor expansion of \( f \) around \( p \) up to order one is
\[ f(x) = f(p) + Df(p)(x - p) + O(\|x - p\|^2) := A(x - p) + O(\|x - p\|^2), \]
so that the linearized flow around \( p \) is
\[ \dot{x} = A(x - p). \]
The eigenvalues of \( A \) are the *exponents* of the fixed point \( p \).
Linear behavior around a fixed point: flows

Assume $\lambda \in Spec(A)$, $\lambda \neq 0$, $A\mathbf{v} = \lambda \mathbf{v}$, $\mathbf{v} \neq 0$.

- If $\lambda \in \mathbb{R}$, consider $\varphi(t) = \mathbf{p} + e^{\lambda t} \mathbf{v}$. Then:
  - $\varphi(t)$ satisfies the system of ODE of the linearized flow,
    \[
    \varphi'(t) = \lambda e^{\lambda t} \mathbf{v} = e^{\lambda t} (A\mathbf{v}) = A(e^{\lambda t} \mathbf{v}) = A(\varphi(t) - \mathbf{p}).
    \]

- If $\lambda > 0$, then $\varphi(t) \to \mathbf{p}$ when $t \to +\infty$, so that it gives a stable manifold of the linearized flow.
- If $\lambda < 0$, then $\varphi(t) \to \mathbf{p}$ when $t \to -\infty$, so that it gives an unstable manifold of the linearized flow.

The existence of a stable or unstable manifold of the full (nonlinear) dynamical system is ensured by the stable manifold theorem for flows.
Invariant objects: fixed points, periodic orbits, tori,…

Linear behavior around a fixed point: flows

Assume $\lambda \in \text{Spec}(A)$, $\lambda \neq 0$, $Av = \lambda v$, $v \neq 0$.

- If $\lambda = i\omega$, for $\omega \in \mathbb{R}$, let $v_1 + iv_2$ be a corresponding eigenvector, with $v_1, v_2 \in \mathbb{R}^n$. Then

$$Av_1 + iAv_2 = A(v_1 + iv_2) = i\omega v_1 - \omega v_2.$$ 

Therefore, if we define

$$\varphi_\gamma(t) = p + \gamma((\cos \omega t)v_1 + (\sin \omega t)v_2),$$

we have

$$\varphi'_\gamma(t) = A(\varphi_\gamma(t) - p).$$

so that $\varphi_\gamma(t)$ satisfies the linearized system of ODE and gives a one-parametric family of $2\pi/\omega$-periodic solutions of the linear system.

The Lyapunov center theorem guarantees, under suitable non-resonance conditions, the existence of a family of periodic orbits for the full nonlinear system, with limiting period $2\pi/\omega$.

- The case $\lambda = a + i\omega$, for $a, \omega \in \mathbb{R}$, $a, \omega \neq 0$ is not possible for a Hamiltonian system and will be not considered.
Linear behavior around a fixed point: maps

Consider a discrete dynamical system given by

$$x \rightarrow F(x),$$

with a fixed point $p$. The Taylor expansion of $F$ around $p$ up to order one is

$$F(x) = F(p) + DF(p)(x - p) + O(\|x - p\|^2),$$

so, if $A := DF(p)$, the linearized dynamical system around $p$ is

$$x \rightarrow L_F(x) := p + A(x - p).$$

The eigenvalues of $A$ are also called the multipliers of the fixed point $p$. For a symplectic map,

$$\lambda \in \text{Spec}A \iff 1/\lambda \in \text{Spec}A.$$
Linear behavior around a fixed point: maps

Assume $\lambda \in \text{Spec}(A)$, $\lambda \neq 0$, $Av = \lambda v$, $v \neq 0$.

- If $\lambda \in \mathbb{R}$, $|\lambda| \neq 1$, consider $\varphi(\xi) = p + \xi v$. Then,

$$L_F(\varphi(\xi)) = \varphi(\lambda \xi),$$

so that $\{\varphi(\xi)\}_{\xi \in \mathbb{R}}$ parametrizes an invariant set of the linear approximation. Moreover,

- $|\lambda| < 1$ then $L_F^n(\varphi(\xi)) = \varphi(\lambda^n \xi) \to \varphi(0) = p$, when $n \to +\infty$. Therefore, $\varphi$ parametrizes a stable manifold of $p$ for the linearized map.
- $|\lambda| > 1$ then $L_F^n(\varphi(\xi)) = \varphi(\lambda^n \xi) \to \varphi(0) = p$, when $n \to -\infty$. Therefore, $\varphi$ parametrizes a unstable manifold of $p$ for the linearized map.

The existence of a stable or unstable manifold of the full (nonlinear) dynamical system is ensured by the stable manifold theorem for maps.
Invariant objects: fixed points, periodic orbits, tori,...

Linear behavior around a fixed point: maps

Assume $\lambda \in \text{Spec}(A)$, $\lambda \neq 0$, $A \mathbf{v} = \lambda \mathbf{v}$, $\mathbf{v} \neq 0$.

- If $\lambda \in \mathbb{C}$, $|\lambda| = 1$, assume $\lambda = \cos \rho + i \sin \rho$ and let $\mathbf{v}_1 + i \mathbf{v}_2$ be an associated eigenvector.

We define

$$\varphi(\xi) = \mathbf{p} + \gamma ((\cos \xi) \mathbf{v}_1 - (\sin \xi) \mathbf{v}_2).$$

Then, if we apply the linearized map to $\varphi(\xi)$, we get

$$\varphi(\xi) \rightarrow \varphi(\xi + \rho).$$

so $\varphi$ parametrizes an invariant closed curve of the linear flow.

We therefore have a one-parametric family (with parameter $\gamma$) of curves, with rotation number $\rho$, invariant by the linearized map.

Under generic nonr-degeneracy conditions, there is a one-parameter family of invariant curves for the full (nonlinear) map with varying rotation number, that converges to $\rho$ as we approach the fixed point.
Linear behavior around a periodic orbit

Consider an autonomous Hamiltonian system

- An initial condition $x_0$ of a $T$–periodic orbit is also a fixed point of $\phi_T$
- Although this is not useful to numerically find the p.o., as we have already seen, it is useful to study the linear behaviour around it
- Consider the RTBP (or any autonomous Hamiltonian system), and let $x_0$ be an initial condition of a $T$–periodic orbit. Then, its monodromy matrix,

$$M := D\phi_T(x_0)$$

has 1 as double eigenvalue.
- Moreover, $M = D\phi_T(x_0)$ is a symplectic matrix, which implies that: if $\lambda$ is an eigenvalue of $M$, then $1/\lambda$ is also eigenvalue. Then,

$$\text{Spec}(M) = \{1, 1, \lambda_1, \lambda_1^{-1}, \lambda_2, \lambda_2^{-1}\}$$

We will assume that $|\lambda_i| \leq |\lambda_i^{-1}|$
Invariant objects: fixed points, periodic orbits, tori,...

