Global Representation of Invariant Manifolds in the CR3BP by Differential Algebra

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Talk outline:

1. Introductions
   1. The circular restricted three body problem (CR3BP)
   2. Short introduction to DA
   3. Manifolds

2. Invariant manifolds with DA
   1. Initial local manifold generation
   2. Manifold globalization through propagation
   3. Quality control, evaluation, and use

3. Applications and further study
The CR3BP

Recap on the Circular Restricted Three Body Problem (CR3BP):

- Two bodies moving on circular orbits around common center of mass (Keplerian)
- Third “massless” body moving under their gravity
- Change into co-rotating coordinate frame makes problem autonomous (time-independent)
- 4D planar CR3BP with $z \equiv 0$
- Has 5 fixed points (Lagrange points) with surrounding periodic orbits
- Their stable and unstable manifolds can be used for cheap transfers to and from the orbits

In the following, $x_P$ is a point on a periodic orbit with period $T$ in the CR3BP.
The CR3BP
Differential Algebra (DA) is an automatic differentiation technique that replaces operations on numbers by the same operation on polynomials. This approach allows for the algebraic manipulation of equations, providing insights into the behavior of dynamical systems.

DA can be implemented in a computer environment (COSY INFINITY, Berz and Makino, 1998; DACE, Dinamica SRL, 2014). For instance, given a sufficiently regular function $f$, DA enables the expansion of the flow $\varphi_t(x)$ of the ODE

$$\frac{dx}{dt} = f(x(t), t)$$

up to arbitrary order in both time $t$ and initial conditions $x$.

Already presented in detail in Zielona Gora and Milano.
Expansion of the flow $\varphi_t(x) \approx P(t_0 + \delta t, x_0 + \delta x)$ in both time $\delta t$ (black) and initial conditions $\delta x$ (red) around a reference point $(x_0, t_0)$. 
Definition (Topological Manifold)

An $n$ dimensional topological manifold $M$ is a second-countable Hausdorff topological space that is locally homeomorphic to open subsets of $\mathbb{R}^n$.

Definition (Chart and Atlas)

A chart is a homeomorphism $\Phi_\alpha : U_\alpha \rightarrow \mathbb{R}^n$ of an open subset $U_\alpha \subset M$. A $C^k$-atlas $\mathcal{A}$ of charts $\Phi_\alpha : U_\alpha \rightarrow \mathbb{R}^n$ is such that $M \subset \bigcup_\alpha U_\alpha$ and the transition function $\Phi_{\alpha\beta}$ is $C^k$:

$$\Phi_{\alpha\beta} = \Phi_\alpha \circ \Phi_\beta^{-1} : \Phi_\beta(U_\alpha \cap U_\beta) \rightarrow \mathbb{R}^n.$$ 

Definition (Differential Manifold)

An $n$ dimensional $C^k$ differential manifold $M$ is a topological manifold with an atlas $\mathcal{A}$ of charts $\alpha : U_\alpha \rightarrow \mathbb{R}^n$ such that $M \subset \bigcup_\alpha U_\alpha$. 

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Short Intro to Manifolds

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Where is the south pole?

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Short Intro to Manifolds
Problem

A single set of coordinates (only one map) is singular!

Similar problem well known to space engineers:

- *Gimbal lock* in attitude control
- also appears regularly in other classical mechanics problems

Manifolds are a mathematical tool to solve the problem:

- (Differential) Manifolds are covered by an *atlas of maps*
- On each *map* the behavior is “similar” to euclidean space
Summary:

- In a real atlas, each page shows a map covering part of the world, taken together the entire globe is covered.
- In a manifold atlas, each map covers part of the manifold, together they cover the entire manifold.

Differential Algebra Representation

Each single map $\Phi_\alpha$ in the atlas $\mathcal{A}$ of the manifold $M$ is represented by a DA polynomial with an associated domain.

