Mission and Navigation Design of Integrated Trajectories to L4 and L5 in the Sun-Earth System

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Mission and Navigation Design of Integrated Trajectories to $L_{4,5}$ in the Sun-Earth System

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We explore the mission and navigation designs of trajectories generated using a high fidelity model in the Sun-Earth system to study the Sun. A detailed examination of selected end-to-end trajectories is described from the science point of view. The geometry of the Trojan orbits around $L_4$ and $L_5$ is analyzed as potential precursor orbits to perform scientific observations at a triangular point. The corresponding trajectory correction maneuvers and orbit determination for both transfer and Trojan orbits are obtained and compared with previous results.

Nomenclature

- $\Delta V_{TOI}$: Delta-V for Transfer Orbit Insertion (Injection) maneuver, km/s
- $\Delta V_{LOI}$: Delta-V for Libration Orbit Insertion maneuver, km/s
- $\Delta V_{TOTAL}$: Total maneuver delta-V, km/s
- $\Delta V_{99\%}$: 99% confidence level for total delta-V, m/s, cm/s or mm/s
- $TOF$: Time of Flight, days
- $r_p$: Radius of periapsis at parking orbit, km
- $inc$: Inclination at parking orbit, degrees
- $L_4, L_5$: Triangular libration points
- $\mu_{\oplus}$: Mass parameter of the Earth, km$^3$/s$^2$

I. Introduction

We describe a technique using a high-fidelity model to generate an end-to-end trajectory from a low Earth parking orbit at 200-km altitude and 28.5° inclination to an orbit around $L_5$ in the Sun-Earth system for a space weather forecast mission. An $L_5$ mission will be ideal for early space warnings to detect Coronal Mass Ejections (CMEs) and observe Corotating Interaction Regions (CIRs) before they hit Earth. These CIRs are aligned to the Parker spiral (see figure 1) and rotate in the direction of planetary motion arriving first at $L_5$, then 3 to 5 days later at...
Earth, and finally at $L_4$ after another 3 to 5 days. Hence, a mission at $L_5$ can provide up to 5 days of advanced warning of solar storms which can wreak havoc on Earth’s telecommunications infrastructure as compared to only a one-hour warning provided by missions at $L_1$ such as the Advanced Composition Explorer (ACE). There are other reasons why $L_5$ is a very attractive point for in-situ scientific measurements that can clarify some of the unknown questions about CMEs and CIRs. Some of these questions are: How do CMEs and CIRs relate to solar magnetism? What is the source of the energy of these CMEs? How are particles accelerated by CMEs and what is the influence of solar flare reconnection? Where and when are shocks formed in the corona and what are the mechanisms that trigger their evolution? What are the processes of CMEs and CIRs that cause magnetic storms? Therefore, in order to better understand some of these unknowns, we need to take accurate measurements from the solar interior to the atmosphere and into the heliosphere.

As seen from the $L_5$ point, there is an east limb of the sun and a west limb of the sun. The Geoeffective (Geosynchronous Earth Orbiter (GEO) in Earthward direction) CMEs are in front of the limb and the Solar Energetic Particles (SEP) CMEs are behind the limb as seen from the Earth. Both GEO CMEs and SEP CMEs can be measured with minimal projection effect.

Solar magnetic field lines of the CIRs in the interplanetary medium would rotate to Earth in approximately 3 to 5 days causing geomagnetic storms that can interrupt communications between satellites in orbit around Earth. The geometry of some of our orbits provides an additional day of solar weather prediction, yielding up to 6 days of advanced warnings of solar storms at Earth. Furthermore, information obtained will enable a reduction of radiation risks in human space flights.

Figure 1. Sketch of a mission to $L_5$ (yellow lagging orbit) and to $L_4$ (magenta leading orbit). The Sun and the heliocentric current sheet carrying solar particles are depicted in orange. The curved red arrows point the direction of the motion of the corotating regions arriving first at $L_5$, then 3 to 5 days later at Earth, and finally at $L_4$ after another 3 to 5 days.

In addition to study the Sun’s magnetic field, this mission will allow searching for Near Earth Objects (NEOs) that may be located at the Sun-Earth triangular points. In 2010, the Wide Infrared Survey Spacecraft (WISE) discovered the first Earth Trojan Asteroid (2010 TK7). In 2011, this asteroid was confirmed to be the first Earth Trojan Asteroid. This Earth co-orbital asteroid (ECA) is in a 1:1 mean motion resonance with the Earth, that is, it goes around the Sun in the same amount of time that the Earth does. It has an approximately 395-year cycle. Due to its eccentric and inclined orbit with respect to the Earth’s orbit, the asteroid seems to orbit the...
triangular point \( L_4 \). Instead, it orbits in tadpole shaped loops around \( L_4 \). These orbits are rather large going as close as 20 million km from Earth (about 50 times the distance from the Earth to the Moon) and nearly as far as the opposite side of the Sun from the Earth. The motion of 2010 TK7 will reverse direction and return to its current position. The irregular motion of this type of orbit will be described in detail in this work.

Our technique generates 3D integrated trajectories with the JPL ephemeris \(^{3}\) DE421. In this paper, we exploit the mission design for several trajectories to the triangular points in the Sun-Earth system. Our emphasis will be based on trajectories to \( L_5 \), but we will also describe selected trajectories to \( L_4 \), with both subject to the following high level mission requirements \(^{6, 8}\):

1. Optimize the transfer time and the propulsion requirement to achieve a 5-year mission in orbit about \( L_5 \).

2. Launch into a circular parking orbit at 200-km altitude with 28.5\(^{\circ}\) inclination about the Earth as would be achieved by a launch from the Kennedy Space Center.

3. Decouple the \( L_5 \) orbit insertion maneuver (LOI) from the start of the science mission at the "arrival" to the vicinity of \( L_5 \).