Linear behavior around a periodic orbit

If

$$\text{Spec}(M) = \{1, 1, \lambda_1, \lambda_1^{-1}, \lambda_2, \lambda_2^{-1}\},$$

the linear behaviour around the p.o. is better studied in terms of its stability parameters, \(s_1\) and \(s_2\), which are defined as

$$s_1 = \lambda_1 + \frac{1}{\lambda_1}, \quad s_2 = \lambda_2 + \frac{1}{\lambda_2}$$

It is easy to check that

- \(s_i \in \mathbb{R},\ |s_i| > 2 \iff \lambda_i \in \mathbb{R} \setminus \{\pm 1\}\)
- \(s_i \in \mathbb{R},\ |s_i| \leq 2 \iff \lambda_i \in \mathbb{C},\ |\lambda_i| = 1\)
- \(s_i \in \mathbb{C} \setminus \mathbb{R},\ \iff \lambda_i \in \mathbb{C} \setminus \mathbb{R},\ |\lambda_i| \neq 1\)
Linear behavior around a periodic orbit

In an autonomous Hamiltonian system, let \( x_0 \) be such that \( \phi_T(x_0) = x_0 \), with \( M = D\phi_T(x_0) \), \( \text{Spec}(M) = \{1, 1, \lambda_1, \lambda_1^{-1}, \lambda_2, \lambda_2^{-1} \} \) and
\[
 s_1 = \lambda_1 + 1/\lambda_1, \quad s_2 = \lambda_2 + 1/\lambda_2
\]

- **Hiperbolic case:** \( s_i \in \mathbb{R}, \ |s_i| > 2 \Rightarrow \lambda_i \in \mathbb{R} \setminus \{\pm 1\} \)
  - There is a stable manifold of the fixed point of \( \phi_T \), tangent to the \( \lambda_i \)–eigendirection of \( M \) at \( x_0 \).
  - There is a unstable manifold of the fixed point of \( \phi_T \), tangent to the \( \lambda_i^{-1} \)–eigendirection of \( M \) at \( x_0 \).

In terms of the periodic orbit:

- There is a stable manifold of the periodic orbit whose section through the \( \lambda_i, \lambda_i^{-1} \)–eigenplane is tangent to the \( \lambda_i \)–eigendirection.
- There is an unstable manifold of the periodic orbit whose section through the \( \lambda_i, \lambda_i^{-1} \)–eigenplane is tangent to the \( \lambda_i^{-1} \)–eigendirection.
Invariant objects: fixed points, periodic orbits, tori, ...

Linear behavior around a periodic orbit

- **Elliptic case:** $s_i \in \mathbb{R}, \quad |s_i| \leq 2 \Rightarrow \lambda_i \in \mathbb{C}, \quad |\lambda_i| = 1$.

  Let $\lambda_i = \cos \rho + i \sin \rho \quad (\Rightarrow s_i = 2 \cos \rho)$, and let $v \neq 0$ be such that

  $$Mv = \lambda_i v = \lambda_i (v_1 + iv_2)$$

- There is a continuous, one-parametric family of closed curves invariant by the linearization of $\phi_T$ around $x_0$ in the $\{x_0 + \alpha_1 v_1 + \alpha_2 v_2\}_{\alpha_1, \alpha_2 \in \mathbb{R}}$ plane, with rotation number $\rho$.

- Under generic non-degeneracy conditions, there is a Cantorian family of invariant curves around $x_0$, with limiting rotation number $\rho$. When transported by the flow, these invariant curves generate 2–dimensional invariant tori.

- Rational values (times $2\pi$) for $\rho$ also give rise to bifurcated p.o., with period $2\pi/\rho$.

- The particular rational values $\rho = 2\pi$ ($s_i = 2$) and $\rho = \pi$ ($s_i = -2$), are known as the **parabolic case**.
Numerical computation of a family of periodic orbits

In a rotating, barycentric, dimensionless coordinate system, the differential equations of motion for the circular three-dimensional restricted problem can be written as

\[
\begin{align*}
\ddot{X} - 2\dot{Y} &= \Omega_X, \\
\ddot{Y} + 2\dot{X} &= \Omega_Y, \\
\ddot{Z} &= \Omega_Z,
\end{align*}
\]

with

\[
\Omega = \frac{1}{2}(X^2 + Y^2) + (1 - \mu)r_1^{-1} + \mu r_2^{-1},
\]

and

\[
\begin{align*}
r_1^2 &= (X - \mu)^2 + Y^2 + Z^2, \\
r_2^2 &= (X - \mu + 1)^2 + Y^2 + Z^2.
\end{align*}
\]
Numerical computation of a family of periodic orbits

- From the inspection of the terms appearing in the differential equations it follows that if for a fixed value of the mass parameter $\mu$, $(x(t), y(t), z(t))$ is a particular solution, then $(x(-t), -y(-t), \pm z(-t))$ is also a solution for the same value of $\mu$.

- This property is useful when we compute periodic solutions with some symmetry.

- We shall look, for instance, for orbits symmetrical with respect to the $(x, z)$-plane (halo orbits).
Numerical computation of a family of periodic orbits

Consider an initial point

$$P_0 = (x_0, y_0, z_0, x_0, \dot{y}_0, \dot{z}_0) = (x, y, z, \dot{x}, \dot{y}, \dot{z})_0,$$

with $y_0 = \dot{x}_0 = \dot{z}_0 = 0$.

Let

$$P_f = (x_f, y_f, z_f, x_f, \dot{y}_f, \dot{z}_f) = (x, y, z, \dot{x}, \dot{y}, \dot{z})_f,$$

with $y_f = 0$, be the first intersection of the orbit which passes by $P_0$ with $y = 0$. In order to have a periodic orbit, due to the symmetries, it is sufficient that

$$\dot{x}_f = f^1(x_0, z_0, \dot{y}_0) = 0,$$

$$\dot{z}_f = f^2(x_0, z_0, \dot{y}_0) = 0,$$

We define $X = (x, z, \dot{y})^T$ and $F = (f^1, f^2)^T$, then the solution of the above nonlinear system of equations are the zeros of

$$F(X_0) = 0.$$
Numerical computation of a family of periodic orbits

- Assume that the rank of the Jacobian matrix of $F, G = DF$, is two. Then $F(X_0) = 0$ describes a curve, usually called the characteristic curve, in the space $(x, z, \dot{y})_0$.

- In order to compute a family of periodic orbits we must follow the characteristic curve of the family using, for instance, a continuation method.

- The idea of that method is to integrate that curve along the arc parameter $s$, defined in this case by $ds^2 = dx^2 + dz^2 + d\dot{y}^2$. That means to integrate the following system of differential equations:

  \[
  \frac{dx}{ds} = \frac{A_1}{A_0}, \quad \frac{dy}{ds} = \frac{A_2}{A_0}, \quad \frac{dz}{ds} = \frac{A_3}{A_0},
  \]

  where

  \[
  A_1 = f_z^1 f_y^2 - f_y^1 f_z^2, \quad A_2 = -(f_x^1 f_y^2 - f_y^1 f_x^2),
  \]

  \[
  A_3 = f_x^1 f_z^2 - f_z^1 f_x^2, \quad A_0 = \left(A_1^2 + A_2^2 + A_3^2\right)^{1/2},
  \]

  and $f^i_\alpha$ are the partial derivatives of the functions $f^i$ with respect to their arguments.
Numerical computation of a family of periodic orbits

- If we have available some points on the characteristic curve, $X_0, X_1, \ldots, X_n$, the integration of the above equations, with some linear multistep method such as an Adams method, produces a new point $X_{n+1}^0$ near the curve.

