Computation and some new instances of Darmon points

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Euler Systems Seminar, Essen

Outline

1. Computing algebraic points on elliptic curves
2. Heegner points
3. Darmon points (curves over $\mathbb{Q}$)
4. Explicit computations
5. Some generalizations: arbitrary base fields

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- Compute the rank r?
  - Related to the Birch and Swinnerton–Dyer Conjecture

The Birch and Swinnerton–Dyer Conjecture (BSD)

Hasse–Weil L-function

$$L(E/K,s) = \sum_{\mathfrak{n}\subseteq\mathcal{O}_K} a_{\mathfrak{n}} n^{-s}, \quad a_{\mathfrak{p}} = |\mathfrak{p}| + 1 - \#E(\mathcal{O}_K/\mathfrak{p})$$

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If $K =$ quadratic imaginary and $\operatorname{ord}_{s=1} L(E/K,s) = 1$, does there exist an efficient algorithm for computing a point of infinite order?

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Question

If $K=$ quadratic imaginary and $\mathrm{ord}_{s=1}L(E/K,s)=1$, does there exist an efficient algorithm for computing a point of infinite order?

- Answer: yes, the Heegner points method
  - Fundamental ingredient in the Gross–Zagier–Kolyvagin theorem

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600 million terms of the L-series. This takes less than a day. We list the x-coordinate of the point on the original elliptic curve. It has numerator

367770537186677506614005642341827170087932269492285584726218770061653546349271015805365134370326743061141306464500052886704651998399766478840791915307861741507
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79038041239611544624083258348136637213230084906083526213683231531105290336750385743792050893130528314337942393060136915457253067727886206663888425022179164712
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- Darmon Points (a.k.a. Stark–Heegner Points)


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K cuadratic imaginary

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  - Modular uniformization $X_0(N) \longrightarrow E$
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Outline

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Conjecture (rationality)

$J_\tau \in E(K^{\mathrm{ab}})$ and $\mathrm{Tr}(J_\tau)$ of infinite order if $\mathrm{ord}_{s=1} L(E/K,s) = 1$

Where does the evidence come from?

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- Next: explain the algorithm for $D > 1$
  - the homology class attached to $\tau \in K \cap \mathcal{H}_p$
  - the cohomology class attached to E
  - the integration pairing

  present some numerical evidence for the conjecture with $D > 1$

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- Computing this explicitly boils down to compute Γ_{ab}
  - Computing generators
  - Given $\gamma \in [\Gamma, \Gamma]$ write it explicitly as a product of commutators

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The homology class

- $\tau \in \mathcal{H}_p \cap K \rightsquigarrow \Delta_\tau \in H_1(\Gamma, \mathrm{Div}^0\mathcal{H}_p)$
- $B \otimes \mathbb{Q}_p \simeq \mathrm{M}_2(\mathbb{Q}_p)$ so $\Gamma \subset B$ acts on $\mathcal{H}_p$ via linear fractional transf.
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- How to compute effectively with rigid analytic differentials?
- How to compute $\int_{\tau_1}^{\tau_2} \omega$?

Rigid analytic differentials and measures

Concrete realization of *p*-adic differentials (Schneider)

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- The isomorphism is explicit (it is essentially Shapiro's Lemma). So we can recover μ_E from φ_E.

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Integrals and overconvergent cohomology (I)

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- This is what we can compute using overconvergent cohomology.

Integrals and overconvergent cohomology (II)

- Let $\mathcal{D}$ be the module of locally analytic distributions on $\mathbb{Z}_p$
  - If $h(t)$ locally analytic function on $\mathbb{Z}_p$ and $\psi \in \mathcal{D}$ then $\psi(h(t)) \in \mathbb{Z}_p$

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There exists unique $\Phi_E \in H^1(\Gamma_0(pM), \mathcal{D})$ lifting φ_E s.t. $U_p\Phi_E = a_p\Phi_E$.

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- Moreover, it can be explicitly computed:
  - Take $\tilde{\varphi} \in \mathrm{Maps}(\Gamma_0(pM), \mathcal{D})$ any lift
  - Iterate U_p: compute $\frac{1}{a_p^n} U_p^n(\tilde{\varphi})$
  - The limit when $n \to \infty$ converges to $\Phi_E \in H_1(\Gamma_0(pM), \mathcal{D})$

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- So we can compute the integrals by lifting φ_E and integrating U_p
- Each iteration of U_p increases the accuracy of the computation in one p-adic digit

Outline

$p = 13$, $D = 6$, prec $= 13^{60}$

$$E_{78}: \quad y^2 + xy = x^3 + x^2 - 19x + 685$$

d_K	P
5	$1 \cdot 48 \cdot \left(-2, 12\sqrt{5} + 1\right)$
149	$1 \cdot 48 \cdot \left(1558, -5040\sqrt{149} - 779\right)$
197	$1 \cdot 48 \cdot \left(\frac{310}{49}, \frac{720}{343}\sqrt{197} - \frac{155}{49}\right)$
293	$1 \cdot 48 \cdot \left(40, -15\sqrt{293} - 20\right)$
317	$1 \cdot 48 \cdot \left(382, -420\sqrt{317} - 191\right)$
437	$1 \cdot 48 \cdot \left(\frac{986}{23}, \frac{7200}{529}\sqrt{437} - \frac{493}{23}\right)$
461	$1 \cdot 48 \cdot \left(232, -165\sqrt{461} - 116\right)$
509	$1 \cdot 48 \cdot \left(-\frac{2}{289}, -\frac{5700}{4913}\sqrt{509} + \frac{1}{289}\right)$
557	$1 \cdot 48 \cdot \left(\frac{75622}{121}, \frac{882000}{1331}\sqrt{557} - \frac{37811}{121}\right)$