4. Design orbits that move 5\(^{\circ}\) above and below the ecliptic plane. This required number of degrees may be different from 5\(^{\circ}\) and will be defined by the mission requirements.

5. Limit the total (upper bound) \( \Delta V \) to less that 300 m/s for all Trajectory Correction Maneuvers (TCMs) and Station-Keeping Maneuvers (SKMs).

6. Limit the number of maneuvers to 6 TCMs for the transfer orbit and to less than 6 SKMs after insertion into the Trojan orbit.

7. Employ the \( \Delta V \)-optimal maneuver sequence subject to mission and operational constraints.

8. Provide delivery accuracy at insertion into the Trojan orbit of about 10 \( \mu \)rad\(^{a}\), that is, the ratio of the orbit determination error to the distance to the Sun will set the minimum instrument resolution. The instrument resolution will be determined by the mission requirements.

**II. Results on Mission and Navigation Design**

In this section, we present some of the promising trajectories for a future mission to \( L_5 \).

Looking at the XZ and YZ projections of figure 2, we see that, after the data\(^b\) are converted to dimensional units, the transfer trajectories for missions launched in April and January are off the ecliptic plane by about \( 1.35 \times 10^5 \) km and \( 1.2 \times 10^5 \) km, respectively. These excursions off the ecliptic plane are less pronounced for the transfer trajectories launched in October or July with about \( 2 \times 10^4 \) km and 7500 km, respectively. Although a July launch requires slightly less \( \Delta V \) than an April launch by about 30 m/s with almost the same TOF, a mission to \( L_5 \) launched in April will provide more interesting science results since the space probe will be able to obtain data from the Sun from higher inclinations with respect to the ecliptic. Ideally, the spacecraft could start science as soon as leaving the Earth. Once it arrives at the Trojan orbit, the spacecraft can continue taking data for a few more years, while orbiting \( L_5 \). Figure 3 shows the projection of the trajectory of figure 2 into the xy plane.

\(^{a}\)We assume that the most accurate pointing requirement is about 10 \( \mu \)rad per pixel (typical imaging camera)

\(^{b}\)These data plots are for non-dimensional units while the text is expressed in dimensional units.
Figure 2. One-year integrated trajectory to $L_5$. a: 3D trajectory (EARTH denotes the location of Earth at injection). b: XZ projection. c: YZ projection. d: XZ zoomed in projection. e: YZ zoomed in projection. The January, April, July and October orbits are in blue, green, magenta and brown, respectively.
Figure 3. XY projection one-year integrated trajectory to L5. The January, April, July and October orbits are in blue, green, magenta and brown, respectively. The orbit in gold is the solution obtained in the Circular Restricted Three-Body Problem (CRTBP).

Figure 4 illustrates another example that analyzed how the different times of launch affect the mission performance. As in the first case analyzed, launching a space probe in July gives the lowest total $\Delta V$ for the mission followed by an April launch ($\sim 80$ m/s more $\Delta V$). The total $\Delta V$ of the mission is less than the one obtained with the CRTBP solution. This trajectory has a TOF of a little less than 2 years. We illustrate only a 2-year Trojan orbit in figure 4 but the time of science of the spacecraft could be extended for a few more years. For this case, the April trajectory reaches excursions of about $2.2 \times 10^6$ km off the ecliptic plane, while the July trajectory reaches excursions of about $6 \times 10^4$ km. Either trajectory will provide good data collection, but picking one or the other trajectory will depend on the scientific mission requirements. A two-year transfer trajectory yields not only a slightly lower $\Delta V_{TOI}$ than a one-year transfer trajectory, but also the $\Delta V_{LOI}$ is reduced to half at the expense of increasing the TOF one more year. Figure 5 shows the projection of the trajectory of figure 4 into the xy-plane.

In figure 6, we show a trajectory with a TOF $\approx 2.4$ years with injection into the transfer orbit at different times in the following months: January, April and October. In the previous cases analyzed, a July launch gave the lowest $\Delta V$ for the mission. However, in this case, April yields the lowest total $\Delta V = 4.066$ km/s for the mission followed by a January launch with $\Delta V = 4.126$ km/s, a July launch with $\Delta V = 4.162$ km/s, and a October launch with $\Delta V = 4.292$ km/s. These trajectories were targeted to the desired altitude and inclination. In this case, we observe that the time of flight is about half a year longer than in the previous case analyzed for comparable $\Delta V_{LOI}$ and about 430 m/s less $\Delta V_{TOI}$ at injection if the launch occurs in July. For this case, the spacecraft will be about $1.1 \times 10^4$ km above and $9 \times 10^3$ km below the ecliptic. By launching the space probe in April, the excursions above and below the ecliptic are of the order of $3.4 \times 10^4$ km. Similarly, excursions of about $2.5 \times 10^4$ km off the ecliptic plane are obtained for the January case. Notice that the insertion burn into the Trojan orbit occurs outside the path of the Earth (see figure 7) whereas for the other two cases (see figures 3 and 5) previously analyzed, the libration insertion occurs inside the path of the Earth. This fact is not conclusive but indicates that launching at different epochs may be favored by the location of the insertion maneuver into the Trojan orbit.
Figure 4. Two-years integrated trajectory to $L_5$. a: 3D trajectory plot. b: XZ projection. c: YZ projection. d: XZ zoomed in projection. e: YZ zoomed in projection. The January, April, July and October orbits are in blue, green, magenta and brown, respectively.

Figure 8 shows the comparison of two trajectories launched at the same epoch (January). The
difference between these two trajectories is the location of the insertion burn. One trajectory (dark blue) has a Trojan insertion burn $\Delta V_{LOI} = 0.372$ km/s inside the path of the Earth around the Sun with a TOF of 2.8 years, while the other trajectory (blue) has an insertion burn $\Delta V_{LOI} = 0.621$ km/s outside the path of the Earth with a TOF of 2.4 years. More details of these two trajectories are given in table 1.