- Due to the errors of the numerical integration, this point must be refined in order to obtain a new periodic orbit, $X_{n+1}$.

- To do that we must use a modified Newton method because the number of variables, 3, is higher than the number of equations, 2. So if $X_{k-1}^k$ is an approximation of the solution, we define the following iteration

$$X^k = X^{k-1} - (DF(X^{k-1}))^{-1} F(X^{k-1}) = X^{k-1} + \Delta X^{k-1},$$

with the additional condition that the euclidean norm of the correction $\|\Delta X^{k-1}\|$ be minimum.
Numerical computation of a family of periodic orbits

This is a problem of conditioned extrema with Lagrange function

\[ L = (\Delta X^{k-1})^T \Delta X^{k-1} + \lambda^T (F(X^{k-1}) + DF(X^{k-1}) \Delta X^{k-1}), \]

whose solution is

\[ \Delta X^{k-1} = -\left(DF(X^{k-1})\right)^T \left(DF(X^{k-1}) \cdot (DF(X^{k-1})^T)^{-1} F(X^{k-1})\right). \]

After some iterations of the procedure we get the periodic orbit with a given precision.
The multiple shooting method

- Sometimes, due to high instabilities, it is not possible to refine a periodic orbit using the standard correction method (single shooting).
- In other situations, a reference orbit is computed in some simplified model (RTBP) and must be "refined" in a more realistic force field model for a relatively long time span.
- In both cases, the use of a multiple shooting method is required.
- In the multiple shooting method, the total time span (the period, if we want to refine a periodic orbit) is splitted into a number of shorter subintervals selecting, for instance, $N$ equally spaced points $t_1, t_2, \ldots, t_N$.
- Different time intervals can also be used.
- We will denote by $\Delta t = t_{i+1} - t_i$ and by

$$Q_i = (t_i, x_i, y_i, z_i, \dot{x}_i, \dot{y}_i, \dot{z}_i)^T, \quad i = 1, 2, \ldots, N$$

the points on the initial reference orbit.
The multiple shooting method

- Let $\phi(Q_i)$ be the image, after the time interval $\Delta t$, of the point $Q_i$ under the flow associated to the equations in which the refinement must be done.
- As, in this way, the epochs $t_i$ are fixed, $(t_i = t_1 + (i - 1)\Delta t)$, we can write $Q_i$ as $Q_i = (x_i, y_i, z_i, \dot{x}_i, \dot{y}_i, \dot{z}_i)^T$.
- If all the points $Q_i$ would belong to the same orbit, then
  \[ \phi(Q_i) = Q_{i+1}, \quad i = 1, \ldots, N - 1. \]
  If this is not the case, a change of the starting values is needed in order to fulfill the matching conditions.
- In this way, one must solve a set of $N - 1$ nonlinear equations, which can be written as
  \[ F \begin{pmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_N \end{pmatrix} = \begin{pmatrix} \phi(Q_1) \\ \phi(Q_2) \\ \vdots \\ \phi(Q_{N-1}) \end{pmatrix} - \begin{pmatrix} Q_2 \\ Q_3 \\ \vdots \\ Q_N \end{pmatrix} = \Phi \begin{pmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_{N-1} \end{pmatrix} - \begin{pmatrix} Q_2 \\ Q_3 \\ \vdots \\ Q_N \end{pmatrix} = 0. \]
The multiple shooting method

- A Newton’s method is used to solve the above system
  \[ F(Q_1, \ldots, Q_N)^T = 0. \]

- If \( Q^{(j)} = \left( Q_1^{(j)}, Q_2^{(j)}, \ldots, Q_N^{(j)} \right)^T \), denotes the \( j \)-th iterate of the procedure, Newton’s equations can be written as
  \[ DF(Q^{(j)}) \cdot (Q^{(j+1)} - Q^{(j)}) = -F(Q^{(j)}), \]
  where the differential of the function \( F \) has the following structure
  \[
  DF = \begin{pmatrix}
  A_1 & -I \\
  A_2 & -I \\
  \vdots & \vdots \\
  A_{N-1} & -I 
  \end{pmatrix},
  \]
  with \( D\Phi = \text{diag}(A_1, A_2, \ldots, A_{N-1}) \).
The multiple shooting method

- Since each of the transition matrices, $A_i$, that appear in $D\Phi$ are $6 \times 6$, at each step of the method we have to solve a system of $(N - 1) \times 6$ equations with $6 \times N$ unknowns, so some additional conditions must be added.

- As additional equations we could fix some initial and final conditions at $t = t_1$ and $t = t_N$. In this case one must take care with the choice because, since the matrix $DF(Q)$ can have a very large condition number, the problem can be ill conditioned from the numerical point of view.

- To avoid this bad conditioning, we can choose a small value for $\Delta t$, but in this case, as the number of points $Q_i$ increases (if we want to cover the same time span) the instability is transferred to the procedure for solving the linear system.

- Also, extra boundary conditions can force the solution in a non natural way giving convergence problems when we try to compute the orbit for a long time interval.

- To avoid this, we can apply Newton’s method directly.
The multiple shooting method

- As the system has more unknowns than equations, we have (in general) an hyperplane of solutions.
- From this set of solutions we try to select the one closer to the initial orbit used to start the procedure. This is done by requiring the correction to be minimum with respect to some norm (i.e. the euclidean norm).
- Denoting by $\Delta Q^{(j)}$

$$\Delta Q^{(j)} = Q^{(j+1)} - Q^{(j)},$$

requiring $\|\Delta Q^{(j)}\|_2$ to be minimum, one gets the Lagrange function $L(\Delta Q, \mu)$ with (vector) multiplier $\mu$

$$L(\Delta Q, \mu) = \Delta Q^T \cdot \Delta Q + \mu^T \cdot (F(Q) + DF(Q) \cdot \Delta Q),$$

from which we get

$$\Delta Q^{(j)} = -DF(Q^{(j)})^T \cdot \left[DF(Q^{(j)}) \cdot DF(Q^{(j)})^T\right]^{-1} \cdot F(Q^{(j)}). \quad (1)$$

- Since the matrix $DF(Q^{(j)})$ is usually very big, a special factorization in blocks is suitable to get the solution (1) in a computationally and efficient way.
The multiple shooting method

$(x, y)$ projections of the orbits obtained with the multiple shooting procedure at different steps. The figure on the left is the orbit of the RTBP, computed with LP expansions, from which the initial points $Q_i$ are taken. The orbit with large jumps discontinuities is the one obtained after the first two iterations. The figure on the right is the orbit computed after 8 iterations. The initial orbit is a quasihalo orbit with $\beta = 0.2$ and $\gamma = 0.08$. 
Numerical computation of invariant 2D tori

We develop the methodology for an autonomous Hamiltonian system.

