Evaluation of Bianchi Rigid Meromorphic Cocycles at Big ATR Points

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- If F is imaginary quadratic the role played by $e^{2\pi i z}$ is played by $j(z)$

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- Darmon-Gerhmann-Lipnowski: RMC on general orthogonal gps.

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- In these cases $\gamma_\tau \in \mathrm{SL}_2(\mathbb{Z}) \subseteq \mathrm{SL}_2(\mathcal{O}_K)$
  - Simplifies the computations: only $J(0,\infty)$ needed

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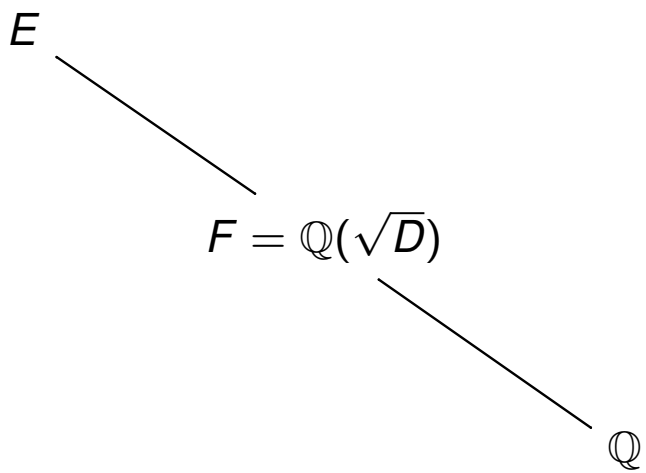

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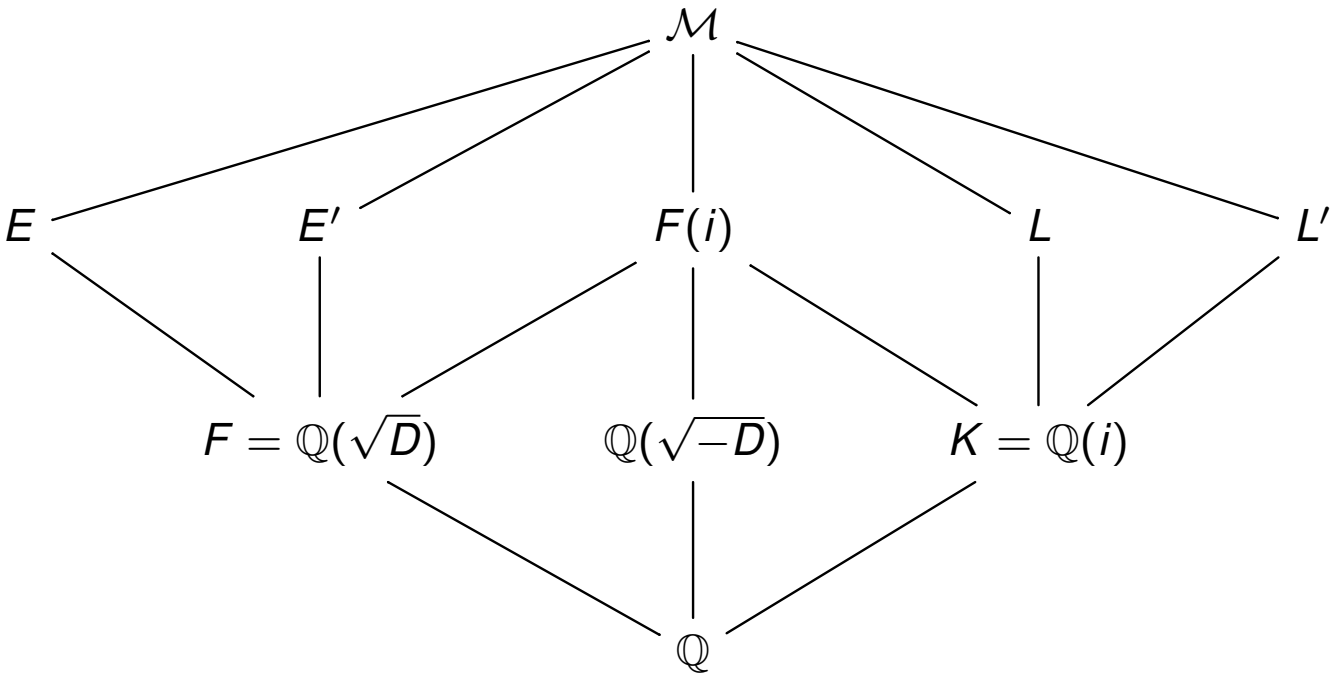