Table 1. $\Delta V$ vs. TOF comparison between trajectories launched during January, case L5

<table>
<thead>
<tr>
<th>$T_{epoch}$</th>
<th>$\Delta V_{TOI}$</th>
<th>$\Delta V_{LOI}$</th>
<th>$\Delta V_{TOTAL}$</th>
<th>TOF</th>
<th>inc.</th>
<th>$r_p$</th>
</tr>
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<tr>
<td>Jan (blue)</td>
<td>3.505</td>
<td>0.621</td>
<td>4.126</td>
<td>877.6</td>
<td>28.48</td>
<td>6578.136000</td>
</tr>
<tr>
<td>Jan (dark – blue)</td>
<td>4.141</td>
<td>0.372</td>
<td>4.513</td>
<td>1125.0</td>
<td>28.49</td>
<td>6578.135977</td>
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For all the orbits investigated in this work, we observe that the spacecraft motion shows resonant natural frequencies. Right after departing from the Earth parking orbit at 200-km altitude, the spacecraft undergoes smooth oscillations but they grow in time and become even more accentuated when the spacecraft is close to Earth approximately one year later. The Earth is oscillating and, therefore, the satellite orbit is oscillating around the Earth-Moon barycenter, because of the motion of the Moon with respect to the center of mass. These fluctuations show a chaotic growth during the rest of the transfer phase before inserting into the Trojan orbit where the oscillations are less pronounced.

We also examined a few trajectories to Trojan orbits around $L_4$. The transfer orbits are trajectories inside the path of the Earth around the Sun (TIPES) although trajectories outside the path of the Earth around the Sun (TOPES) to $L_4$ can also be designed.\footnote{In figure 9, we observe different trajectories launched at different times from an Earth parking orbit having a 200-km altitude and}
Figure 6. 2.4-years transfer trajectory to $L_5$. a: 3D trajectory. b: XY projection. c: XZ projection. d: YZ projection. The January, April, July and October orbits are in blue, green, magenta and brown, respectively. The orbit in gold (see figure 7) is the solution obtained in the CRTBP.

28.5° inclination to the other triangular point, $L_4$. From table 2, we notice that, in this case, launching the spacecraft in October yields the lowest total $\Delta V$ for the mission with 4.162 km/s, which is about 1.064 km/s less than the solution obtained in the CRTBP. The TOF is about 13 days longer for a launch in October than launching at a different epoch. The excursions above and below the ecliptic plane are of the order of $10^5$ km for July and October launches. These excursions are of the order of $5 \times 10^4$ km above the ecliptic for April and January launches but the spacecraft spends very little time below the ecliptic. Therefore, the October launch is a more attractive trajectory for science purposes since we can achieve higher excursions above and below the ecliptic for science-data-taking purposes. The July launch would yield slightly lower excursions than the October launch with a TOF of only 13 days longer and incur a penalty of 616 m/s compared to the October case, that is, 423 m/s more at injection and 193 more at insertion.
Figure 7. XY projection 2.4-years integrated trajectory to $L_5$. The January, April, July and October orbits are in blue, green, magenta and brown, respectively. The orbit in gold is the solution obtained in the CRTBP.

Figure 8. Comparison of two orbits launched in same epoch time (January): 2.4-years transfer trajectory (dark-blue) and 2.8-years transfer orbit (blue) to a Trojan orbit around $L_5$.

III. Maneuver Analysis of Integrated Trajectories

In this section, we present the procedure to obtain the hyperbolic orbital elements and the hyperbolic osculating conic near the Earth. Once the B-plane is defined, we can determine the classical and hyperbolic orbital elements and provide the relevant parameters of the departure trajectory from a circular 200-km altitude parking orbit with 28.5° degrees inclination.
Figure 9. Year-and-one-half transfer trajectories to $L_4$.  

(a) 3D trajectory plot.  

(b) XY projection.  

(c) XZ projection.  

d) YZ projection.  

The January, April, July and October orbits are in blue, green, magenta and brown, respectively. The orbit in gold (see figure 9(b)) is the solution obtained in the CRTBP.

Table 2. $\Delta V$ vs. TOF comparison between trajectories launched at different times of the year, case L4

<table>
<thead>
<tr>
<th>Epoch</th>
<th>$\Delta V_{TOI}$ (km/s)</th>
<th>$\Delta V_{LOI}$ (km/s)</th>
<th>$\Delta V_{TOTAL}$ (km/s)</th>
<th>TOF (days)</th>
<th>inc. (deg)</th>
<th>$r_p$ (km)</th>
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<td>Jan</td>
<td>4.468</td>
<td>0.744</td>
<td>5.212</td>
<td>560.3</td>
<td>28.38</td>
<td>6578.135966</td>
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<tr>
<td>Apr</td>
<td>4.039</td>
<td>0.830</td>
<td>4.869</td>
<td>559.3</td>
<td>28.60</td>
<td>6578.136034</td>
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<tr>
<td>Jul</td>
<td>3.875</td>
<td>0.903</td>
<td>4.778</td>
<td>559.2</td>
<td>28.51</td>
<td>6578.136005</td>
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<tr>
<td>Oct</td>
<td>3.452</td>
<td>0.710</td>
<td>4.162</td>
<td>573.4</td>
<td>28.42</td>
<td>6578.135974</td>
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<tr>
<td>CRTBP</td>
<td>4.462</td>
<td>0.764</td>
<td>5.226</td>
<td>612.9</td>
<td>na</td>
<td>6578.136000</td>
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A. Hyperbolic Orbital Elements

Given the state vector of the spacecraft as $\vec{X}_i(t_i)=[\vec{r}_i \ \vec{v}_i]$, the hyperbolic set of orbital elements $\{B, \theta, T_p, V_\infty, \alpha_\infty, \delta_\infty\}$ can be obtained as follows:

1. Obtain the pole vector or angular momentum, $\vec{P} = \vec{r} \times \vec{v}$ where $P = \sqrt{P_x^2 + P_y^2 + P_z^2}$.