- We look for a parametrization of a 2D torus,

\[ \psi : \mathbb{R}^2 \rightarrow \mathbb{R}^6 \]

\[ (\theta_1, \theta_2) \rightarrow \psi(\theta_1, \theta_2) \]

with \( \psi \) being a \( 2\pi \)-periodic function in \( \theta_1, \theta_2 \), solving

\[ \psi(\theta_1 + t\omega_1, \theta_2 + t\omega_2) = \phi_t(\psi(\theta_1, \theta_2)), \quad \forall t \in \mathbb{R}, \quad \forall \theta_1, \theta_2 \in [0, 2\pi], \]

where \( \omega_1, \omega_2 \) are the frequencies of the torus.
Numerical computation of invariant 2D tori

- To reduce the dimension of the problem, we observe that

\[ \varphi(\xi) = \psi(\xi, 0), \]

is an invariant curve invariant of \( \phi_{2\pi/\omega_2} \), and satisfies

\[ \varphi(\xi + \rho) = \phi_{2\pi/\omega_2}(\varphi(\xi)), \]

for \( \rho = 2\pi \omega_1/\omega_2 \).

- Once we have \( \varphi \), we can recover \( \psi \) using

\[ \psi(\theta_1, \theta_2) = \phi_{\frac{\theta_2}{2\pi} \frac{2\pi}{\omega_2}} \left( \varphi(\theta_1 - \frac{\theta_2}{2\pi} \rho) \right) \]
Numerical computation of invariant 2D tori

- Note that

\[ \varphi(\xi + \rho) = \phi_{2\pi/\omega_2}(\varphi(\xi)), \]

s a functional equation: we have "infinite equations" (one for each value of \( \phi \in [0, 2\pi) \)) and "infinite unknowns" (we cannot describe a general function \( \phi \) by a finite number of parameters).

- We discretize space of functions by looking for \( \varphi \) as a truncated Fourier series

\[ \varphi(\xi) = A_0 + \sum_{k=1}^{N_f} (A_k \cos(k\xi) + B_k \sin(k\xi)) \]

- We will discretize parameter space by looking for \( \varphi \) satisfying

\[ \varphi(\xi_i + \rho) = \phi_{2\pi/\omega_2}(\varphi(\xi_i)), \]

for

\[ \xi_i = i \frac{2\pi}{1 + 2N_f}, \quad i = 1, \ldots, 2N_f. \]
Indeterminations

We have two indeterminations to cope with:

- **Invariant curve indetermination:** Assuming there exists a parametrization of a 2D torus

  \[ \psi(\theta_1 + t\omega_1, \theta_2 + t\omega_2) = \phi_t(\psi(\theta_1, \theta_2)), \quad \forall t \in \mathbb{R}, \quad \forall \theta_1, \theta_2 \in [0, 2\pi], \]

  not only \( \phi(\xi) = \psi(\xi, 0) \) satisfies

  \[ \phi(\xi + \rho) = \phi_{2\pi/\omega_2}(\phi(\xi)), \]

  but any \( \phi(\xi) = \psi(\xi, \eta_0), \) for \( \eta_0 \in [0, 2\pi) \) also does.

- This indetermination can be avoided by fixing a curve on the torus.
- This can be done by prescribing a value for a coordinate of \( A_0 \).
- It must be chosen by geometrical considerations.
Indeterminations

- **Phase shift indetermination:** If \( \varphi(\xi) \) satisfies

\[
\varphi(\xi + \rho) = \phi_{2\pi/\omega_2}(\varphi(\xi)),
\]

then, for any \( \xi_0 \in \mathbb{R} \),

\[
\varphi_{\xi_0}(\xi) = \varphi(\xi - \xi_0),
\]

also does.

- This indetermination can be avoided by prescribing a coordinate of \( A_1 \) to be zero.
- Assume that \( A_1 = (A_1^{1}, \ldots, A_1^{6}), B_1 = (B_1^{1}, \ldots, B_1^{6}) \). If \( (A_1^k, B_1^k) \neq (0, 0) \), since

\[
A_1^k \cos(k(\xi - \xi_0)) + B_1^k \sin(k(\xi - \xi_0)) = A_1^k \cos(k\xi_0) - B_1^k \sin(k\xi_0) \cos(k\xi) + \left( A_1^k \sin(k\xi_0) + B_1^k \cos(k\xi_0) \right) \sin(k\xi)
\]

\[
:= \tilde{A}_1^k \cos(k\xi) + \tilde{B}_1^k \sin(k\xi)
\]

we can always choose \( \xi_0 \) such that \( \tilde{A}_1^k = 0 \).
The system of equations

We want to set a system of equations such that:

- We want to prescribe values of the energy (Hamiltonian). For that, we add an additional equation.
- We want to overcome high instability. For that, we will implement multiple shooting.
The system of equations

We look for $\varphi_0, \ldots, \varphi_m$ satisfying

\[
\begin{aligned}
H(\varphi_0(0)) - h &= 0, \\
\varphi_{j+1}(\xi_i) - \phi_{\delta/m}(\varphi_j(\xi_i)) &= 0, \quad j = 0, \ldots, m - 2, \quad i = 0, \ldots, 2N_f \\
\varphi_0(\xi_i + \rho) - \phi_{\delta/m}(\varphi_{m-1}(\xi_i)) &= 0, \quad i = 0, \ldots, 2N_f
\end{aligned}
\]

where $\xi_i = (i \, 2\pi)/(1 + 2N_f)$, $i = 1, \ldots, 2N_f$, $\delta = 2\pi/\omega_2$ and the unknowns are

\[ h, \delta, \rho, A^0_0, A^0_1, B^0_1, \ldots, A^0_{N_f}, B^0_{N_f}, \ldots, A^{m-1}_{N_f}, B^{m-1}_{N_f}, \]

with $h, \delta, \rho \in \mathbb{R}$, $A^i_j, B^i_j \in \mathbb{R}^6$ and

\[ \varphi_j(\xi) = A^i_0 + \sum_{l=0}^{N_f} \left( A^i_l \cos l\xi + B^i_l \sin l\xi \right). \]

This system is $(1 + 6m(1 + 2N_f)) \times (3 + 6m(1 + 2N_f))$. 
Computation of a torus

The tori we are looking for are embedded in 2–parametric families, which can be parametrized by 2 parameters among $h, \delta, \rho$. Therefore, in order to compute a torus, we

- eliminate one coordinate of $A_{0}^{0}$, in order to fix a curve on the torus,
- set a coordinate of $A_{1}^{0}$ equal to zero and eliminate it, in order to get rid of the phase shift indetermination.
- eliminate two unknowns among $h, \delta, \rho$, in order to fixate a particular torus.

When applying Newton’s method, we end up with a

$$(1 + 6m(1 + 2N_{f})) \times (3 + 6m(1 + 2N_{f}) - 4)$$

system of linear equations, with unique solution but which has more equations than unknowns.
This is not a problem, as long as we use the general routine we have described, specifying the kernel dimension to be zero.
Numerical approximation of invariant manifolds

We start with the simplest example in the case of invariant manifolds of a fixed point of a diffeomorphism.

1. Assume that \( p = (x^*, y^*) \) is a fixed point of

\[
P : \mathcal{U} \subset \mathbb{R}^2 \rightarrow \mathbb{R}^2,
\]

with characteristic exponents \( \lambda > 1, \mu \in (0, 1) \). We want to compute \( W^u(p) \) (for \( W^s(p) \) just use \( P^{-1} \)).