$p = 11$, $D = 10$, prec $= 11^{60}$

$$E_{110}: \quad y^2 + xy + y = x^3 + x^2 + 10x - 45.$$

d_K	P
13	$2 \cdot 30 \cdot \left(\frac{1103}{81} - \frac{250}{81}\sqrt{13}, -\frac{52403}{729} + \frac{13750}{729}\sqrt{13}\right)$
173	$2 \cdot 30 \cdot \left(\frac{1532132}{9025}, -\frac{1541157}{18050} - \frac{289481483}{1714750}\sqrt{173}\right)$
237	$2 \cdot 30 \cdot \left(\frac{19096654883784207386 7}{40166486596584126 49} - \frac{10722443619184119320}{40166486596584126 49}\sqrt{237}, -\frac{3505590193011437142853233857149}{804999791382984541 1423756107} + \frac{2354484601305645209913203722 00}{8049997913829845411423756107}\sqrt{237}\right)$
277	$2 \cdot 30 \left(\frac{46317716623881}{12553387541776}, -\frac{58871104165657}{25106775083552} - \frac{2091276933523905524 3}{44477606117965542976}\sqrt{277}\right)$
293	$2 \cdot 30 \cdot \left(\frac{7088486530742}{2971834657801}, -\frac{10060321188543}{5943669315602} - \frac{591566427769149607}{10246297476835603402}\sqrt{293}\right)$
373	$2 \cdot 30 \cdot \left(\frac{298780258398}{62087183929}, -\frac{360867442327}{124174367858} - \frac{19368919551426449}{30940899762281434}\sqrt{373}\right)$

$p = 19$, $D = 6$, $\text{prec} = 19^{60}$

$$E_{110}: \quad y^2 + xy = x^3 - 8x$$

d_K	P
29	$1 \cdot 72 \cdot \left(-\frac{6}{25}\sqrt{29} - \frac{38}{25}, -\frac{18}{125}\sqrt{29} + \frac{86}{125}\right)$
53	$1 \cdot 72 \cdot \left(-\frac{1}{9}, \frac{7}{54}\sqrt{53} + \frac{1}{18}\right)$
173	$1 \cdot 72 \cdot \left(-\frac{3481}{13689}, \frac{347333}{3203226}\sqrt{173} + \frac{3481}{27378}\right)$
269	$1 \cdot 72 \cdot \left(\frac{1647149414400}{23887470525361}\sqrt{269} - \frac{43248475603556}{23887470525361}, \frac{2359447648611379200}{116749558330761905641}\sqrt{269} + \frac{268177497417024307564}{116749558330761905641}\right)$
293	$1 \cdot 72 \cdot \left(\frac{21289143620808}{4902225525409}, \frac{4567039561444642548}{10854002829131490673}\sqrt{293} - \frac{10644571810404}{4902225525409}\right)$
317	$1 \cdot 72 \cdot \left(-\frac{25}{9}, -\frac{5}{54}\sqrt{317} + \frac{25}{18}\right)$
341	$1 \cdot 72 \cdot \left(\frac{3449809443179}{499880896975}, \frac{3600393040902501011}{3935597293546963250}\sqrt{341} - \frac{3449809443179}{999761793950}\right)$
413	$1 \cdot 72 \cdot \left(\frac{59}{7}, \frac{113}{98}\sqrt{413} - \frac{59}{14}\right)$

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There is a harmonic differential form $\omega_E \in H^2(\Gamma \backslash \mathcal{H} \times \mathbb{H}_3, \mathbb{C})$ with

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- In practice, one can compute ω_E!

- ω_E has a "Fourier-Bessel expansion":

$$\omega_E(z,x,y) = \sum_{\substack{\alpha\in\mathcal{O}_F\\ \alpha_0>0}} \frac{a_{(\alpha)}}{N_{F/\mathbb{Q}}(\alpha)} \frac{\alpha_0}{\delta_0} \exp\left(-2\pi i\left(\frac{\alpha_0\bar{z}}{\delta_0}+\frac{\alpha_1 x}{\delta_1}+\frac{\alpha_2\bar{x}}{\delta_2}\right)\right)\mathbb{K}\left(\frac{\alpha_1 y}{\delta_1}\right)\cdot\begin{pmatrix}\frac{-dx}{y}\wedge d\bar{z}\\ \frac{dy}{y}\wedge d\bar{z}\\ \frac{d\bar{x}}{y}\wedge d\bar{z}\end{pmatrix}$$

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- We can compute the $a_{(\alpha)}$ by counting points on $E(\mathcal{O}_F/\mathfrak{p})$

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- The image of $J_\tau \in \mathbb{C}/\Lambda_E \simeq E(\mathbb{C})$ coincides (up to 32 digits of accuracy) with $10P$, where

$$P = \left(r - 1 : w - r^2 + 2r : 1\right) \in E(K)$$

is a point of infinite order!

Computation and some new instances of Darmon points

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