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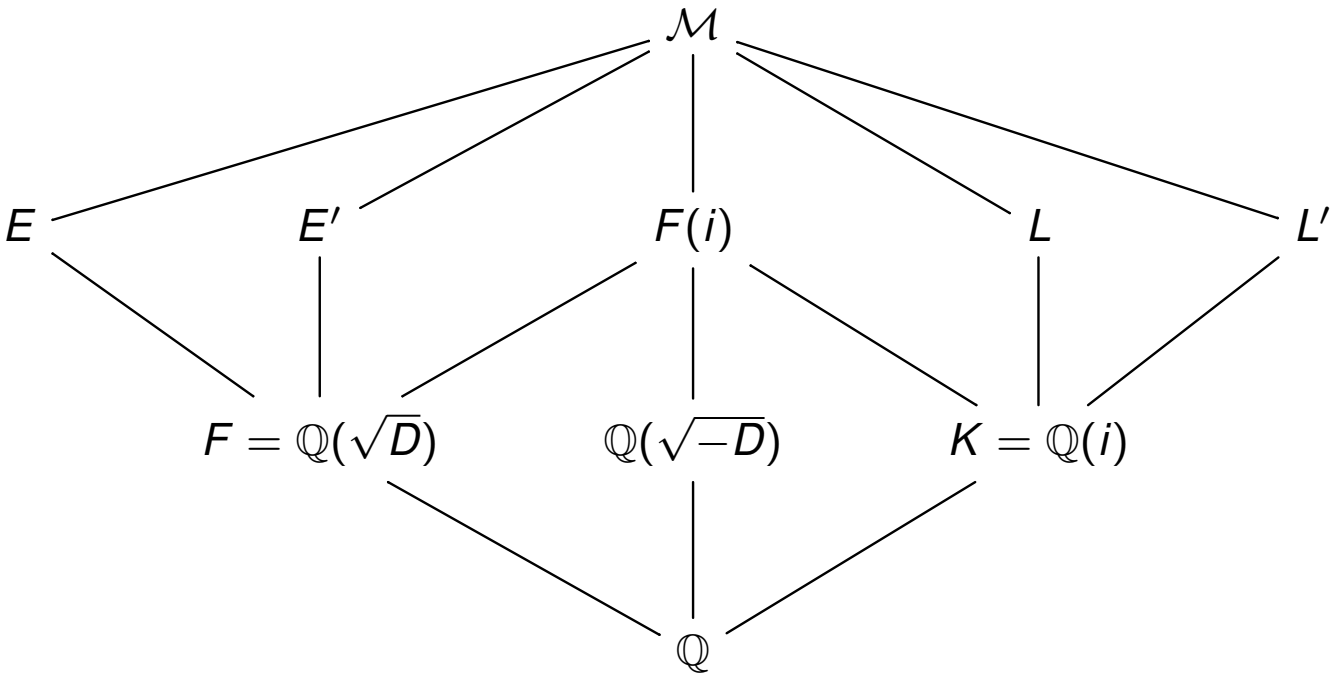