2. Let \( (x(t), y(t)) \) be a parametric representation of the manifold around \( p \), with \( p = (x(0), y(0)) \) and

\[
P(x(t), y(t)) \approx (x(\lambda t), y(\lambda t)).
\]

This can be as simple as the linear approximation

\[
(x(t), y(t)) = (x^* + tv_1, y^* + tv_2),
\]

where \( v = (v_1, v_2) \) is a normalized eigenvector of the eigenvalue \( \lambda \) or any higher order approximation.
Numerical approximation of invariant manifolds

- First we check up to which value of $t$ the difference
  \[ \| P(x(t), y(t)) - (x(\lambda t), y(\lambda t)) \| \]
  is less than some tolerance.
- Let $t_0$ be this value. In general it is not needed to be extremely restrictive concerning the tolerance because the small errors tend to decrease (at least locally) due to the compression along the stable direction.
- Then we take a fundamental domain generated by $t$ ranging in $(t_0/\lambda, t_0)$. A simple method generates a big piece of $W^u(p)$ by taking $n$ points in this domain and performing iterates of those points.
- Some care must be taken also to prevent from large bendings of the manifold. This can be done choosing adequately a step, $\Delta t$, on the fundamental domain (which obviously can not be kept fix on the full domain) and performing a given number of iterations, say $k$, with this step.
- When the value of $t$ is greater than $t_0$, then both the parameter $t$ and the step, $\Delta t$, must be divided by the eigenvalue and $k$ can be replace by $k + 1$. 

Gómez - Masdemont (UB & IEEC, UPC & IEEC)
Numerical approximation of invariant manifolds

- This same method can be applied to obtain unstable invariant manifolds of periodic orbits with one unstable direction by using Poincaré sections.
- A more difficult question is to obtain higher dimensional invariant manifolds.
- The key difficulty is that if we have $k$ unstable directions with related eigenvectors $v_j$, $j = 1, \ldots, k$ and eigenvalues $\lambda_j$, $j = 1, \ldots, k$, $|\lambda_j| > 1$, starting at a point
  \[ p + \sum_{j=1}^{k} t_j v_j, \]
  with $\sum_{j=1}^{k} t_j^2$ small near the fixed point $p$, we shall escape essentially in the direction of the dominant eigenvalue. That is, if $|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \ldots$ for almost any $k$–ple of values $(t_1, \ldots, t_k)$ we obtain points very close to the ones generated by iteration starting at $p + t_1 v_1$.
- A general rule to overcome this behaviour consists on the use of high order expansions allowing to start the computations at a relatively big distance of $p$. 
Local approximation of the stable manifold of a quasiperiodic halo orbit

- We are going to see how to compute the linear approximation of the stable manifold of a quasiperiodic halo orbit (qpo).

- We know that if an orbit is periodic, the eigenvectors of the variational matrix computed over one period of the orbit, that is the monodromy matrix, give the directions of the tangent spaces to the invariant manifolds at the starting point.

- Now our orbits are not periodic but quasiperiodic, with small deviations from the halo periodic orbit of the restricted three body problem. So, the eigenvector associated to the eigenvalue with smallest absolute value of the variational matrix computed, over a revolution approximates quite well the stable direction when the orbit crosses $y = 0$.

- We want to use the stable manifold of quasiperiodic orbits computed for a given time span, covering several revolutions. If we want to use this manifold for the transfer, we shall be interested in the stable manifold associated to the initial revolutions.
Local approximation of the stable manifold of a quasiperiodic halo orbit

- Instead of computing the stable direction associated to a selected revolution, the variational matrix $\bar{A}$ associated to the whole quasiperiodic orbit is taken as a monodromy matrix.

- We recall that $\bar{A}$ is a $7 \times 7$ matrix because time is added in the equations of motion, in order to get an autonomous system, and its pattern is:

$$
\bar{A} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
g_1 & g_2 & g_3 & g_4 & g_5 & g_6 & A
\end{pmatrix},
$$

where $A$ is the part of $\bar{A}$ associated to positions and velocities. We note that if $\nu$ is an eigenvector of $A$, then $(0, \nu)$ is an eigenvector of $\bar{A}$. We shall compute the stable direction for the $6 \times 6$ matrix $A$. 
**Local approximation of the stable manifold of a quasiperiodic halo orbit**

- If $A_i$ is the $6 \times 6$ matrix related to positions and velocities of a certain variational matrix $\bar{A}_i$ associated to the $i$-th revolution of a numerical qpo made of $N$ revolutions, we have:

$$A = A_N \times A_{N-1} \times \ldots \times A_1.$$ 

- Because of possible overflows and rounding errors, due to the fact that the eigenvalue of each $A_i$ associated to the unstable manifold is of the order of 1700, the above product must not be computed. With the power method applied to

$$A^{-1} = A_1^{-1} \times A_2^{-1} \times \ldots \times A_N^{-1},$$

we can compute the eigenvector $v_1$ of the matrix $A$ associated to the stable manifold. Then the eigendirections $v_j$ related to the beginning of each $j$-th revolution are computed by means of:

$$v_j = \frac{A_j^{-1} v_{j+1}}{|A_j^{-1} v_{j+1}|}, \quad j = N, \ldots, 2,$$
Local approximation of the stable manifold of a quasiperiodic halo orbit

When we have the direction of the stable manifold at the beginning of a revolution, we can obtain the stable direction at any intermediate point in the revolution transporting this vector by means of the $6 \times 6$ differential matrix of the flow corresponding to the coordinates of position and velocity. That is, if $v_j(t_0) = v_j$ is the initial vector at time $t_0$ we have

$$v_j(t) = A_j(t)v_j(t_0),$$

where $A_j(t) = Id$. 
Globalization of the stable manifold of the QPO

Once the local approximation is available, the next thing to do is the globalization of the stable manifold.

Given a displacement, $D$, in kilometers from the selected point in the qpo, in the right sense of the stable manifold (for instance the one which gives approaches to the Earth if we want to use the manifold for transfer), initial conditions $X_{ws}^0$ in the linear approximation of the manifold are given by means of:

$$X_{ws}^0 = X_{qpo} + D \cdot V_{ws},$$

where $X_{qpo}$ is the selected point of the qpo and $V_{ws}$ is the scaled stable direction in the point $X_{qpo}$.