2. Compute the magnitude of position and velocity vectors, $R = \sqrt{x^2 + y^2 + z^2}$ and $V = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$.

3. Find the hyperbolic excess velocity, $V_\infty = \sqrt{V^2 - \frac{2\mu}{R}}$.

4. Compute the semilatus rectum, $p = \frac{P^2}{\mu}$.

5. Obtain the energy, $\varepsilon = \frac{V^2}{2}$.

6. Compute the eccentricity, $e = \sqrt{1 + \frac{p(2e)}{\mu}}$.

7. Compute the semimajor axis, $a = \frac{\mu}{V_\infty^2}$ (Note that, in the convention followed here, the semimajor axis $a$ and seminor axis $b$ are greater than zero).

8. Obtain the impact parameter, $B = \sqrt{ap}$.

9. Compute the time of periapsis from epoch time, $T_p = t - \sqrt{\frac{a^3}{\mu} \left( e \sinh F - F \right)}$ where $t$ denotes the integrated transfer time, $\sinh F = \frac{\vec{r} \cdot \vec{v}}{\mu}$, $\cosh F = \frac{a + \vec{r}}{ae}$ and $F = \ln(\sinh F + \cosh F)$.

10. Find the $\vec{S} = \sin(\eta - \eta_l)\vec{r} \times \vec{P} - \cos \eta_l \vec{r}$ along the hyperbolic asymptote where $\eta_l = \cos^{-1} \left( \frac{-1}{e} \right)$, $\sin \eta = \frac{\vec{v} \cdot \vec{r}}{e} \sqrt{\frac{2}{\mu}}$, $\cos \eta = \frac{p-r}{ae}$ so that $\eta = \tan^{-1}(\sin \eta, \cos \eta)$.

11. Compute the right ascension and declination of the incoming asymptote as $\alpha_\infty = \tan^{-1}(S_y, S_x)$, $\delta_\infty = \sin^{-1} \left( \frac{S_z}{\sqrt{S_x^2 + S_y^2}} \right)$, respectively.

12. Compute $\hat{T} = \frac{1}{\sqrt{S_x^2 + S_y^2}} (S_y, -S_x, 0)$ and $\hat{R} = (-T_y S_z, S_z T_x, T_y S_x - T_x S_y)$. These expressions will be proved in section B.

13. Compute the hyperbolic true anomaly, $\theta = \tan^{-1}(\sin \theta, \cos \theta)$ where $\cos \theta = -\hat{R} \cdot \hat{P}$ and $\sin \theta = \hat{T} \cdot \hat{P}$.

14. Assemble the six hyperbolic elements $(B, \theta, T_p, V_\infty, \alpha_\infty, \delta_\infty)$ that map one to one into the Cartesian position and velocity vectors.

Note that $\theta$ and $\eta$ are determined in four quadrants.

B. Hyperbolic Osculating Conic near the Earth

We can specify the orientation of the satellite conic by using the following set of vectors: $\xi$ which is parallel to the apse line of the departure hyperbola and $\zeta$ is the normal vector to the orbital

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plane defined as $\vec{\zeta} = \frac{R \times V}{|R \times V|}$. Let’s define another vector $\hat{z} = \vec{\zeta} \times \frac{R}{R} = \frac{R^2 \hat{V} - (\hat{R} \cdot \hat{V}) \hat{R}}{R \cdot h}$. The $\hat{\zeta}$ can be obtained as follows:

$$\hat{z} = \frac{R^2 \cdot \hat{V} - (\hat{R} \cdot \hat{V}) \hat{R}}{|R \cdot h|} = \frac{R^2 \cdot \hat{V}}{|R||R \times V|} - \frac{\hat{R}(\hat{R} \cdot \hat{V})}{|R||R \times V|} \tag{1}$$

$$= -\frac{\hat{R} \cdot (RV \cos \alpha)}{|R||R \times V|} + \frac{RV}{|R||V| \sin \alpha} \tag{2}$$

$$= -\frac{\hat{R}}{|R|} \cdot \frac{\cos \alpha}{\sin \alpha} + \frac{V}{|V| \sin \alpha} \tag{3}$$

where $\alpha = \cos^{-1}\left(\frac{1}{\rho} \left(\frac{p}{R} - 1\right)\right)$ is the polar angle or true anomaly. Also, multiplying the previous expression by $\sin \alpha$, we finally obtain an expression for $\hat{\zeta}$:

$$\hat{\zeta} = \hat{R} \cos \alpha - \hat{z} \sin \alpha \tag{4}$$

The vector $\eta$ can be easily obtained given the fact that $\hat{\eta}$ is perpendicular to $\hat{\zeta}$. Thus,

$$\hat{\eta} = \hat{R} \sin \alpha + \hat{z} \cos \alpha \tag{5}$$

The three unit vectors that define the B-plane are $\vec{S}$, $\vec{T}$ and $\vec{R}$. The $\vec{S}$ is the vector in the direction of the incoming hyperbolic asymptote; $\vec{T}$ is the vector (perpendicular to $\vec{S}$) parallel to the line of intersection between the B-plane and the J2000 Earth Mean Ecliptic plane being positive in the direction of decreasing right ascension. The $\vec{R}$ completes the set of orthogonal unit vectors for a right-handed system. Also, we will use the $\vec{B}$ defined as the impact parameter, which is the vector from the center of the target (Earth) perpendicular to the incoming asymptote of the hyperbola. The physical meaning of the B vector represents the miss distance of the satellite if the Earth was
massless (see figure 10). Therefore,

\[ \bar{S} = \cos \alpha \hat{e} + \sin \alpha (\hat{\zeta} \times \hat{e}) \]  