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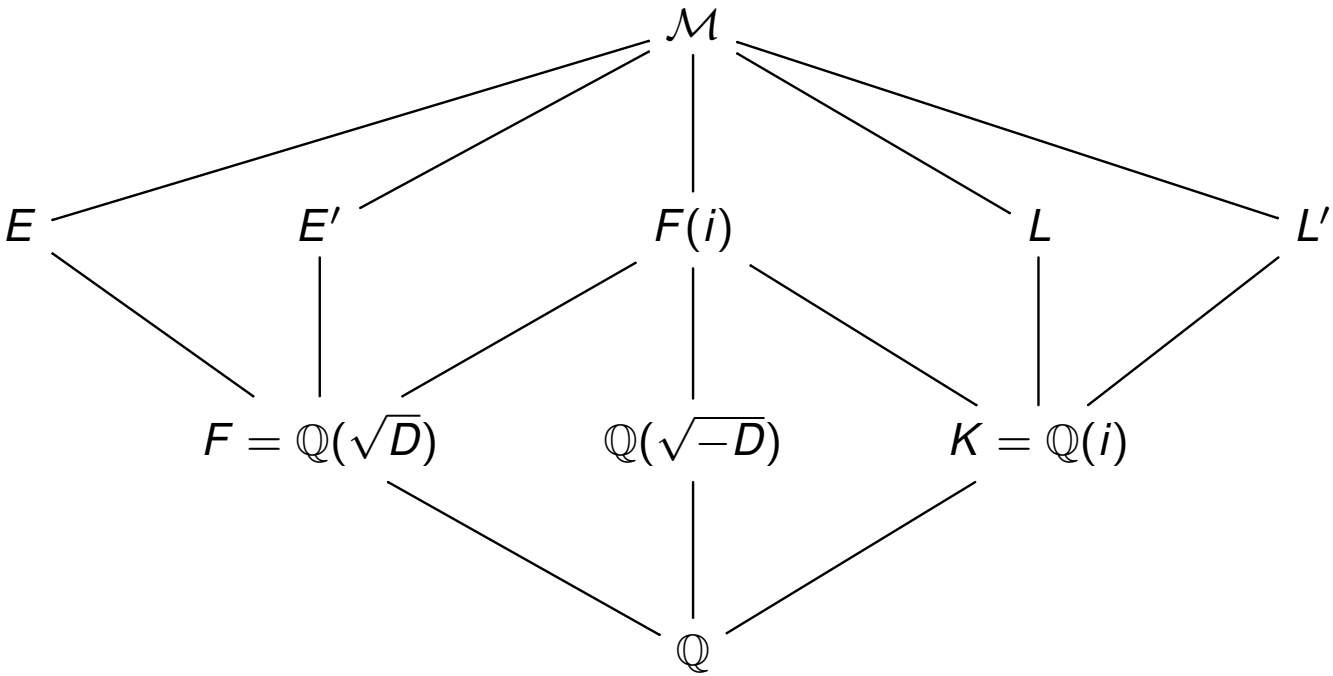

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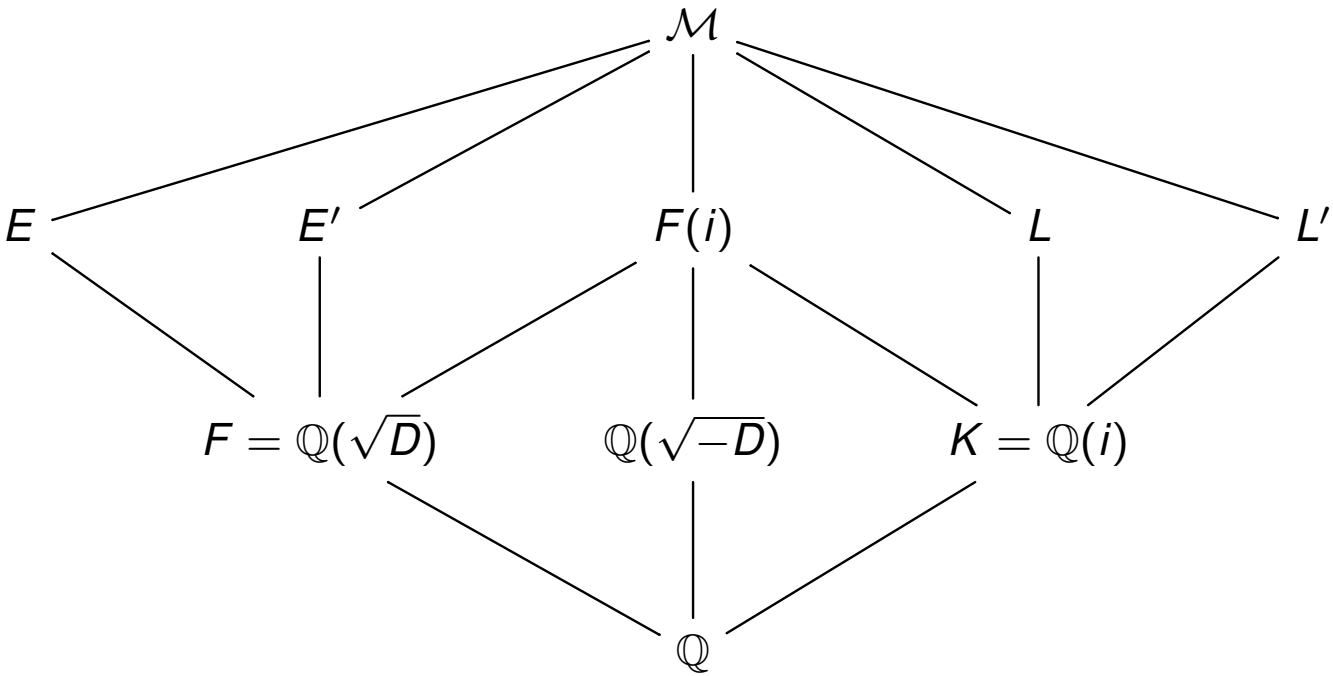