It must be noted that the magnitude $D$ can not be too small, in absolute value, in order to prevent rounding errors and large integration periods of time when globalizing the manifold. However, it can not be too large because the linear approximation is good near the point $X_{qpo}$. Values between 200 and 250 km are adequate.
The restricted $n$ and three body problems
The \( n \)-body problem

The inertial Newton equations of motion of \( n \) punctual masses \( m_1, m_2, \ldots, m_n \) are

\[
m_k \ddot{r}_k = G \sum_{j=1, j \neq k}^{n} \frac{m_j m_k}{r_{jk}^3} (r_j - r_k), \quad k = 1, \ldots, n,
\]

where \( r_{jk} = |r_j - r_k| \).

\[\text{Inertial and relative coordinates}\]

\[\text{Gómez - Masdemont (UB & IEEC, UPC & IEEC)}\]

\[\text{Invariant Manifold Dynamics}\]

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The $n$-body problem as a perturbation of Kepler’s problem

To get the equation of relative motion of $m_2$ with respect to $m_1$, we take $k = 1, 2$ in the preceding equation, to get

$$\frac{d^2 r_1}{dt^2} = G \frac{m_2}{r_{21}^3} (r_2 - r_1) + G \sum_{j=3}^{n} \frac{m_j}{r_{j1}^3} (r_j - r_1),$$
$$\frac{d^2 r_2}{dt^2} = G \frac{m_1}{r_{12}^3} (r_1 - r_2) + G \sum_{j=3}^{n} \frac{m_j}{r_{j2}^3} (r_j - r_2).$$

Subtracting these two equations we get the one of the relative motion

$$\frac{d^2 r}{dt^2} + \frac{\mu}{r^3} = -G \sum_{j=3}^{n} m_j \left( \frac{d_j}{d_j^3} + \frac{\rho_j}{\rho_j^3} \right),$$

where $r = r_2 - r_1$, $\rho_j = r_j - r_1$, $d_j = r - \rho_j$ and $\mu = G(m_1 + m_2)$.

If $m_3 = \ldots = m_n = 0$, then either (3) as (4) are the equations of the two body problem.
The restricted three body problem

The inertial equations of the three body problem are

\begin{align}
\ddot{r}_1 &= -Gm_2 \frac{r_1 - r_2}{r_{21}} - Gm_3 \frac{r_1 - r_3}{r_{31}}, \\
\ddot{r}_2 &= -Gm_1 \frac{r_2 - r_1}{r_{12}} - Gm_3 \frac{r_2 - r_3}{r_{32}}, \\
\ddot{r}_3 &= -Gm_1 \frac{r_3 - r_1}{r_{13}} - Gm_2 \frac{r_3 - r_2}{r_{23}},
\end{align}

(5)

where $r_{jk} = |r_j - r_k|$. If one of the masses can be neglected, in front of the other two, the above equations can be simplified. Taking, for instance, $m_3 = 0$ we get

\begin{align}
\ddot{r}_1 &= -Gm_2 \frac{r_1 - r_2}{r_{21}}, \\
\ddot{r}_2 &= -Gm_1 \frac{r_2 - r_1}{r_{12}}, \\
\ddot{r}_3 &= -Gm_1 \frac{r_3 - r_1}{r_{13}} - Gm_2 \frac{r_3 - r_2}{r_{23}}.
\end{align}

(6)
The restricted three body problem

The first two equations of system (6) describe the Keplerian motion of \( m_1 \) and \( m_2 \), which are usually known as \textbf{primaries} and the third equation defines the \textbf{restricted three body problem} (RTBP).

\[ \begin{align*}
P_1 (m_1) & \quad \mathbf{r}_1 \\
\mathbf{r}_B & \quad \mathbf{r}_2 \\
\mathbf{r}_3 & \quad P_3 (m_3=0) \\
\end{align*} \]

\[ \begin{align*}
P_1 (m_1) & \quad \mathbf{r}_1 \\
\mathbf{r}_B & \quad \mathbf{r}_2 \\
\mathbf{r}_3 & \quad P_3 (m_3=0) \\
\end{align*} \]

\textit{Barycentric coordinates}

Since the equations are independent of any particular inertial frame, we can set the origin at any point, for instance at the barycentre of \( m_1 \) and \( m_2 \) \((m_3 = 0)\).
The restricted three body problem

One particular case of the RTBP is the **circular restricted three body problem**, in which $m_1$ and $m_2$ describe circular orbits around their barycentre.

- We will use a **synodic** reference system, with origin at the barycentre, in which the two primaries remain at rest on the $x$–axis.
- If $\rho$ denotes the synodical position and $r$ the corresponding inertial (sidereal) one

$$\rho = R(t)r,$$

where $R(t)$ is the transformation (rotation) between both reference frames, defined by the constant angular velocity $\omega = (0, 0, n)^T$ of the rotating frame with respect to the fixed one.

Recall that

$$\frac{d^2 r}{dt^2} = \frac{d^2 \rho}{dt^2} + 2\omega \wedge \frac{d\rho}{dt} + \omega \wedge (\omega \wedge \rho), \quad (7)$$
The restricted three body problem

The synodic reference frame

$m_3 = 0$

$P = (x, y, z)$

$m_1 = 1 - \mu$

$P_1 = (\mu, 0, 0)$

$m_2 = \mu$

$P_2 = (\mu - 1, 0, 0)$

$r_1$

$r_2$

$L_1$

$L_2$
The restricted three body problem

Denoting by $\rho = (x, y, z)^T$ the synodic position of $m_3$, its Coriolis and centripetal accelerations are

$$\omega \wedge \frac{d\rho}{dt} = n \begin{pmatrix} \dot{y} \\ -\dot{x} \end{pmatrix}, \quad \omega \wedge (\omega \wedge \rho) = n^2 \begin{pmatrix} x \\ y \\ 0 \end{pmatrix},$$

and the synodic equations of the RTBP become

$$\ddot{x} - 2n\dot{y} - n^2 x = -Gm_1 \frac{x - x_1}{r_{13}^3} - Gm_2 \frac{x - x_2}{r_{23}^3},$$

$$\ddot{y} + 2n\dot{x} - n^2 y = -Gm_1 \frac{y}{r_{13}^3} - Gm_2 \frac{y}{r_{23}^3},$$

$$\ddot{z} = -Gm_1 \frac{z}{r_{13}^3} - Gm_2 \frac{z}{r_{23}^3}.$$

where $(x_1, 0, 0)$ and $(x_2, 0, 0)$ denote the two fixed synodic positions of $m_1$ and $m_2$, respectively.
It is useful to introduce a suitable set of units in such a way that:

- the distance between the primaries is equal to one,
- \( n = 1 \),
- the sum of the masses of the primaries is also one.

So \( m_1 = 1 - \mu, \quad m_2 = \mu \), with \( \mu \in [0, 1] \). With this choice \( G = 1 \) and we can set \( \rho_1 = (\mu, 0, 0)^T, \quad \rho_2 = (\mu - 1, 0, 0)^T \), and get

\[
\begin{align*}
    r_{13} &= \sqrt{(x - \mu)^2 + y^2 + z^2}, \\
    r_{23} &= \sqrt{(x - \mu + 1)^2 + y^2 + z^2}.
\end{align*}
\]
The restricted three body problem

Using the auxiliary function

$$\Omega = \frac{1}{2} (x^2 + y^2) + \frac{1 - \mu}{r_1} + \frac{\mu}{r_2},$$

where $r_1 = r_{13}$ and $r_2 = r_{23}$, the RTBP equations become

$$\ddot{x} - 2\dot{y} = \Omega_x,$$
$$\ddot{y} + 2\dot{x} = \Omega_y,$$
$$\ddot{z} = \Omega_z.$$  \hspace{1cm} (8)

System (8) has a first integral, the *Jacobian integral*, which can be obtained multiplying equations (8) by $\dot{x}$, $\dot{y}$ and $\dot{z}$, adding the results and integrating; in this way we get

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 - 2\Omega(x, y, z) = C_J.$$  \hspace{1cm} (9)
The Jacobian integral of the RTBP can be used to obtain the velocity \( V = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 \) of the third body at any arbitrary position.