(6)

\[ \bar{B} = b (\bar{S} \times \hat{\zeta}) \]  

(7)

Knowing that the eccentricity vector, \( \bar{e} \), is parallel to \( \bar{\xi} \), \( \bar{S} \) and \( \bar{R} \) can be expressed in the following form:

\[ \bar{S} = \frac{1}{e} \hat{\xi} + \sqrt{\left(1 - \frac{1}{e^2}\right)} \hat{\eta} \]  

(8)

\[ \bar{B} = b \left( \frac{1}{e} (\hat{\xi} \times \hat{\zeta}) + b \sqrt{1 - \frac{1}{e^2}} \right) \hat{\eta} \times \hat{\zeta} \]  

\[ = \frac{b}{e} \hat{\eta} + b \sqrt{1 - \frac{1}{e^2}} \hat{\xi} \]  

\[ = b \left( \sqrt{1 - \frac{1}{e^2}} \hat{\xi} - \frac{1}{e} \hat{\eta} \right) \]  

(9)

where \( b \) is the magnitude of \( \bar{B} \) and can be obtained from \( |\bar{B}| = b = \sqrt{\mu/a} \). Therefore, if we know \( \bar{S} \), we know \( \bar{T} \) and \( \bar{R} \).

We also know that \( \bar{T} \) and \( \bar{S} \) are perpendicular to each other; therefore, \( T_x = \sqrt{1 - \frac{1}{e^2}} = S_y \) and \( T_y = -\frac{1}{e} = -S_x \) and \( T_z = 0 \). By normalizing the \( \bar{T} \) vector, we obtain:

\[ \hat{T} = \frac{1}{\sqrt{S_x^2 + S_y^2}} (S_y, -S_x, 0) \]  

(10)

Finally, we can obtain the \( \bar{R} \) vector as \( \bar{R} = \bar{T} \times \bar{S} \), so:

\[ \bar{R} = (-T_y S_x, S_z T_x, T_y S_x - T_x S_y) \]  

(11)

where \( \bar{R} \) is pointing in the direction of \( \hat{\xi} \) which is normal to the Earth equatorial plane, and is usually negative. Since \( \bar{R} \) usually points in the negative direction with respect to the Earth equatorial plane, then to use a coordinate system where \( \bar{R} \) points up, we can replace \( \bar{R} \) by \( -\bar{R} \).

In table 3, we show the relevant parameters for the departure trajectory for each case that we have examined. Column 1 gives the launch month for each trajectory. Column 2 gives the C3, which is the main performance parameter that represents the minimum energy requirement to accomplish the mission. The data in column 3 are the \( V_\infty \) or magnitude of the vector difference between the departure velocity of the spacecraft along the hyperbolic trajectory and the orbital velocity of the Earth. Columns 4 and 5 give the declination and right ascension of the departure hyperbolic asymptote. Column 6 contains the semiminor axis of the departure hyperbola which equals the magnitude of the impact parameter \( \bar{B} \). Column 7 gives the angle of the asymptote of the departure hyperbolic trajectory from the line of apsides, that is, the true anomaly at infinity.

Numerically, we can obtain the osculating classical elements and the hyperbolic elements for each trajectory as follows:

1. From the eccentricity and semimajor axis of the departure asymptote, we can obtain the point of injection which represents the location of the injection maneuver TOI at which the spacecraft achieves the the velocity required to enter the hyperbolic departure trajectory \( r_p = a(e - 1) \).

2. Obtain the circular velocity, \( V_c = \sqrt{\frac{\mu}{r_p}} \) of the spacecraft prior to TOI.
3. Determine the departure velocity at injection, \( V_p = \sqrt{\mu \left( \frac{2}{r_p} + \frac{1}{a} \right)} \).

4. Compute \( \Delta V_{inj} = \Delta V_{TOI} = V_p - V_c \).

5. The \( \Delta V_{LOI} \) is computed iteratively by numerical integration.

6. The TOF is computed iteratively by numerical integration.

7. The hyperbolic excess velocity \( V_{HE} \) can be determined since it is the same as \( V_\infty \) on the departure hyperbola, \( V_{HE} = V_\infty = \sqrt{V_p^2 - V_{esc}^2} \), where \( V_{esc} = \sqrt{\frac{2\mu}{r_p}} \) denotes the escape velocity.

8. Compute \( C_3 \) from the hyperbolic excess velocity, \( C_3 = V_{HE}^2 \).

9. Calculate the semiminor axis, \( b \), of the departure hyperbola, \( b = r_p \sqrt{\frac{2\mu}{V_{HE}^2}} + 1 \), which verifies the value of \( b \) obtained earlier.

10. Determine the angle of the departure asymptote, \( \beta_{hyp} = \tan^{-1} \left( \frac{bV_{HE}^2}{\mu} \right) \).

11. The \( \delta_\infty \) and \( \alpha_\infty \) are obtained as explained in subsection \( \text{A} \).

For the first January orbit \( c \) and for the second January orbit \( d \), the maneuver analysis at injection, cruise and arrival is shown in table \( 4 \) and table \( 5 \) for both cases, respectively.