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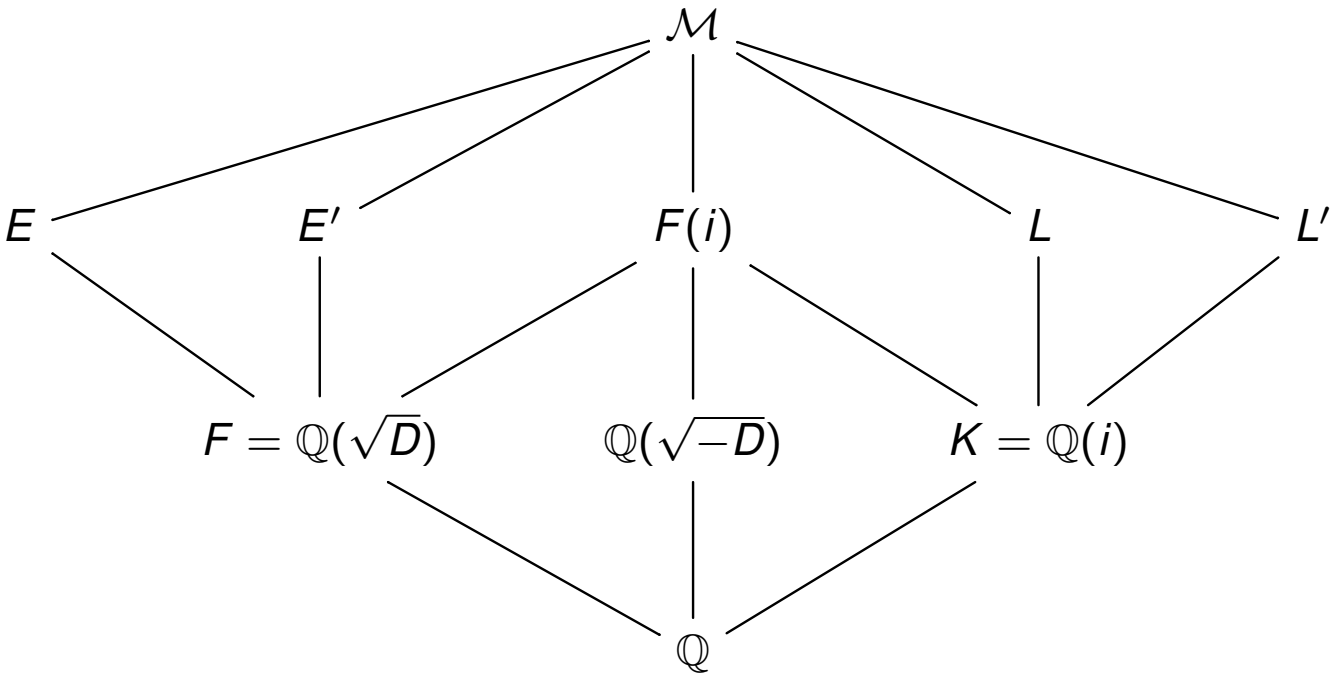