For a fixed value of \( C_J \) one can have different values of \( V \) at the same position of the small particle.

In the case when \( V = 0 \) the Jacobian integral becomes

\[
x^2 + y^2 + \frac{2(1 - \mu)}{(x - \mu)^2 + y^2 + z^2}^{1/2} + \frac{2\mu}{(x - \mu + 1)^2 + y^2 + z^2}^{1/2} = C_J,
\]

and gives the geometrical location of points at which the velocity of the infinitesimal body is zero.

On one side of this surface the velocity will be real and on the other complex. Though, when the velocity is real, we can say nothing about the orbit but, at least, we can be sure that in that region motions are possible.
Zero velocity surfaces

The shape of the zero velocity surfaces depends on the value of the Jacobi constant.
Zero velocity curves

Zero velocity curves (the intersection of the zero velocity surfaces with the $z = 0$ plane) for $\mu > 0$. The motion is forbidden in the filled areas. The tick marks on the horizontal axis show the position of the primaries.
The restricted three body problem. Equilibrium points

- The equilibrium points of the RTBP play an important role in the shape of the zero velocity surfaces/curves.
- These points (also called Lagrangian points) are the solutions of the system of equations

\[
\frac{\partial \Omega}{\partial x} = 0, \quad \frac{\partial \Omega}{\partial y} = 0, \quad \frac{\partial \Omega}{\partial z} = 0.
\]

- If the third particle is placed in an equilibrium point with zero velocity, its acceleration will be zero as well, so it will stay there.
- There are five equilibrium points, all of them in the \( z = 0 \) plane, usually labelled \( L_1, L_2, L_3, L_4 \) and \( L_5 \). The first three points, also called collinear libration points, are on the horizontal synodical axis, while the last two, called triangular libration points, form equilateral triangles with the primaries.
The restricted three body problem. Equilibrium points

The five equilibrium points

\[ L_2 \quad L_1 \quad L_4 \quad L_3 \quad L_5 \]

\[ m_2 = \mu \quad m_1 = 1 - \mu \]
Lyapunov’s theorem applied to the RTBP equilibrium points

In the restricted three body problem the eigenvalues at the three collinear libration points are one pair real and the other two pairs pure imaginary:

\[ \pm \omega_r, \pm \sqrt{-1} \omega_x, \pm \sqrt{-1} \omega_z. \]

Since \( \omega_x/\omega_z \notin \mathbb{Z} \) and \( \omega_z/\omega_x \notin \mathbb{Z} \), Lyapunov’s theorem implies there are two one-parameter families of periodic solutions emanating from each of these libration points. They are the so called \textit{planar and vertical Lyapunov families of periodic orbits}. 
Lyapunov’s theorem applied to the RTBP equilibrium points

- At the equilateral triangular libration points of the planar RTBP, the roots of the characteristic equation for $0 < \mu < \mu_1 = (1 - \sqrt{69}/9)/2 \approx 0.0385$, are distinct pure imaginary numbers $\pm iw_1, \pm iw_2$, with $0 < w_1 < w_2$.

- Since $w_2/w_1$ is less than one, Lyapunov’s theorem implies that there is a family of periodic orbits emanating from the triangular points with period approaching $2\pi/w_1$ for all $0 < \mu < \mu_1$. This family is called the short period family.

- Define $\mu_n$ to be the value of $\mu$ for which $w_1/w_2 = n \in \mathbb{Z}$. If $0 < \mu < \mu_1$ and $\mu \neq \mu_n$, $n = 1, 2, \ldots$ then the Lyapunov’s theorem implies that there is a family of periodic orbits from the triangular points with period approaching $2\pi/w_2$. This family is called the long period family.
Consider Newton’s equation for the motion of an infinitesimal body in the force field created by \( n \) punctual masses.

These masses can be assumed to be the bodies of the Solar System, which will be denoted by \( S \).

In an inertial reference frame, the equation writes

\[
\ddot{\mathbf{r}} = G \sum_{i \in S} m_i \frac{\mathbf{r}_i - \mathbf{r}}{||\mathbf{r} - \mathbf{r}_i||^3},
\]

where \( \mathbf{r}_i \) are the inertial coordinates of the bodies in \( S \) and \( \mathbf{r} \) the ones of the infinitesimal body.

The prime denotes derivative with respect to some dynamical time \( t^* \).

We choose two bodies \( I, J \in S \) which will play the role of the primaries.

Assuming \( m_I > m_J \), the mass parameter, \( \mu \), is defined as

\[
\mu = \frac{m_J}{m_I + m_J}.
\]
The \( n \)-body problem as a perturbation of the RTBP

- Next, we must introduce the synodic reference frame such that
  - The origin of this system is set at the barycentre of \( I, J \),
  - The positions of the primaries are fixed at \((\mu, 0, 0)\) and \((\mu - 1, 0, 0)\).
- The transformation from synodical coordinates \( \rho \) to inertial (sidereal) ones, \( r \), is defined by
  \[
  r = b + kC\rho, \tag{10}
  \]
- The translation \( b \), to put the barycentre of the primaries at the origin, is given by
  \[
  b = \frac{m_I r_I + m_J r_J}{m_I + m_J}.
  \]
- The orthogonal matrix \( C = (e_1, e_2, e_3) \), sets the primaries on the \( x \)-axis and turns the instantaneous plane of motion of the primaries into the \( xy \) plane:
  \[
  e_1 = \frac{r_{JI}}{||r_{JI}||}, \quad e_3 = \frac{r_{JI} \times r'_{JI}}{||r_{JI} \times r'_{JI}||}, \quad e_2 = e_3 \times e_1.
  \]
- \( k = ||r_{JI}|| \) is a scaling factor which makes the distance between the primaries to be constant and equal to 1.
The $n$-body problem as a perturbation of the RTBP

- We want to use the same time units as the one usual for the RTBP, where $2\pi$ time units correspond to one revolution of the primaries. If $t^*$ is the dynamical time and $n$ is the mean motion of $J$ with respect to $I$, we introduce the *adimensional time* $t$ by

$$t = n(t^* - t_0^*),$$

where $t_0^*$ is a fixed epoch.

Using the above transformations, the equations of motion become

$$\ddot{x} = b_1 + b_4 \dot{x} + b_5 \dot{y} + b_7 x + b_8 y + b_9 z + b_{13} \Omega_x$$
$$\ddot{y} = b_2 - b_5 \dot{x} + b_4 \dot{y} + b_6 \dot{z} - b_8 x + b_{10} y + b_{11} z + b_{13} \Omega_y$$
$$\ddot{z} = b_3 - b_6 \dot{y} + b_4 \dot{z} + b_9 x - b_{11} y + b_{12} z + b_{13} \Omega_z$$

where $b_i$ are suitable time-dependent functions and $\Omega$.
The \( n \)-body problem as a perturbation of the RTBP

\[
\Omega = \frac{1 - \mu}{\sqrt{(x - \mu)^2 + y^2 + z^2}} + \frac{\mu}{\sqrt{(x - \mu + 1)^2 + y^2 + z^2}} + \sum_{i \in S^*} \frac{\mu_i}{\sqrt{(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2}}.
\]

Note that setting \( b_i = 0 \) for \( i \neq 5, 7, 10, 13, b_5 = 2, b_7 = b_{10} = b_{13} = 1 \) and skipping the sum over \( S^* = S \setminus \{I, J\} \) in \( \Omega \), the equations of motion become the usual RTBP equations with mass parameter \( \mu \).