![Figure 11. The 99% confidence level for a fixed a priori position error. The magenta and black lines correspond to the \( \Delta V_{99\%} \) of selected transfer and Trojan orbits, respectively. The solid lines represent Case L5a and the dashed lines Case L5b. The red dashed line represents the assumed upper bound confidence level of 150 m/s for both the transfer and Trojan orbits (individually).](image-url)

For this analysis, we limit the upper bound of the total \( \Delta V \) of the TCMs to less than 150 m/s for the transfer phase. This limit is the value obtained from the pre-launch thruster performance estimates for the Genesis mission.\(^{[^9]}\) Our maneuver strategy includes another 150 m/s for SKMs after insertion into the Trojan orbit. We studied two scenarios. For each scenario, we examined the

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\(^{[^9]}\) \( \Delta V_{LOI} \) is performed outside the path of the Earth around the Sun (TOPES).

\(^{[^9]}\) \( \Delta V_{LOI} \) is performed inside the path of the Earth around the Sun (TIPES).
Table 3. Relevant parameters for departure trajectory

<table>
<thead>
<tr>
<th>Launch Epoch</th>
<th>$C^3$ ($km^2/s^2$)</th>
<th>$V_\infty$ ($km/s$)</th>
<th>$\delta_\infty$ (deg)</th>
<th>$\alpha_\infty$ (deg)</th>
<th>$b$ ($km$)</th>
<th>$\beta_{hyp}$ (deg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>L5a: One-year transfer to $L_5$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Jan</td>
<td>26.97</td>
<td>5.193</td>
<td>-18.46</td>
<td>44.58</td>
<td>15419.0</td>
<td>46.21</td>
</tr>
<tr>
<td>Apr</td>
<td>23.60</td>
<td>4.858</td>
<td>-24.89</td>
<td>112.91</td>
<td>16293.2</td>
<td>43.97</td>
</tr>
<tr>
<td>Jul</td>
<td>21.03</td>
<td>4.585</td>
<td>-17.96</td>
<td>20.27</td>
<td>17107.8</td>
<td>42.07</td>
</tr>
<tr>
<td>Oct</td>
<td>24.00</td>
<td>4.899</td>
<td>-25.6</td>
<td>304.2</td>
<td>16179.7</td>
<td>44.25</td>
</tr>
<tr>
<td>L5b: Two-year transfer to $L_5$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Jan</td>
<td>22.91</td>
<td>4.786</td>
<td>-21.99</td>
<td>33.22</td>
<td>16498.9</td>
<td>43.47</td>
</tr>
<tr>
<td>Apr</td>
<td>20.36</td>
<td>4.512</td>
<td>-26.60</td>
<td>106.00</td>
<td>17346.1</td>
<td>41.53</td>
</tr>
<tr>
<td>Jul</td>
<td>17.63</td>
<td>4.198</td>
<td>-13.95</td>
<td>190.54</td>
<td>18461.8</td>
<td>39.22</td>
</tr>
<tr>
<td>Oct</td>
<td>19.89</td>
<td>4.460</td>
<td>-19.01</td>
<td>188.02</td>
<td>17518.7</td>
<td>41.16</td>
</tr>
<tr>
<td>L5c: Two-years and four months transfer to $L_5$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Jan$^c$</td>
<td>6.26</td>
<td>2.503</td>
<td>-28.42</td>
<td>359.82</td>
<td>29671.8</td>
<td>25.00</td>
</tr>
<tr>
<td>Jan$^d$</td>
<td>21.03</td>
<td>4.586</td>
<td>-20.42</td>
<td>28.60</td>
<td>17107.4</td>
<td>42.07</td>
</tr>
<tr>
<td>Apr</td>
<td>4.01</td>
<td>2.003</td>
<td>-4.92</td>
<td>70.19</td>
<td>36754.5</td>
<td>20.29</td>
</tr>
<tr>
<td>Jul</td>
<td>7.64</td>
<td>2.763</td>
<td>-4.11</td>
<td>172.60</td>
<td>27019.4</td>
<td>27.37</td>
</tr>
<tr>
<td>Oct</td>
<td>11.80</td>
<td>3.434</td>
<td>-28.49</td>
<td>274.03</td>
<td>22087.6</td>
<td>33.17</td>
</tr>
<tr>
<td>L4: One and a half year transfer to $L_4$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Jan</td>
<td>28.92</td>
<td>5.379</td>
<td>-22.37</td>
<td>217.73</td>
<td>14983.9</td>
<td>47.40</td>
</tr>
<tr>
<td>Apr</td>
<td>18.62</td>
<td>4.315</td>
<td>-26.47</td>
<td>302.58</td>
<td>18025.6</td>
<td>40.10</td>
</tr>
<tr>
<td>Jul</td>
<td>14.74</td>
<td>3.840</td>
<td>-26.23</td>
<td>30.22</td>
<td>19972.7</td>
<td>36.46</td>
</tr>
<tr>
<td>Oct</td>
<td>24.57</td>
<td>4.957</td>
<td>-24.01</td>
<td>117.01</td>
<td>16021.5</td>
<td>44.64</td>
</tr>
</tbody>
</table>

effect of the a priori position and velocity errors on the 99% confidence level $\Delta V_{99\%}$, assuming a 1% execution error. First, we held the a priori position error fixed while varying the error in a priori velocity for both the transfer (magenta line) and the Trojan (black line) as illustrated in figure 11.

Then we compare the trajectory correction maneuver analysis of the January case for which the insertion maneuver was performed outside the path of the Earth around the Sun (see table 4) with the maneuver analysis of the January case where the insertion maneuver was performed inside of
99% Confidence Level (A priori Velocity fixed)

Figure 12. The 99% confidence level $\Delta V_{99\%}$ for a fixed a priori velocity error. The magenta and black lines correspond to the $\Delta V_{99\%}$ of selected transfer and Trojan orbits, respectively. The solid lines represent Case L5a and the dashed lines Case L5b. The red dashed line represents the assumed upper bound confidence level of 150 m/s for both the transfer and Trojan orbits (not combined). Note that the assumed a priori velocity errors ($\sim 5$ m/s) for the transfer orbit are very large, which is equivalent to about 5% error of the value of the first TCM. For the Trojan orbit, these a priori velocity errors ($\sim 1$ m/s) are also large or about 10% of the first SKM for the studied Trojan orbits.