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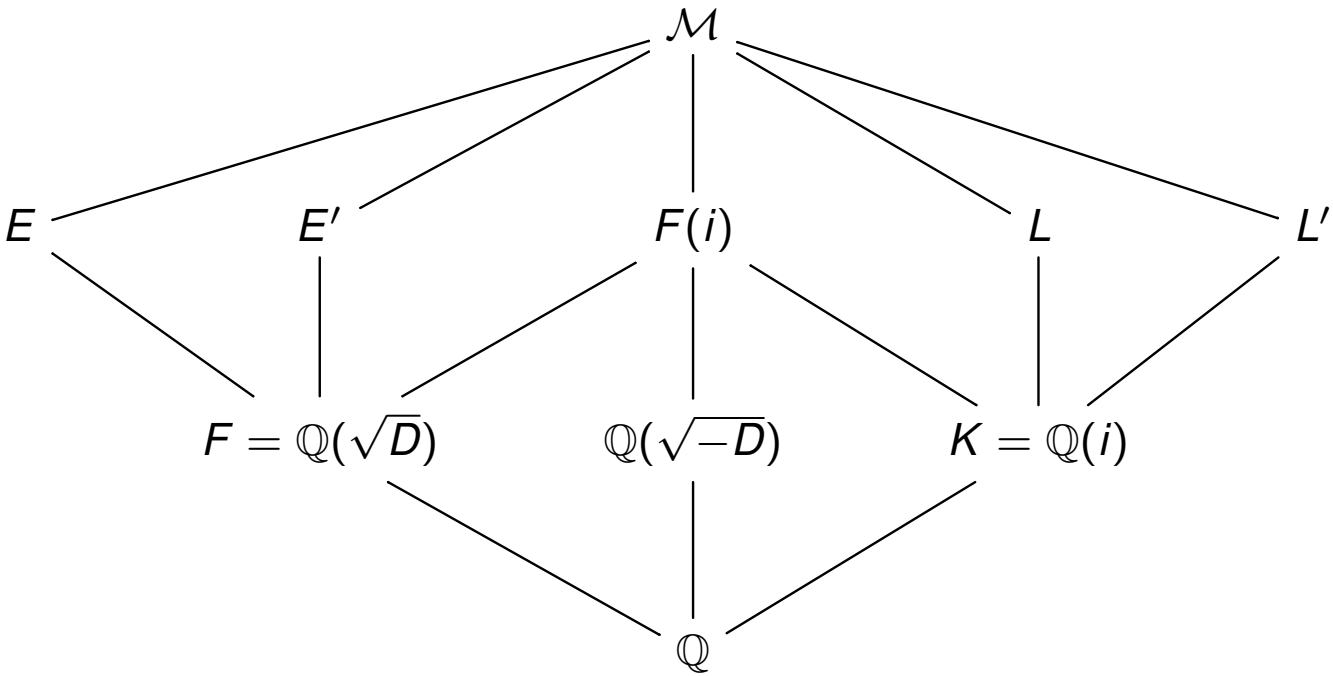

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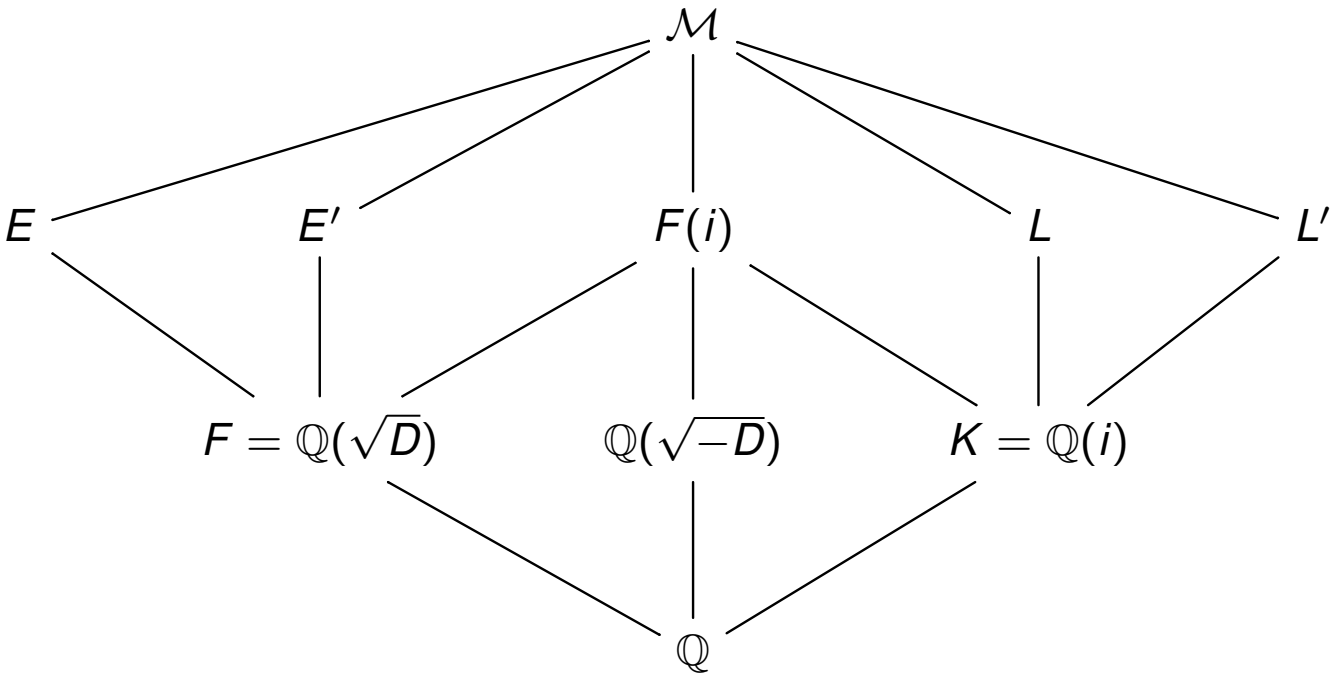