Once a set of primaries is fixed, we get an idea of the order of magnitude of the perturbation by looking at the first coefficient of the Fourier expansions of the \( b_i \) functions.
Dynamical substitutes of the equilibrium points

- The dynamical substitutes of the equilibrium points are defined as those solutions of the equations of motion that have as basic frequencies only those of the perturbing bodies.

- These substitutes are not unique since they depend, for instance, on the initial epoch at which they are computed.
Dynamical substitutes of the $L_{1,2,3}$ libration points in a simplified Earth-Moon model.

Dynamical substitutes for a RnBP model of the three collinear equilibrium points. The two primaries are the Earth and the Moon, and the only frequency that has been taken in the Fourier expansions of the $b_i$ functions is the mean elongation of the Moon from the Sun. This is the frequency of the time-dependent part of the Bicircular Problem and the Quasi-Bicircular Problem. For this model, the libration points can be continued to periodic orbits which have the same period as the perturbation.
Dynamical substitutes of the Earth-Moon $L_1$ point in a complete Solar System model

Coordinate projections of the dynamical substitute of the $L_1$ point of the Earth–Moon system in the RnBP. Only the first 5 years of the computed orbits are displayed (approx diameter 0.38 km)
Dynamical substitutes of the Earth-Moon $L_2$ point in a complete Solar System model

Coordinate projections of the dynamical substitute of the $L_2$ point of the Earth–Moon system in the RnBP. Only the first 5 years of the computed orbits are displayed (approx diameter 3.8 km)
Dynamical substitutes of the Earth-Moon $L_3$ point in a complete Solar System model

Coordinate projections of the dynamical substitute of the $L_3$ point of the Earth–Moon system in the RnBP. Only the first 5 years of the computed orbits are displayed (approx. diameter 380 km)
Basic perturbing frequencies in the Earth–Moon system

- $\omega_B^1 = 1.0$: mean longitude of the Moon
- $\omega_B^2 = 0.925195997455093$: mean elongation of the Moon from the Sun
- $\omega_B^3 = 8.45477852931292 \times 10^{-3}$: mean longitude of the lunar perigee
- $\omega_B^4 = 4.01883841204748 \times 10^{-3}$: longitude of the mean ascending node of the lunar orbit on the ecliptic
- $\omega_B^5 = 3.57408131981537 \times 10^{-6}$: Sun’s mean longitude of perigee

Basic frequencies appearing in Moon’s motion
(P.R. Escobal: Methods of Astrodynamics)
The restricted $n$ and three body problems

Dynamical substitutes of the equilibrium points

Dynamical substitutes of the $L_{1,2,3}$ libration points

| $L_2$ | Freq.       | Amp.      | $k_1$ | $k_2$ | $k_3$ | $k_4$ | $|\epsilon|$         |
|-------|-------------|-----------|-------|-------|-------|-------|------------------------|
| $\omega_x$ | 1.784043   | 1.5054E−6 | −1    | 3     | 1     | 0     | 2.4302E−7              |
|        | 2.775587   | 1.2430E−6 | 0     | 3     | 0     | 0     | 5.3703E−7              |
|        | 1.916741   | 1.0243E−6 | 1     | 1     | −1    | 0     | 5.4424E−7              |
|        | 1.858843   | 2.7049E−7 | 0     | 2     | 1     | 0     | 3.3354E−6              |
| $\omega_z$ | 1.771569   | 6.4862E−6 | −1    | 3     | 0     | −1    | 3.3679E−7              |
|        | 1.929215   | 8.7039E−7 | 1     | 1     | 0     | 1     | 3.0645E−7              |
|        | 2.763114   | 5.5294E−7 | 0     | 3     | −1    | −1    | 6.4479E−7              |
|        | 0.780024   | 5.1575E−7 | −2    | 3     | 1     | −1    | 4.2804E−8              |

Fourier analysis results of the $x$ and $z$ functions of the $L_2$ dynamical substitutes (200 years). The frequencies have been adjusted as linear combinations, $\sum_{i=1}^{4} k_i \omega_{B_i}$, of the basic frequencies.
The Numerical Refinement Procedure

Initial Guess from CR3BP
ΔT = ΔT₀

Parallel Shooting

N > N_min

Dense Sampling

End Nodes Removal

Refined Fourier Analysis

ΔT ≥ ΔT*

Y

Node Generation
ΔT = γ_pΔT, γ_p > 1

N

Stop

Y

Notes

- An initial epoch should be fixed a priori
- ΔT: time-span covered by a certain set of nodes
- ΔT₀: the time-span of the initial guess produced in CR3BP
- ΔT*: the desired final time-span we want to achieve
- γ_p: a given ratio parameter
- δT: a given constant time step
The Numerical Refinement Procedure: First step

1. Initial seed from CR3BP
2. Multiple shooting procedure
3. Refined Fourier analysis

A solution of the equations of RTBP:

\[
\begin{align*}
\ddot{x} - 2\dot{y} &= \Omega_x, \\
\ddot{y} + 2\dot{x} &= \Omega_y, \\
\ddot{z} &= \Omega_z.
\end{align*}
\]
1. Initial seed from CRTBP
2. **Multiple shooting procedure**
3. Refined Fourier analysis

- The initial seed are the states (nodes, equally or unequally spaced in time) along a libration point orbit
- The algorithm solves the matching equations using a modified Newton method, in which the norm of the correction is minimised at each step
- The result of the procedure is a series of nodes along the refined trajectory
- The time interval between nodes is 0.43 day, introducing over 50000 nodes for a 60-year time span
- The matching errors are controlled to be less than 1 mm in position and 0.1 mm/day in velocity throughout this work
The restricted $n$ and three body problems

Dynamical substitutes of the equilibrium points

The Numerical Refinement Procedure: Third step

1. Initial seed from CRTBP
2. Multiple shooting procedure
3. **Refined Fourier analysis**

A collocation method for solving the following problem: given $N_f$ samples \( \{f(jT/N_f)\}_{j=0}^{N_f-1} \) of a real-valued quasi-periodic function $f(t)$, equally spaced in $[0, T]$, determine a trigonometric polynomial,

\[
Q_f(t) = A_0^c + \sum_{l=1}^{N_f} \left[ A_l^c \cos(\omega_l t) + A_l^s \sin(\omega_l t) \right]
\]

whose **frequencies** $\{\omega_l\}_{l=1}^{N_f}$, and **amplitudes**, $\{A_l^c\}_{l=0}^{N_f}$, $\{A_l^s\}_{l=1}^{N_f}$, are a good approximation of the ones of $f(t)$.

For our problem, the Fourier analysis is done to the three time series associated to the coordinates $(x(t), y(t), z(t))$ of the trajectory determined by the multiple shooting.

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