Table 4. Maneuver Analysis for January Case $\Delta V_{LOI}$ outside of Earth path

<table>
<thead>
<tr>
<th>Transfer</th>
<th>$T_0 + 10d$</th>
<th>$T_0 + 20d$</th>
<th>$T_0 + 150d$</th>
<th>$T_f - 30d$</th>
<th>$T_f - 3d$</th>
<th>$T_f$</th>
<th>$\Delta V_{99%}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>ErrPos. = 10 km</td>
<td>51.44</td>
<td>0.54</td>
<td>0.52 cm/s</td>
<td>0.67 cm/s</td>
<td>0.15 cm/s</td>
<td>0.60 mm/s</td>
<td>113.7</td>
</tr>
<tr>
<td>ErrVel. = 5 m/s</td>
<td>45.13</td>
<td>0.47</td>
<td>0.46 cm/s</td>
<td>0.67 cm/s</td>
<td>0.15 cm/s</td>
<td>0.60 mm/s</td>
<td>99.8</td>
</tr>
<tr>
<td>ErrPos. = 10 km</td>
<td>67.47</td>
<td>0.70</td>
<td>0.68 cm/s</td>
<td>0.67 cm/s</td>
<td>0.15 cm/s</td>
<td>0.60 mm/s</td>
<td>149.2</td>
</tr>
<tr>
<td>ErrVel. = 10 m/s</td>
<td>71.84</td>
<td>0.74</td>
<td>0.72 cm/s</td>
<td>0.67 cm/s</td>
<td>0.15 cm/s</td>
<td>0.60 mm/s</td>
<td>158.8</td>
</tr>
<tr>
<td>ErrPos. = 15 km</td>
<td>33.73</td>
<td>0.35</td>
<td>0.35 cm/s</td>
<td>0.67 cm/s</td>
<td>0.15 cm/s</td>
<td>0.60 mm/s</td>
<td>74.6</td>
</tr>
<tr>
<td>ErrVel. = 5 m/s</td>
<td>69.65</td>
<td>0.71</td>
<td>0.75 cm/s</td>
<td>0.67 cm/s</td>
<td>0.15 cm/s</td>
<td>0.60 mm/s</td>
<td>154.0</td>
</tr>
<tr>
<td>ErrPos. = 5 km</td>
<td>4.35</td>
<td>0.05</td>
<td>0.2 cm/s</td>
<td>0.67 cm/s</td>
<td>0.15 cm/s</td>
<td>0.60 mm/s</td>
<td>9.6</td>
</tr>
</tbody>
</table>

the path of the Earth around the Sun (see table 5).

For a fixed a priori error in position of 15 km on the transfer orbit (L5a), the $\Delta V_{99\%}$ is between...
Table 5. Maneuver Analysis for January Case $\Delta V_{\text{LOI}}$ inside of Earth path

<table>
<thead>
<tr>
<th>Transfer</th>
<th>$T_0 + 10d$</th>
<th>$T_0 + 20d$</th>
<th>$T_0 + 150d$</th>
<th>$T_f - 30d$</th>
<th>$T_f - 3d$</th>
<th>$T_f$</th>
<th>$\Delta V_{99%}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Err Pos. = 2^{km}$</td>
<td>70.22</td>
<td>0.15</td>
<td>0.27$^{cm/s}$</td>
<td>0.79$^{cm/s}$</td>
<td>0.19$^{cm/s}$</td>
<td>0.81$^{mm/s}$</td>
<td>154.8</td>
</tr>
<tr>
<td>$Err Vel. = 1^{m/s}$</td>
<td>$Err Pos. = 2^{km}$</td>
<td>62.09</td>
<td>0.13</td>
<td>0.27$^{cm/s}$</td>
<td>0.79$^{cm/s}$</td>
<td>0.19$^{cm/s}$</td>
<td>0.81$^{mm/s}$</td>
</tr>
<tr>
<td>$T_0 + 5d$</td>
<td>$T_0 + 10d$</td>
<td>$T_0 + 150d$</td>
<td>$T_f - 30d$</td>
<td>$T_f - 3d$</td>
<td>$T_f$</td>
<td>$\Delta V_{99%}$</td>
<td></td>
</tr>
<tr>
<td>$Err Pos. = 10^{km}$</td>
<td>62.10</td>
<td>3.51</td>
<td>0.62$^{cm/s}$</td>
<td>0.80$^{cm/s}$</td>
<td>0.19$^{cm/s}$</td>
<td>0.81$^{mm/s}$</td>
<td>139.7</td>
</tr>
<tr>
<td>$Err Vel. = 5^{m/s}$</td>
<td>$Err Pos. = 10^{km}$</td>
<td>54.31</td>
<td>3.11</td>
<td>0.57$^{cm/s}$</td>
<td>0.80$^{cm/s}$</td>
<td>0.19$^{cm/s}$</td>
<td>0.81$^{mm/s}$</td>
</tr>
<tr>
<td>$T_0 + 2d$</td>
<td>$T_0 + 6d$</td>
<td>$T_0 + 150d$</td>
<td>$T_f - 30d$</td>
<td>$T_f - 3d$</td>
<td>$T_f$</td>
<td>$\Delta V_{99%}$</td>
<td></td>
</tr>
<tr>
<td>$Err Pos. = 10^{km}$</td>
<td>54.35</td>
<td>0.67</td>
<td>0.73$^{cm/s}$</td>
<td>0.80$^{cm/s}$</td>
<td>0.19$^{cm/s}$</td>
<td>0.81$^{mm/s}$</td>
<td>120.3</td>
</tr>
<tr>
<td>$Err Vel. = 5^{m/s}$</td>
<td>$Err Pos. = 14^{km}$</td>
<td>66.44</td>
<td>0.82</td>
<td>0.81$^{cm/s}$</td>
<td>0.80$^{cm/s}$</td>
<td>0.19$^{cm/s}$</td>
<td>0.81$^{mm/s}$</td>
</tr>
</tbody>
</table>