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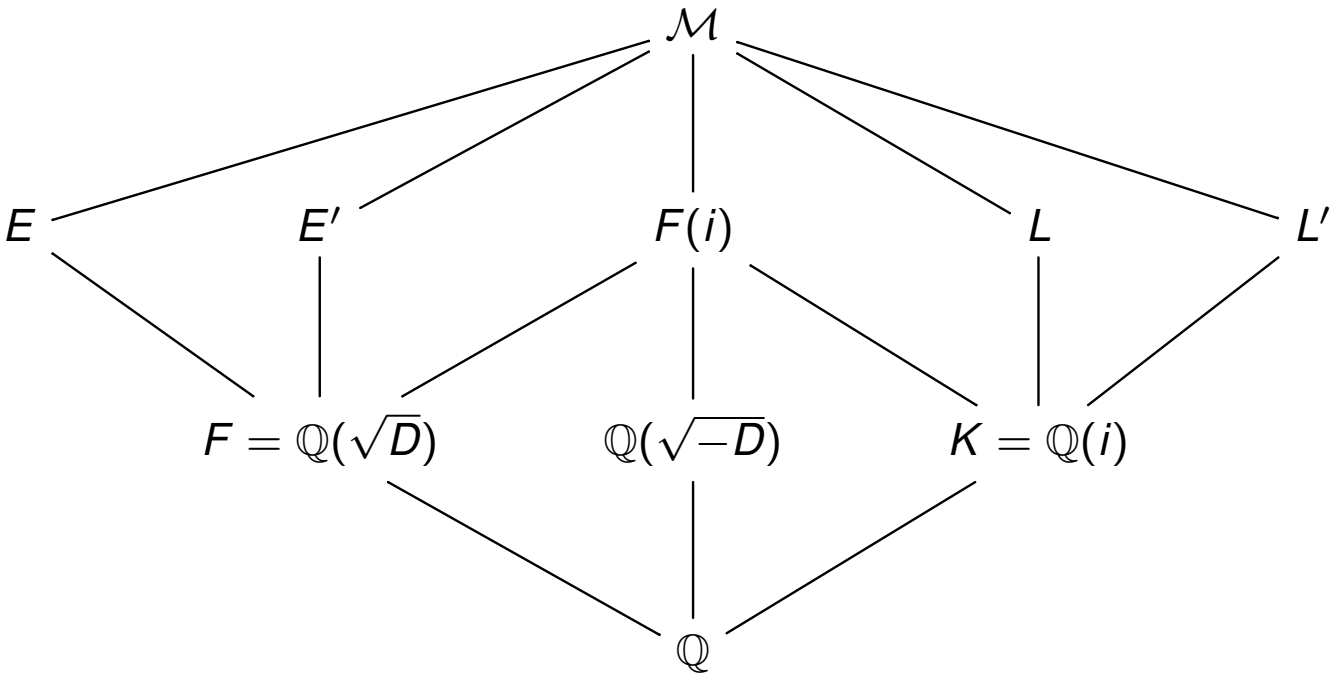

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  - Identified 190 algebraic nontrivial quantities

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- Analog of Bruinier–Yang on support of HMF at CM points

Evaluation of Bianchi Rigid Meromorphic Cocycles at Big ATR Points

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Aveiro 2026