Simplifications for applications:

- Store $\Phi_\alpha^{-1}$ instead of $\Phi_\alpha$.
- The domain of all $\Phi_\alpha^{-1}$ is taken to be the same, e.g. $[-1, 1]^n$. 
Goal

Obtain a representation of the (unstable) invariant manifold of the periodic orbit as an atlas $\mathcal{A}$ of polynomial maps $P_i : [0, 1] \times [-1, 1] \to \mathbb{R}^4$. Each of the maps represents a small patch of the global manifold structure.
Differential Algebra Manifold Representation

Goal

Obtain a representation of the (unstable) invariant manifold of the periodic orbit as an atlas $\mathcal{A}$ of polynomial maps $P_i : [0, 1] \times [-1, 1] \rightarrow \mathbb{R}^4$. Each of the maps represents a small patch of the global manifold structure.

Overview of the process:

1. Reduce the continuous time problem with 2D manifold to a discrete one with 1D manifold
2. From that, generate a specially chosen curve transversal to the flow and in the manifold of the periodic orbit
3. Globalize that local curve by forward integration using a DA-based flow expansion (Taylor integrator)
1. Compute the one turn map \( \hat{F}(\delta x) = \phi_T(x_P + \delta x) - x_P \) around a point on the periodic orbit.

- \( \hat{F}(\delta x) \) is origin preserving, i.e. \( \hat{F}(0) = 0 \) since it is computed around the periodic orbit.
- Derivative \( D\hat{F}|_0 \) can be diagonalized with eigenvalues \( \lambda_1 > 1, \lambda_2 < 1, \lambda_{3,4} = 1 \)

2. Use diagonalized (conjugated) map \( F(\delta x) \) to compute local polynomial expansion \( W^u_{loc} \) of 1D unstable manifold around origin.

- \( F \) only converges in small neighborhood around periodic orbit (\( \sim 10^{-5} \))
- order-by-order manifold construction (see probably Zubin’s next talk)
Local Manifold Expansion

Orbit of $q$ in the local unstable manifold $W^{u}_{loc}$. 
Local Manifold Expansion

Globalization of the local unstable manifold $W^u_{\text{loc}}$ by iteration.
Local unstable manifold $W^u_{loc}$ in CR3BP coordinates.
Local Manifold Expansion

Globalization of the local unstable manifold $W^u_{loc}$ by iteration.
Globalization of the Manifold

Strategy:

1. Select initial piece $x_i \subset W^u_{loc}$ as described
2. Propagate $x_i$ using a DA flow expansion based Taylor integrator with automatic domain splitting to manage exponential length growth
3. Store flow expansion $\phi_t(\delta x) = P(t, \delta x)$ of each time step in the atlas

Observations:

- $W^u_{loc}$ (and $x_i$) are transversal to the flow by construction
- $W^u_{loc}$ (and $x_i$) are contained in unstable manifold of periodic orbit
- By construction $x_i$ sweeps out entire manifold without overlap or gaps
- Forward integration (at least locally) strongly stretches $x_i$
Local Manifold Expansion

Resulting polynomial approximation $\gamma$ of local unstable manifold $W_u^{loc}$ around $x_p$ in RC3BP coordinates:

$$\gamma(\delta x) = 10^{-3} \cdot \begin{pmatrix} -1.61 \\ 0.497 \\ -4.36 \\ 1.76 \end{pmatrix} \delta x + 10^{-6} \cdot \begin{pmatrix} -2.13 \\ 3.21 \\ 7.12 \\ 9.34 \end{pmatrix} \delta x^2 + 10^{-8} \cdot \begin{pmatrix} 1.52 \\ -0.625 \\ -3.74 \\ -6.12 \end{pmatrix} \delta x^3 + \ldots$$

Note the non-linear corrections to the manifold approximation obtained only with the linear eigenvectors (blue).
Globalization of the Manifold
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Result and Quality Control
Result and Quality Control

Statistics:

- Computation Order: 20
- Forward integration of $x_i$ for $t = 3 \cdot T$ ($T = 2.77$)
- Computation time: $1m32s$ on iMac with 2.9 GHz Intel i5
- Number of polynomial maps $P_i$: 4335
- Number of splits: 55

Quality Control:

- Backward integration of final curve for $t = -4 \cdot T$
  $\Rightarrow$ stable for at least one extra turn
- Backward integration of final curve for $t = -3 \cdot T$
  error relative to $x_i < 10^{-6}$
- Jacobi constant on manifold constant up to about $10^{-5}$
Problem

- Initial curve $x_i$ grows very quickly
- Single polynomial expansion cannot accurately represent entire curve

Solution: *Automatic Domain Splitting*

1. After each integration step, check convergence
2. Automatically divide the curve into several smaller, convergent pieces
3. Continue integration with each piece separately

![Diagram showing automatic domain splitting](image)
Automatic Domain Splitting

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Global Manifold Parametrization

Besides local parametrization by maps $P_i$, construction introduces \emph{global} parametrization of the manifold:

$$M = \left\{ \mathbb{R}^+ \times [-1, 1] \rightarrow \mathbb{R}^4 \right\}$$

\begin{align*}
(t, \delta x) &\mapsto \varphi_t(x_i(\delta x))
\end{align*}

- Each point on manifold identified by point $\delta x$ on the initial piece $x_i$ and integration time $t$
- Not 1 : 1 (manifold is a cylinder, not a plane!)
- Mapping from global parametrization to phase space: evaluate correct map $P_i(t, \delta x)$
- Global parametrization useful in many applied problems
Tiling of the \((t, \delta x)\) parameter space due to splitting
Global Manifold Parametrization

Problem

Stretching of the curve $x_i$ under flow $\varphi_t$ is not uniform! Instead logarithmic scale along curve.

The region around the periodic orbit is the cause of the length growth!

- Head of curve ($\delta x = +1$) leaves region quickly $\Rightarrow$ not much stretching
- Tail of curve ($\delta x = -1$) stays in region very long $\Rightarrow$ lots of stretching

Solution

Undo logarithmic scale by adjusting $\delta x$ in global parametrization via (time dependent) exponential scaling $\delta \hat{x} = S_t(\delta x)$
Tiling of the \((t, \delta x)\) parameter space due to splitting after exponential correction
DA Manifold

The result of the DA Manifold computation is a list of polynomials covering the manifold.

- Very accurate global manifold representation
- Efficient computation of globalized manifold atlas (few minutes), can be performed once offline and stored
- Evaluation of functions on manifold is simple (e.g. Jacobi constant)
Advantages of DA Manifold representation:

- Very fast evaluation of points on the manifold
  (parametrized by global parameters $t$ and $\delta x$)
- Derivatives, normals, tangent planes, etc. of manifold readily available
- Quick determination of closest point in manifold (by various measures)
  - am I in the manifold?
  - how far to the manifold?
  - what $\Delta v$ do I need to get into the manifold?
Possible Applications: Hybrid Propulsion

- Use continuous low thrust to raise orbit until "near" manifold
- Use single small kick to correct velocity and enter manifold

not a real orbit!
(in case you couldn't tell)
Other Application of DA Manifolds: DA Manifold Propagator

- Single polynomial cannot represent propagated set
- Automatically detect when and how to split (ADS)
- Result: DA manifold covering final set by convergent polynomial expansions
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DA Manifold Propagator with ADS

Comparison of split vs. single polynomial representation:

Applied successfully to uncertainty propagation in resonant close encounter motion (e.g. Apophis).
Other Research

- Automatic domain splitting for DA propagation to overcome convergence problems
  - Uncertainty propagation (Apophis, INTEGRAL)
  - Identification and classification of dynamical regimes
- Lagrangian Coherent Structures
  - Elliptic restricted three body problem
    (non-autonomous $\Rightarrow$ no manifolds)
  - Computation of the weak stability boundary
- DA high order transfer map propagation method for quasi-periodic orbits
- Averaging techniques for perturbed 2-body dynamics with DA
  - Combining analytical averaging techniques with differential algebra
  - Long term density propagation for study of e.g. space debris evolution
- GPU asteroid sequencing algorithm (originally developed for GTOC7)
Thank You

Questions?