83 m/s and 105 m/s when assuming variations in the a priori velocity errors of 10 cm/s and 10 m/s, respectively (see figure 11). The $\Delta V_{99\%}$ can be reduced for lower errors in the a priori position. For the corresponding Trojan orbits, the $\Delta V_{99\%}$ ranges between 94 m/s and 150 m/s, assuming an a priori position error of 3 km. By decreasing this a priori error to 1 km, we notably reduce the $\Delta V_{99\%}$ down to values between 31 m/s and 136 m/s.

For the second orbit (L5b), the $\Delta V_{99\%}$ of the transfer segment is between 33 m/s and 133 m/s for an a priori position error of 1 km whereas for the Trojan leg, the $\Delta V_{99\%}$ falls between 18 m/s and 46 m/s when keeping a constant a priori error in position of 5 km. The transfer leg is depicted by the magenta-dashed curve and the Trojan segment by the black-dashed curve.

Secondly, we vary the a priori error in position while maintaining the a priori velocity error constant as displayed in figure 12. As in the first scenario, the dot-dashed red line denotes the limit requirement for the $\Delta V_{99\%}$. Our analysis shows that the correction maneuvers (TCMs and SKMs) for an $L_5$ mission can be achieved with less than 300 m/s as specified by requirement 5 in section I. The TCM units are given in m/s unless otherwise specified in the table.

In table 4, we observe the TCMs for the transfer leg at injection, cruise and arrival at the Trojan orbit for different a priori errors in position and velocity. Note that the largest TCMs performed a few days after injection are TCM-1 and TCM-2 as expected. At arrival, TCM-6 is less than one mm/s so this maneuver will probably be eliminated in operations. The $\Delta V_{99\%}$ confidence level is about or much less than the assumed upper bound $\Delta V = 150$ m/s for TCMs during the transfer trajectory. For example, the $\Delta V_{99\%}$ confidence level displayed in table 4 is 158.8 m/s but it is slightly reduced to 154.0 m/s when the maneuver is moved to a few days closer to the transfer orbit injection assuming an a priori error in position of 15 km and an a priori error in velocity of 5 m/s. The smallest assumed a priori errors in position (1 km) and velocity (1 cm/s) yield a $\Delta V_{99\%}$ confidence level of only 9.6 m/s.

In this second case (see table 5), we found similar $\Delta V_{99\%}$ confidence levels for the different
cases analyzed. Moreover, maneuver relocation during early stages of the transfer leg can reduce significantly the total cost of the $\Delta V$ for TCMs. For example, the sum of $\Delta V$s for TCM-1 and TCM-2, assuming an a priori error in position of 10 km and an a priori error in velocity of 5 m/s is about 65 m/s whereas this value is reduced by 10 m/s after relocating the maneuver at $T_0 + 2d$ and $T_0 + 6d$ for TCM-1 and TCM-2, respectively, instead of performing the maneuvers at $T_0 + 5d$ and $T_0 + 10d$. The $\Delta V_{99\%}$ confidence level when maneuver relocation occurs is reduced by about 20 m/s.

C. Orbit Determination

The delivery accuracy for the first January orbit where $\Delta V_{LOI}$ is performed outside the path of the Earth around the Sun (TOPES) is 1200 km and 21.58 cm/s for both position and velocity at $T_0 + 2d$ for TCM-1, 172.0 km and 3.1 cm/s for TCM-2 and 31.8 km and 0.54 cm/s for TCM-3. At the end of the transfer leg ($TOF \approx 2.4$ years), the covariance analysis yields 3 km and 0.43 mm/s for TCM-6. For comparison, in the second January orbit where $\Delta V_{LOI}$ is performed inside the path of the Earth around the Sun (TIPES), the delivery accuracy is about 2245.1 km and 40.3 cm/s for TCM-1, 490.5 km and 8.8 cm/s for TCM-2, and 38.0 km and 0.7 cm/s for TCM-3. At the end of the transfer leg ($TOF \approx 2.8$ years), the delivery accuracy yields an error in position of 3 km and $0.58 \text{ mm/s}$ in velocity. The orbit determination accuracies for the January launch cases are very similar to one another.

The knowledge error in position and velocity at the location where TCM-1 is performed are 207 km and 23.4 cm/s, respectively. The error in position increases slightly to 243.7 km when TCM-3 is performed but the error in velocity is lowered to 2 cm/s. After one year in the Trojan orbit around $L_5$, the knowledge error for both position and velocity is 174.6 km and 6.4 mm/s, respectively. The spacecraft can achieve a pointing accuracy of 1.4 $\mu$rad per pixel when orbiting the Trojan orbit. Therefore, the knowledge uncertainties are small enough that they do not affect the total $\Delta V$ required for adjusting the trajectory.

Conclusions

By exploiting selected integrated trajectories in the Sun-Earth system, we found that the spacecraft can be placed in certain orbits around $L_5$ which provide a continuous monitoring of the solar events and make it possible to anticipate space weather in some scenarios from 3 to 6 days before the arrival of ejecta at Earth. Trajectory correction maneuver analysis of selected integrated orbits showed that between 70 m/s and 150 m/s were required for the transfer phase and less than 50 m/s for maintenance of the Trojan orbit for most of the cases analyzed. Overall, the total (TCMs and SKMs) budget for an $L_5$ mission can be achieved with less than the required upper bound of 300 m/s.

Acknowledgments

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References


